Title: An elementary construction of the representations of $SL(2, GF(q))$

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Citation: Osaka Journal of Mathematics. 6(2) P.329–P.338

Issue Date: 1969

Text Version: publisher

URL: https://doi.org/10.18910/8521

DOI: 10.18910/8521

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1. Introduction

Let $GF(q)$ be a field containing $q$ elements, $q$ odd. Let $GL(2, GF(q))$, the group of non-singular two-by-two matrices with entries in $GF(q)$, and let $G$ denote $SL(2, GF(q))$, the subgroup of $GL$ consisting of matrices with determinant one. In this paper, assuming a knowledge of certain of the characters of $GL$, we construct all the irreducible unitary representations of $G$. Our construction involves essentially no technique beyond the theory of induced representations and the orthogonality relations on a finite group. For a similarly elementary computation of the characters of $GL$ we refer the reader to [3]. In future papers we shall generalize the methods employed in this paper to construct the representations of the $n \times n$ matrix groups $GL(n, GF(q))$ and $SL(n, GF(q))$.

Kloosterman [2] was the first to describe all the irreducible matrix representations of $SL(2, GF(q))$. Weil in [5] generalizes and gives an alternative construction for Kloosterman's representations. In [4] Tanaka uses Weil's theory to construct representations and presents a complete and unified description of the representations of $G$. We also mention the paper [1] of Gelfand-Graev, which classifies but does not detail the actual construction of all the representations of $G$.

2. The representations

Let $B$ be the upper unipotent, $D$ the diagonal, and $T$ the upper triangular subgroups of $G$. Then $T=DB$. $G$ has order $q(q^2-1)$ and contains an abelian subgroup $R$ (unique up to conjugacy) of order $q+1$. Except for plus-or-minus the identity of $G$ elements of $R$ have characteristic roots in $GF(q^2)-GF(q)$. $R$ is isomorphic to the subgroup of $GF(q^2)^\times$ comprised of elements of norm one.

The $q+4$ equivalence classes of irreducible representations of $G$ break up roughly into two main classifications. The $\frac{1}{2}(q+5)$ representations of the
principal series” all contain $B$-invariant vectors. Those $\frac{1}{2}(q+3)$ inequivalent representations which do not contain $B$-invariant vectors we call discrete series. More precisely, the principal series include:

1. The trivial representation of degree 1, $U \equiv 1$;
2. A $q$-dimensional representation $U^1_{\alpha}$ which occurs with $U \equiv 1$ in the induced representation $\text{ind}_{\alpha}$;
3. $\frac{1}{2}(q-3)$ irreducible induced representations $U^\alpha = \text{ind}_{\alpha}$, where $\alpha$ is a one-dimensional representation of $T$ which is not real-valued. $U^\alpha$ has degree $q+1$ and $U^{\alpha'}$ is equivalent to $U^\alpha$ if and only if $\alpha' = \alpha$ or $\alpha^{-1}$.
4. Let $\alpha = \text{sgn}$, where $\text{sgn} \equiv 1$ and $\text{sgn}^2 \equiv 1$. Then $\text{ind}_{\alpha} = U^\alpha_{\alpha} = U^\alpha_{\text{sgn}} + U^\alpha_{\text{sgn}}$, the direct sum of two inequivalent irreducible representations, each of degree $\frac{1}{2}(q+1)$.

The discrete series are as follows:

5. If $\pi$ is a non-trivial character of $R$, then there is a representation $U^\pi$ of $G$ of degree $q-1$ associated with $\pi$. $U^\pi$ is characterized by the fact that it does not occur in $\text{ind}_{\pi}$. $U^\pi$ is irreducible if and only if $\pi$ is not real-valued. $U^\pi$ is equivalent to $U^{\pi'}$ if and only if $\pi' = \pi$ or $\pi^{-1}$, so there are $\frac{1}{2}(q-1)$ inequivalent irreducible representations of degree $q-1$.
6. If $\pi \equiv 1$, $\pi^2 \equiv 1$, then $U^\pi U^\pi_1 + U^\pi_2$, the direct sum of inequivalent representations of degree $\frac{1}{2}(q-1)$.

3. The construction of principal series

The construction of the representations of the principal series as induced representations is well-known. For completeness we discuss this problem in detail.

Let $\alpha$ be a one-dimensional representation of $T$. Since $B$ is the commutator subgroup of $T$, $\alpha(btb') = \alpha(t)$ for any $b$ and $b' \in B$ and $t \in T$. $T/B$ is canonically $D$, so $\alpha$ is the extension to $T$ of a character of the abelian group $D$. The mapping which identifies $d \in D$ with its upper diagonal entry regarded as an element of the multiplicative group $GF(q)^\times$ is an isomorphism. In this section, when convenient, we regard $\alpha$ as a function on $GF(q)^\times$ via this identification. Let $U^\alpha$ denote the representation of $G$ induced from $\alpha$.

By the definition of $U^\alpha$, $G$ acts by right translation in the space $V^\alpha$ which consists of complex-valued functions $\psi$ on $G$ satisfying

$$\psi(tg) = \alpha(t)\psi(g)$$

for all $t \in T$ and $g \in G$. Any such function is determined by its restriction to a set of representatives of $T \setminus G$. Since two matrices in $G$ with the same lower entries differ only by a left factor in $B$, $\psi \in V^\alpha$ implies $\psi(g) = \psi(g_{21}, g_{22}), g_{21}$ and
Representations of $SL(2, GF(q))$  

$g_{31}$ the lower entries of $g \in G$. Equation (3.1) entails

$$\psi(d^{-1}g_{31}, d^{-1}g_{32}) = \alpha(d)\psi(g_{31}, g_{32})$$

for $d \in GF(q)^*$, $g_{31}$ and $g_{32}$ as before, so $\psi$ is actually determined by its values, which may be chosen arbitrarily, on a set of representatives for the projective line over $GF(q)$.

**Theorem 3.1.** Let $\alpha$ be a one-dimensional representation of $T$. Let $U^*$ be the representation of $G$ induced from $\alpha$. $U^*$ is right translation in the space $V^*$ defined by relations (3.1) and (3.2).

1. The degree of $U^*$ is $q+1$.
2. $U^*$ is irreducible if and only if $\alpha^2 \equiv 1$.
3. $U^{*'}$ is equivalent to $U^*$ if and only if $\alpha' = \alpha$ or $\alpha^{-1}$.
4. $U^1$ decomposes into the direct sum of an irreducible representation of degree $q$ and the unique one-dimensional representation of $G$.
5. $U^{*n}$, where $\text{sgn} \equiv 1$ but $\text{sgn}^n \equiv 1$, decomposes into the direct sum of two inequivalent representations of degree $\frac{1}{2}(q+1)$.

**Proof.**

(1) A set of representatives for the projective line over $GF(q)$ (e.g. $\{(0, 1), (-1, z) | z \in GF(q)\}$) has cardinality $q+1$. In view of the above remarks this proves that $V^*$ has dimension $q+1$.

(2) The proofs of the remaining parts of this theorem depend upon an analysis of the commuting algebra of $U^*$.

Let $C^*$ be the convolution algebra of all complex-valued functions $f$ on $G$ satisfying $f(tgt') = \alpha(tt')f(g)$ for any $t, t' \in T$ and $g \in G$. Then $U^*(g_0)(f*\psi) = f^*U^*(g_0)\psi$ for any $g_0 \in G$ and $\psi \in V^*$, since $f \in C^*$ acting from the left by convolution keeps $V^*$ stable and commutes with right translation. Frobenius' reciprocity theorem says precisely that $C^*$ is large enough to be the full commuting algebra of $U^*$.

$f \in C^*$ is determined by its values on a set of representatives for the double cosets $T \backslash G / T$, e.g. $\{e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}$. Clearly, dim $C^* \leq 2$. Dim $C^* = 2$ if and only if $f(w) \neq 0$ for some $f \in C^*$, if and only if $\alpha(t)f(w) = f(tw) = f(wt^{-1}) = \alpha^{-1}(t)f(w)$ for all $t \in D$. Thus dim $C^* = 2$ if and only if $\alpha^2(t) \equiv 1$, so (2) is true.

(3) The space of intertwining operators between $V^*$ and $V'^*$, $\alpha \neq \alpha'$, is canonically the vector space $T^* \cdot a'$ of complex-valued functions on $G$ satisfying $f(tgt') = \alpha(t)f(g)\alpha'(t')$ for all $t, t' \in T$ and $g \in G$. It is spanned by any function $f$ which satisfies $\alpha(t)f(w) = f(w)\alpha'(t^{-1})$ for all $t \in D$. $f(w) \neq 0$ implies $\alpha' = \alpha^{-1}$.

(4) $V^1$ contains the constant functions on $G$ as a stable subspace. The orthogonal complement of this one dimensional module must be an irreducible $q$-dimensional representation space for $G$. 

(5) By the analysis in (2) we know that $U^\text{sgn}$ decomposes into the direct sum of two inequivalent representations, $U_1^\text{sgn}+U_2^\text{sgn}=U^\text{sgn}$. By Frobenius' reciprocity theorem $\text{res}_{G/F}\ U^\text{sgn}$, for $\nu=1$ or 2, contains sgn and no other one-dimensional representation of $T$. Since $G/\{\pm e\}$ is a simple group, $G$ has no non-trivial one-dimensional representations. Therefore, Lemma (4.3) implies that the degree of $U^\text{sgn}_\nu$ is $\frac{1}{2}(q+1)$, $\nu=1$ or 2.

REMARK. To complete our description of the representations of the principal series we need to be more specific about the $G$-stable subspaces $V^\text{sgn}_1$ and $V^\text{sgn}_2$ of $V^\text{sgn}$. Set $\phi(-1, z)=\Phi(z)$ for $z \in GF(q)$, where $\Phi$ is an additive character of $GF(q)$; let $\phi(0, 1)=0$. Then $\Phi$ extends uniquely to a function in $V^\text{sgn}$ and $U^\text{sgn}(b(u))\phi=\Phi(u)\phi$, where $u$ is the super diagonal entry of $b(u) \in B$. Moreover, $U^\text{sgn}(d)\phi(-1, z)=\text{sgn}(d)\phi(-1, d^{-2}z)=\text{sgn}(d)\Phi(d^{-2}z)$ for all $z \in GF(q)$, $d \in D$ (identified with $GF(q)^*$); $U^\text{sgn}(d)\phi(0, 1)=0$. Let $\Phi \equiv 1$. Then the $\frac{1}{2}(q-1)$ functions $\phi'$ which correspond to characters $\Phi'$ such that $\Phi'(d^{-2}z)=\Phi(z)$ for some $d \in GF(q)^*$ belong to $V^\text{sgn}$; the other non-trivial additive characters of $GF(q)$ must correspond to elements of $V^\text{sgn}_\nu$, $1 \leq \nu \neq \nu' \leq 2$. $V^\text{sgn}$ also contains a vector $\psi$ satisfying $\psi(\text{tgt'})=\text{sgn}(tt')\psi(g)$ for all $t, t' \in T$, $g \in G$. In fact $\psi$ may be chosen to be an idempotent in $C^\text{sgn}$.

**Proposition 3.2.** Set

$$
\psi(g) = \frac{1}{|G|} \sum_{t \in T} \text{sgn}(t) , \quad \text{if } g = t \in T ;
$$

$$
= \frac{1}{2} \frac{|G|}{|T|} \left( q \text{sgn}(-1) \right)^{-1/2} \text{sgn}(t) ,
$$

if $g = t \omega b$, with $t \in T$, $b \in B$, and $\omega = \left| \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right|$.

Then $\psi$ is an idempotent in the algebra $C^\text{sgn}$. There are two choices $\psi$, $\psi'$ depending on the sign of $[\text{sgn}(-1)]^{1/2}$. Clearly, $\psi + \psi'$ is the identity in $C^\text{sgn}$. The function $\phi$ defined in the preceding remark and corresponding to the non-trivial character $\Phi$ of $GF(q)$ belongs to the same $G$-irreducible subspace of $V^\text{sgn}$ as $\psi$ if and only if $\sum_{x \in G} \text{sgn}(x)\Phi(x) = [q \text{sgn}(-1)]^{1/2}$ (with the same choice for the sign of the right hand side as in the definition of $\psi$).

**Proof.** To show that $\psi$ is an idempotent in $C^\text{sgn}$ it suffices to show that $\psi*\psi(e)=\psi(e)$ and $\psi*\psi(w)=\psi(w)$. We have

$$
\psi*\psi(g) = \frac{1}{|G|} \sum_{x \in G} \psi(x)\psi(x^{-1}g) = \frac{|T|}{|G|} \sum_{x \in G/T} \psi(x)\psi(x^{-1}g)
$$

$$
= \frac{|T|}{|G|} \{ \psi(e)\psi(g) + \sum_{x \in GF(q)} \psi(w)\psi(w^{-1}b^{-1}(u)g) \} .
$$
Therefore,

\[ \psi^* \psi(e) = \frac{|T|}{|G|} \cdot \frac{1}{4} \frac{|G|^2}{|T|^2} \{1 + [q \text{ sgn}(-1)]^{-1} \text{sgn}(-1)q\} \]

\[ = \psi(e). \]

\[ \psi^* \psi(w) = \frac{|T|}{|G|} \cdot \frac{1}{4} \frac{|G|^2}{|T|^2} \{[q \text{ sgn}(-1)]^{-1/2} + [q \text{ sgn}(-1)]^{-1/2} \]

\[ + [q \text{ sgn}(-1)]^{-1} \sum_{x \in GF(q) \setminus G} \text{sgn}(-u) \}. \]

The last term on the right, being a character sum, is zero. It arises from the relation

\[ w^{-1} b^{-1}(u) w = \begin{bmatrix} 0 & -1 & 1 & -u \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \text{if } u \neq 0. \]

Thus, \( \psi^* \psi(w) = \psi(w) \).

Finally, since \( \psi \) is a minimal idempotent in \( C^{\text{sgn}} \), \( \psi^* \phi = \phi \), if \( \psi \in V_{\text{sgn}}^* \) and \( \phi \in V_{\text{sgn}}^* \). If \( \phi \in V_{\text{sgn}}^* \), then \( \phi \in V_{\text{sgn}}^* \), so \( \psi^* \phi = 0 \).

\[ \psi^* \phi(w) = \frac{|T|}{|G|} \sum_{x \in GF(q)} \psi(x) \phi(x^{-1} w) \]

\[ = \frac{|T|}{|G|} \{ \psi(e) \phi(w) + \psi(w) \sum_{x \in GF(q) \setminus G} \phi(w^{-1} b^{-1}(u) w) \}. \]

Using relation * as well as the definitions of \( \psi \) and \( \phi \), we obtain

\[ \psi^* \phi(w) = \phi(w) \left\{ \frac{1}{2} + \frac{1}{2} [q \text{ sgn}(-1)]^{-1/2} \sum_{x \in GF(q) \setminus G} \text{sgn}(-u) \Phi(u) \right\}, \]

which implies the last part of the proposition.

4. **The construction of discrete series for \( GL(2, GF(q)) \)**

Let \( \Pi \) be a character of \( GF(q^2)^x \) whose restriction to the elements of norm one is non-trivial. Then \( \Pi \) corresponds to a representation \( Q_{\Pi} \) of the discrete series of \( G=GL(2, GF(q)) \) (i.e. \( \text{res } Q_{\Pi} \equiv 1 \)). It turns out that \( \text{res } Q_{\Pi} \) is an irreducible representation of \( \mathcal{G} \), the triangle subgroup of \( G \). To determine a space of functions which transforms under \( \mathcal{G} \) as \( Q_{\Pi} \) we find an irreducible representation \( m \) of \( \mathcal{G} \) such that \( m= \text{res } Q_{\Pi} \). Then, using the trace of \( Q_{\Pi} \) (which we assume known) we extend the matrix coefficients of \( m \) to \( \mathcal{G} \). To determine the discrete series of \( G \) we study \( \text{res } Q_{\Pi} \).

Let \( \mathcal{D} \) be the diagonal subgroup of \( G \) and let \( \alpha \) be a character of \( \mathcal{D} \). Ind \( \alpha = M^* \) is right translation in the space of complex-valued functions on \( \mathcal{D} \)
which satisfy \( \psi(dt) = \alpha(d) \psi(t) \) for all \( d \in \mathcal{D} \) and \( t \in \mathcal{I} \). Since \( B \) represents \( \mathcal{D} \setminus \mathcal{I} \), we may consider \( M^* \) as acting in a vector space \( B^* \) of complex-valued functions on \( B \). We write \( \psi \in B^* \) as a function of the super diagonal entries of elements of \( B \). Then

\[
(4.1) \quad M^*(db(u))\psi(x) = \alpha(d)\psi(d_{11}^{-1}d_{22}x + u)
\]

for any \( d \in \mathcal{D} \) and \( b(u) \) the element of \( B \) with superdiagonal entry \( u \in GF(q) \), \( d_{11} \) and \( d_{22} \) the non-zero entries of \( d \).

To see how \( M^* \) decomposes take as an orthonormal basis of \( B^* \) the \( q \) characters of \( B \). The operators \( M^*(b) \) for \( b \in B \) obviously diagonalize with respect to this basis. Let \( \Phi_0 \) be the trivial character of \( B \). Clearly \( \Phi_0 \) transforms under \( M^* \) as the one-dimensional representation \( \alpha \) of \( \mathcal{I} \). Now let \( \Phi \) be a fixed non-trivial character of \( B \). For \( i \in GF(q)^\times \) set \( \Phi_i(x) = \Phi(ix) \) for all \( x \in GF(q) \). Then \( \Phi_i \) is a non-trivial character of \( B \) and every non-trivial character of \( B \) is of the form \( \Phi_i \), for some \( i \in GF(q)^\times \). (4.1) entails that, except for scalar factors, \( \mathcal{D} \) acts transitively on the non-trivial characters of \( B \). Since \( M^* \) is completely reducible, we see that the \((q-1)\)-dimensional subspace of \( B^* \) spanned by the non-trivial characters of \( B \) must be irreducible. Call the resulting representation \( m_\alpha \).

**Lemma 4.1.** An irreducible representation of \( \mathcal{I} \) is either of degree one or \( q-1 \). An irreducible \((q-1)\)-dimensional representation of \( \mathcal{I} \) is determined by its restriction to the center of \( \mathcal{I} \).

Proof. If an irreducible representation of \( \mathcal{I} \) is not one-dimensional, it is equivalent to a representation \( m_\alpha \) for some character \( \alpha \) of \( \mathcal{D} \). Thus it is \((q-1)\)-dimensional. By Frobenius' reciprocity theorem characters \( \alpha' \) which occur in \( \text{res} \ m_\alpha \) occur with multiplicity one. Since \( m_\alpha \) is irreducible, every \( \alpha' \) contained in \( \text{res} \ m_\alpha \) must have the same values on the center of \( \mathcal{I} \) (i.e. the scalars). There are \( q-1 \) distinct characters of \( \mathcal{D} \) which agree on the scalars, so they must all occur in \( \text{res} \ m_\alpha \). By Frobenius' theorem, \( m_\alpha \) is equivalent to \( m_{\alpha'} \), for all such \( \alpha' \).

**Lemma 4.2.** Let \( \Phi \) be a non-trivial character of \( B \). For \( i \in GF(q)^\times \) set \( \Phi_i(x) = \Phi(ix) \) for all \( x \in GF(q) \) (considered as super-diagonal entries of elements of \( B \)). The matrix coefficients of the representation \( m_\alpha \) with respect to the basis for \( B^* \) consisting of the \( q-1 \) non-trivial characters \( \{\Phi_i\}_{i \in GF(q)^\times} \) of \( B \) are the \((q-1)^2 \) functions

\[
(4.2) \quad m_{ij}^*(t) = \langle m_\alpha(t)\Phi_j, \Phi_i \rangle, \quad i \text{ and } j \in GF(q)^\times ,
\]

\[
= \alpha(d)\Phi_j(u), \quad \text{if } \Phi_j(d_{11}^{-1}d_{22}x) = \Phi_i(x) \text{ for all } x \in GF(q);
\]

\[
= 0, \quad \text{otherwise}.
\]
In (4.2) \( t = db(u) \), where \( u \in GF(q) \) is the super-diagonal entry of the matrix \( b(u) \in B \) and \( d_{11} \) and \( d_{22} \) are the diagonal entries of \( d \in D \).

Proof. Immediate from equation (4.1).

**Lemma 4.3.** Let \( m_\alpha \) be an irreducible representation of \( G \) of degree \( q-1 \). Then \( \text{res} \ m_\alpha \) decomposes into inequivalent representations of degree \( \frac{1}{2}(q-1) \). Any irreducible representation of \( T \) is either one-dimensional or \( \frac{1}{2}(q-1) \)-dimensional.

Proof. \( \text{Res} \ m_\alpha \) decomposes simply; if \( \text{res} \ m_\alpha \) decomposes, the component representations must be inequivalent. By (4.1) \( M^\alpha(d) \Phi(x) = \alpha(d) \Phi(d^{-2} x) \) for \( d \in D \), so two characters \( \Phi \) and \( \Phi' \) of \( B \) occur in the restriction to \( B \) of the same irreducible subrepresentation of \( \text{res} \ m_\alpha \) if and only if \( \Phi'(x) = \Phi(a^2 x) \) for some \( a \in GF(q)^* \) and all \( x \in GF(q) \). Since half the characters of \( B \) satisfy this relation and half do not, \( \text{res} \ m_\alpha \) contains two irreducible representations, each of degree \( \frac{1}{2}(q-1) \). The last statement in Lemma (4.3) follows from the fact that any irreducible representation of \( T \) occurs in the restriction to \( T \) of some irreducible representation of \( G \).

**Lemma 4.4.** Let \( G \) be a finite group and \( H \) a subgroup of \( G \). Let \( U \) be a unitary representation of \( G \) whose degree is \( d \) and character is \( \chi \). Assume \( \text{res} \ U \) is irreducible. Then, for any matrix coefficient \( u_{ij} \) of \( U \), \( 1 \leq i, j \leq d \), and any \( g \in G \)

\[
u_{ij}(g) = \frac{d}{|H|} \sum_{h \in H} u_{ij}(h) X(h^{-1} g)
\]

Proof.

\[
\frac{d}{|H|} \sum_{h \in H} u_{ij}(h) X(h^{-1} g) = \frac{d}{|H|} \sum_{h \in H} u_{ij}(h) \sum_{k=1}^{d} u_{kh}(h^{-1} g)
\]

\[
= \frac{d}{|H|} \sum_{h \in H} u_{ij}(h) \sum_{k=1}^{d} \sum_{i=1}^{d} u_{ki}(h^{-1}) u_{ih}(g)
\]

\[
= \frac{d}{|H|} \sum_{i=1}^{d} u_{ih}(g) \sum_{k=1}^{d} u_{ij}(h) \bar{u}_{ih}(h)
\]

by Schur's orthogonality relations on \( G \).

Lemma (4.1) implies that for any representation \( \Upsilon \) of the discrete series of \( G \), \( \text{res} \ \Upsilon \) is equivalent to an irreducible representation \( m_\alpha \), where \( m_\alpha \) is, up to equivalence, the unique irreducible \((q-1)\)-dimensional representation of \( G \) which agrees with \( \Upsilon \) on the scalars. Since \( G = G \cup \tau w B \), \( w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), it
suffices to compute the matrix coefficients for \( Q^\Pi \) at \( w \) in order to extend them from \( \mathcal{D} \) to all of \( G \). For this purpose we need the character \( X^\Pi \) of \( Q^\Pi \) (To find directions for the easy computation of \( X^\Pi \) consult [3], p. 227.). Figure 1 presents \( X^\Pi \).

<table>
<thead>
<tr>
<th>Conjugacy Classes on ( G )</th>
<th>Values of ( X^\Pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} )</td>
<td>( \Pi(\lambda)(q-1) )</td>
</tr>
<tr>
<td>( \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} )</td>
<td>(-\Pi(\lambda))</td>
</tr>
<tr>
<td>( \begin{pmatrix} 0 &amp; 0 \ 1 &amp; 1 \end{pmatrix} )</td>
<td>(0)</td>
</tr>
<tr>
<td>( \begin{pmatrix} \alpha &amp; 0 \ 0 &amp; \beta \end{pmatrix} )</td>
<td>(-\Pi(\epsilon)+\Pi(\epsilon^t))</td>
</tr>
</tbody>
</table>

\( \epsilon, \lambda \in GF(q)^*, \ t+1; \ \epsilon=\alpha+\beta\sqrt{\zeta}, \ \alpha, \beta, \zeta \in GF(q) \) with \( \zeta \) not a square and \( \beta \neq 0 \). * Matrices with the same characteristic roots are conjugate.

Figure 1.

**Lemma 4.5.** Let \( X^\Pi \) be the character of a representation \( Q^\Pi \) of the discrete series of \( \mathcal{G} \). Let \( \alpha \) be a character of \( \mathcal{D} \) such that \( \alpha(\lambda)=\Pi(\lambda) \) for any scalar matrix \( \lambda \in \mathcal{D} \). Then \( m_\alpha \) is equivalent to \( \text{res} \ Q^\Pi \). Fix a non-trivial character \( \Phi \) of \( B \). Let \( \{m_{ij}^\alpha\}_{i,j \in GF(q)^*} \) be the matrix coefficients of \( m_\alpha \) with respect to the basis \( \{\Phi_f\}_{f \in GF(q)^*} \) of \( B^\alpha \) (see Lemma (4.2) and relation (4.2)). The matrix coefficients \( m_{ij}^\alpha \) are the restrictions to \( \mathcal{D} \) of matrix coefficients \( u_{ij}^\Pi \) of \( Q^\Pi \). For \( w=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( \delta(a,b) \) the diagonal matrix with diagonal entries \( \delta_{11}=a \) and \( \delta_{22}=b \),

\[
(4.3) \quad u_{ij}^\Pi(w) = -\alpha^{-1}(\delta(i,j))q^{-1} \sum_{\epsilon \in GF(q)^*} \Pi(\epsilon) \Phi(\epsilon+\epsilon^t) .
\]

**Proof.** By Lemma (4.4) and relation (4.2)

\[
u_{ij}^\Pi(w) = \frac{q-1}{|\mathcal{D}|} \sum_{t \in \mathcal{D}} m_{ij}^\alpha(t) X^\Pi(t^{-1}w)
= q^{-1} \sum_{u \in GF(q)^*} \alpha(\delta) \Phi_f(u) X^\Pi(b^{-1}(u)\delta^{-1}w) ,
\]

where \( \delta=\delta(j,i) \) and \( b(u) \in B \) has super diagonal entry \( u \). Use of the explicit formula for \( X^\Pi \) easily yields (4.3).

**Theorem 4.6.** Let \( \Pi \) be a character of \( GF(q'^*) \) whose restriction to the elements of norm one is not trivial. Let \( X^\Pi \) be the character of the irreducible representation of \( \mathcal{G} \) associated with \( \Pi \). Let \( \alpha \) be any character of \( \mathcal{D} \) which agrees
with $\Pi$ on the scalar matrices. Then $m_a$ is res $Q^\Pi$ and $B^\sigma$, the representation space of $m_a$, is a representation space for $Q^\Pi$. Fix a non-trivial character $\Phi$ of $B$ and write it as a function of the super-diagonal entries of elements of $B$. Take as a basis for $B^\sigma$ the $q-1$ non-trivial characters $\{\Phi_i\}_{i \in GF(q)^*}$, where $\Phi_i(x) = \Phi(ix)$ for all $x \in GF(q)$. Matrix coefficients for $Q^\Pi$ acting in $B^\sigma$ are as follows. For $i, j \in GF(q)^*$ set $u_{ij}^\Pi = \langle Q^\Pi (g) \Phi_j, \Phi_i \rangle$. If $g = db(u)$, where $d \in D$ has diagonal entries $d_1$ and $d_2$, and $b(u) \in B$ has super-diagonal entry $u \in GF(q)$, then

$$u_{ij}^\Pi (g) = \alpha(d) \Phi_j(u), \quad \text{provided } d_1^{-1} d_2 = j^{-1} i;$$

$$= 0, \quad \text{otherwise.}$$

If $g = b(v) wd(b(u))$, where $d \in D$ has diagonal entries $d_1$ and $d_2$, $w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $b(u)$ and $b(v) \in B$ have super-diagonal entries $u$ and $v$ respectively then

$$u_{ij}^\Pi (g) = \Phi(iv+ju)[ - \Pi(d_1) \alpha^{-1}(\delta(i,j)) q^{-1} \sum_{\varepsilon \in GF(q)^*} \Pi(\varepsilon) \Phi(\varepsilon+\varepsilon^q) ]$$

where $\delta(i,j)$ is the diagonal matrix with upper entry $i$ and lower entry $j$ and $l = ij d_1 d_2$.

Proof. Relation (4.4) is the same as (4.2), so no proof is needed. To prove (4.5) note first that $u_{ij}^\Pi (b(v) gb(u)) = \Phi_i(v) u_{ij}^\Pi (g) \Phi_j(u)$. Moreover, $u_{ij}^\Pi (wd) = \alpha(d) u_{ij}^\Pi d_1^{d_2} (w)$. Use of (4.3) to express $u_{ij}^\Pi (g)$ as an exponential sum leads to a proof of (4.5).

5. Discrete series of $G$

Let $\Pi$ be a character of $GF(q)^*$ whose restriction to $N^1$, the elements of norm one in $GF(q)^*$, is not trivial. Let $\pi$ be $\Pi$ restricted to $N^1$. Let $Q^\Pi$ be the representation of the discrete series of $G$ associated with $\Pi$. Set $U^\pi = \text{res } Q^\Pi$. The trace $X^\pi$ of $U^\pi$ is the restriction to $G$ of $X^{Q^\Pi}$, so, up to equivalence, $U^\pi$ depends only on the values of $\Pi$ restricted to $N^1$. Furthermore, $U^\pi$ and $U^{\pi'}$ are equivalent if and only if $\pi' = \pi$ or $\pi^{-1}$, since, if $\pi'$ is the restriction to $N^1$ of a character $\Pi'$ of $GF(q)^*$, $X^\pi = X^{\pi'}$ if and only if $\pi' = \pi$ or $\pi^{-1}$.

We may take as representatives for the conjugacy classes in $G$ those representatives for conjugacy classes in $Q$ which lie in $G$ (see Figure 1.). However, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix}$, $\xi$ a non-square, are not conjugate in $G$; similarly $- \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $- \begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix}$.

Theorem 5.1. Let $\pi$ be a non-trivial character of $N^1$ and let $U^\pi$ be the corresponding representation of $G$ defined above. $U^\pi$ is irreducible if and only if $\pi^2 \equiv 1$. If $\pi^2 \equiv 1$, $U^\pi = U^\pi_1 + U^\pi_2$, the direct sum of inequivalent $\frac{1}{2}(q-1)$-dimensional representations.
Proof. It suffices to show that \(|G|^{-1} \sum_{g \in G} |X^\pi(g)|^2 = 1\), if \(\pi^2 = 1\), and 2, otherwise. The computation is easy and we omit it. In the case that \(U^\pi\) is reducible, the components are \(\frac{1}{2}(q-1)\)-dimensional and inequivalent, since, according to Lemma (4.3), this statement holds already for \(\text{res}_{q^{1/2}} U^\pi\). We may use Lemma (4.3) to obtain representation spaces for \(U^\pi_1\) and \(U^\pi_2\).

There are \(q+4\) conjugacy classes in \(G\) and we have accounted for this many equivalence classes of irreducible representations, so our description of the irreducible representations of \(SL(2, GF(q))\) is complete.

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References