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ON THE DISTRIBUTION OF THE NORMS OF THE HYPERBOLIC TRANSFORMATIONS

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The elements of a Fuchsian group Γ of the first kind which operates properly discontinuously on the complex upperhalf plane, can be divided into four classes, one of which consists only of the identity element, while the others are formed respectively of the hyperbolic, the elliptic and the parabolic elements. The elliptic and parabolic elements are closely related to the fundamental domain of Γ and have been fully studied, but it seems to us that the hyperbolic elements have been paid no attention.

A hyperbolic transformation P has two fixed points on the real axis. When these are sent to 0 and ∞ by a suitable change of variable, the transformation takes the normal form:

$$w' = \alpha w\,, \qquad \alpha>1\,;$$

i.e. it is a dilation of magnitude α with center at the origin. We call α the norm of P and also the norm of the hyperbolic conjugacy class $\{P\}_\Gamma$, and denote it by $N\{P\}$, leaving the subscript Γ out. We shall call a hyperbolic element P primitive, if it is not a power with exponent >1 of any other element in the group Γ; correspondingly we say that the conjugacy class $\{P\}_\Gamma$ is primitive. When we write the primitive hyperbolic classes as $\{P_\alpha\}_\Gamma$ $(\alpha=1, 2, \cdots)$, the hyperbolic classes can be expressed as $\{P_\alpha^k\}$ $(\alpha=1, 2, \cdots; k=1, 2, \cdots)$.

In this paper we shall show that if the fundamental domain $\mathcal{D}$ of Γ is compact, then we have

$$\frac{1}{2}\sum_{N\{P_\alpha\}^n<x}\log N\{P_\alpha\}\frac{\left(\log\dfrac{x}{N\{P_\alpha\}^n}\right)^2}{1-N\{P_\alpha\}^{-n}} = x+\sum_{\nu=1}^{M}\frac{x^{a_\nu}}{a_\nu^3}+O(x^{1/2})\,, \tag{1}$$

where on the left hand side the sum is taken over α and n such that $N\{P_\alpha\}^n<x$, and on the right the a_ν $(\frac{1}{2}<a_\nu<1)$ are numbers which depend on Γ.

Using (1), we can estimate roughly the distribution of the norms of the hyperbolic elements. Let $\theta(x)$ be the number of the hyperbolic conjugacy classes whose norms are less than x. Then we have

$$\frac{x}{(\log x)^3} < \theta(x) < 6x .$$

Taking as Γ the unit group of an indefinite division quaternion algebra, we shall deduce from (1) a rough estimation of the sum of hR in the order of magnitude of units, where h and R are the class number and the regulator, respectively, of each real quadratic field.

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1. In this paper we assume that Γ has a compact fundamental domain $\mathscr{D}$. The Selberg's zeta function attached to Γ is defined by

$$Z_\Gamma(s) = \prod_{\alpha} \prod_{n=0}^{\infty} (1 - N\{P_\alpha\}^{-s-n}) \tag{2}$$

for s the real part of which is greater than 1, where the product converges absolutely(*). For the sake of simplicity we write $Z(s) = Z_\Gamma(s)$. From (2) we have

$$\frac{Z'}{Z}(s) = \sum_{\alpha} \sum_{n=1}^{\infty} \frac{\log N\{P_\alpha\}}{1 - N\{P_\alpha\}^{-n}} \frac{1}{N\{P_\alpha\}^{ns}} .$$

We shall integrate $\dfrac{x^s}{s^3} \dfrac{Z'}{Z}(s)$ on the straight line $\sigma = 2$ $(s = \sigma + it)$. We need the following

Lemma. 1) $\displaystyle \left| \int_{2-iT}^{2+iT} \frac{y^s}{s^3} ds - \pi i (\log y)^2 \right| \leqslant \frac{2}{T^3} \frac{y^2}{|\log y|} \qquad (y > 1)$,

2) $\displaystyle \left| \int_{2-iT}^{2+iT} \frac{y^s}{s^3} ds \right| \leqslant \frac{2}{T^3} \frac{y^2}{|\log y|} \qquad (0 < y < 1)$.

Proof. One can prove these by the same argument as Landau [3], Satz 449.

By this lemma we get

(*) As for the properties and notations on Selberg zeta functions, we refer to Selberg [4], and Kuga [2].

$$\left|\int_{2-iT}^{2+iT}\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds-2\pi i\frac{1}{2}\sum_{N\{P_\alpha\}^n<x}\frac{\log N\{P_\alpha\}}{1-N\{P_\alpha\}^{-n}}\left(\log\frac{x}{N\{P_\alpha\}^n}\right)^2\right|$$
$$\leqslant\frac{2x^2}{T^3}\sum_{\alpha,n}\frac{\log N\{P_\alpha\}}{1-N\{P_\alpha\}^{-n}}\frac{1}{N\{P_\alpha\}^{2n}}\frac{1}{\left|\log\frac{x}{N\{P_\alpha\}^n}\right|},$$

hence

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds=\frac{1}{2}\sum_{N\{P_\alpha\}^n<x}\frac{\log N\{P_\alpha\}}{1-N\{P_\alpha\}^{-n}}\left(\log\frac{x}{N\{P_\alpha\}^n}\right)^2. \qquad (3)$$

We denote by $L^2(\mathcal{D})$ the Hilbert space defined by the inner product

$$(F_1, F_2)=\int_{\mathcal{D}}\bar{F}_1(z)F_2(z)dz.$$

The non-trivial zeros of $Z(s)$ are the numbers $\frac{1}{2}+ir_k$, where the r_k are the values for which there is a solution of the equation

$$-y^2\Delta F(z)=\lambda F(z), \qquad \lambda=\frac{1}{4}+r^2$$

with $F(z)$ in $L^2(\mathcal{D})$. The non-trivial zeros ρ have thus real part equal to $\frac{1}{2}$, with the exception of a finite number of zeros that are real and lie in the interval $0\leqslant s\leqslant 1$, which we denote by

$$0=a_{N+1}<a_N\leqslant\cdots\leqslant a_1<a_0=1.$$

Particularly $s=1$ is the zero of multiplicity one, as follows from the fact that the eigenvalue zero is of multiplicity one. The trivial zeros of $Z(s)$ are $s=-n$ $(n=0, 1, 2, \cdots)$, whose multiplicity is

$$N_n=\frac{A(\mathcal{D})}{2\pi}(2n+1)-\sum_\beta\sum_{k=1}^{m_\beta-1}\frac{1}{m_\beta\sin\frac{k\pi}{m_\beta}}\sin\frac{k\pi}{m_\beta}(2n+1),$$

where β runs through all the primitive elliptic classes.

Let m be any integer $\geqslant 1$. We apply the Cauchy's theorem to the integral

$$\int\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds$$

on the rectangle with the vertices $2\pm Ti$, $-m-\frac{1}{2}\pm Ti$. That gives

$$\frac{1}{2\pi i}\int\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds=x+\sum_{\nu=1}^{N}\frac{x^{a_\nu}}{a_\nu^3}+\sum_{0<|Im\rho|<T}\frac{x^\rho}{\rho^3}+\omega+\sum_{n=1}^{m}N_n\frac{x^{-n}}{(-n)^3}, \qquad (4)$$

where ω is the residue of $\frac{x^s}{s^3}\frac{Z'}{Z}(s)$ at $s=0$. From the functional equation of $Z(s)$, namely

$$Z(s) = Z(1-s)\exp\left\{A(\mathscr{D})\int_0^{s-(1/2)} v\tan v\pi dv - 2\pi\sum_{\beta}\sum_{k=1}^{m_\beta-1}\frac{1}{m_\beta\sin\frac{k\pi}{m_\beta}}\right.$$
$$\left.\times\int_0^{s-(1/2)}\frac{e^{-2\pi i v(k/m_\beta)}}{1+e^{-2\pi i v}}dv\right\},$$

we see easily that the integrals

$$\int_{2+Ti}^{-\infty+Ti}\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds, \qquad \int_{-\infty-Ti}^{2-Ti}\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds$$

exist and

$$\int_{-m-(1/2)-Ti}^{-m-(1/2)+Ti}\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds$$

tends to zero as $m\to\infty$. From (4) thus follows

$$\frac{1}{2\pi i}\int_{2-Ti}^{2+Ti}\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds+\frac{1}{2\pi i}\int_{-\infty-Ti}^{2-Ti}\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds+\frac{1}{2\pi i}\int_{2+Ti}^{-\infty+Ti}\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds$$
$$= x+\sum_{\nu=1}^{N}\frac{x^{a_\nu}}{a_\nu^3}+\sum_{0<|Im\rho|<T}\frac{x^\rho}{\rho^3}+\omega+\sum_{n=1}^{\infty}N_n\frac{x^{-n}}{(-n)^3}. \tag{5}$$

Using again the functional equation of $Z(s)$, we see readily

$$\left|\int_{-\infty+Ti}^{-1+Ti}\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds\right|<\frac{C_1}{T^2x\log x}. \tag{6}$$

Next we shall estimate the integral

$$\int_{-1+Ti}^{2+Ti}\frac{x^s}{s^3}\frac{Z'}{Z}(s)ds.$$

Since $Z(s)$ is an integral function of genus 2, its canonical product is

$$Z(s) = s^{N_0+1}e^{b_0+b_1s+b_2s^2}(1-s)\prod_{n=1}^{\infty}\left\{\left(1+\frac{s}{n}\right)e^{-(s/n)+(1/2)(s/n)^2}\right\}^{N_n}$$
$$\times\prod_{\substack{\rho\\ \rho\neq 0,1}}\left(1-\frac{s}{\rho}\right)e^{s/\rho+(1/2)(s/\rho)^2},$$

hence

$$\frac{Z'}{Z}(s) = \frac{N_0+1}{s}+b_1+2b_2s+\frac{1}{s-1}+\sum_{n=1}^{\infty}N_n\left(\frac{1}{s+n}-\frac{1}{n}+\frac{s}{n^2}\right)$$
$$+\sum_{\substack{\rho\\ \rho\neq 0,1}}\left(\frac{1}{s-\rho}+\frac{1}{\rho}+\frac{s}{\rho^2}\right),$$

$$\left|\frac{Z'}{Z}(s)-\sum_{0<|\gamma-T|<1}\frac{1}{s-\rho}\right|\leqslant\left|\frac{N_0+1}{s}+b_1+2b_2s+\frac{1}{s-1}\right|_{(a)}$$
$$+\left|\sum_{n=1}^{\infty}N_n\Big(\frac{1}{s+n}-\frac{1}{n}+\frac{s}{n^2}\Big)\right|_{(b)}+\left|\sum_{0<|\gamma-T|<1}\Big(\frac{1}{\rho}+\frac{s}{\rho^2}\Big)\right|_{(c)}$$
$$+\left|\sum_{|\gamma-T|\geqslant 1}\Big(\frac{1}{s-\rho}+\frac{1}{\rho}+\frac{s}{\rho^2}\Big)\right|_{(d)},\qquad (\gamma=Im\rho).$$

Here we see easily that the term $(a)<C_2T$ and the term $(b)<C_3T\log T$. To evaluate the terms (c) and (d) we need the following fact about the asymptotic distribution of the eigenvalues:

Theorem.(*) *On the eigenvalue problem*:

$$-y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)F(z)=\lambda F(z),\qquad F(z)\in L^2(\mathcal{D}),$$

we denote by $\alpha(\lambda)$ the number of eigenvalues that are less than λ. Then

$$\alpha(\lambda)\sim\frac{A(\mathcal{D})}{4\pi}\lambda.$$

Proof. In the trace formula [4, p. 74]:

$$\sum_k h(r_k)=\frac{A(\mathcal{D})}{2\pi}\int_{-\infty}^{\infty}r\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}h(r)dr+\sum_{\beta}\sum_{k=1}^{m_\beta-1}\frac{1}{m_\beta\sin\frac{k\pi}{m_\beta}}\int_{-\infty}^{\infty}\frac{e^{-2\pi r(k/m_\beta)}}{1+e^{-2\pi r}}h(r)dr$$
$$+2\sum_{\alpha}\sum_{k=1}^{\infty}\frac{\log N\{P_\alpha\}}{N\{P_\alpha\}^{k/2}-N\{P_\alpha\}^{-k/2}}g(k\log N\{P_\alpha\}),$$

we put

$$h(r)=e^{-\{(1/4)+r^2\}t},\qquad t>0.$$

The left side turns out be

$$2\int_0^{\infty}e^{-t\lambda}d\alpha(\lambda).$$

Next we examine the behavior of the right side as $t\to 0$.
(i) identity element:

$$\frac{A(\mathcal{D})}{2\pi}\int_{-\infty}^{\infty}r\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}e^{-tr^2}dr$$
$$=\frac{A(\mathcal{D})}{2\pi}\cdot 2\Big[\int_0^{\infty}re^{-tr^2}dr-2\int_0^{\infty}r\frac{e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}e^{-tr^2}dr\Big]$$
$$=\frac{A(\mathcal{D})}{2\pi}\frac{1}{t}+[\text{bounded as } t\to 0].$$

(*) This theorem in more general situation has been announced in [1]. The following proof is due to S. Tanaka.

(ii) elliptic element:

$$\lim_{t\to 0}\int_{-\infty}^{\infty}\frac{e^{-2\pi r(k/m_\beta)}}{1+e^{-2\pi r}}e^{-tr^2}dr=\int_{-\infty}^{\infty}\frac{e^{-2\pi r(k/m_\beta)}}{1+e^{-2\pi r}}dr<\infty\,.$$

(iii) hyperbolic element:
Put

$$g_t(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iru}e^{-tr^2}dr=\frac{1}{2\sqrt{\pi t}}e^{-u^2/4t}\,.$$

Since $\log N\{P_\alpha\}\geqslant\varepsilon_0>0$,

$$g_t(k\log N\{P_\alpha\})\leqslant\frac{1}{2\sqrt{\pi t}}e^{-\varepsilon_0^2/8t}e^{-k^2\log^2 N\{P_\alpha\}/8t}\,.$$

Therefore

$$\begin{aligned}&2\sum_\alpha\sum_{k=1}^{\infty}\frac{\log N\{P_\alpha\}}{N\{P_\alpha\}^{k/2}-N\{P_\alpha\}^{-k/2}}g_t(k\log N\{P_\alpha\})\\&<\frac{1}{\sqrt{\pi t}}e^{-\varepsilon_0^2/8t}\sum_\alpha\sum_k\frac{\log N\{P_\alpha\}}{N\{P_\alpha\}^{k/2}-N\{P_\alpha\}^{-k/2}}e^{-k^2\log^2 N\{P_\alpha\}/8t}\,.\end{aligned}$$

Here $\frac{1}{\sqrt{\pi t}}e^{-\varepsilon_0^2/8t}$ and the sum on the right hand side tend to zero as $t\to 0$.

Summarizing, we conclude

$$\int_0^\infty e^{-t\lambda}d\alpha(\lambda)\sim\frac{A(\mathcal{D})}{4\pi}\frac{1}{t}\,,\qquad\text{as}\quad t\to 0\,.$$

Hence we get by the theorem of Karamata [7, p. 192]

$$\alpha(\lambda)\sim\frac{A(\mathcal{D})}{4\pi}\lambda\,,\qquad\text{as}\quad\lambda\to\infty\,.$$

Thus we have proved the theorem.

Now putting

$$N(T)=\#\{\rho\,;\;0<\gamma<T\}\,,$$

we get by the above theorem

$$N(T)\sim\frac{A(\mathcal{D})}{4\pi}T^2\,,$$

and consequently

$$N(T)<C_4T^2\,.$$

Since $Z(s)$ is an integral function of order 2, it follows that, for any $\delta>0$, we have

$$\sum_{0<\gamma<T} \frac{1}{\gamma^2} < T^{\delta}$$

for sufficiently large T.

From these it can be easily proved that

$$(c) < C_5 T \quad \text{and} \quad (d) < C_6 T^{2+(1/2)}.$$

Therefore we get

$$\left| \frac{Z'}{Z}(s) - \sum_{0<|\gamma-T|<1} \frac{1}{s-\rho} \right| < C_7 T^{2+1/2},$$

and consequently

$$\left| \int_{-1+Ti}^{2+Ti} \frac{x^s}{s^3} \frac{Z'}{Z}(s) ds - \sum_{0<|\gamma-T|<1} \int_{-1+Ti}^{2+Ti} \frac{x^s}{s^3} \frac{1}{s-\rho} ds \right| < C_8 \frac{x^2}{T^{1/2}}.$$

Again using $N(T) < C_4 T^2$, we see that

$$\left| \sum_{0<|\gamma-T|<1} \int_{-1+Ti}^{2+Ti} \frac{x^s}{s^3} \frac{1}{s-\rho} ds \right| < C_9 \frac{x^2}{T}.$$

Thus we have the wanted estimation:

$$\left| \int_{-1+Ti}^{2+Ti} \frac{x^s}{s^3} \frac{Z'}{Z}(s) ds \right| < C_{10} \frac{x^2}{T^{1/2}}. \tag{7}$$

If T tends to ∞ in (5), then we get, on account of (3), (6), (7),

$$\frac{1}{2} \sum_{N\{P_\alpha\}^n<x} \frac{\log N\{P_\alpha\}}{1-N\{P_\alpha\}^{-n}} \left(\log \frac{x}{N\{P_\alpha\}^n} \right)^2$$
$$= x + \sum_{\nu=1}^{N} \frac{x^{a_\nu}}{a_\nu^3} + \sum_{0<|Im\rho|} \frac{x^\rho}{\rho^3} + \omega + \sum_{n=1}^{\infty} N_n \frac{x^{-n}}{(-n)^3}.$$

From this, putting $M = \mathrm{Max}\,\{\nu\,;\ \frac{1}{2} < a_\nu\}$, we obtain the required equality

$$\frac{1}{2} \sum_{N\{P_\alpha\}^n<x} \log N\{P_\alpha\} \frac{\left(\log \frac{x}{N\{P_\alpha\}^n} \right)^2}{1-N\{P_\alpha\}^{-n}} = x + \sum_{\nu=1}^{M} \frac{x^{a_\nu}}{a_\nu^3} + O(x^{1/2}). \tag{1}$$

Evaluating the terms in the sum, we have

$$\frac{x}{(\log x)^3} < \sum_{N\{P_\alpha^n\}<x} 1 < 6x.$$

But $\sum_{N\{P_\alpha^n\}<x} 1$ is just the distibution $\theta(x)$ of the norms of the hyperbolic conjugacy classes.

2. Let A be an indefinite division quaternion algebra with discriminant Φ over the rational number field, and J a maximal order in A. Denote by Γ the group of the units of J with positive reduced norm. As A has a faithful representation by real matrices of degree 2, Γ is considered as a Fuchsian group with compact fundamental domain. So we shall apply the above results to this case.

The norm of a hyperbolic element in Γ is η^2, where η is a unit with positive norm in a real quadratic field $\boldsymbol{Q}(\sqrt{d})$ with discriminant d contained in A. Denote by $\nu(\eta)$ the number of the primitive hyperbolic conjugacy classes which have η^2 as their norm. Then owing to the ideal theory of quaternion algebra (see Shimizu [5](*)), we find

$$\nu(\eta) = \left\{ \prod_{p\mid\Phi}\left(1-\left(\frac{d}{p}\right)\right)\right\}\cdot\left\{ \sum_{f,(f,\Phi)=1} h(f^2 d)\right\}.$$

Here $h(f^2 d)$ is the narrow class number of the order with conductor f contained in $\boldsymbol{Q}(\sqrt{d})$, $\left(\frac{d}{p}\right)$ stands for the Kronecker symbol, and the sum is taken over all the orders of those conductors f which have η as their fundamental unit and are prime to the discriminant Φ of the quaternion algebra A.

Thus if we take the unit group as a Fuchsian group Γ, the formula (1) turns out to be

$$\sum_{\eta^{2n}<x} \nu(\eta)\log\eta \frac{\left(\log\frac{x}{\eta^{2n}}\right)^2}{1-\eta^{-2n}} = x+\sum_{\nu=1}^{M}\frac{x^{a_\nu}}{a_\nu^3}+O(x^{1/2}). \tag{8}$$

Here the sum is taken over all the units $\eta>1$ with positive norm of all the real quadratic fields contained in the algebra A and over all the natural numbers n such that $\eta^{2n}<x$.

By a simple argument, we can deduce from (8)

$$C_{11}\frac{x^2}{(\log x)^3}<\sum_{\eta<x}\nu(\eta)\log\eta<C_{12}x^2. \tag{9}$$

Now let h_D be the narrow class number of primitive binary quadratic forms $Q(x, y)=ax^2+bxy+cy^2$ with positive discriminant $D=b^2-4ac$, and define $\eta_D=\frac{t+u\sqrt{D}}{2}$, where t, u are the smallest positive integral solutions of $t^2-Du^2=4$. Then Siegel has proved in [6] that

(*) In this paper imaginary quadratic fields contained in quaternion algebra are treated, but we can similarly manage the real case by a partial modification.

$$\sum_{D<x} h_D \log \eta_D = \frac{\pi^2}{18\zeta(3)} x^{3/2} + 0(x \log x) .$$

Now

$$\eta_D = \frac{t+u\sqrt{D}}{2} = \frac{t+\sqrt{t^2-4}}{2} \qquad (t \geqslant 3)$$

is almost equal to t. Therefore if we sum $h \log \eta$ in the order of magnitude of units, how large is it?

Partial answers to this question are deduced from (9):

$$C_{13}\frac{x^2}{(\log x)^3} < \sum_{\substack{\eta<x \\ \eta \in \boldsymbol{Q}(\sqrt{d}) \subset A}} \Big\{ \sum_{\substack{f_\eta \\ (f_\eta, \Phi)=1}} h(f_\eta^2 d)\Big\} \log \eta < C_{14} x^2 ,$$

where the second sum is taken over all the orders of those conductors f_η which have η as their fundamental unit and $(f_\eta, \Phi)=1$. The condition that quadratic fields $\boldsymbol{Q}(\sqrt{d})$ are contained in the algebra A is given by the congruence about d. For examples:

i) $\Phi=2\cdot 3$, $d \equiv 0, 5, 8, 12, 20, 21 \pmod{24}$

ii) $\Phi=5\cdot 7$, $d \equiv 0, 3, 5, 7, 10, 12, 13, 17, 20, 27, 28, 33 \pmod{35}$

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