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<th>An elementary construction of the representations of $\text{SL}(2, \text{GF}(q))$</th>
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<tr>
<td><strong>Author(s)</strong></td>
<td>Silberger, Allan J.</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 6(2) P.329–P.338</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1969</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/8521">https://doi.org/10.18910/8521</a></td>
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<tr>
<td><strong>DOI</strong></td>
<td>10.18910/8521</td>
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AN ELEMENTARY CONSTRUCTION OF THE REPRESENTATIONS OF SL(2, GF(q))

ALLAN J. SILBERGER

(Received February 10, 1969)

1. Introduction

Let $GF(q)$ be a field containing $q$ elements, $q$ odd. Let $G$ denote $GL(2, GF(q))$, the group of non-singular two-by-two matrices with entries in $GF(q)$, and let $G$ denote $SL(2, GF(q))$, the subgroup of $G$ consisting of matrices with determinant one. In this paper, assuming a knowledge of certain of the characters of $G$, we construct all the irreducible unitary representations of $G$. Our construction involves essentially no technique beyond the theory of induced representations and the orthogonality relations on a finite group. For a similarly elementary computation of the characters of $G$ we refer the reader to [3]. In future papers we shall generalize the methods employed in this paper to construct the representations of the $n \times n$ matrix groups $GL(n, GF(q))$ and $SL(n, GF(q))$.

Kloosterman [2] was the first to describe all the irreducible matrix representations of $SL(2, GF(q))$. Weil in [5] generalizes and gives an alternative construction for Kloosterman’s representations. In [4] Tanaka uses Weil’s theory to construct representations and presents a complete and unified description of the representations of $G$. We also mention the paper [1] of Gelfand-Graev, which classifies but does not detail the actual construction of all the representations of $G$.

2. The representations

Let $B$ be the upper unipotent, $D$ the diagonal, and $T$ the upper triangular subgroups of $G$. Then $T=DB$. $G$ has order $q(q^2-1)$ and contains an abelian subgroup $R$ (unique up to conjugacy) of order $q+1$. Except for plus-or-minus the identity of $G$ elements of $R$ have characteristic roots in $GF(q^2)-GF(q)$. $R$ is isomorphic to the subgroup of $GF(q^2)^\times$ comprised of elements of norm one.

The $q+4$ equivalence classes of irreducible representations of $G$ break up roughly into two main classifications. The $\frac{1}{2}(q+5)$ representations of the
"principal series" all contain $B$-invariant vectors. Those $\frac{1}{2}(q+3)$ inequivalent representations which do not contain $B$-invariant vectors we call discrete series.

More precisely, the principal series include:

1. The trivial representation of degree 1, $U \equiv 1$;
2. A $q$-dimensional representation $U^1$ which occurs with $U \equiv 1$ in the induced representation $\text{ind}_{T+G}

3. $\frac{1}{2}(q-3)$ irreducible induced representations $U^\alpha = \text{ind}_{T+G} \alpha$, where $\alpha$ is a one-dimensional representation of $T$ which is not real-valued. $U^\alpha$ has degree $q+1$ and $U^{\alpha'}$ is equivalent to $U^\alpha$ if and only if $\alpha' = \alpha$ or $\alpha^{-1}$.

4. Let $\alpha = \text{sgn}$, where $\text{sgn} \equiv 1$ and $\text{sgn}^2 \equiv 1$. Then $\text{ind}_{T+G} \text{sgn} = U^\text{sgn} = U_1^\text{sgn} + U_2^\text{sgn}$, the direct sum of two inequivalent irreducible representations, each of degree $\frac{1}{2}(q+1)$.

The discrete series are as follows:

5. If $\pi$ is a non-trivial character of $R$, then there is a representation $U^\pi$ of $G$ of degree $q-1$ associated with $\pi$. $U^\pi$ is characterized by the fact that it does not occur in $\text{ind}_\pi$. $U^\pi$ is irreducible if and only if $\pi$ is not real-valued. $U^\pi$ is equivalent to $U^{\pi'}$ if and only if $\pi' = \pi$ or $\pi^{-1}$, so there are $\frac{1}{2}(q-1)$ inequivalent irreducible representations of degree $q-1$.

6. If $\pi \equiv 1$, $\pi^2 \equiv 1$, then $U^\pi = U_1^\pi + U_2^\pi$, the direct sum of inequivalent representations of degree $\frac{1}{2}(q-1)$.

3. The construction of principal series

The construction of the representations of the principal series as induced representations is well-known. For completeness we discuss this problem in detail.

Let $\alpha$ be a one-dimensional representation of $T$. Since $B$ is the commutator subgroup of $T$, $\alpha(bb') = \alpha(t)$ for any $b$ and $b' \in B$ and $t \in T$. $T/B$ is canonically $D$, so $\alpha$ is the extension to $T$ of a character of the abelian group $D$. The mapping which identifies $d \in D$ with its upper diagonal entry regarded as an element of the multiplicative group $GF(q)^\times$ is an isomorphism. In this section, when convenient, we regard $\alpha$ as a function on $GF(q)^\times$ via this identification. Let $U^\alpha$ denote the representation of $G$ induced from $\alpha$.

By the definition of $U^\alpha$, $G$ acts by right translation in the space $V^\alpha$ which consists of complex-valued functions $\psi$ on $G$ satisfying

$$\psi(tg) = \alpha(t)\psi(g)$$

for all $t \in T$ and $g \in G$. Any such function is determined by its restriction to a set of representatives of $T/G$. Since two matrices in $G$ with the same lower entries differ only by a left factor in $B$, $\psi \in V^\alpha$ implies $\psi(g) = \psi(g_{21}, g_{22})$, $g_{21}$ and
\( g_{21} \) the lower entries of \( g \in G \). Equation (3.1) entails
\[
\varphi(d^{-1}g_{21}, d^{-1}g_{22}) = \alpha(d)\varphi(g_{21}, g_{22})
\]
for \( d \in GF(q)^* \), \( g_{21} \) and \( g_{22} \) as before, so \( \varphi \) is actually determined by its values, which may be chosen arbitrarily, on a set of representatives for the projective line over \( GF(q) \).

**Theorem 3.1.** Let \( \alpha \) be a one-dimensional representation of \( T \). Let \( U^* \) be the representation of \( G \) induced from \( \alpha \). \( U^* \) is right translation in the space \( V^* \) defined by relations (3.1) and (3.2).

1. The degree of \( U^* \) is \( q+1 \).
2. \( U^* \) is irreducible if and only if \( \alpha^2 \equiv 1 \).
3. \( U^* \) is equivalent to \( U^* \) if and only if \( \alpha' = \alpha \) or \( \alpha^{-1} \).
4. \( U^* \) decomposes into the direct sum of an irreducible representation of degree \( q \) and the unique one-dimensional representation of \( G \).
5. \( U^s \), where \( sgn \equiv 1 \) but \( sgn^2 \equiv 1 \), decomposes into the direct sum of two inequivalent representations of degree \( \frac{q}{2}(q+1) \).

**Proof.**

1. A set of representatives for the projective line over \( GF(q) \) (e.g. \( \{(0, 1), (-1, z) | z \in GF(q)\} \)) has cardinality \( q+1 \). In view of the above remarks this proves that \( V^* \) has dimension \( q+1 \).
2. The proofs of the remaining parts of this theorem depend upon an analysis of the commuting algebra of \( U^* \).

Let \( C^* \) be the convolution algebra of all complex-valued functions \( f \) on \( G \) satisfying \( f(tg) = \alpha(t)f(g) \) for any \( t \), \( t' \in T \) and \( g \in G \). Then \( U^*(g_0)(f \ast \varphi) = f \ast U^*(g_0)\varphi \) for any \( g_0 \in G \) and \( \varphi \in V^* \), since \( f \in C^* \) acting from the left by convolution keeps \( V^* \) stable and commutes with right translation. Frobenius’ reciprocity theorem says precisely that \( C^* \) is large enough to be the full commuting algebra of \( U^* \).

\( f \in C^* \) is determined by its values on a set of representatives for the double cosets \( T \setminus G/T \), e.g. \( \{e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \} \). Clearly, \( \dim C^* \leq 2 \). \( \dim C^* = 2 \) if and only if \( f(w) = 0 \) for some \( f \in C^* \), if and only if \( \alpha(t)f(w) = f(tw) = f(wt^{-1}) = \alpha^{-1}(t)f(w) \) for all \( t \in D \). Thus \( \dim C^* = 2 \) if and only if \( \alpha(t) \equiv 1 \), so (2) is true.

3. The space of intertwining operators between \( V^* \) and \( V^* \), \( \alpha \equiv \alpha' \), is canonically the vector space \( T^{s, s'} \) of complex-valued functions on \( G \) satisfying \( f(tg) = \alpha(t)f(g) \alpha'(t') \) for all \( t, t' \in T \) and \( g \in G \). It is spanned by any function \( f \) which satisfies \( \alpha(t)f(w) = f(w)\alpha'(t^{-1}) \) for all \( t \in D \). \( f(w) = 0 \) implies \( \alpha' = \alpha^{-1} \).

4. \( V^* \) contains the constant functions on \( G \) as a stable subspace. The orthogonal complement of this one dimensional module must be an irreducible \( q \)-dimensional representation space for \( G \).
(5) By the analysis in (2) we know that $U^\text{sgn}$ decomposes into the direct sum of two inequivalent representations, $U_1^\text{sgn} + U_2^\text{sgn} = U^\text{sgn}$. By Frobenius' reciprocity theorem $\text{res} \ U_\nu^\text{sgn}$, for $\nu=1$ or $2$, contains $\text{sgn}$ and no other one-dimensional representation of $T$. Since $G/\{\pm e\}$ is a simple group, $G$ has no non-trivial one-dimensional representations. Therefore, Lemma (4.3) implies that the degree of $U_\nu^\text{sgn}$ is $\frac{1}{2}(q+1)$, $\nu=1$ or $2$.

Remark. To complete our description of the representations of the principal series we need to be more specific about the $G$-stable subspaces $V_1^\text{sgn}$ and $V_2^\text{sgn}$ of $V^\text{sgn}$. Set $\phi(-1, z) = \Phi(z)$ for $z \in GF(q)$, where $\Phi$ is an additive character of $GF(q)$; let $\phi(0, 1) = 0$. Then $\phi$ extends uniquely to a function in $V^\text{sgn}$ and $\text{Usgn}(b(u))\phi = \Phi(u)\phi$, where $u$ is the super diagonal entry of $b(u) \in B$. Moreover, $\text{Usgn}(d)\phi(-1, z) = \text{sgn}(d)\phi(-1, d^{-2}z) = \text{sgn}(d)\Phi(d^{-2}z)$ for all $z \in GF(q)$, $d \in D$ (identified with $GF(q)^*$); $\text{Usgn}(d)\phi(0, 1) = 0$. Let $\Phi \equiv 1$. Then the $\frac{1}{2}(q-1)$ functions $\phi'$ which correspond to characters $\Phi'$ such that $\Phi'(d^{-2}z) = \Phi(z)$ for some $d \in GF(q)^*$ belong to $V_2^\text{sgn}$; the other non-trivial additive characters of $GF(q)$ must correspond to elements of $V_1^\text{sgn}$, $1 \leq \nu = \nu' \leq 2$. $V_2^\text{sgn}$ also contains a vector $\psi$ satisfying $\psi(tt') = \text{sgn}(tt')\psi(g)$ for all $t, t' \in T, g \in G$. In fact $\psi$ may be chosen to be an idempotent in $C^\text{sgn}$.

Proposition 3.2. Set

$$\psi(g) = \begin{cases} \frac{1}{2} \frac{|G|}{|T|} \text{sgn}(t), & \text{if } g = t \in T; \\ \frac{1}{2} \frac{|G|}{|T|} (q \text{sgn}(-1))^{-1/2} \text{sgn}(t), & \text{if } g = twb, \text{with } t \in T, b \in B, \text{ and } w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{cases}$$

Then $\psi$ is an idempotent in the algebra $C^\text{sgn}$. There are two choices $\psi, \psi'$ depending on the sign of $(\text{sgn}(-1))^{1/2}$. Clearly, $\psi + \psi'$ is the identity in $C^\text{sgn}$. The function $\phi$ defined in the preceding remark and corresponding to the non-trivial character $\Phi$ of $GF(q)$ belongs to the same $G$-irreducible subspace of $V^\text{sgn}$ as $\psi$ if and only if $\sum_{x \in G} \Phi(-x) = [q \text{sgn}(-1)]^{1/2}$ (with the same choice for the sign of the right hand side as in the definition of $\psi$).

Proof. To show that $\psi$ is an idempotent in $C^\text{sgn}$ it suffices to show that $\psi \ast \psi(e) = \psi(e)$ and $\psi \ast \psi(w) = \psi(w)$. We have

$$\psi \ast \psi(g) = \frac{1}{|G|} \sum_{x \in G} \psi(x)\psi(x^{-1}g) = \frac{|T|}{|G|} \sum_{x \in G/T} \psi(x)\psi(x^{-1}g)$$

$$= \frac{|T|}{|G|} \left\{ \psi(e)\psi(g) + \sum_{x \in GF(q)} \psi(w)\psi(w^{-1}b^{-1}(u)g) \right\}.$$
Therefore,
\[
\psi^*\psi(e) = \frac{|T|}{|G|} \cdot \frac{1}{4} \frac{|G|^2}{|T|^2} \left( 1 + [q \text{ sgn}(-1)]^{-1} \text{sgn}(-1)q \right)
\]
\[= \psi(e). \]

\[
\psi^*\psi(w) = \frac{|T|}{|G|} \cdot \frac{1}{4} \frac{|G|^2}{|T|^2} \left( [q \text{ sgn}(-1)]^{-1/2} + [q \text{ sgn}(-1)]^{-1/2} \right)
\]
\[+ [q \text{ sgn}(-1)]^{-1} \sum_{x \in GF(q)} \text{sgn}(-u). \]

The last term on the right, being a character sum, is zero. It arises from the relation
\[
w^{-1}b^{-1}(u)w = \begin{bmatrix} 0 & -1 & 1 & -u \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \text{if } u \neq 0.
\]

Thus, \(\psi^*\psi(w) = \psi(w).\)

Finally, since \(\psi\) is a minimal idempotent in \(C^r\), \(\psi^*\phi = \phi\), if \(\psi \in V^*\text{sgn}\) and \(\phi \in V^*\text{sgn}\). If \(\phi \in V^*\text{sgn}\), then \(\phi \in V_{sgn}\), so \(\psi^*\phi = 0\).

\[
\psi^*\phi(w) = \frac{|T|}{|G|} \sum_{x \in GF(q)} \psi(x) \phi(x^{-1}w)
\]
\[= \frac{|T|}{|G|} \left( \psi(e) \phi(w) + \psi(w) \sum_{x \in GF(q)} \phi(w^{-1}b^{-1}(u)w) \right).
\]

Using relation \(*\) as well as the definitions of \(\psi\) and \(\phi\), we obtain
\[
\psi^*\phi(w) = \phi(w) \left\{ \frac{1}{2} + \frac{1}{2} [q \text{ sgn}(-1)]^{-1/2} \sum_{x \in GF(q)} \text{sgn}(-u)\Phi(u) \right\},
\]
which implies the last part of the proposition.

4. The construction of discrete series for \(GL(2, GF(q))\)

Let \(\Pi\) be a character of \(GF(q)^x\) whose restriction to the elements of norm one is non-trivial. Then \(\Pi\) corresponds to a representation \(\mathcal{U}^\Pi\) of the discrete series of \(G=GL(2, GF(q))\) (i.e. \(\text{res } \mathcal{U}^\Pi \equiv 1\)). It turns out that \(\text{res } \mathcal{U}^\Pi\) is an irreducible representation of \(\mathcal{G}\), the triangle subgroup of \(G\). To determine a space of functions which transforms under \(\mathcal{G}\) as \(\mathcal{U}^\Pi\) we find an irreducible representation \(m\) of \(\mathcal{G}\) such that \(m= \text{res } \mathcal{U}^\Pi\). Then, using the trace of \(\mathcal{U}^\Pi\) (which we assume known) we extend the matrix coefficients of \(m\) to \(\mathcal{G}\). To determine the discrete series of \(G\) we study \(\text{res } \mathcal{U}^\Pi\).

Let \(\mathcal{D}\) be the diagonal subgroup of \(G\) and let \(\alpha\) be a character of \(\mathcal{D}\). \(\text{Ind } \mathcal{D} \downarrow G = M^\alpha\) is right translation in the space of complex-valued functions on \(\mathcal{D}\).
which satisfy \( \psi(dt) = \alpha(d) \psi(t) \) for all \( d \in D \) and \( t \in I \). Since \( B \) represents \( D \setminus I \), we may consider \( M^* \) as acting in a vector space \( B^* \) of complex-valued functions on \( B \). We write \( \psi \in B^* \) as a function of the super diagonal entries of elements of \( B \). Then

\[
(4.1) \quad M^*(db(u))\psi(x) = \alpha(d)\psi(d_{11}^{-1}d_{22}x + u)
\]

for any \( d \in D \) and \( b(u) \) the element of \( B \) with superdiagonal entry \( u \in GF(q) \), \( d_{11} \) and \( d_{22} \) the non-zero entries of \( d \).

To see how \( M^* \) decomposes take as an orthonormal basis of \( B^* \) the \( q \) characters of \( B \). The operators \( M^*(b) \) for \( b \in B \) obviously diagonalize with respect to this basis. Let \( \Phi_\delta \) be the trivial character of \( B \). Clearly \( \Phi_\delta \) transforms under \( M^* \) as the one-dimensional representation \( \alpha \) of \( I \). Now let \( \Phi \) be a fixed non-trivial character of \( B \). For \( i \in GF(q)^* \) set \( \Phi_i(x) = \Phi(ix) \) for all \( x \in GF(q) \). Then \( \Phi_i \) is a non-trivial character of \( B \) and every non-trivial character of \( B \) is of the form \( \Phi_i \) for some \( i \in GF(q)^* \). (4.1) entails that, except for scalar factors, \( D \) acts transitively on the non-trivial characters of \( B \). Since \( M^* \) is completely reducible, we see that the \( (q-1) \)-dimensional subspace of \( B^* \) spanned by the non-trivial characters of \( B \) must be irreducible. Call the resulting representation \( m_\alpha \).

**Lemma 4.1.** An irreducible representation of \( I \) is either of degree one or \( q-1 \). An irreducible \((q-1)\)-dimensional representation of \( I \) is determined by its restriction to the center of \( I \).

Proof. If an irreducible representation of \( I \) is not one-dimensional, it is equivalent to a representation \( m_\alpha \) for some character \( \alpha \) of \( D \). Thus it is \((q-1)\)-dimensional. By Frobenius' reciprocity theorem characters \( \alpha ' \) which occur in \( \text{res } m_\alpha \) occur with multiplicity one. Since \( m_\alpha \) is irreducible, every \( \alpha ' \) contained in \( \text{res } m_\alpha \) must have the same values on the center of \( I \) (i.e., the scalars). There are \( q-1 \) distinct characters of \( D \) which agree on the scalars, so they must all occur in \( \text{res } m_\alpha \). By Frobenius' theorem, \( m_\alpha \) is equivalent to \( m_{\alpha '}, \) for all such \( \alpha ' \).

**Lemma 4.2.** Let \( \Phi \) be a non-trivial character of \( B \). For \( i \in GF(q)^* \) set \( \Phi_i(x) = \Phi(ix) \) for all \( x \in GF(q) \) (considered as super-diagonal entries of elements of \( B \)). The matrix coefficients of the representation \( m_\alpha \) with respect to the basis for \( B^* \) consisting of the \( q-1 \) non-trivial characters \( \{ \Phi_i \}_{i \in GF(q)^*} \) of \( B \) are the \((q-1)^2\) functions

\[
(4.2) \quad m^*_{ij}(t) = \langle m_\alpha(t)\Phi_j, \Phi_i \rangle, \quad i \text{ and } j \in GF(q)^*,
\]

\[
= \alpha(d)\Phi_j(u), \quad \text{if } \Phi_j(d_{11}^{-1}d_{22}x) = \Phi_i(x) \text{ for all } x \in GF(q);
\]

\[
= 0, \quad \text{otherwise}.
\]
In (4.2) \( t = db(u) \), where \( u \in GF(q) \) is the super-diagonal entry of the matrix \( b(u) \in B \) and \( d_{11} \) and \( d_{22} \) are the diagonal entries of \( d \in \mathcal{D} \).

Proof. Immediate from equation (4.1).

**Lemma 4.3.** Let \( m_\alpha \) be an irreducible representation of \( \mathcal{D} \) of degree \( q-1 \). Then \( \text{res } m_\alpha \) decomposes into inequivalent representations of degree \( \frac{1}{2}(q-1) \). Any irreducible representation of \( T \) is either one-dimensional or \( \frac{1}{2}(q-1) \)-dimensional.

Proof. \( \text{Res } m_\alpha \) decomposes simply; if \( \text{res } m_\alpha \) decomposes, the component representations must be inequivalent. By (4.1) \( M^\alpha(d) \Phi(x) = \alpha(d) \Phi(d^{-1} x) \) for \( d \in D \), so two characters \( \Phi \) and \( \Phi' \) of \( B \) occur in the restriction to \( B \) of the same irreducible subrepresentation of \( \text{res } m_\alpha \) if and only if \( \Phi'(x) = \Phi(a^2 x) \) for some \( a \in GF(q)^* \) and all \( x \in GF(q) \). Since half the characters of \( B \) satisfy this relation and half do not, \( \text{res } m_\alpha \) contains two irreducible representations, each of degree \( \frac{1}{2}(q-1) \). The last statement in Lemma (4.3) follows from the fact that any irreducible representation of \( T \) occurs in the restriction to \( T \) of some irreducible representation of \( \mathcal{D} \).

**Lemma 4.4.** Let \( G \) be a finite group and \( H \) a subgroup of \( G \). Let \( U \) be a unitary representation of \( G \) whose degree is \( d \) and character is \( \chi \). Assume \( \text{res } U \) is irreducible. Then, for any matrix coefficient \( u_{ij} \) of \( U \), \( 1 \leq i, j \leq d \), and any \( g \in G \)

\[
\begin{align*}
\quad u_{ij}(g) &= \frac{d}{|H|} \sum_{h \in H} u_{ij}(h) X(h^{-1} g) .
\end{align*}
\]

Proof.

\[
\begin{align*}
\frac{d}{|H|} \sum_{h \in H} u_{ij}(h) X(h^{-1} g) &= \frac{d}{|H|} \sum_{h \in H} u_{ij}(h) \sum_{k=1}^d u_{kk}(h^{-1} g) \\
&= \frac{d}{|H|} \sum_{h \in H} u_{ij}(h) \sum_{k=1}^d \sum_{l=1}^d u_{kl}(h^{-1}) u_{lk}(g) \\
&= \frac{d}{|H|} \sum_{l, k} u_{lk}(g) \sum_{h \in H} u_{ij}(h) \bar{u}_{lk}(h) \\
&= u_{ij}(g) ,
\end{align*}
\]

by Schur's orthogonality relations on \( G \).

Lemma (4.1) implies that for any representation \( \mathcal{U}^\Pi \) of the discrete series of \( \mathcal{G} \), \( \text{res } \mathcal{U}^\Pi \) is equivalent to an irreducible representation \( m_\alpha \), where \( m_\alpha \) is, up to equivalence, the unique irreducible \( (q-1) \)-dimensional representation of \( \mathcal{D} \) which agrees with \( \mathcal{U}^\Pi \) on the scalars. Since \( \mathcal{G} = \mathcal{D} \cup \mathcal{D} wB \), \( w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), it
suffices to compute the matrix coefficients for $\mathcal{Q}^\Pi$ at $w$ in order to extend them from $\mathcal{G}$ to all of $\mathcal{G}$. For this purpose we need the character $X^\Pi$ of $\mathcal{Q}^\Pi$ (To find directions for the easy computation of $X^\Pi$ consult [3], p. 227.). Figure 1 presents $X^\Pi$.

<table>
<thead>
<tr>
<th>Conjugacy Classes on $G$</th>
<th>Values of $X^\Pi$</th>
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<tbody>
<tr>
<td>$\lambda \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$\Pi(\lambda)(q-1)$</td>
</tr>
<tr>
<td>$\lambda \begin{pmatrix} 1 &amp; 0 \ 1 &amp; 1 \end{pmatrix}$</td>
<td>$-\Pi(\lambda)$</td>
</tr>
<tr>
<td>$\lambda \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\begin{pmatrix} \varepsilon &amp; 0 \ 0 &amp; \varepsilon \end{pmatrix} \sim \begin{pmatrix} \alpha &amp; \beta \ \beta &amp; \alpha \end{pmatrix}$</td>
<td>$-(\Pi(\varepsilon)+\Pi(\varepsilon^t))$</td>
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$t, \lambda \in GF(q)^\times, t \neq 1; \varepsilon = \alpha + \beta \sqrt{\zeta}, \alpha, \beta, \zeta \in GF(q)$ with $\zeta$ not a square and $\beta \neq 0$. * Matrices with the same characteristic roots are conjugate.

**Lemma 4.5.** Let $X^\Pi$ be the character of a representation $\mathcal{Q}^\Pi$ of the discrete series of $\mathcal{G}$. Let $\alpha$ be a character of $\mathcal{D}$ such that $\alpha(\lambda) = \Pi(\lambda)$ for any scalar matrix $\lambda \in \mathcal{D}$. Then $m_\alpha$ is equivalent to res $\mathcal{G}^\downarrow \mathcal{G}$. Fix a non-trivial character $\Phi$ of $B$. Let $\{m_{ij}^\alpha\}_{i,j \in GF(q)^\times}$ be the matrix coefficients of $m_\alpha$ with respect to the basis $\{\Phi_i\}_{i \in GF(q)^\times}$ of $\mathcal{B}^\alpha$ (see Lemma (4.2) and relation (4.2)). The matrix coefficients $m_{ij}^\alpha$ are the restrictions to $\mathcal{G}$ of matrix coefficients $u_{ij}^\Pi$ of $\mathcal{Q}^\Pi$.

For $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\delta(a, b)$ the diagonal matrix with diagonal entries $\delta_{11} = a$ and $\delta_{22} = b$,

$$u_{ij}^\Pi(w) = -\alpha^{-1}(\delta(i, j))q^{-1} \sum_{x \in GF(q)^\times} \Pi(\varepsilon) \Phi(\varepsilon + \varepsilon^t).$$

**Proof.** By Lemma (4.4) and relation (4.2)

$$u_{ij}^\Pi(w) = \frac{(q-1)}{|\mathcal{G}|} \sum_{t \in \mathcal{G}} m_{ij}^\alpha(t) X^\Pi(t^{-1}w)$$

$$= q^{-1} \sum_{x \in GF(q)^\times} \alpha(\delta) \Phi_j(u) X^\Pi(b^{-1}(u) \delta^{-1}w),$$

where $\delta = \delta(j, i)$ and $b(u) \in B$ has super diagonal entry $u$. Use of the explicit formula for $X^\Pi$ easily yields (4.3).

**Theorem 4.6.** Let $\Pi$ be a character of $GF(q^2)^\times$ whose restriction to the elements of norm one is not trivial. Let $X^\Pi$ be the character of the irreducible representation of $\mathcal{G}$ associated with $\Pi$. Let $\alpha$ be any character of $\mathcal{D}$ which agrees...
with Π on the scalar matrices. Then \( m_a \) is \( \text{res } \mathcal{Q}^\Pi \) and \( B^\sigma \), the representation space of \( m_a \), is a representation space for \( \mathcal{Q}^\Pi \). Fix a non-trivial character \( \Phi \) of \( B \) and write it as a function of the super-diagonal entries of elements of \( B \). Take as a basis for \( B^\sigma \) the \( q-1 \) non-trivial characters \( \{ \Phi_i \}_{i \in \mathbb{F}_q^\times} \), where \( \Phi_i(x) = \Phi(ix) \) for all \( x \in \mathbb{F}_q \). Matrix coefficients for \( \mathcal{Q}^\Pi \) acting in \( B^\sigma \) are as follows. For \( i, j \in \mathbb{F}_q^\times \) set \( u_{i,j}^{\Pi,\sigma} = \langle \mathcal{Q}^\Pi(g), \Phi_i, \Phi_j \rangle \). If \( g = db(u) \), where \( d \in \mathcal{D} \) has diagonal entries \( d_{11} \) and \( d_{22} \) and \( b(u) \in B \) has super-diagonal entry \( u \in \mathbb{F}_q \), then

\[
(4.4) \quad u_{i,j}^{\Pi,\sigma}(g) = \alpha(d) \Phi_j(u), \quad \text{provided } d_{11}^{-1}d_{22} = j^{-1}i; \\
= 0, \quad \text{otherwise.}
\]

If \( g = b(v)wd(b(u)) \), where \( d \in \mathcal{D} \) has diagonal entries \( d_{11} \) and \( d_{22} \), \( w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), and \( b(u) \) and \( b(v) \in B \) have super-diagonal entries \( u \) and \( v \) respectively then

\[
(4.5) \quad u_{i,j}^{\Pi,\sigma}(g) = \Phi(iw + ju)\left[ -\Pi(d_{11})^{-1}(\delta(i,j))q^{-1} \sum_{\xi \in \mathbb{F}_q^\times} \Pi(\xi) \Phi(\xi \pm \varepsilon^\sigma) \right]
\]

where \( \delta(i,j) \) is the diagonal matrix with upper entry \( i \) and lower entry \( j \) and \( l = ij d_{11}^{-1}d_{22}^{-1} \).

Proof. Relation (4.4) is the same as (4.2), so no proof is needed. To prove (4.5) note first that \( u_{i,j}^{\Pi,\sigma}(b(v)gb(u)) = \Phi_i(v)u_{i,j}^{\Pi,\sigma}(g)\Phi_j(u) \). Moreover, \( u_{i,j}^{\Pi,\sigma}(wd) = \alpha(d)u_{i,j}^{\Pi,\sigma}d_{11}^{-1}d_{22}^{-1} \). Use of (4.3) to express \( u_{i,j}^{\Pi,\sigma}d_{11}^{-1}d_{22}^{-1} \) as an exponential sum leads to a proof of (4.5).

5. Discrete series of \( G \)

Let \( \Pi \) be a character of \( \mathbb{F}_q^\times \times \), whose restriction to \( N^1 \), the elements of norm one in \( \mathbb{F}_q^\times \times \), is not trivial. Let \( \pi \) be \( \Pi \) restricted to \( N^1 \). Let \( \mathcal{Q}^\Pi \) be the representation of the discrete series of \( G \) associated with \( \Pi \). Set \( U^\pi = \text{res } \mathcal{Q}^\Pi \). The trace \( X^\pi \) of \( U^\pi \) is the restriction to \( G \) of \( X^\Pi \), so, up to equivalence, \( U^\pi \) depends only on the values of \( \Pi \) restricted to \( N^1 \). Furthermore, \( U^\pi \) and \( U^\pi' \) are equivalent if and only if \( \pi' = \pi \) or \( \pi^{-1} \), since, if \( \pi' \) is the restriction to \( N^1 \) of a character \( \Pi' \) of \( \mathbb{F}_q^\times \times \), \( X^\pi = X^{\pi'} \) if and only if \( \pi' = \pi \) or \( \pi^{-1} \).

We may take as representatives for the conjugacy classes in \( G \) those representatives for conjugacy classes in \( \mathcal{Q} \) which lie in \( G \) (see Figure 1.). However, \( \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \) and \( \begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix} \), \( \xi \) a non-square, are not conjugate in \( G \); similarly \( -\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \) and \( -\begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix} \).

Theorem 5.1. Let \( \pi \) be a non-trivial character of \( N^1 \) and let \( U^\pi \) be the corresponding representation of \( G \) defined above. \( U^\pi \) is irreducible if and only if \( \pi^2 \neq 1 \). If \( \pi^2 = 1 \), \( U^\pi = U_1^\pi + U_2^\pi \), the direct sum of inequivalent \( \frac{1}{2}(q-1) \)-dimensional representations.
Proof. It suffices to show that \( |G|^{-1} \sum_{x \in G} |X^x(g)|^2 = 1 \), if \( \pi \equiv 1 \), and 2, otherwise. The computation is easy and we omit it. In the case that \( U^x \) is reducible, the components are \( \frac{1}{2}(q-1) \)-dimensional and inequivalent, since, according to Lemma (4.3), this statement holds already for \( \text{res} U^x \). We may use Lemma (4.3) to obtain representation spaces for \( U^x \) and \( U^y \).

There are \( q+4 \) conjugacy classes in \( G \) and we have accounted for this many equivalence classes of irreducible representations, so our description of the irreducible representations of \( SL(2, GF(q)) \) is complete.

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References


