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ON UPPERBOUNDS OF VIRTUAL MORDELL-WEIL RANKS

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0. Introduction

0.0. Let $f: X \rightarrow C$ be a relatively minimal fibration of curves of genus $g \geq 1$ over a smooth projective curve C of genus b defined over an algebraically closed field k . Let $K = k(C)$ be the field of rational functions on C . In the theory of Mordell-Weil lattices due to Shioda (cf. [17], [18]) the following conditions are assumed:

- (0.1) (i) f admits a global section (O) as zero-section,
(ii) K/k -trace of the Jacobian J_F of the generic fibre F/K of f is trivial.

Under these conditions the Mordell-Weil group $J(K)$ of K -rational points of J is finitely generated. The rank r of its free part is called the Mordell-Weil rank. We shall be concerned with characteristic zero case (in this case the second assumption in (0.1) is equivalent to $q(X) = b$). In [14, Theorem 1.3] an upperbound of r via the invariants of f is given. In particular, for the case of rational surfaces X it was shown in a joint paper ([15]) that $r \leq 4g + 4$. Moreover the structure of fibrations with maximal rank $r = 4g + 4$ and the structure of corresponding Mordell-Weil lattices are completely determined in [15] (a such fibration is obtained as a blowing up of a linear pencil of hyperelliptic curves on a Hirzebruch surface Σ_e with $0 \leq e \leq g$ ($g \geq 2$)).

In this note we consider a similar problem for locally non-trivial fibrations, not necessarily satisfying conditions (0.1). Let $\text{NS}(X)$ be the Néron-Severi group of X . Then $\text{NS}(X)/\text{torsion}$ admits the lattice structure with the intersection pairing. Hodge's index theorem asserts that its signature is $(1, \rho - 1)$, where $\rho := \text{rank NS}(X)$ is the Picard number of X .

DEFINITION 0.2 (cf. [11]). The virtual Mordell-Weil rank r of f is defined to be the rank of the essential sublattice of the Néron-Severi lattice (cf. [17], [18]), i.e.,

as

$$(0.3) \quad r = \rho - 2 - \sum_{t \in C} (n_t - 1),$$

where n_t is the number of irreducible components of $X_t := f^{-1}(t)$.

If f satisfies conditions (0.1) then this is nothing but the well-known formula for the Mordell-Weil rank r (*loc. cit.*). This justifies our definition.

0.4. A natural question arising here is to give a best possible upperbound of virtual Mordell-Weil rank and we are interested in knowing when it becomes the real Mordell-Weil rank. In a similar way as in [14, Theorem 1.3] by using Xiao's inequality ([20]), one can have the following bound for a locally non-trivial fibration $f: X \rightarrow C$

$$(0.5) \quad r \leq (6 + 4/g)d + 2(q - b) + 2g(b - 1),$$

where $d = \deg(f_*\omega_{X/C})$, $q = q(X)$. Moreover we can show that the equality holds only if f is a hyperelliptic fibration, all fibres of f are irreducible and $q = b$.

In the non-hyperelliptic case with $f_*\omega_{X/C}$ semi-stable we have a sharper bound for r due to Konno's stronger version of the slope inequality ([5, Lemma 2.5]). In the light of new results of Konno (personal communication) we know that the case of equality implies that $\text{Cliff}(f) = 1$, i.e., general fibres of f are trigonal or plane quintic (see also his recent paper [6] where he treats the non-semistable case with $\text{Cliff}(f) = 1$). From the point of view of Mordell-Weil lattices the "computable" case $p_g = q = 0$ is most interesting. In this case $r \leq 3g + 6$ (also for the number of singular fibres $s: s \leq 7g + 6$). We give two examples showing that bounds actually are sharp. It should be very interesting to get a complete description as in [15] for the maximal case.*

We can also have more precise structure theorem for the following pencils ($b = 0$):

(I) *Pencils with $\chi(\mathcal{O}_X) = 1$.* In this case the bound (0.5) can be read as $r \leq 4g + 4 + 2q$. We remark that the equality $r = 4g + 4 + 2q$ leads us to the maximal case studied in [15] by using [11, Lemma 3.1.2]. Also $s \leq 8g + 4$ and the equality $s = 8g + 4$ gives us Lefschetz pencils in the constructions of [15].

(II) *Pencils with $c_1^2(X) = -4(g - 1) < 0$.* Here we have $r \leq 4g + 4 - 8q$ (*resp.* $s \leq 8g + 4 - 12q$). The maximal case is obtained as a blowing up of a pencil (*resp.* a Lefschetz pencil, except for $g = 2q$) on a ruled surface $\Sigma_e^q \rightarrow E$, $g(E) = q$ ($-q \leq e \leq g - 2q$) whose general members are double coverings of curve E . In the maximal case the structure of the essential sublattice in the Néron-Severi lattice is uniquely determined. The proof uses the fact that in this case X is double

* A full account of this situation is now in preparation ([12]).

covering of $E \times \mathbf{P}^1$ whose branch locus is a smooth irreducible curve of numerical type $(2g+2-4q, 2)$ (cf. [11, Theorem 3.1 and Lemma 3.1.2]).

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1. Bounds of virtual Mordell-Weil ranks

1.1. We use the following notation:

$X(Y, \gamma, B) := \text{Spec}(\mathcal{O}_Y \oplus \mathcal{O}_Y(\gamma)^\vee)$: the double covering of a smooth surface Y with branch locus $B \sim 2\gamma$;

$f: X \rightarrow C$: a relatively minimal fibration, not locally trivial, of curves of genus $g \geq 1$;

C : a smooth projective curve with genus b .

S : the finite set of critical points on C , s = the number of S .

$\omega_{X/C}$: the relative dualizing sheaf, $d := \text{deg}(f_*\omega_{X/C})$.

ω : the relative canonical class $K_{X/C}$.

$\lambda(f) := \omega^2/d$: the slope of f .

$\rho' := h^{1,1} - \rho$: the difference of the middle Hodge number $h^{1,1}$ and Picard number ρ of X . Note that ρ' is a non-negative number by virtue of Lefschet'z theorem on algebraic cycles.

$\chi(X_t) :=$ the topological Euler number of $X_t := f^{-1}(t)$ for $t \in C$.

$e_t(X) := \chi(X_t) - (2 - 2g)$: the local Euler number over $t \in C$.

n_t : the number of irreducible components of X_t .

$g(\tilde{X}_t)$: the genus of the normalization of X_t .

Recall that the ground field k is the field of complex numbers \mathbf{C} .

Proposition 1.2. *Let $f: X \rightarrow C$ be a relatively minimal fibration as in (1.1) and r denote the virtual Mordell-Weil rank of f . Then we have*

$$(1.2.1) \quad r \leq (4 + \frac{4}{g})[\chi(\mathcal{O}_X) - (g-1)(b-1)] + 2[\chi(\mathcal{O}_X) + q - 1],$$

or equivalently,

$$(1.2.2) \quad r \leq (6 + \frac{4}{g})d + 2(q-b) + 2g(b-1).$$

Moreover the equality in the bounds above holds if and only if

- 1) f is a hyperelliptic fibration with lowest slope $\lambda(f) = 4 - 4/g$,
- 2) all fibers of f are irreducible,
- 3) $\rho' = 0$.

Proof (cf. [14, Theorem 1.3]). First in view of Xiao's inequality $\lambda(f) \geq 4 - 4/g$ ([20, Theorem 2]) one can put

$$(1.2.3) \quad \omega^2 = \omega_f + \left(4 - \frac{4}{g}\right)d$$

with non-negative ω_f .

By an easy calculation using Leray's spectral sequence and Riemann-Roch we have

$$(1.2.4) \quad \chi(\mathcal{O}_X) = d + (g-1)(b-1).$$

Next since $\chi(X) = c_2(X) = 2 - 2b_1 + b_2$ and $b_1 = 2q$, where b_i is the i -th Betti number we infer from Noether's formula

$$h^{1,1} = b_2 - 2p_g = \left(4 + \frac{4}{g}\right)[\chi(\mathcal{O}_X) - (g-1)(b-1)] + 2[\chi(\mathcal{O}_X) + q] - \omega_f.$$

Taking into account (0.3) one obtains

$$(1.2.5) \quad r = \left(4 + \frac{4}{g}\right)[\chi(\mathcal{O}_X) - (g-1)(b-1)] + 2[\chi(\mathcal{O}_X) + q - 1] - \omega_f - \sum_{t \in C} (n_t - 1) - \rho',$$

or equivalently (in view of (1.2.4))

$$(1.2.6) \quad r = \left(6 + \frac{4}{g}\right)d + 2(q-b) + 2g(b-1) - \omega_f - \sum_{t \in C} (n_t - 1) - \rho'.$$

Bounds (1.2.1)–(1.2.2) follow directly from (1.2.5)–(1.2.6). Moreover the equality holds if and only if

- 1) $\omega_f = 0$, or equivalently, $\lambda(f) = 4 - 4/g$,
- 2) $n_t = 1, \forall t \in C$, i.e., all fibres of f are irreducible,
- 3) $\rho' = 0$.

It remains to use Konno's result stating that fibrations with lowest slope $\lambda(f) = 4 - 4/g$ are hyperelliptic ([5, Proposition 2.6]). \square

REMARK 1.3. As a consequence of this proposition we obtain $q = b$ in the case of the equality of bounds above ([20, Corollary 1]). If we assume moreover that f admits a global section, then this is equivalent to the triviality of K/k -trace of the Jacobian J_F . So r becomes the real Mordell-Weil rank and it makes sense to study the structure of Mordell-Weil lattices in this case.

Corollary 1.4. *Let $f: X \rightarrow C$ be as in Proposition 1.2. Assume that r is*

maximal, i.e., $r = (6 + 4/g)d + 2(q - b) + 2g(b - 1)$. Then X is a double covering of a ruled surface $\Sigma_e^b \rightarrow C$ with smooth branch locus B and

$$(1.4.1) \quad \omega^2 = 2(g - 1)[2m - (g + 1)e], \quad d = \frac{g}{2}[2m - (g + 1)e],$$

where $B \equiv 2(g + 1)C_0 + 2mF_0$ with C_0, F_0 denoting the minimal section and a fibre on Σ_e^b .

Proof. From Proposition 1.2 and Horikawa's theory it follows that X is the canonical resolution of a double cover of a ruled surface over C with simple singularities. It remains to use standard calculations with double coverings (see, e.g., [4], or [1, V, 22], cf. also §3). The fact that B is smooth follows from the irreducibility of fibres of f . Indeed, if B were singular, a fibre of f could consist of extra curves arising from the resolution of singularities.

1.5. Consider the non-hyperelliptic case and assume that $f_*\omega_{X/C}$ is semi-stable then one can have a sharper bound thanks to Konno's stronger version of the slope inequality ([5, Lemma 2.5]). In particular if $p_g = q = 0$ then it is easy to see that $f_*\omega_{X/P^1}$ is semi-stable (cf. [11, A.4.4]). So we have $r \leq 3g + 6$. We give here examples which show that this bound is sharp. Take a Lefschetz pencil of curves of degree m in the projective plane P^2 , considered in [19]. By blowing up m^2 distinct base points from P^2 one obtains a smooth rational surface X with natural morphism $f : X \rightarrow P^1$. The fact that f is non-hyperelliptic if $m > 3$ is obvious. An easy computation shows that we have the following invariants:

- 1) $g = (m - 1)(m - 2)/2$,
- 2) $\omega^2 = 3m^2 - 12m + 9, \omega_f = (m - 3)^2$,
- 3) $r = m^2 - 1, s = 3(m - 1)^2$.

Thus the case of $m = 4, 5$ gives us the equality in the bound above. It should be very interesting to describe all such fibrations (see footnote in the Introduction).

2. Pencils with $\chi(\mathcal{O}_X) = 1$

Proposition 2.1. *Let $f : X \rightarrow P^1$ be a relatively minimal fibration of curves of genus $g \geq 1$ (having a section if $g = 1$). Assume that $\chi(\mathcal{O}_X) = 1$. Then we have*

$$(2.1.1) \quad r \leq 4g + 4 + 2q.$$

Moreover the equality $r = 4g + 4 + 2q$ implies that X is a rational surface (hence $q = 0$) and f has a section. In particular, r gives actually the Mordell-Weil rank and if $g \geq 2$ we obtain the known constructions with Hirzebruch surfaces $\Sigma_e, 0 \leq e \leq g$, as in [15].

Proof. In fact bound (2.1.1) can be easily followed from (1.2.1)–(1.2.2). In our special case we have $d=g$ and

$$(2.1.2) \quad \omega_f = \omega^2 - 4(g-1),$$

is a non-negative integer.

Next (1.2.5)–(1.2.6) can be rewritten as

$$(2.1.3) \quad r = 4g + 4 + 2q - \omega_f - \sum_{t \in S} (n_t - 1) - \rho'.$$

Consequently $r \leq 4g + 4 + 2q$ and the equality holds if and only if $\omega_f = 0$, all singular fibres are irreducible, and $\rho' = 0$. Since the elliptic case ($g=1$) is obvious we can assume $g \geq 2$. Then the condition $\omega_f = 0$ implies that X is a ruled surface ([11, Theorem 3.1 and Lemma 3.1.2]), in particular $p_g = q = 0$, and by the same token the rationality of X . (Note that the fact $p_g = q = 0$ also follows from Remark 1.3). It remains to refer to [15, Theorem 4.1] for the rest of the Proposition. \square

Lemma 2.2. (i) For $t \in S$ one has

$$(2.2.1) \quad e_t(X) > 0$$

except X_t is a non-singular elliptic curve with some multiplicity (the case $e_t(X) = 0$).

(ii) Moreover $e_t(X) = 1$ if and only if either X_t is irreducible with at most one node as its singularity, or X_t is a curve with two smooth irreducible components C_1, C_2 meeting at one point transversally such that $g(C_1) + g(C_2) = g$.

The first statement is nothing but Theorem 7 in [16, IV]. The proof of the second statement is purely technical and can be followed from the arguments in the proof of that Theorem and Lemma 4 (*loc. cit.*)

Corollary 2.3. Under the assumptions of Proposition 1.2 we have

$$(2.3.1) \quad s \leq (8 + 4/g)d.$$

In the case of equality we also have (1.4.1)

Proof. First note that

$$(2.3.2) \quad \chi(X) = c_2(X) = \sum_{t \in S} e_t(X) + 4(g-1)(b-1)$$

(see, e.g., [16, IV, §4] or [1, III, 11.4]). Furthermore from this, (1.2.3) and Noether's formula we have

$$(2.3.3) \quad \sum_{t \in S} e_t(X) = (8 + \frac{4}{g})d - \omega_f.$$

It remains to use (2.2.1) to get (2.3.1). The case of equality implies that $\omega_f = 0$. The same arguments as in the proofs of Proposition 1.2 and Corollary 1.4 show that one has (1.4.1). □

Corollary 2.3.4. $e_t(X) \leq 2(g - g(\tilde{X}_t)) + 2(n_t - 1)$.

Proposition 2.4. *In the situation of Proposition 2.1 we have*

$$(2.4.1) \quad s \leq 8g + 4.$$

Furthermore every fibration with maximal number $s = 8g + 4$ is a rational hyperelliptic Lefschetz pencil with a section such that $\omega^2 = 4(g - 1)$.

Proof. Since $d = g$ it follows from (2.3.3) that

$$(2.4.2) \quad \sum_{t \in S} e_t(X) + \omega_f = 8g + 4.$$

So (2.4.2) together with (2.2.1) implies the bound (2.4.1). Moreover $s = 8g + 4$ holds if and only if:

- 1) $\omega_f = 0$,
- 2) $e_t(X) = 1, \quad \forall t \in S$.

As in the proof of Proposition 2.1, $\omega_f = 0$ implies that f is a rational hyperelliptic pencil. Furthermore since X is the canonical resolution of a double cover of $P^1 \times P^1$ with simple singularities, singular fibres with two smooth irreducible components in the second statement of Lemma 2.2 can not occur. Thus we have a Lefschetz pencil. □

Corollary 2.4.3. *Let $f: X \rightarrow P^1$ be as in Proposition 2.1. If $p_g = q > 0$ then $r \leq 4g + 2q$ (resp. $s \leq 8g$) and the equality is possible only in case $p_g = q = 1$.*

Proof. 1) From [20, Corollary 1] and the assumption $q > 0$ it follows that $\lambda(f) \geq 4$. Furthermore $\lambda(f) = 4$ implies $q = 1$ ([20, Theorem 3]). It remains to use (2.1.3) and (2.4.2). □

REMARK 2.4.4. For the detailed construction of Lefschetz pencils in Proposition

2.4 we refer to [15]. Note that those fibrations are irregular in the sense of [11, §3] if $g \geq 2$.

3. Pencils with $c_1^2(X) = -4(g-1) < 0$

3.1. In this section we consider the class of pencils with $c_1^2(X) = -4(g-1) < 0$. First recall some facts from the theory of double covering of surfaces. Let B be an even reduced effective divisor on a smooth surface Y . Consider the double covering $X(Y, \gamma, B)$ with branch locus B and γ such that $B \sim 2\gamma$ (cf. 1.1). Let $X_{CR} = X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 = X(Y, \gamma, B)$ be the canonical resolution of X_0 . Putting $Y_0 = Y, B_0 = B$ then

- 1) each X_i is a double covering of Y_i with branch locus B_i ,
- 2) Y_i is a blowing up of Y_{i-1} at a singular point of B_i with multiplicity $m_i, i \leq n-1$,
- 3) B_n is non-singular.

Recall that X_0 has at most rational double points as its singularities if and only if all m_i are less than 4 ([4, Lemma 5]).

Lemma 3.2. *In the notation above we have*

$$(3.2.1) \quad \chi(\mathcal{O}_{X_{CR}}) = 2\chi(\mathcal{O}_Y) + \frac{1}{2}\gamma(K_Y + \gamma) - \frac{1}{2}\sum_i \left[\frac{m_i}{2} \right] \left(\left[\frac{m_i}{2} \right] - 1 \right),$$

$$(3.2.2) \quad c_1^2(X_{CR}) = 2(K_Y + \gamma)^2 - 2\sum_i \left(\left[\frac{m_i}{2} \right] - 1 \right)^2.$$

Proof. See [4, Lemma 6].

Theorem 3.3. *Let $f: X \rightarrow \mathbb{P}^1$ be a relatively minimal fibration with $g \geq 2$. Assume that*

- 1) $c_1^2(X) = -4(g-1)$,
- 2) $\chi(\mathcal{O}_X) \geq 3-g$.

Then X is a ruled surface defined as a family $\pi: X \rightarrow E, g(E) = q$. The morphisms f and π define a 2-to-1 map from X to $Y = E \times \mathbb{P}^1$ with a branch locus B . The second projection of Y induces f and B is of numerical type $(2g+2-4q, 2)$. Moreover X is the canonical resolution of $X(Y, \gamma, B)$ with rational double singularities. In particular, if $q=0$ then X is a rational surface and f is hyperelliptic.

Proof. From [11, Theorem 3.1 and Lemma 3.1.2] we have known that X is a ruled surface $\pi: X \rightarrow E, g(E) = q$. The morphisms f and π define a 2-to-1 map from X to Y . So it can be easily seen that the branch locus has the desired numerical

type. It remains to show that X is the canonical resolution of $X(Y, \gamma, B)$ with rational double singularities. Indeed arguments similar to those in [8, §2] show that the natural morphism $h: X_{CR} \rightarrow X$ is a contraction of all (-1) -curves on X_{CR} . Now calculating $\chi(\mathcal{O}_{X_{CR}})$, $c_1^2(X_{CR})$ by (3.2.1)–(3.2.2) we obtain:

$$\chi(\mathcal{O}_{X_{CR}}) = 1 - q - \frac{1}{2} \sum_i \left[\frac{m_i}{2} \right] \left(\left[\frac{m_i}{2} \right] - 1 \right),$$

$$c_1^2(X_{CR}) = -4(g-1) - 2 \sum_i \left(\left[\frac{m_i}{2} \right] - 1 \right)^2.$$

Therefore

$$\sum_i \left[\frac{m_i}{2} \right] \left(\left[\frac{m_i}{2} \right] - 1 \right) = 0,$$

$$\sum_i \left(\left[\frac{m_i}{2} \right] - 1 \right)^2 = 0.$$

That is $X = X_{CR}$ and all m_i are less than 4 as desired. □

Corollary 3.3.1. *Under the assumptions of Theorem 3.3 let m be the number of critical points of π . We have*

$$(m = 4g + 4 - 8q) \Leftrightarrow (B \text{ is a smooth irreducible curve})$$

Theorem 3.4. *Let $f: X \rightarrow P^1$ be as in Theorem 3.3. Then we have*

$$(3.4.1) \quad r \leq 4g + 4 - 8q \quad \text{and} \quad s \leq 8g + 4 - 12q.$$

Moreover the equality $r = 4g + 4 - 8q$ implies that it is obtained by blowing up from a pencil on a ruled surface $\pi': \Sigma_e^q \rightarrow E$, $g(E) = q$ ($-q \leq e \leq g - 2q$) whose general members are double coverings of curve E . Furthermore in the case $2q \neq g$ the equality $s = 8g + 4 - 12q$ implies the same conclusions with a Lefschetz pencil. In the maximal cases the structure of the essential sublattice in the Néron-Severi lattice is uniquely determined.

Proof. 1) The first bound in (3.4.1) is obvious since $\rho = b_2 = 4g + 6 - 8q$ and by using (0.3). The second one follows immediately from Lemma 2 and

$$(3.4.2) \quad \sum_{i \in S} e_i(X) = 8g + 4 - 12q$$

(cf. (2.4.2)). As a consequence we obtain $g + 1 \geq 2q$.

2) Now assume $r = 4g + 4 - 8q$. Then all singular fibres are irreducible. Since by Theorem 3.3, X is the canonical resolution of $X(Y, \gamma, B)$ with rational double singularities it follows that B is a smooth irreducible curve, so that $m = 4g + 4 - 8q$. Let us denote by $\{E_i^\pm\}_{i=1}^{4g+4-8q}$ the irreducible components of corresponding singular fibres of π . Obviously E_i^\pm s are sections of f . After a succession of blowings down (each time one of E_i^\pm which we denote simply by E_i) we obtain a standard ruled surface $\pi': \Sigma_e^q \rightarrow E$. Surface Σ_e^q has degree $-e$ and a section C_0 such that $C_0^2 = -e$. For invariant e we know that $e \geq -q$ ([9, 7]).

Thus we have a birational morphism $\varphi: X \rightarrow \Sigma_e^q$. Setting $F' = \varphi(F)$ the image of a smooth general fibre we may assume that F' is also smooth and birational to F . An easy computation shows that

$$F' \equiv 2C_0 + aF_0,$$

where $a = g + 1 + e - 2q$ and $F_0 \simeq P^1$ is a fibre of π' . It means that we obtain a linear pencil of a linear system $|2C_0 + \bar{a}F_0|$ on Σ_e^q with $\deg \bar{a} = a$ and X is obtained as a blowing up of the base points of this linear pencil. We have to consider two cases.

(+) If $e \geq 0$ then from [2, V, 2.20] it follows that $a \geq 2e$, or equivalently, $e \leq g + 1 - 2q$. Assume that $e = g + 1 - 2q$ then $F' \cdot C_0 = 0$. Let C'_0 be the proper transform of C_0 by φ , one can see that $F \cdot C'_0 = 0$ and $C_0'^2 = C_0^2 = -e = 2q - g - 1$. In view of the irreducibility of fibres of f it is possible only if $e = 2q - g - 1 = 0$. Hence π is a smooth fibering, that is, $X = \Sigma_e^q$. On one hand C_0 is fibre of f by the above. On the other hand C_0 is a section of π . This contradicts the fact that $F \cdot F_0 = 2$. We have proved $e \leq g - 2q$.

(++) If $e < 0$ then from [2, V, 2.21] we have known that $a \geq e$, or equivalently, $g + 1 \geq 2q$.

3) The Néron-Severi group $NS(X)$ in the maximal case is as follows.

$$(3.4.3) \quad NS(X) \simeq \mathbf{Z} \cdot C_0 \oplus \mathbf{Z} \cdot F_0 \oplus (\oplus_{i=1}^{4g+4-8q} \mathbf{Z} \cdot E_i)$$

where we denote total transforms of C_0, F_0 under φ by the same letters. We have a relation

$$(3.4.4) \quad F \sim 2C_0 + \bar{a}F_0 - \sum_{i=1}^{4g+4-8q} E_i.$$

4) One can show easily the assertions for s with adding the Lefschetz property to the pencils. In fact since $e_i(X) = 1$ and $g \neq 2q$ we see that a singular fibre with two smooth irreducible components C_1, C_2 with $g(C_1) + g(C_2) = g$ (cf. Lemma 2.2) does not appear. Since X is the canonical resolution of $X(Y, \gamma, B)$, one obtains $g(C_1) = g(C_2) = q$, that is impossible by the assumption $g \neq 2q$. Thus arguing as above we get decomposition (3.4.3). Therefore in the case $s = 8g + 4 - 12q$ (even without the condition $g \neq 2q$) the structure of the essential sublattice in the

Néron-Severi lattice is uniquely determined. □

Corollary 3.4.5. *Let $f: X \rightarrow \mathbf{P}^1$ be as in Theorem 3.3. Assume that all singular fibres are irreducible, then*

$$s \geq 8 + \frac{4q-4}{g+1-2q}.$$

In particular we have $s \geq 7$ and if either $q \geq 1$, or $q=0$ and $g \geq 4$, then $s \geq 8$.

Proof. By virtue of the Riemann-Hurwitz formula one sees that $g(\tilde{X}_t) \geq 2q-1$ (Note that since S is not empty we get another proof of estimate $g+1 > 2q$). The corollary now follows easily from (3.4.2) and Corollary 2.3.4. □

3.5. The maximal case with rational base $q=0$ leads us to known constructions with Hirzebruch surfaces Σ_e . As a rule for constructing examples with maximal numbers s, r we need the very ampleness of linear system $|2C_0 + \bar{a}F_0|$ on Σ_e^q (cf. [15]). In general one can construct certain examples with maximal numbers $r=4g+4-8q, s=8g+4-12q$ under some conditions with respect to e .

Note that linear system $|2E + \bar{a}P^1|$ on $\mathbf{P}^1 \times E$ with $\deg \bar{a} = a = g+1-2q$ is very ample if $a \geq 2q+1$. This gives an example with $e=0$. In fact one can prove the following proposition.

Proposition 3.5.1. *If $4q \leq g$, then for $-q \leq e \leq \frac{(g+1)}{2} - 2q$ linear systems $|2C_0 + \bar{a}F_0|$ with $\deg \bar{a} = a = g+1+e-2q$ are very ample on Σ_e^q .*

Proof. Denote by $\mathcal{L} = \mathcal{O}(2C_0 + \bar{a}F_0)$ and consider any two (possibly coinciding) points P_1, P_2 on Σ_e^q being contained in fibres F_1, F_2 respectively. As is well known, to prove the very ampleness of \mathcal{L} it suffices to show:

$$(*) \quad H^1(m_i \otimes \mathcal{L}) = H^1(m_1 m_2 \otimes \mathcal{L}) = 0,$$

where m_i is the ideal sheaf of P_i .

On the other hand due to two exact sequences

$$0 \rightarrow \mathcal{L}(-F_i) \rightarrow m_i \otimes \mathcal{L} \rightarrow m_i \otimes \mathcal{L} / \mathcal{L}(-F_i) \rightarrow 0$$

$$0 \rightarrow \mathcal{L}(-F_1 - F_2) \rightarrow m_1 m_2 \otimes \mathcal{L} \rightarrow m_1 m_2 \otimes \mathcal{L} / \mathcal{L}(-F_1 - F_2) \rightarrow 0$$

one can see easily that the vanishing statement (*) follows from the following two statements:

1) the corresponding to (*) vanishing statement for \mathcal{L} restricted on fibres of π' : it is easy since $\deg \mathcal{L}|_{F_i} = 2$ and $F_i \simeq \mathbf{P}^1$ so that $\mathcal{L}|_{F_i}$ is very ample on F_i ,

2) $H^1(\mathcal{L}(-\bar{b}F_0))=0$ for any effective divisor \bar{b} on E with $\deg \bar{b} \leq 2$.

For the second statement by virtue of the Kodaira-Ramanujam vanishing theorem ([13]) it suffices to verify the numerical positivity of divisor $D \equiv 2C_0 + (a-2)F_0 - K_{\Sigma_g}$. Here are standard calculations using [2, V, 2.20–2.21].

(i) Since $g \geq 4q$ by the assumption we have $D^2 = 8(g+1-4q) > 0$.

(ii) Case $e \geq 0$: let $C' \equiv bC_0 + cF_0$ be an irreducible curve ($b > 0, c \geq be$) then from the condition for e we have

$$D \cdot C_0 = g + 1 - 4q - 2e \geq 0,$$

$$D \cdot C' = -4be + b(g + 1 + 2e - 4q) + 4c \geq -4be + b(g + 1 + 2e - 4q) + 4be > 0.$$

(iii) Case $e < 0$: for an irreducible curve $C' \equiv bC_0 + cF_0$ we have

(+) either $b = 1, c \geq 0$, so that

$$D \cdot C' = -4e + (g + 1 + 2e - 4q) + 4c > -4e + 2e = -2e > 0,$$

(+) or $b \geq 2, 2c \geq be$, and here

$$D \cdot C' = -4be + b(g + 1 + 2e - 4q) + 4c > -4be + 2be + 2be = 0.$$

This completes the proof. □

REMARK 3.5.2. If $q = 1$ then using [2, V, exer.2.12] it is easy to verify that the linear systems above are very ample if $-1 \leq e \leq g - 6$.

REMARK 3.5.3. In fact the method in the proof of Proposition 3.5.1 enables us to establish the very ampleness of the divisor $a_0C_0 + \bar{a}F_0$ on Σ_g^q with

$$a_0 > 0, \quad a \geq \max\left\{\frac{a_0e + 1}{2} + 2q, (a_0 + 1)e + 2q\right\}.$$

This is not sharp in general. The only case where it gives sharp estimates is $e = q = 0$ (cf. [2, V, 2.18]).

3.6. From the very ampleness of $|2C_0 + \bar{a}F_0|$ in a similar way as in [15] one can find a Lefschetz pencil of this linear system such that $4g + 4 - 8q$ distinct base points $p_1, \dots, p_{4g+4-8q}$ do not lie on C_0 and any two of them are not on the same fibre of π' . By blowing up the base points we obtain the desired fibration $f: X \rightarrow \mathbb{P}^1$ with maximal numbers r and s . Indeed if we denote by $\{E_{ij}\}_{i=1}^{4g+4-8q}$ the exceptional curves dominating points $\{p_i\}_{i=1}^{4g+4-8q}$ respectively then for the Néron-Severi group $\text{NS}(X)$ we have decomposition (3.4.3) with relation (3.4.4). Therefore $\rho = 4g + 6 - 8q$, and by (0.3) $r = 4g + 4 - 8q$. Furthermore it is easy to see that $c_1^2(X) = -4(g - 1)$. So (3.4.2) implies $s = 8g + 4 - 12q$.

REMARK 3.6.1. Examples with $q = 1$ give us the best upper bound of the

self-intersection of curves on irrational ruled surfaces (cf. [3]).

Note Added in Revision. A part of results presented here remains valid in positive characteristic (at least, $\neq 2$) due to Moriwaki's version of the Cornalba-Harris-Xiao inequality in any characteristic (cf. [10]). We will come back to this theme in the other occasion.

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