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SOME REMARKS ON AFFINE HOMOGENEOUS SPACES

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1. Introduction. Throughout this article, we fix an algebraically closed field k of characteristic zero as the ground field and mean by algebraic groups linear algebraic groups. Let G be a connected algebraic group and let H be a closed subgroup of G. We are interested in characterizing those subgroups H for which the indiction functor ind_H^G is exact, though complete answers have not been yet obtained. Note that the exactness condition on ind_H^G is equivalent to the condition that the quotient G/H is affine (cf. [11; Th. (4.3)]). Since G/H is the quotient variety of a quasi-projective variety G/H^0 under the natural free action of a finite group H/H^0, G/H is affine if and only if so is G/H^0, where H^0 is the identity component of H. Therefore, we may and shall assume that H is connected. Concerning this problem, the following result is well-known.

Theorem 1.1. (cf. [1], [7]) *Suppose G is reductive. Then the following conditions are equivalent:*

(1) *G/H is affine;*

(2) *H is reductive;*

(3) *For an arbitrary finite-dimensional rational H-module M, there exists a finite-dimensional rational G-module N such that M is an H-submodules of N and $M^H=N^G$, where M^H (resp. N^G) is the set of elements of M (resp. N) invariant under the H-action (resp. the G-action).*

Even when the characteristic of k is positive, the conditions (1) and (2) are sitll equivalent (cf. [5]). If we drop the equality $M^H=N^G$ from the condition (3), it is equivalent to the condition that G/H is quasi-affine (cf. [2]).

Since the reductive case is known by the above theorem, we shall consider the general case where G is non-reductive. Since the characteristic of k is zero, G has a "Levi decomposition". Namely, G is a semi-direct product of a reductive subrgoup L called a Levi factor of G and the unipotent radical $U:=R_u(G)$ of G, where L is uniquely determined up to conjugations by elements of U (cf. [3]).

Set $V:=R_u(H)$. Then Theorem 1.1 can be rephrased as follows.

Theorem 1.2. *If G is reductive, G/H is affine if and only if the following* *equivalent conditions are satisfied:*

(1′) *G/V is affine;*

(2′) *V is trivial;*

(3′) *For an arbitrary finite-dimensional rational V-module M, there exists a finite-dimensional rational G-module N such that M is a V-submidule of N and $M^V=N^G$.*

Our purpose of this paper is to consider the relationships among the following conditions which generalize the conditions in Theorem (1.2):

(0) $V\subseteqq U$;

(I) G/V is affine;

(II) For an arbitrary finite-dimensional rational V-module M, there exists a finite-dimensional rational G-module N such that M is a V-submodule of N and $M^V=N^G$;

(III) For any element u of U, the equality

$$uVu^{-1}\cap L=\{e\}$$

holds.

Note that if G is reductive, each of the conditions (0) and (III) coincides with the condition (2′). Furthermore, the condition (III) implies that V intersects trivially any Levi factor of G. We also note that, by replacing H by its unipotent radical, we may assume that H is unipotent. Indeed, if $G/R_u(H)$ is affine, then G/H is the quotient of an affine variety $G/R_u(H)$ with respect to a reductive algebraic group $H/R_u(H)$, hence G/H is affine (cf. [9; Th. (1.1)]). In the section 2, we prove the implications (0)$\Rightarrow$(I)$\Rightarrow$(II)$\Rightarrow$(III) and observe that (I)$\Rightarrow$(0) is false. In the section 3, we shall prove the equivalence of the conditions (I), (II) and (III) when G is a direct product of a Levi factor and the unipotent radical U. In the section 4, we consider the case where U is commutative. In this case, we may assume that a Levi factor is isomorphic to $GL(U)$ with U regarded as a vector group. When either $\dim V=1$ or V satisfies additional hypotheses, we can prove the equivalence of the conditions (I), (II) and (III). In the section 5, we prove two theorems, one of which concerns the vanishing of the Hochscild cohomology groups and the other does the non-vanishing of them. In the final section, we consider the case where, with the above notations, H is a commutative unipotent group. The exactness of H, i.e., that G/H is affine, can be interpreted in terms of derivations on the coordinate ring $K[G]$ of G associated to the natural H-action on G.

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2. General case. We work in the previous situation posed in the section 1. We sasume that H is connected. As a preliminary result, we shall prove the following :

Lemma 2.1. *With the above notations, the following two conditions are equivalent:*

(1) *For an arbitrary finite-dimensional rational H-module M, there exists a finite-dimensional rational G-module N such that M is an H-submodule of N and $M^H = N^G$.*

(2) *For an arbitrary finite-dimensional rational H-module M, there exists a finite-dimensional rational G-module N such that M is an H-submodule of N and $M^H \subseteqq N^G$.*

Proof. The implication (1)⇒(2) is obviuos. We prove the other direction. Let M be an arbitrary finite-dimensional rational H-module. By virtue of the condition (2), we can take a finite-dimensional rational G-module N satisfying the condition of (2). By induction on $\dim N - \dim M$, we shall show that there exists a finite-dimensional rational G-module N' such that M is an H-submodule of N' and $N'^G = M^H$. If $\dim N - \dim M = 0$, i.e., $M = N$, then $M^H = N^G$ obviously. So, we can take N as N'. If $\dim N - \dim M > 0$, we write $N^G = M^H \oplus N_1$, where the symbol $\oplus$ stands for direct sum. If $N_1 = 0$, we can take N as N'. If $N_1 \neq 0$, since $N_1 \cap M = 0$ and N_1 is a G-submodule of N, we have $M \subseteqq \tilde{N} := N/N_1$ and $M^H \subseteqq \tilde{N}^G$. Moreover, we have $\dim \tilde{N} - \dim M < \dim N - \dim M$. So we can find a finite-dimensional rational G-module N' by the induction hypothesis. Q.E.D.

Next, we shall consider the relationships among the conditions (0)~(III) given in the section 1. We have:

Theorem 2.2. *The following implications hold.*

$$(0) \Rightarrow (\mathrm{I}) \Rightarrow (\mathrm{II}) \Rightarrow (\mathrm{III})$$

Proof. (0)⇒(I) Since $G/U \simeq L$ is affine and U/V is also affine (cf. [4; Prop. 2]), G/V is affine by virtue of [1; Cor. 1].

(I)⇒(II) Since V is unipotent and G/V is affine, G splits as a variety $G \simeq (G/V) \times V$ (cf. [4; Prop. 1], [10; Th. 10]). Hence we can regard $k[V]$ as a subring of $k[G]$ via the comorphism of the second projection. Let M be an arbitrary finite-dimensional rational V-module and let $\{v_1, \cdots, v_n\}$ be a k-basis of M. We can take a_{ij}'s in $k[V]$ so that the following equality holds for any $h \in V$,

$$h \cdot v_j = \sum_{i=1}^{n} a_{ij}(h) v_i .$$

Put

$$u_j := (a_{1j}, \cdots, a_{nj}) \in k[V]^n$$

and

$$M' := \mathrm{Span}_k (u_1, \cdots, u_n) \quad \text{in} \quad k[V]^n$$

Condider a rational V-action on $k[V]$ defined by

$$h \cdot a(x) = a(x \cdot h)\,,$$

where $a \in k[V]$ and $h, x \in V$.

We claim that M' is a V-submodule of $k[V]^n$ and $M' \simeq M$n as V-modules. In fact, define a map $\varphi: M' \to M$ by $\varphi(\sum_{i=1}^n b_i u_i) = \sum_{i=1}^n b_i v_i$. Then φ is clearly an isomorphism of k-vector spaces. From the choice of a_{ij}'s, we have

$$a_{ij}(xh) = \textstyle\sum_{k=1}^n a_{ik}(x) a_{kj}(h)\,.$$

Hence, we obtain

$$\begin{aligned} h \cdot u_j &= (h \cdot a_{1j}, \cdots, h \cdot a_{nj}) \\ &= \textstyle\sum_{k=1}^n a_{kj}(h) u_k\,. \end{aligned}$$

This implies that φ is V-equivariant.

Let N be a finite-dimensional rational G-module which is generated by M' in $k[G]^n$ (cf. [6; Prop. (8.6)]). Clearly, N contains M' as a V-submodule and $M'^V = k^n \cap M' \subseteqq N^G$. Hence the implication (I)$\Rightarrow$(II) follows from Lemma 2.1.

(II)$\Rightarrow$(III) Take an arbitrary $u \in U$ and set $L' := uLu^{-1}$ and $K := L' \cap V$. Since V/K is affine (cf. [4; Prop. 2]), given any finite-dimensional rational K-module M, the implication (I)$\Rightarrow$(II) implies that there exists a finite-dimensional rational V-module N' such that M is a K-submidule of N' and $M^K = N'^V$. Furthermore, by the condition (II), we can take a finite-dimensional rational G-module N such that N' is a V-submodule of N and $N'^V = N^G$. The G-module N is viewed as a finite-diminsional rational L'-module which contains M as a K-submodule. Moreover, the inclusion $M^K = N'^V = N^G \subseteqq N^{L'}$ holds. Hence $K = \{e\}$ by virtue of Lemma 2.1 and Theorem 1.1. Q.E.D.

As for the converses of the implications in Theorem 2.2, we can only find a counter-example to the implication (I)$\Rightarrow$(0), which is given in the Proposition 2.3 below. We consider a direct product $G := SL(2, k) \times G_a$ and a closed subgroup

$$H := \left\{\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, x\right) \in G \,|\, x \in k\right\}.$$

where G_a is the additive group. We then have:

Proposition 2.3. *With the above notations ,we have:*
(1) *$G/H \simeq SL(2,k)$, whence G/H is affine;*
(2) *$R_u(H)$ is not contained in $R_u(G)$.*

Proof. (1) Let $\pi: G \to G/H$ be the canonical quotient morphism. It is enough to show that $\pi' := \pi|_{SL(2,k)\times\{0\}}$ is bijective, for π' is then birational since the characteristic of k is zero and π' is biregular by Zariski's Main Theorem.

But this is clear since $SL(2, k)\times\{0\}$ is a cross-section of the morphism π.

(2) Since $R_u(H)=H$ and $R_u(G)=\{e\}\times G_a$, the assertion is clear. Q.E.D.

In the subsequent sections, we shall consider to what extent the converses of the implications in Theorem 2.2 hold if we put additional restrictions.

3. Case $G\simeq L\times U$. In this section, we shall show that the conditions (I)$\sim$(III) are equivalent if G is a direct product as algebraic groups of its Levi factor L and its unipotent radical U.

Theorem 3.1. *If $G\simeq L\times U$, the conditions (I)$\sim$(III) are equivalent.*

Proof. It is enough to show the implication (III)$\Rightarrow$(I) in view of Theorem 2.2. Let $f:=p_2|_V\colon V\to U$, where $p_2\colon L\times U\to U$ denotes the second projection, and let $K:=\operatorname{Im} f$. By the condition (III), f is injective. So, we can define a homomorphism $g\colon K\to L$ by $g:=p_1|_V\cdot f^{-1}$, where $p_1{:}L\times U\to L$ denotes the first projection. Since the natural morphism $U\to U/K$ is a trivial principal K-bundle (cf. [4; Prop. 1], [10; Th. 10]), we have $U\simeq(U/K)\times K$ by choosing a cross-section $\sigma\colon U/K\to U$ to the natural morphism $U\to U/K$. We assume $\sigma[e]=e$, where $[e]$ is the class of the identity e in U/K. Our assertion follows from the next claim:

Claim. $$G/V\simeq L\times(U/K).$$

Proof. Let $\pi\colon G\to G/V$ be the natural quotient morphism. Noting that $G\simeq L\times(U/K)\times K$, we have only to show that the restriction π' of π onto $L\times(U/K)\times\{e\}$ is an isomorphism. For this purpose, as in Proposition 2.3, it is enough to show that π' is a bijection.

Injectivity of π': This can be easily shown if one notes that V is expressed as

$$V=\{(g(x), [e], x)\in L\times(U/K)\times K \mid x\in K\}.$$

Surjectivity of π': Choosing $l\in L$, $u\in U$ and $x\in K$ arbitrarily, we have $\pi(l, [u], x)=\pi((l, [u], x)\cdot(g(x)^{-1}, [e], x^{-1}))=\pi(lg(x)^{-1}, [u], e)\in\operatorname{Im}\pi'$. Hence π' is surjective. Q.E.D.

REMARK. The implication (I)$\Rightarrow$(0) does not hold even if G is a direct product of L and U. In fact, in the counter-example in the section 1, the given Levi decomposition of G is a direct product.

4. Case U is commutative. In this section, we consider a special case where the unipotent radical U of G is commutative, i.e., U is a vector space. We often identify elements of U with column vectors of length $\dim U$. First of all, we reduce the problem to the following special case. Let $G_U:=$a semi-direct product of $GL(U)$ and U, whose multiplication is defined by $(A,u)\cdot(B,v):=(AB, B^{-1}u+v)$, where $A, B\in GL(U)$ and $u, v\in U$. Write $G=L\cdot U$ as a semi-direct product of a Levi factor L and U and define a homomorphism of algebraic groups $\varphi\colon G\to G_U$ by

$$\varphi(l, u) := ((\operatorname{Int} l)|_U, u) \quad (l\in I, u\in U),$$

where

$$(\operatorname{Int} l)(x) = lxl^{-1}.$$

We consider new conditions $(\tilde{\mathrm{I}})\sim(\widetilde{\mathrm{III}})$ which are stated as:

$(\tilde{\mathrm{I}})$ $G_U/\varphi(V)$ is affine;

$(\widetilde{\mathrm{II}})$ For an arbitrary finite-dimensional rational $\varphi(V)$-module M, there exists a finite-dimensional rational G_U-module N such that M is a $\varphi(V)$-submodule of N and $M^{\varphi(V)}=N^{G_U}$;

$(\mathrm{I\tilde{I}I})$ For any element u of U, we have $u\varphi(V)u^{-1}\cap GL(U)=\{e\}$.

The conditions (I)$\sim$(III) and $(\tilde{\mathrm{I}})\sim(\mathrm{I\tilde{I}I})$ are related to each other as follows.

Theorem 4.1. (1) *Among the conditions* (I), $(\tilde{\mathrm{I}})$, $\cdots$, (III) *and* $(\mathrm{I\tilde{I}I})$, *we have the following implications*:

$$(\mathrm{I}) \Leftrightarrow (\tilde{\mathrm{I}}) \Rightarrow (\widetilde{\mathrm{II}}) \quad \textit{and} \quad (\mathrm{II}) \Rightarrow (\mathrm{III}) \Rightarrow (\mathrm{I\tilde{I}I}).$$

(2) *If $V\cap L=\{e\}$, we have further implications* $(\mathrm{I\tilde{I}})\Rightarrow(\mathrm{II})$ *and* $(\mathrm{I\tilde{I}I})\Rightarrow(\mathrm{III})$.

Proof. (1) (I)$\Rightarrow(\tilde{\mathrm{I}})$. Consider the isomorphisms $G_U/\varphi(G)\simeq(G_U/U)/(\varphi(G)/U)\simeq GL(U)/\varphi(L)$. Since L is reductive, $\varphi(L)$ is reductive, too. Hence $GL(U)/\varphi(L)$ is affine by virtue of [9; Th. 1.1]. Hence $G_U/\varphi(G)$ is affine. On the other hand, consider the isomorphisms $\varphi(G)/\varphi(V)\simeq G/(V\cdot\operatorname{Ker}\varphi)\simeq \operatorname{Ker}\varphi\backslash(G/V)$. Here, note that G/V is affine by the condition (I) and that $\operatorname{Ker}\varphi$ is a reductive algebraic group as it is a normal subgroup of the reductive algebraic group L. Hence by [9; Th. (1.1)], $\operatorname{Ker}\varphi\backslash(G/V)$ is affine, hence so is $\varphi(G)/\varphi(V)$. Therefore $G_U/\varphi(V)$ is affine (cf. [1; Cor. 1]). $(\tilde{\mathrm{I}})\Rightarrow$(I). It is clear that $\varphi(G)/\varphi(V)$ is a closed subset of $G_U/\varphi(V)$. So $\varphi(G)/\varphi(V)$ is affine because $G_U/\varphi(V)$ is affine by the condition $(\tilde{\mathrm{I}})$. Hence $G/(V\cdot\operatorname{Ker}\varphi)\simeq\varphi(G)/\varphi(V)$ is affine. On the other hand, $V\cdot\cdot\operatorname{Ker}\varphi/V\simeq\operatorname{Ker}\varphi$ is affine. Therefore G/V is affine (cf. [1; Cor. 1]). $(\bar{\mathrm{I}})\Rightarrow(\mathrm{I\tilde{I}})$. This follows from Theorem 2.2. (II)$\Rightarrow$(III). We have already proved this in Theorem 2.2. (III)$\Rightarrow(\mathrm{I\tilde{I}I})$. By the definition of φ, we have $\varphi^{-1}(GL(U))=L$ and $\varphi(uVu^{-1})=u\varphi(V)u^{-1}$ for any $u\in U$. Hence, for any $u\in U$, we have

$$\begin{aligned} GL(U)\cap u\varphi(V)u^{-1} &= GL(U)\cap\varphi(uVu^{-1}) \\ &= \varphi(\varphi^{-1}(GL(U))\cap uVu^{-1}) \\ &= \varphi(L\cap uVu^{-1}) = \{e\}. \end{aligned}$$

(2) $(\tilde{\mathrm{II}})\Rightarrow(\mathrm{II})$. Since $\operatorname{Ker}\varphi\subset L$, the condition $V\cap L=\{e\}$ implies that $\varphi|_V\colon V\to\varphi(V)$ is an isomorphism. Hence the sets {finite-dimensional rational V-module} and {finite-dimensional rational $\varphi(V)$-module} are obviously in one-to-one correspondence. Let M be a finite-dimensional rational V-module. Then M is thought of as a finite-dimensional rational $\varphi(V)$-module. Hence we can take a finite-dimensional rational G_U-module N which satisfies the requirements set in the condition $(\tilde{\mathrm{II}})$. We can regard this N as a rational G-module through φ. Then M is clearly a V-submodule of N with respect to this G-module structure on N and $M^V=M^{\varphi(V)}=N^{G_U}\subseteqq N^G$. Hence our assertion follows from Lemma 2.1. $(\widetilde{\mathrm{III}})\Rightarrow(\mathrm{III})$.
Consider the equalities:

$$\begin{aligned}\{e\} &= GL(U)\cap u\varphi(V)u^{-1}\\ &= GL(U)\cap\varphi(uVu^{-1})\\ &= \varphi(\varphi^{-1}(GL(U))\cap uVu^{-1})\\ &= \varphi(L\cap uVu^{-1})\,.\end{aligned}$$

Hence we have an inclusion $L\cap uVu^{-1}\subseteqq\operatorname{Ker}\varphi$. Now we can finish as $L\cap uVu^{-1}\subseteqq u(\operatorname{Ker}\varphi\cap V)u^{-1}=\{e\}$. Q.E.D.

Making use of Theorem 4.1, we can prove the equivalence of the conditions (I)$\sim$(III) under some special situations.

Theorem 4.2. *If* $\dim V=1$, *the conditions* (I)$\sim$(III) *are equivalent.*

Theorem 4.3. *If V satisfies the following two conditions, three conditions* (I)$\sim$(III) *are equivalent*:

(i) $\varphi(V)$ *is commutative*;

(ii) *There exists an element of* $\operatorname{Int}\varphi(V)$ *which has the minimal polynomial of degree equal to* $\dim U$.

By virtue of Theorem 2.2 and Theorem 4.1, it is enough to show the implication $(\widetilde{\mathrm{III}})\Rightarrow(\tilde{\mathrm{I}})$, assuming that the condition (III) holds.

Proof of Theorem 4.2. Put $V':=\varphi(V)$, where $\varphi(V)\simeq V$ because $V\cap L=\{e\}$, and take an element $v'=(A,u)\in V'\setminus\{e\}$, where $A\in GL(U)$ and $u\in U$. Put $U':=\operatorname{Im}(A^{-1}-E)$, where elements of $GL(U)$ acts on U from the left, U being identified with the space of column vectors of length $\dim U$.

Claim. The element u is not contained in U'.

Proof. Assuming the contrary, write $u=A^{-1}u'-u'$ with $u'\in U$. Then we have

$$\begin{aligned}u'^{-1}(A,u)u' &= (E,-u')\cdot(A,u)\cdot(E,u')\\ &= (A,0)\in GL(U)\setminus\{E\}\,.\end{aligned}$$

Thus we obtain a non-trivial element of $u'^{-1}V'u' \cap GL(U)$. This is a contradiction to the condition ($\widetilde{\text{III}}$).

Take a basis $\{e_1, \cdots, e_n\}$ of U' and extend this basis to a basis $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+m}\}$ of U. If we write $u = \sum_{i=1}^{m+n} c_i e_i$, the above claim implies that $(c_{n+1}, \cdots, c_{n+m}) \neq (0, \cdots, 0)$. Suppose $c_{n+i} \neq 0$. Put $U'' := \mathrm{Span}_k(e_1, \cdots, e_n, e_{n+1}, \cdots, \hat{e}_{n+i}, \cdots, e_{n+m})$, where e_{n+i} is deleted. It is easy to see that U'' satisfies the following three conditions:

(1) U'' is V'-stable, where V' acts on U by conjugation;

Note that V' is a unipotent group of dimension 1. Hence the condition (1) is ascertained by the following computation for the nonzero element v' of V' and $e_i \in U''$:

$$\begin{aligned} v'e_iv'^{-1} &= (A, u)\cdot(E, e_i)\cdot(A, u)^{-1} \\ &= (E, (A^{-1}-E)(-Ae_i)+e_i) \\ &\in \mathrm{Im}(A^{-1}-E)+U'' = U''. \end{aligned}$$

(2) $0 \in U''$;

(3) The multiplication morphism in G_U induces an isomorphism $U'' \times p_2(V') \xrightarrow{\sim} U$, where $p_2\colon GL(U)\times U \to U$ is the second projection, which is not necessarily a group homomorphism.

Hence our assertion follows from the next lemma.

Lemma 4.4. *The notations and assumptions being as above, we assume the following conditions*:

(i) $V \cap L = \{e\}$;

(ii) *There exists an irreducible closed subvariety W of U such that*

(a) *W is $p_1(V)$-stable, where $p_1(V)$ acts on U by conjugation*;

(b) $e \in W$;

(c) *the multiplication morphism of G induces an isomorphism $W \times p_2(V) \xrightarrow{\sim} U$. Then G/V is affine. (Note here that we don't have to assume that U is commutative.)*

Proof. Let $\pi\colon G \to G/V$ be the canonical quotient morphism. Noting that $G \simeq L \times W \times p_2(V)$ as varieties, let π' be the restriction of π onto $L \times W \times \{e\}$. It is enough to show that $\pi'\colon L \times W \to G/V$ is bijective. Put $V_2 := p_2(V)$ and $\tau := p_1 \cdot (p_2|_V)^{-1}\colon V_2 \to L$. We identify G with $L \times W \times V_2$ as a variety.

Injectivity of π'. Assume $\pi(x, y, e) = \pi(x', y', e)$ with $x, x' \in L$ and $y, y' \in W$. Then $(x', y', e) = (x, y, e)\cdot(\tau(v), e, v)$ for $v \in V_2$. The right hand side can be written as

$$(x\tau(v),\ \tau(v)^{-1}y\tau(v),\ v)\,.$$

By comparison of both sides, we obtain $v = e$ and $\tau(v) = e$ by the definition of τ. Hence $(x', y', e) = (x, y, e)$. Thus π' is injective.

Surjectivity of π'. It is enough to show $\pi(x, y, z) \in \operatorname{Im}\pi'$ for arbitrarily chosen $x \in L$, $y \in W$ and $z \in V_2$. Since $V = \{(\tau(z), e, z) \mid z \in V_2\}$, we can take an element $z' \in V_2$ such that $(\tau(z), e, z)^{-1} = (\tau(z'), e, z')$, where $(\tau(z), e, z)^{-1}$ lies in V because V is a group. From the equation $(e, e, e) = (\tau(z), e, z) \cdot (\tau(z'), e, z') = (\tau(z)\tau(z'), e, \tau(z')^{-1} z \tau(z') z')$, we have $\tau(z')^{-1} z \tau(z') z' = e$. Hence,

$$\begin{aligned}(x, y, z) \cdot (\tau(z'), e, z') &= (x\tau(z'), \tau(z')^{-1} y \tau(z'), \tau(z')^{-1} z \tau(z') z') \\ &= (x\tau(z'), \tau(z')^{-1} y \tau(z'), e).\end{aligned}$$

Therefore we obtain $\pi(x, y, z) = \pi(x\tau(z'), \tau(z')^{-1} y \tau(z'), e) \in \pi(L \times W) = \operatorname{Im}\pi'$.

Q.E.D.

Corollary 4.5. *In the same situation as in Lemma* 4.4, *where we don't have to assume that* U *is commutative,* G/V *is affine provided* $p_2(V) = U$ *and* $V \cap L = \{e\}$.

Proof. We can take $\{e\}$ as W in Lemma 4.4. Q.E.D.

Proof of Theorem 4.3. Put $V' := \varphi(V)$. For any element x of V', there is a unique homomorphism $\mathcal{E}\colon G_a \to V'$ of algebraic groups such that $\mathcal{E}(1) = x$ (cf. [6; p. 96]). We use the notation x^t with $t \in k$ to denote $\mathcal{E}(t)$. By the condition (ii), we can choose an element v' of V' such that the minimal polymomial of $\operatorname{Int} v'$ acting on U has degree equal to $\dim U$. Let V_1 be the one-dimensional closed subgroup $\{(v')^t \mid t \in k\}$, which is generated by v'. Since V' is commutative, there exist one-dimensional connected closed subgroups $V_2 \cdots, V_r$, such that

$$V' \simeq V_1 \times V_2 \times \cdots \times V_r, \quad \text{where} \quad r \leq n := \dim U.$$

Let v_i be a generator of V_i for each $1 \leq i \leq r$. We choose v' as v_1. Write $v_i = (v_i', v_i'')$ with $v_i' \in GL(U)$ and $v_i'' \in U$. The conditions (i) and (ii) enable us to write $v_1'^{-1}, \cdots, v_r'^{-1}$ as $E+N, E+N_2, \cdots, E+N_r \in U(n, k)$ after a suitable change of bases of U, where N is a nilpotent matrix of the following form

$$N = \begin{bmatrix} 0 & 1 & 0 \\ & \ddots & \ddots\ 1 \\ 0 & & 0 \end{bmatrix}.$$

Furthermore, since V' is commutative, we conclude by a straightforward computation that N_i has the following form

$$N_i := a_1^{(i)} N + \cdots + a_{n-1}^{(i)} N^{n-1} \quad \text{with} \quad a_j^{(i)} \in k.$$

Write $v_i'' = {}^t(u_1^{(i)}, \cdots, u_n^{(i)})$ as a column vector and let

$$k_i := \max \{k \mid u_k^{(i)} \neq 0\} \quad \text{and} \quad j_i := \min \{j \mid a_1^{(i)} \neq 0\}.$$

If $k_i = k_j \ (i < j)$, replacing v_j by a suitable element of the form $v_i^p v_j^q$ with $p, q \in k$

and V_j by the closure of $\langle v_i^p v_j^q \rangle$, the group generated by $v_i^p v_j^q$, we may assume that $k_1, \cdots, k_r$ are different from each other.

Claim. v_i'' does not belong to $\operatorname{Im} N_i$ for $1 \leq i \leq r$.

Proof. Assuming the contrary, take $u \in U$ so that $v_i''=N_i u$. It is easy to see that $e \neq u^{-1} v_i u \in u^{-1} V_i u \cap GL(U) \subseteq u^{-1} V' u \cap GL(U)$. This contradicts the condition $(\mathrm{I}\tilde{\mathrm{I}}\mathrm{I})$.

By the above claim, we obtain $k_i + j_i \geq n+1$ and $k_1 = n$, in particular. After a suitable permutation of the indices $\{2, \cdots, r\}$, if necessary, we may assume $n = k_1 > k_2 > \cdots > k_r$.

Put

$$W := \{{}^t(u_1, \cdots, u_n) \in U \mid u_{k_i} = 0 \quad (1 \leq i \leq r)\}$$

and write the second factor of v_i^s as a column vector ${}^t(u_1^{(i)}(s), \cdots, u_n^{(i)}(s))$. Then it is easy to check that $u_k^{(i)}(s)$ is a polynomial in s of degree 1, for $k = k_i$. Hence, for any $u \in U$, there is a unique $(s_1, \cdots, s_r) \in k^n$ such that

$$u \cdot v_1^{s_1} \cdots v_r^{s_r} \in W .$$

This implies that $\pi|_{GL(U) \times W} : GL(U) \times W \to G_U / V'$ is bijective, where $\pi : G_U \to G_U / V'$ is the canonical quotient morhism. Therefore G_U / V' is affine.

5. Some results on Hochschild cohomology groups.

In this section, we consider the Hochschild cohomology groups of an algebraic group G with coefficients in a rational G-module. We shall prove two theorems, one of which concerns the vanishing of the Hochschild cohomology groups, and the other does the non-vanishing of them.

Theorem 5.1. *Let G be an algebraic group and let V be a rational G-module. Then we have :*

(1) *If V is finite-dimensional, so is $H^i(G, V)$ for every i.*

(2) *$H^i(G, V) = 0$ for every $i > \dim R_u(G)$.*

Proof. Since $H^i(G, \operatorname{ind\,lim} V_\lambda) = \operatorname{ind\,lim} H^i(G, V_\lambda)$, we may assume V is finite-dimensional. Consider the following exact sequence of algebraic groups.

$$1 \to R_u(G) \to G \to G' := G/R_u(G) \to 1 ,$$

where $R_u(G)$ is the unipotent radical of G. This gives a spectral sequence (cf. [11; Th. (2.9)]):

$$H^p(G'\ H^q(R_u(G), V)) \Rightarrow H^{p+q}(G, V) .$$

Since G' is fully reducible, $\operatorname{Hom}_{G'}(k, ?)$ is an exact functor. Hence we have:

$$H^p(G', H^q(R_u(G), V)) = 0 \quad \text{for every } p>0 .$$

So, we have:

$$H^q(G, V) = H^0(G', H^q(R_u(G), V)) = H^q(R_u(G), V)^{G'} .$$

By these arguments, we are reduced to the case where G is unipotent. In this case, the proof proceeds by indiction on $\dim G$.

Case 1. If $\dim G=0$, the assertions are trivial.

Case 2. If $\dim G=1$, we show the assertion by the induction on $\dim V$. If $\dim V=1$, we have $H^q(G, V)=0\,(q\geq 2)$, $=k\,(q=0, 1)$ by [12; p. 71]. If $\dim V>1$, we have $V^G\neq 0$, by Lie-Kolchin's Theorem. Look at the following exact sequence of rational G-modules,

$$0 \to V^G \to V \to V/V^G \to 0$$

If $V=V^G$, we have $H^q(G, V)=H^q(G, k)^n$ where $n=\dim V$, and we are done. So, we may assume $V\neq V^G$. Then by the above exact sequence, we have an exact sequence

$$H^q(G, V^G) \to H^q(G, V) \to H^q(G, V/V^G) ,$$

and the indiction hypothesis completes the proof.

Case 3. If $\dim G>1$, G has a normal subgriup H different from $\{e\}$ and G. Condider the following exact sequence and the associated spectral sequence:

$$1 \to H \to G \to G/H \to 1 ,$$
$$H^p(G/H, H^q(H, V)) \Rightarrow H^{p+q}(G, V) .$$

The induction hypothesis implies $\dim H^{p+q}(G, V)<\infty$. We prove the second assertion. If $p+q>\dim G$, we have $p>\dim G/H$ or $q>\dim H$. By the indiction hypothesis, we have $H^p(G/H, H^q(H, V))=0$. Hence $H^{p+q}(G, V)=0$. Q.E.D.

For the next teorem, we need the following:

Lemma 5.2. *Let V be a finite-dimensional G_a-module. Then we have*

$\dim H^1(G_a, V)$ = the number of G_a-indecomposable components of V.

Proof. Since $H^1(G_a, V)=\sum_{i=1}^r H^i(G_a, V_i)$, where each V_i is an indecomposable component of V, we may assume V is indecomposable. We argue by induction on $\dim V$. If $n:=\dim V=1$, the assertion follows [12; p.71]. Assume $\dim V>1$. Choosing a suitable coordinate of G_a and a suitable k-basis $e_1, \cdots, e_n$ of V, we may also assume

$$x\cdot e_i = e_i+xe_{i-1}(2\leq i\leq n) \quad \text{and} \quad x\cdot e_1 = e_1$$

for every $x\in G_a$. Put $W:=\mathrm{Span}_k(e_1, \cdots, e_{n-1})$. It is easy to check that W is an indecomposable rational G_a-module. We consider the cohomology exact sequence:

$$\begin{aligned} 0 &\to W^{G_a} \to V^{G_a} \to (V/W)^{G_a} \to H^1(G_a, W) \\ &\to H^1(G_a, V) \to H^1(G_a, V/W) \to H^2(G_a, W) = 0 . \end{aligned}$$

The last equality follows from Theorem 5.1. Our induction hypothesis implies $\dim H^1(G_a, W)=1$. From the result of the case $n=1$, we have $\dim H^1(G_a, V/W)=1$. From the result of the case $n=1$, we have $\dim H^1(G_a, V/W)=1$. Furthermore, it is clear that W^{G_a}, V^{G_a} and $(V/W)^{G_a}$ has the same dimension 1. Hence we have $\dim H^1(G_a, V)=1$. Q.E.D.

Theorem 5.3. *Let G be a unipotent algbraic group and let V be a non-zero, finite-dimensional rational G-module. Then we have $H^r(G, V)\neq 0$, where $r=\dim G$,*

Proof. We use the induction on r. If $r=1$, we know that $\dim H^r(G, V)$ is the number of G-indecomposable components of V by Lemma 5.2. Hence $H^r(G, V)\neq 0$. If $r>1$, we can find a normal closed subgroup H of G such that $H\neq \{e\}, G$. Consider a spectral sequence:

$$H^p(G/H, H^q(H, V)) \Rightarrow H^{p+q}(G, V) .$$

By virtue of Theorem 5.1, we have:

$$H^r(G, V) = H^s(G/H, H^t(H, V)) ,$$

where $s:=\dim G/H$ and $t:=\dim H$. The induction hypothesis insures us that $H^t(H, V)\neq 0$. Furthermore, Theorem 5.1 implies that $\dim H^t(H, V)<\infty$. So, the induction hypothesis implies that $H^r(G, V)$ is a non-zero, finite-dimensional vector space. Q.E.D.

6. Unipotent exact subgroups in terms of derivations.

In this section, we shall give an interpretation of commutative unipotent exact subgroups (cf. [11; p.6]) of an algebraic group in terms of derivations.

Theorem 6.1. *Let G be an algebraic group and let H be a commutative, unipotent, closed subgroup of G. Then the following conditions are equivalent:*

(1) *H is exact in G,* i.e., *G/H is affine;*

(2) *$H^1(H, k[G]) = 0$;*

(3) *Let $X_1, \cdots, X_n$ be a k-basis of the Lie algebra $L(H)$ of H. For any n-tuple $f=(f_i)\in k[G]^n$ such that $X_i(f_j)=X_j(f_i)$ for any pair (i, j), there exists an element f of $k[G]$ such that $X_i(f)=f_i (1\leq i\leq n)$.*

Proof. The equivalence of (1) and (2) follows from [11; Remark (2.5)b].

We shall rewrite the condition (2) in terms of $X_1, \cdots, X_n$. Let $\alpha := (\alpha_1, \cdots, \alpha_n)$ be a row vector whose entries are elements of $k[x_1, \cdots, x_n, y_1, \cdots, y_n]$. Let $I=(i_1, \cdots, i_n)$ be a multi-index with integers as entries. The symbol α^I denotes

$$\alpha_1^{i_1} \cdots \alpha_n^{i_n}$$

We also denote by the symbol $\rho_x\colon k[G] \to k[G]$ the right translation by an element x of H, $\rho_x(f)(g)=f(gx)$ (cf. [6; p.62]). We often identify H with a k-vector space consisting of row vectors of size n by choosing a suitable coordinate-system.

Consider the Hochschild complex of a rational H-module $k[G]$:

$$0 \to k[G] \xrightarrow{\delta^0} k[G][x_1, \cdots, x_n] \xrightarrow{\delta^1} k[G][x_1 \cdots, x_n, y_1, \cdots, y_n],$$

where

$$\delta^0(f) := \rho_x(f) - f \quad \text{and} \quad \delta^1(\Sigma_I f_I x^I) := \Sigma_I \rho_x(f_I) y^I - \Sigma_I f_I (x+y)^I + \Sigma_I f_I x^I .$$

It suffices to show the following lemma.

Lemma 6.2. *Let Z be the set*

$$\{(f^{(i)}) \in k[G]^n \mid X_j(f^{(i)}) = X_i(f^{(j)}) \quad \textit{for any} \quad (i, j)\}$$

let $e(i)$ be the n-ple row vector with 1 *in the i-th entry and* 0 *elsewhere. Then the mapping $\Phi\colon Ker\,\delta^1 \to Z$, defined by $\Sigma_I f_I x^I \mapsto (f_{e(1)}, \cdots, f_{e(n)})$, is well-defined and it gives rise to an isomorphism of k-vector spaces.*

Proof. We divide our proof into the subsequent several steps:

(i) For any $f \in k[G]$, write $\rho_x(f) = \Sigma_I f[I] x^I$. Then we have $f[e(i)] = X_i(f)$ $(1 \leq i \leq n)$ and $f[0]=f$.

(ii) For any $f \in k[G]$, we have $(f[I])[J] = \binom{I+J}{I} f[I+J]$, where $\binom{I+J}{I} = \Pi_{\lambda=1}^n \binom{i(\lambda)+j(\lambda)}{i(\lambda)}$ if $I=(i(1), \cdots, i(n))$ and $J=(j(1), \cdots, j(n))$.

(iii) Suppose $\Sigma_I f_I x^I \in \operatorname{Ker} \delta^1$. Then we have:

$$f_I[K] = \binom{I+K}{I} f_{I+K} \qquad (\text{if } I \neq 0) \quad \text{and} \quad f_0 = 0 .$$

(iv) For any i and $j (1 \leq i, j \leq n)$, we have $X_j(f_{e(i)}) = X_i(f_{e(j)})$. Hence the map Φ is well-defined.

(v) Φ is bijective.

Proof of (i). Put $x=0$ in the equality $\rho_x(f) = \Sigma_I f[I] x^I$. Then we get $f = \rho_0(f) = f[0]$. Next, let $\mu\colon G \times G \to G$ be the multiplication and write $\mu^*(f) = \Sigma_i f_i g_i$. Write $g_i|_H = \Sigma a_I^{(i)} x^I$. For every $g \in G$, we have:

$$X_i(f)(g) = (\partial/\partial x_i)_e(\lambda_{g^{-1}}(f)\,|_H) = (\partial/\partial x_i)_e(\Sigma_j f_j(g) g_j\,|_H)$$
$$= \Sigma_j f_j(g)(\partial/\partial x_i)_e(g_j\,|_H) = \Sigma_j f_j(g) a^{(j)}_{e(i)}\,,$$

where λ_g stands for the left-translation by the element g of G_a (cf. [6; p. 62]), and $(\partial/\partial x_i)_e$ is the evalutaion at e after applying $\partial/\partial x_i$.
Hence we have: $X_i(f)=\Sigma_j a^{(j)}_{e(i)} f_j$. On the other hand, we have: $f[e(i)]=\Sigma_j a^{(j)}_{e(i)} f_j$. So, $f[e(i)]=X_i(f)$.

Proof of (ii). By the definition, we have $\rho_\alpha(f)=\Sigma_I f[I]\alpha^I$(for $\alpha\in H$). Hence we have $\rho_\beta\cdot\rho_\alpha(f)=\rho_\beta(\Sigma_I f[I]\alpha^I)=\Sigma_I \rho_\beta(f[I])\alpha^I=\Sigma_{I,K}(f[I])[K]\alpha^I\beta^K$ for α, $\beta\in H$.

On the other hand, we have $\rho_{\alpha+\beta}(f)=\Sigma_I f[I](\alpha+\beta)^I$. Since α and β are chosen arbitrarily, comparing the above equalities, we can verify the assertion.

Proof of (iii). We have $\delta^1(\Sigma_I f_I x^I)=\Sigma_I \rho_x(f_I)y^I-\Sigma_I f_I(x+y)^I+\Sigma_I f_I x^I=0$. Comparing the coefficients of $x^K y^I$, we have $f_I[K]=\binom{I+K}{I}f_{I+K}$. We also have $\Sigma_K f_0[K]x^K=0$, by comparing the coefficients of y^0. Hence $f_0=f_0[0]=0$.

Proof of (iv). We get the following two equalities by virtue of (i) and (iii):

$$X_j(f_{e(i)}) = X_j(f_0[e(i)]) = (f_0[e(i)])[e(j)] \qquad \text{and}$$
$$X_i(f_{e(j)}) = X_i(f_0[e(j)]) = (f_0[e(j)])[e(i)]\,.$$

The both sides of these equalities are equal to $f_0[e(i)+e(j)]$ by virtue of (ii).

Proof of (v). First of all, we prove that Φ is injective. Suppose $\Phi(\Sigma_I f_I x^I)=(f[e(1)],\cdots,f[e(n)])=0$. By (iv), we have $f_0=0$. If $I=(i_1,\cdots,i_n)\neq 0$, say $i_j>0$, we get the following equality by virtue of (iv):

$$0=f_{e(j)}[I-e(j)]=\binom{I}{I-e(j)}f_I\,.$$

Hence $f_I=0$. These arguments imply that $\Sigma_I f_I x^I=0$. So, Φ is injective.

Finally, we prove that Φ is surjective. For any $(f^{(i)})\in Z$, we define the following elements of $k[G]$ by

$$f_0 := 0\,,$$
$$f_I := f^{(j)}[I-e(j)]/i_j$$

where i_j is the j-th entry of I, which is assumed to be non-zero. We must verify that the above definition is independent of the choice of j. For this purpose, it suffices to check that the following equality holds whenever $i_p i_q\neq 0$:

$$i_q f^{(p)}[I-e(p)] = i_p f^{(q)}[I-e(q)]\,.$$

We may assume $p<q$. By virtue of the above (i) and (ii), we can rewrite the

both sides as follows:

$$\begin{aligned}\text{the left side } &= (f^{(p)}[e(q)])[I-e(p)-e(q)]\\ &= (X_q(f^{(p)}))[I-e(p)-e(q)], \qquad \text{and}\\ \text{the right side} &= (f^{(q)}[e(p)])[I-e(p)-e(q)]\\ &= (X_p(f^{(q)}))[I-e(p)-e(q)].\end{aligned}$$

Since $X_q(f^{(p)})=X_p(f^{(q)})$, we obtain the required equality.

Next we compute:

$$\delta^1(\Sigma_I f_I x^I) = \Sigma_I \rho_x(f_I)y^I - \Sigma_I f_I(x+y)^I - \Sigma_I f_I x^I .$$

The coefficient of $x^I y^0$ is $f_0[I]=0$ since $f_0=0$. If the p-th entry i_p of I is non-zero, the coefficient of $x^K y^I$ is:

$$\begin{aligned}&1/i_p\times(f^{(p)}[I-e(p)])[K]-\binom{I+K}{I}f_{I+K}\\ &= 1/i_p\times(f^{(p)}[I-e(p)])[K]\\ &\quad -(i_p+k_p)/i_p\times\binom{I+K-e(p)}{i-e(p)}\times 1/(i_p+k_p)\times f^{(p)}(I+K-e(p))\\ &= 0 \quad \text{(by (ii))}.\end{aligned}$$

Hence $\delta^1(\Sigma_I f_I x^I)=0$. It is clear that $\Phi(\Sigma_I f_I x^I)=(f^{(i)})$. So, Φ is bijective.

Q.E.D.

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