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On the Complete Cohomology Theory of Frobenius Algebras

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As has been observed by Artin, Cartan, Eilenberg and Tate, homology groups of a finite group, over its module, may be put into one whole sequence of positive-, 0-, and negative-dimensional cohomology (homology) groups in whose back lies a certain complete complex of the group. The fact has been found by Artin and Tate to be very useful in cohomological formulation of class field theory [14]. Also Chevalley has based his foundation of cohomology theory of finite groups upon the fact [4]. One of the interesting features in this complete cohomology theory is the generlization of Hochschild's reduction theorem [6] to the case of mixed dimensions, which is proved by means of a certain "contraction" procedure and which reflects the "homogenuity" with respect to dimension.

Now, in the present paper[1] we shall show that a similar theory can be developed for Frobenius algebras. Thus, let A be a Frobenius algebra, over a field (firstly, for the sake of simplicity), and M a double-module over A. By means of a certain automorphism $x \rightarrow x^*$ of A introduced formerly by the writer in connection with his structural study of Frobenius algebras and uniquely determined by A up to inner automorphisms ([10] cf. also [1], [12]) we shall define a certain modification of homology groups of A in M. Interpreting these modified homology groups as negative-dimensional cohomology groups, we shall then connect their sequence with that of ordinary positive dimensional cohomology groups through the 0-dimensional cohomology group defined as the residue module $\mathfrak{N}(M) = \{u \in M \mid au = ua (a \in A)\} / \delta_0 M$, where δ_0 is the map $u \rightarrow \sum_\iota a_\iota u b_\iota$ defined by some bases (a_ι), (b_ι) of A "dual" with respect to the automorphism $*$; the module $\mathfrak{N}(M)$, which is a natural analogy to the so-called normalized 0-dimensional cohomology group, has been observed formerly by Shimura [13] in the special case $M = A$. The same effect

1) The main portion of the present paper was presented at the Colloque Henri Poincaré, Paris, October 1954.

is achieved indeed also by extending the ordinary standard, say, complex to a complete complex whose augmentation is specified by the automorphism in a certain manner. The cohomology group $H^p(A, M)$, with $M = 1M1$, of any dimension $p \gtreqless 0$ can then be interpreted as $\mathfrak{N}(C^p(A, M))$, where $C^p(A, M)$ is the A-double-module of p-cochains under Hochschild's operation. This is however a special case of the *general reduction theorem* $H^p(A, C^q(A, M)) = H^{p+q}(A, M)$, which we shall prove in two ways, one rather constructive and one axiomatic. The reduction theorem gives a back ground to the fact that the cohomological dimension of a Frobenius algebra is ∞ as soon as it is $\neq 0$; the fact has been observed by Ikeda-Nagao-Nakayama as an easy consequence of their structural determination of the cohomological dimension of an algebra [8] and has been considered by Eilenberg-Nakayama from a somewhat more general standpoint [5].

We shall briefly consider, in our final section, the case of Frobenius algebras over a commutative ring too. Thus, in this generalized formulation our treatment covers the case of (complete) cohomology of finite groups. Anyway, it clarifies, as it seems to the writer, a back ground of the last. It also seems to the writer rather interesting that Frobenius algebras and (quasi-) Frobenius rings, which were introduced and studied in a different context, have proved to have cohomological significances as are seen in the present paper and [5].

In closing the introduction the writer wants to say that he admits, and is quite aware of, that the exposition of the paper is old-fashioned, particularly in view of the appearance of the book [3] by Cartan-Eilenberg. But this old-fashioned way, by which he was led to the result and which he is too lazy to revise, might be of some use too.

§ 1. Cohomology and modified homology groups of an algebra.

Let A be an algebra over a field K. For any natural number n we denote by $C_n(A)$, or briefly C_n, the tensor product

$$C_n(A) = C_n = A \otimes A \otimes \cdots \otimes A \tag{1}$$

of n copies of A over K, while we denote by $C_0(A) = C_0$ the field K itself. Let M be a double-module over A. By the n-chain group $C_n(A, M)$ of A in M we understand the tensor product

$$C_n(A, M) = C_n \otimes M \tag{2}$$

over $K (n = 0, 1, \cdots)$. On the other hand we understand by the n-cochain group $C^n(A, M)$ the K-module

(3) $$C^n(A, M) = \mathrm{Hom}_K(C_n, M).$$

For $n \geqq 1$, we denote by ∂_n the K-homomsrphism of $C_n(A, M)$ into $C_{n-1}(A, M)$ such that

(4) $$\partial_n(x_1 \otimes \cdots \otimes x_n \otimes u) = x_2 \otimes \cdots \otimes x_n \otimes ux_1 + \sum_{i=1}^{n-1}(-1)^i x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n \otimes u + (-1)^n x_1 \otimes \cdots \otimes x_{n-1} \otimes x_n u$$

($u \in M$), and by δ_n the K-homomorphism of $C^{n-1}(A, M)$ into $C^n(A, M)$ such that for $f \in C^{n-1}(A, M)$

(5) $$(\delta_n f)(x_1 \otimes \cdots \otimes x_n) = x_1 f(x_2 \otimes \cdots \otimes x_n) + \sum_{i=1}^{n-1}(-1)^i f(x_2 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n) + (-1)^n f(x_1 \otimes \cdots \otimes x_{n-1}) x_n.$$

We have $\partial_n \partial_{n+1} = 0$ and $\delta_{n+1}\delta_n = 0\,(n = 1, 2, \cdots)$. The homology and cohomology groups are introduced as usual:

(6) $$H_n(A, M) = (\text{Kernal } Z_n(A, M) \text{ of } \partial_n)/(\text{Image } B_n(A, M) \text{ of } \partial_{n+1}),$$

(7) $$H^n(A, M) = (\text{Kernel } Z^n(A, M) \text{ of } \delta_{n+1})/(\text{Image } B^n(A, M) \text{ of } \delta_n).$$

Now, $C_0(A, M)$ may be identified with M itself and is thus a double-module over A. We make also $C_n(A, M)$ $(n = 1, 2, \cdots)$ and $C^n(A, M)$ $(n = 0, 1, \cdots)$ into double-modules over A rather than mere K-modules, on putting

(8) $$(x_1 \otimes \cdots \otimes x_n \otimes u)x = x_1 \otimes \cdots \otimes x_n \otimes ux$$

(9) $$x(x_1 \otimes \cdots \otimes x_n \otimes u) = x_1 \otimes \cdots \otimes x_n \otimes ux - \partial_{n+1}(x \otimes x_1 \otimes \cdots \otimes x_n \otimes u)$$

and, for $f \in C^n(A, M)$,

(10) $$(xf)\,(x_1 \otimes \cdots \otimes x_n) = xf(x_1 \otimes \cdots \otimes_n)$$

(11) $$(fx)\,(x_1 \otimes \cdots \otimes x_n) = xf(x_1 \otimes \cdots \otimes x_n) - (\delta_{n+1} f)\,(x \otimes x_1 \otimes \cdots \otimes x_n).$$

Also $C^0(A, M)$ may be identified with M by identifying with $u \in M$ the element f in $C^0(A, M)$ with $f(1) = u$, and our definition of $C^0(A, M)$ as a double-module over A is in accord with this identificaction. Further, we verify readily the relations

(12) $$C_m(A, C_n(A, M)) = C_{m+n}(A, M),$$

(13) $$C^m(A, C^n(A, M)) = C^{m+n}(A, M),$$

for $m, n \geqq 0$, in the sense of the identifications at hand.

Let now

(14) $$x \to x^*$$

be an automorphism of the algebra A. Then we introduce an new boundary operation ∂_n^* as a K-homomorphism of $C_n(A, M)$ into $C_{n-1}(A, M)$ such that

(15) $$\partial_n^*(x_1\otimes\cdots\otimes x_n\otimes u) = x_2\otimes\cdots\otimes x_n\otimes ux_1^* + \sum_{i=1}^{n-1}(-1)^i x_1\otimes\cdots\otimes x_i x_{i+1}\otimes\cdots\otimes x_n\otimes u + (-1)^n x_1\otimes\cdots\otimes x_{n-1}\otimes x_n u$$

$(n=1, 2, \cdots)$. We have $\partial_n^*\partial_{n+1}^*=0$ and we put

(16) $$H_n^*(A, M) = (\text{Kernel } Z_n^*(A, M) \text{ of } \partial_n^*)/(\text{Image } B_n^*(A, M) \text{ of } \partial_{n+1}^*).$$

(An analogous modification of cohomology groups can be introduced, if we put

(17) $$(\delta_n^* f)(x_1\otimes\cdots\otimes x_n) = x_1^* f(x_2\otimes\cdots\otimes x_n) + \sum_{i=1}^{n-1}(-1)^i f(x_1\otimes\cdots\otimes x_i x_{i+1}\otimes\cdots\otimes x_n) + (-1)^n f(x_1\otimes\cdots\otimes x_{n-1})x_n$$

and

(18) $$H_*^n(A, M) = (\text{Kernel } Z_*^n(A, M) \text{ of } \delta_{n+1}^*)/(\text{Image } B_*^n(A, m) \text{ of } \delta_n^*).$$

But we shall not use this modified cohomology in this paper, except perhaps for occasional reference for comparison).

The definition of δ_n is more or less (left-right) symmetric and the definition of $H^n(A, M)$ is quite symmetric. On the other hand, the definition of $C_n(A, M)$ is not symmetric. But we may establish a K-isomorphism between $C_n(A, M)=C_n\otimes M$ and $M\otimes C_n$ by mapping $x_1\otimes\cdots\otimes x_n\otimes u$ onto $u\otimes x_1\otimes\cdots\otimes x_n$. Then ∂_n induces a boundary operation for $M\otimes C_n$, which is more or less symmetric to ∂_n. We are then led to the same (or isomorphic, to be precise) homology groups. Thus also $H_n(A, M)$ are symmetric. As for our modified homology groups $H_n^*(A, M)$, they are not symmetric. But we can obtain the same homology groups $H_n^*(A, M)$ for $M\otimes C_n$ too. Indeed we have merely to introduce a boundary operation (not symmetric to ∂_n^*) which maps $u\otimes x_1\otimes\cdots\otimes x_n \in M\otimes C_n$ to $ux_1^*\otimes x_2\otimes\cdots\otimes x_n + \sum_{i=1}^{n-1}(-1)^i u\otimes\cdots\otimes x_i x_{i+1}\otimes\cdots\otimes x_n + (-1)^n x_n u\otimes x_1\otimes\cdots\otimes x_{n-1}$. We note also that the modified cohomology groups $H_*^n(A, M)$ are not symmetric.

REMARK. We want also to note that another way to define the modified homology group $H_n^*(A, M)$, for instance, is to define on M a new A-double-module structure. Thus retain the original structure of M as a left-module over A, but define the new operation of $x\in A$ on the right-side of M to be the operation, on the right-side of M, of the element x^*. If we denote the A-double-module thus obtained by M^*, then $H_n^*(A, M)$ is nothing but the ordinary homology group $H_n(A, M^*)$. So, there is not much new in $H_n^*(A, M)$. However, our purpose of introducing it is to achieve what has been described in the introduction, by specifying A to be a Frobenius algebra and $*$ to be its particular automorphism as will be precisely prescribed in the next section.

Let now M, P, Q be three double-modules over our (arbitrary) algebra A and there be an exact sequence

$$0 \to M \to P \to Q \to 0 \tag{19}$$

of A–A–homomorphisms. Then we have the exact sequences

$$H^n(A, M) \to H^n(A, P) \to H^n(A, Q) \to H^{n+1}(A, M) \to H^{n+1}(A, P) \tag{20}$$

(and similarly for the modified cohomology groups) and

$$H^*_{n+1}(A, P) \to H^*_{n+1}(A, Q) \to H^*_n(A, M) \to H^*_n(A, P) \to H^*_n(A, Q)\,, \tag{21}$$

proof being habitual.

We now assume that our algebra A possesses a unit element 1. Then we have

$$H^n(A, M) = H^n(A, 1M) = H^n(A, M1) = H^n(A, 1M1) \tag{22}$$

for $n=1, 2, \cdots$ (Hochschild [6]). (Let us repeat the well-known proof under the setting of our generalized cohomology groups $H^n_*(A, M)$. Thus, for $f \in C^n(A, M)$ with $n \geqq 1$, define $g \in C^{n-1}(A, M)$ by $g(x_1 \otimes \cdots \otimes x_{n-1}) = (I-1)f(1 \otimes x_1 \otimes \cdots \otimes x_{n-1})$, where I is the identity operator on M. Then we see

$$(\delta^*_n g)(x_1 \otimes \cdots \otimes x_n) = (1-I)(\delta^*_{n+1} f)(1 \otimes x_1 \otimes \cdots \otimes x_n) + (1-I)f(x_1 \otimes \cdots \otimes x_n). \tag{23}$$

So, if $f \in Z^*_n(A, M)$ then we have $(\delta^*_n g)(x_1 \otimes \cdots \otimes x_n) = (1-I)f(x_1 \otimes \cdots \otimes x_n)$, or

$$\delta^*_n g = 1g - f\,. \tag{24}$$

Now, for any $f \in C^n(A, M)$ with $n \geqq 0$, define $f' \in C^n(A, 1M)$ by setting $f'(x_1 \otimes \cdots \otimes x_n) = 1f(x_1 \otimes \cdots \otimes x_n)$ $(= (1f)(x_1 \otimes \cdots \otimes x_n))$. Then $\delta^*_{n+1} f' = (\delta^*_{n+1} f)'$ and $f \to f'$ induces a homomorphism of $H^n_*(A, M)$ onto $H^n_*(A, 1M)$ for $n=1, 2, \cdots$. It is actually an isomorphism, for we have by (24) $f = f' - \delta^*_n g$ (in somewhat loose usage of symbols) for every $f^n_* \in Z(A, M)$. Thus we have the first equality in (22). A similar computation shows $H^n_*(A, M) = H^n_*(A, M1)$. Hence $H^n_*(A, M) = H^n_*(A, 1M1)$ too.)

We have also

$$H^*_n(A, M) = H^*_n(A, 1M) = H^*_n(A, 1M1) = H^*_n(A, 1M1) \tag{25}$$

for $n=1, 2, \cdots$. (To prove this, put $w = 1 \otimes (v - v1)$ for $v \in C_n(A, M)$ with $n=0, 1, \cdots$. Then we have

(26) $$\partial^*_{n+1}w = 1\otimes(\partial^*_n(v1-v))+(v1-v)\ .$$

If $v\in Z^*_n(A, M)$ then

(27) $$\partial^*_{n+1}w = v1-v\ .$$

Now, $v\to v'=v1$ induces a homomorphism of $H^*_n(A, M)$ onto $H^*_n(A, M1)$. It is in fact an isomorphism, since (27) shows $v=v'-\partial^*_{n+1}w$ for every $v\in Z^*_n(A, M)$. Further, we get $H^*_n(A, M)=H^*_n(A, 1M)$ by a similar computation and the previous consideration that $H^*_n(A, M)$ may be obtained from $M\otimes C_n$ instead of $C_n(A, M)=C_n\otimes M)$.

§ 2. Frobenius algebras.

An algebra A over a field K is called a Frobenius algebra when it is of a finite rank, say k, over K, has a unit element, and is left A-isomorphic to $A^0=\mathrm{Hom}_K(A, K)$, where we consider A^0 as a left-module over A as usual by $(x\alpha)(a)=\alpha(ax)$ $(\alpha\in A^0;\ x, a\in A)$. Let $(a_1, \cdots, a_k)$ be a basis of A, and let $(\beta_1, \cdots, \beta_k)$ be the basis of A^0 which corresponds to $(a_1, \cdots, a_k)$ by our (arbitrary, but fixed) left A-isomorphism of A and A^0. Let $(b_1, \cdots, b_k)$ be the basis of A dual to $(\beta_1, \cdots, \beta_k)$ (i.e. $\beta_\iota(b_\kappa)=\delta_{\iota\kappa}$). Then the left regular represetation $x\to L(x)=(\lambda_{\iota\kappa}(x))$ of A defined by (a_ι) is identical with the right regular representation defined by (b_ι). Thus

(28) $$xa_\iota = \textstyle\sum_\kappa a_\kappa\lambda_{\kappa\iota}(x)\ ,$$

(29) $$b_\iota x = \textstyle\sum_\kappa \lambda_{\kappa\iota}(x)b_\kappa\ .$$

Let P be the matrix in K such that

(30) $$(a_1,\ \cdots,\ a_k) = (b_1,\ \cdots,\ b_k)P',$$

where P' is the transpose of P. If $x\to R(x)=(\rho_{\iota\kappa}(x))$ is the right regular representation of A defined by (a_ι), then $P^{-1}R(x)P$ must be the right regular representation defined by (b_ι), which is however $L(x)$. Thus

(31) $$R(x)P = PL(x)\ .$$

It is well known that such P has a form $P=(\mu(a_\iota a_\kappa))_{\iota\kappa}$ with $\mu\in A^0$; indeed μ is the image of $1\in A$ under our isomornhism of A and A^0. Put for each $x=\sum_\iota a_\iota\xi_\iota\in A$ $(\xi_\iota\in K)$

(32) $$x^* = \textstyle\sum_\iota a_\iota\xi^*_\iota,\quad (\xi^*_1,\ \cdots,\ \xi^*_k) = (\xi_1,\ \cdots,\ \xi_k)P'P^{-1}.$$

Then we see $\mu(x^*y)=\mu(yx)$ for every $x, y\in A$, and $x\to x^*$ is an automorphism of A. Further we have

(33) $$R(x)P' = P'L(x^*)$$

and $x^*(b_1, \cdots, b_k)=(b_1, \cdots, b_k)R(x)$, i.e.

(34) $$x^*b_\iota = \sum_\kappa b_\kappa \rho_{\kappa\iota}(x)$$

along with

(35) $$a_\iota x = \sum_\kappa \rho_{\iota\kappa}(x)a_\kappa\,.$$

(For all these see Nakayama [10]; cf. also Brauer [1], Nesbitt-Thrall [12]).

We note that the automorphism $x \rightarrow x^*$ of A is determined by A up to inner automorphisms of A. Further, a group algebra (of a finite group over a field) is a Frobenius algebra. Indeed, if we take the group elements as its basis $(a_1, \cdots, a_k)$ then we may set $(b_1, \cdots, b_k)=(a_1^{-1}, \cdots, a_k^{-1})$ and P may be chosen so as the automorphism $x \rightarrow x^*$ is simply the identity automorphism; the group algebra is in fact what is called a symmetric algebra, i.e. a Frobenius algebra with $P'=P$ (for a suitable choice of the isomorphism of A and A^0).

Now, with a Frobenius algebra A, over K, and a double-module M over A, we set

(36) $$M^A = \{u \in M \mid xu = ux \quad \text{for all} \quad x \in A\}\,,$$

(37) $$\sigma u = \sum_\iota a_\iota u b_\iota \qquad (u \in M)$$

where $(a_1, \cdots, a_k)$, $(b_1, \cdots, b_k)$ are *dual* bases of A as described above.

Lemma 1. *For every $u \in M$ and $x \in A$ we have*

(38) $$\sigma(ux^* - xu) = 0\,.$$

Proof. The left-hand side is equal to

$$\sum_\iota a_\iota(ux^* - xu)b_\iota = \sum_{\iota,k} a_\iota u b_k \rho_{k\iota}(x) - \sum_{\iota,k} \rho_{\iota k}(x) a_k u b_\iota = 0$$

by (34), (35).

Lemma 2. *We have*

(39) $$\sigma M \subseteq M^A.$$

Proof. For every $x \in A$ and $u \in M$ we have, by (29), (28)

$$(\sigma u)x = \sum_\iota a_\iota u b_\iota x = \sum_{\iota,\kappa} a_\iota u \lambda_{\iota\kappa}(x) b_\kappa = \sum_\kappa x a_\kappa u b_\kappa = x(\sigma u)\,.$$

By this lemma we consider the factor group $M^A/\sigma M$ and denote it by $\mathfrak{N}(M)$:

(40) $$\mathfrak{N}(M) = M^A/\sigma M\,.$$

Lemma 3. *If $M=1M$ then $\mathfrak{N}(M)=\mathfrak{N}(M1)$. Similarly, if $M=M1$ then $\mathfrak{N}(M)=\mathfrak{N}(1M)$.*

Proof. The assumption $M=1M$ readily implies $M^A=(M1)^A$. Further, clearly $\sigma M=\sigma(M1)$ $(=\sigma(1M)=\sigma(1M1))$ (without any assumption).

§ 3. 0– and negative-dimensional cohomology of a Frobenius algdbra.

Taking our automorphism $x\to x^*$ of the Frobenius algebra A, as defined in § 2, we consider the boundary operation ∂_n^* as was introduced in § 1, (15). Further we introduce a K-homomorphism Δ of $C_0(A, M)$ into $C^0(A, M)$ as follows :

(41) $$(\Delta u)(1)=\textstyle\sum_\iota a_\iota u b_\iota \qquad (u\in M=C_0(A, M))\,.$$

By means of Lemmas 1, 2 we verify readily

(42) $$\Delta\partial_1^*=0\,,$$

(43) $$\delta_1\Delta=0\,.$$

Both $C_0(A, M)$ and $C^0(A, M)$ may be identified with M, and Δ may thus be identified with the endomorphism σ of M.

Now, we set

(44) $$C^{-n}(A, M)=C_n(A, M)\,,\quad \delta_{-n}=\partial_n^*$$

for $n=1, 2, \cdots$ and

(45) $$H^{-n}(A, M)=H_{n-1}^*(A, M)$$

for $n=2, 3, \cdots$. We put further

(46) $$\delta_0=\Delta\,,$$

(47) $$H^{-1}(A, M)=(\text{Kernel of } \delta_0(=\Delta))/\text{Image of } \delta_{-1}(=\partial_1^*))\,,$$

(48) $$H_0(A, M)=(\text{Kernel of } \delta_1)/(\text{Image of } \delta_0(=\Delta))\,.$$

Thus we have the operation δ_p and the cohomology group (in generalized sense) $H^p(A, M)$ for every dimension $p\gtreqless 0$. (Observe that, according as $p>$or $=$ or<0, δ_p maps $C^{p-1}(A, M)$ into $C^p(A, M)$, or $C^p(A, M)$ into itself, or $C^p(A, M)$ into $C^{p+1}(A, M)$).

Theorem 1. *If M, P, Q are double-modules over a Frobenius algebra A and if we have an exact sequence* (19) *of A–A-homomorphisms, then we have the exact sequence*

(49) $$H^p(A, M)\to H^p(A, P)\to H^p(A, Q)\to H^{p+1}(A, M)\to H^{p+1}(A, P)$$

for any integer p.

Proof. The theorem asserts indeed the exact sequence infinite in both directions: $\cdots \to H^{-2}(P) \to H^{-2}(Q) \to H^{-1}(M) \to H^{-1}(P) \to H^{-1}(Q) \to H^{0}(M) \to H^{0}(P) \to H^{0}(Q) \to H^{1}(M) \to H^{1}(P) \to \cdots$, where $H^p(P)$ etc. stand for $H^p(A, P)$ etc. The exactness of the parts not explicitly written have been observed in §1. The exactness of the part explicitly written can also be seen in usual manner.

Another way of defining cohomology groups $H^p(A, M)$, in particular those for $p \leqq 0$, is to introduce the augmented standard complex as follows: Let $X_{n-1} = X_{n-1}(A)$, for $n \geqq 1$, denote the tensor product (over K) $A \otimes C_{n-1}(A) \otimes A$ under the ordinary A-double-module structure. The differentiation maps $d_n: X_n \to X_{n-1}$ and the augmentation $\varepsilon: X_0 \to A$ defined by (the K-linearity and)

$$d_n(x_0 \otimes \cdots \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1},$$
$$\varepsilon(x_0 \otimes x_1) = x_0 x_1$$

make $\sum_{n=1}^{\infty} X_{n-1}$ the ordinary standard complex of A. Now, again with $n \geqq 1$, denote by X_{-n} the module $X_{n-1}^0 = \mathrm{Hom}_K(X_{n-1}, K)$ with the ordinary A-double-module structure induced by that of X_{n-1}. d_n and ε induce the maps

$$d_{-n}: X_{-n} \to X_{-n-1},$$
$$\varepsilon' : A^0 \to X_{-1}.$$

Together with d_n, ε the maps d_{-n}, ε' are A–A-homomorphisms. Consider, now, an A-left-isomorphism of A and A^0 and consider the associated automorphism $*$ of A as in §2. We introduce then into X_{-n} a new A-double-module structure on retaining the old left A-operation but defining the new operation of $x \in A$ on the right to be the old operation, on the right, of the element x^*. If we make the same alteration of the A-double-module structure in A^0, then the relations (34), (35) express nothing but that our A-left isomorphism $A \to A^0$ is also an A-right-isomorphism. Denote by d_0 the product of $\varepsilon: X_0 \to A$, this isomorphism of A to A^0, and ε': $A^0 \to X_{-1}$. Altogether we get a complex which we want to call the complete standard complex of A with augmentation:

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & X_1 \arrow[r, "d_1"] & X_0 \arrow[rr, "d_0"] \arrow[dr, "\varepsilon"'] & & X_{-1} \arrow[r, "d_{-1}"] & X_{-2} \arrow[r] & \cdots . \\ & & & A \to A^0 \arrow[ur, "\varepsilon'"'] & & & \end{tikzcd}$$

A^0, X_{-1}, X_{-2}, $\cdots$ are considered under their new A-double-module structure obtained from the old one by the right-hand side modification by $*$, and all the arrows are A–A-homomorphisms.

Then we put, with an A-double-module M,

(50) $$C^p(A, M) = \mathrm{Hom}_{A-A}(X_p, M);$$

This is allowed, since for $p \geqq 1$ the right-hand side is essentially the same thing as that of (3). Moreover, for $p \leqq -1$ $C^p(A, M)$ essentially coincides with the module $C_{-p-1}(A, M)$ as defined in (2). The differentiation maps d_q in our complete complex induces the map $\delta_p : C^{p-1}(A, M) \to C^p(A, M)$ which are old δ_p for $p \geqq 1$ and old ∂_{-p} for $p \leqq -1$. Indeed, for $p \gtreqless 0$ we can readily verify that the cohomology group (Kernel $Z^p(A, M)$ of δ_{p+1})/(Image $B^p(A, M)$ of δ_p) is our $H^p(A, M)$.

A similar construction may be made in order to get a general augmented complete complex which is not the standard one.

§ 4. Interpretation of H^p by $\mathfrak{N}$.

Both $C_0(A, M)$ and $C^0(A, M)$ are identified with M, and $\delta_0 = \Delta$ is nothing but the operation σ in § 2. Further, for $u \in M (= C^0(A, M))$ its coboundary $\delta_1 u$ is the mapping $(\in C^1(A, M))$ $x \to xu - ux$ of A into M. Thus M^A is nothing but the kernel of δ_0. So we have

(51) $$H^0(A, M) = M^A/\sigma M = \mathfrak{N}(M).$$

Now we want to show

(52) $$H^n(A, M) = H^n(A, 1M) = \mathfrak{N}(C^n(A, 1M))$$

$(n = 1, 2, \cdots)$, where we consider $C^n(A, 1M)$ as a double-module over A under the operations (10), (11). Since the first equality has been given in § 1, we can, without loss in generality, assume $M = 1M$ from the beginning. Let $f \in C^n(A, M)$. Then

(53) $$(xf - fx)(x_1 \otimes \cdots \otimes x_n) = (xf)(x_1 \otimes \cdots \otimes x_n) - (xf)(x_1 \otimes \cdots \otimes x_n)$$
$$+ (\delta_{n+1} f)(x \otimes x_1 \otimes \cdots \otimes x_n) = (\delta_{n+1} f)(x \otimes x_1 \otimes \cdots \otimes x_n).$$

So $f \in C^n(A, M)^A$ if and only if $\delta_{n+1} f = 0$. Thus

(54) $$C^n(A, M)^A = Z^n(A, M).$$

(This is independent of $M = 1M$).

Next we prove $\sigma C^n(A, M) = \delta_n C^{n-1}(A, M)$. To do so, let $h \in C^{n-1}(A, M)$ and $g \in C^n(A, M)$ satisfy the relation

(55) $$h(x_1 \otimes \cdots \otimes x_{n-1}) = \textstyle\sum_\iota a_\iota g(b_\iota \otimes x_1 \otimes \cdots \otimes x_{n-1}).$$

Then

$$(\delta_n h)(x_1 \otimes \cdots \otimes x_n) = x_1 h(x_2 \otimes \cdots \otimes x_n) + \textstyle\sum_{i=1}^{n-1} (-1)^i h(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots)$$
$$+ (-1)^n h(x_1 \otimes \cdots \otimes x_{n-1}) x_n = \textstyle\sum_\iota x_\iota a_\iota g(b_\iota \otimes x_2 \otimes \cdots \otimes x_{n-1}) +$$
$$\textstyle\sum_{i=1}^{n-1} \sum_\iota (-1)^i a_\iota g(b_\iota \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots) + (-1)^n \sum_\iota a_\iota g(b_\iota \otimes x_1 \otimes \cdots \otimes x_{n-1}) x_n.$$

Hence

$$\sum_\iota x_1 a_\iota g(b_\iota \otimes x_2 \otimes \cdots \otimes x_{n-1}) = \sum_\iota a_\iota g(b_\iota x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1})$$

by (28), (29). So we have $(\delta_n h)(x_1 \otimes \cdots \otimes x_n) = \sum_\iota (a_\iota g b_\iota)(x_1 \otimes \cdots \otimes x_n)$, or

(56) $$\delta_n h = \sum_\iota a_\iota g b_\iota .$$

Thus, if $f \in \sigma C^n(A, M)$, say $= \sum_\iota a_\iota g b_\iota$, then $f = \delta_n h$ with h defined by (55), whence $f \in \delta_n C^{n-1}(A, M)$.

To prove the converse, let $f = \delta_n h$ with $h \in C^{n-1}(A, M)$. Let, further $(\nu_{\iota\kappa})$ be a non-singular $k \times k$-matrix in K such that

(57) $$(a_1, \cdots, a_k)(\nu_{\iota\kappa}) = (1, \cdots) .$$

Let $(b'_1, \cdots, b'_k)$ be the basis of A such that

(58) $$b_\iota = \sum_\kappa \nu_{\iota\kappa} b'_\kappa .$$

Define now $g \in C^n(A, M)$ so as

$$g(b'_1 \otimes x_1 \otimes \cdots \otimes x_{n-1}) = h(x_1 \otimes \cdots \otimes x_{n-1}) ,$$
$$g(b'_\iota \otimes x_1 \otimes \cdots \otimes x_{n-1}) = 0 \quad \text{for} \quad \iota = 2, \cdots, k .$$

Then, because of $M = 1M$,

$$\begin{aligned} h(x_1 \otimes \cdots \otimes x_{n-1}) &= 1h(x_1 \otimes \cdots \otimes x_{n-1}) = \sum_\iota a_\iota \nu_{\iota 1} h(x_1 \otimes \cdots \otimes x_{n-1}) \\ &= \sum_{\iota,\kappa} a_\iota \nu_{\iota k} g(b'_\kappa \otimes x_1 \otimes \cdots \otimes x_{n-1}) = \sum_\iota a_\iota g(\textstyle\sum_\kappa \nu_{\iota\kappa} b'_\kappa \otimes x_1 \cdots \otimes x_{n-1}) \\ &= \sum_\iota a_\iota g(b_\iota \otimes x_1 \otimes \cdots \otimes x_{n-1}) . \end{aligned}$$

So we have (55) and, therefore, $\delta_n h = \sum_\iota a_\iota g b_\iota$ by (56).

Thus we have proved

(59) $$\sigma C^{n-1}(A, M) = \delta_n C^{n-1}(A, M) = B^n(A, M)$$

(on using $M = 1M$). Combining this with (53), we have $\mathfrak{N}(C^n(A, M)) = H^n(A, M)$ as desidered.

Next we want to prove

(60) $$H^{-n}(A, M) = H^{-n}(A, M1) = \mathfrak{N}(C^{-n}(A, M1))$$

$(n = 1, 2, \cdots)$. Again we may, and shall, assume $M1 = M$. Let $(c_1, \cdots, c_k)$ be any K-basis of A, and consider an arbitrary element

$$w = \sum_{(\iota_i)} c_{\iota_1} \otimes \cdots \otimes c_{\iota_{n-1}} \otimes u_{\iota_1 \cdots \iota_{n-1}} \quad (u_{\iota_1 \cdots \iota_{n-1}} \in M)$$

of $C^{1-n}(A, M) = C_{n-1}(A, M) = C_{n-1} \otimes M$. We put

(61) $$w^\S = \sum_{\iota, (\iota_i)} a \otimes c_{\iota_1} \otimes \cdots \otimes c_{\iota_{n-1}} \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota .$$

Naturally $w^\S \in C_n \otimes M = C^{-n}(A, M)$, and the mapping $w \to w^\S$ is not only a (K-)homomorphism but a monomorphism, since $w^\S = 0$ implies, for each (ι_i), $wb_\iota = c_{\iota_1} \otimes \cdots \otimes c_{\iota_{n-1}} \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota = 0$ for every $\iota = 1, \cdots, k$. Now we have

$$\begin{aligned} xw^\S - w^\S x = \textstyle\sum_{\iota,(\iota_i)} (xa_\iota \otimes c_{\iota_1} \otimes \cdots \otimes c_{\iota_{n-1}} \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota - x \otimes a_\iota c_{\iota_1} \otimes \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota \\ + \textstyle\sum_{i=1}^{n-2} (-1)^{i-1} x \otimes a_\iota \otimes c_{\iota_1} \otimes \cdots \otimes c_{\iota_i} c_{\iota_{i+1}} \otimes \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota + (-1)^{n-1} x \otimes a_\iota \otimes c_{\iota_1} \\ \otimes \cdots \otimes c_{\iota_{n-1}} u_{\iota_1 \cdots \iota_{n-1}} b_\iota - a_\iota \otimes c_{\iota_1} \otimes \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota x) . \end{aligned}$$

Hence $\sum_\iota xa_\iota \otimes c_{\iota_1} \otimes \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota = \sum_\iota a_\iota \otimes c_{\iota_1} \otimes \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota x$ by (28), (29). So we have

$$\begin{aligned} (62) \quad xw^\S - w^\S x = \textstyle\sum_{\iota,(\iota_i)} (-x \otimes a_\iota c_{\iota_1} \otimes \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota + \sum_{i=1}^{n-2} (-1)^{i-1} x \otimes a_\iota \otimes c_{\iota_1} \otimes \\ \cdots \otimes c_{\iota_i} c_{\iota_{i+1}} \otimes \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota + (-1)^{n-1} x \otimes a_\iota \otimes c_{\iota_1} \otimes \cdots \otimes c_{\iota_{n-1}} u_{\iota_1 \cdots \iota_{n-1}} b_\iota) \\ = -x \otimes \textstyle\sum_\iota a_\iota w b_\iota . \end{aligned}$$

If here $n \geqq 2$ then

$$\begin{aligned} \textstyle\sum_\iota a_\iota w b_\iota = \sum_\iota (b_\iota c_{\iota_1} \otimes \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota + \sum_{i=1}^{n-2} (-1)^i x \otimes a_\iota \otimes c_{\iota_1} \otimes \cdots \otimes c_{\iota_i} c_{\iota_{i+1}} \otimes \\ \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota + (-1)^n a_\iota \otimes c_{\iota_1} \otimes \cdots \otimes c_{\iota_{n-1}} u_{\iota_1 \cdots \iota_{n-1}} b_\iota) = \textstyle\sum (a_\iota \otimes c_{\iota_2} \otimes \\ \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} c_{\iota_1}^* b_\iota + \textstyle\sum_{i=1}^{n-2} (-1)^i a_\iota \otimes c_{\iota_1} \otimes \cdots \otimes c_{\iota_i} c_{\iota_{i+1}} \otimes \cdots \otimes u_{\iota_1 \cdots \iota_{n-1}} b_\iota \\ + (-1)^n a_\iota \otimes c_{\iota_1} \otimes \cdots \otimes c_{\iota_{n-1}} u_{\iota_1 \cdots \iota_{n-1}} b_\iota) \end{aligned}$$

by (34), (35). The last sum is, however, nothing but $(\partial_{n+1}^* w)^\S$ in the sense of the mapping § similar as above (with $n-1$ in place of n). Thus

$$(63) \qquad \textstyle\sum_\iota a_\iota w b_\iota = (\partial_{n-1}^* w)^\S = (\delta_{1-n} w)^\S .$$

So, combining this with (62), we have

$$(64) \qquad xw^\S - w^\S x = -x \otimes (\delta_{1-n} w)^\S .$$

Hence $w^\S \in C^{-n}(A, M)^A$ if and only if $\delta_{1-n} w = 0$. This relation holds however also when $n = 1$. For, then $w \in M$ and $xw^\S - w^\S x = x(\sum_\iota a_\iota \otimes wb_\iota) - (\sum_\iota a_\iota \otimes wb_\iota)x = \sum_\iota xa_\iota \otimes wb_\iota - \sum_\iota x \otimes a_\iota wb_\iota - \sum_\iota a_\iota \otimes wb_\iota x = -\sum_\iota x \otimes a_\iota wb_\iota = -x \otimes \sigma w = -x \otimes \delta_0 w$, by (28), (29).

Next, let $w_0 \in C^{-n}(A, M) = C_n(A, M)$. Then

$$\textstyle\sum_\iota a_\iota w_0 b_\iota = (\partial_n^* w_0)^\S = (\delta_{-n} w_0)^\S$$

by (63), with $n+1$ in place of n. Thus, for $w \in c^{-(n-1)}(A, M)$ we have $w^\S \in \sigma C^{-n}(A, M)$ if and only if $w \in \delta_{-n} C^{-n}(A, M)$.

These considerations show that the mapping $w \to w^\S$ of $C^{-(n-1)}(A, M)$ into $C^{-n}(A, M)$ induces a monomorphism of $H^{-n}(A, M) = H_{n-1}^*(A, M)$ into $\mathfrak{N}(C^{-n}(A, M))$. We want to show that it is an epimorphism. For this purpose, write an arbitrary element of $C^{-n}(A, M)=C_n(A, M)$ in the form

$$v=\sum_{\iota,(\iota_i)} a_\iota\otimes c_{\iota_1}\otimes\cdots\otimes c_{\iota_{n-1}}\otimes u_{\iota\iota_1\cdots\iota_{n-1}} \quad (u_{\iota\iota_1\cdots\iota_{n-1}}\in M).$$

Then

$$\begin{aligned}(65)\quad xv-vx=&\sum_{\iota,(\iota_i)}(xa_\iota\otimes c_{\iota_1}\otimes\cdots\otimes u_{\iota\iota_1\cdots\iota_{n-1}}-x\otimes a_\iota c_{\iota_1}\otimes c_{\iota_2}\otimes\cdots\otimes u_{\iota\iota_1\cdots\iota_{n-1}}\\&+\sum_{i=1}^{n-2}(-1)^{i-1}x\otimes a_\iota\otimes c_{\iota_1}\otimes\cdots\otimes c_{\iota_i}c_{\iota_{i+1}}\otimes\cdots\otimes u_{\iota\iota_1\cdots\iota_{n-1}}+(-1)^n x\otimes a_\iota\otimes c_{\iota_1}\\&\otimes\cdots\otimes c_{\iota_{n-1}}u_{\iota\iota_1\cdots\iota_{n-1}}-a_\iota\otimes c_{\iota_1}\otimes\cdots\otimes u_{\iota\iota_1\cdots\iota_{n-1}}x).\end{aligned}$$

Suppose here $v\in C^{-n}(A, M)^A$. Then $xv-vx=0$ for all $x\in A$ and in particular $1v-v1=0$. Then relation (65) with $x=1$ gives then $\sum_{\iota,(\iota_i)}(a_\iota c_{\iota_1}\otimes c_{\iota_2}\otimes\cdots\otimes n_{\iota\iota_1\cdots\iota_{n-1}}+\sum_{i=1}^{n-2}(-1)^i a_\iota\otimes c_{\iota_1}\otimes\cdots\otimes c_{\iota_i}c_{\iota_{i+1}}\otimes\cdots\otimes u_{\iota\iota_1\cdots\iota_{n-1}}+(-1)^{n-1}a_\iota\otimes c_{\iota_1}\otimes\cdots\otimes c_{\iota_{n-1}}u_{\iota\iota_1\cdots\iota_{n-1}})=0$. Putting this back into (65), we get

$$\sum_{\iota,(\iota_i)}(xa_\iota\otimes c_{\iota_1}\otimes\cdots\otimes u_{\iota\iota_1\cdots\iota_{n-1}}-a_\iota\otimes c_{\iota_1}\otimes\cdots\otimes u_{\iota\iota_1\cdots\iota_{n-1}}x)=0.$$

Hence, for each (ι_i), we have

$$(66)\qquad \sum_\iota(xa_\iota\otimes u_{\iota\iota_1\cdots\iota_{n-1}}-a\otimes u_{\iota\iota_1\cdots\iota_{n-1}}x)=0.$$

By (28), this (66) reads $\sum_\iota(\sum_\kappa a_\kappa\otimes\lambda_{\kappa\iota}(x)u_{\iota\iota_1\cdots\iota_{n-1}}-a_\iota\otimes u_{\iota\iota_1\cdots\iota_{n-1}}x)=0$. So

$$\sum_\kappa\lambda_\kappa(x)u_{\kappa\iota_1\cdots\iota_{n-1}}-u_{\iota\iota_1\cdots\iota_{n-1}}x=0$$

for each ι. By (29), this means that for each (ι_i) the mapping $b_\iota\to u_{\iota\iota_1\cdots\iota_{n-1}}$ $(\iota=1,\cdots,k)$ gives a right A-homomorphism of A into M. Let $v_{\iota_1\cdots\iota_{n-1}}$ be the image of 1 in this homomorphism. Then $u_{\iota\iota_1\cdots\iota_{n-1}}=v_{\iota_1\cdots\iota_{n-1}}b_\iota$ for each ι. Thus we have

$$v=\sum_{\iota,(\iota_i)}a_\iota\otimes c_{\iota_1}\otimes\cdots\otimes c_{\iota_{n-1}}\otimes v_{\iota_1\cdots\iota_{n-1}}b_\iota=(\sum_{(\iota_i)}c_{\iota_1}\otimes\cdots\otimes c_{\iota_{n-1}}\otimes v_{\iota_1\cdots\iota_{n-1}})^\S.$$

It follows that our monomorphism maps $H^{-n}(A, M)$ onto $\mathfrak{N}(C^{-n}(A, M))$. The relation (60) is thus proved.

Taking these in somewhat weaker forms we have

Theorem 2. *Let A be a Frobenius algebra and M be a double-module over A with $M=1M1$. Then, for any p.*

$$H^p(A, M)=\mathfrak{N}(C^p(A, M)),$$

where the A-double-module structure of $C^p(A, M)$ is defined by (8), (9) *or* (10), (11).

§ 5. Reduction theorem.

We now want to establish

Theorem 3 (General reduction theorem). *Let A be a Frobenius* *algebra and M be a double-module over A with $1M1=M$. For p, $q\gtreqless 0$ we have*

$$H^{p+q}(A, M) = H^p(A, C^q(A, M)) , \tag{67}$$

where the A-double-module strtcture of $C^q(A, M)$ is given by (10), (11) *or* (8), (9) *according as $q\geqq$ or <0.*

Proof. The case $q=0$ is trivial. The case $p, q>0$ and the case $p<-1$, $q<0$ are well-known (except perhaps for the presence of an automorphism $*$, in the definition, for the case $p<-1$, $q<0$). Indeed, in all these cases the Frobenius algebra property of A has nothing to do; A and $*$ may be any algebra and its arbitrary automorphism. Moreover, the assumption $1M1=M$ is not needed in all these cases. It suffices perhaps to mention, besides (53), the following formulas (in the former of which we place an automorphism though not needed for our present purpose): with $F(x_1)(x_2\otimes\cdots\otimes x_n)=f(x_1\otimes\cdots\otimes x_n)$

$$\begin{aligned}((\delta F^*)(x_0\otimes x_1))(x_2\otimes\cdots\otimes x_n) &= x_0^*F(x_1)(x_2\otimes\cdots\otimes x_n)-(F(x_0x_1))(x_2\otimes\cdots\otimes x_n)\\ &+\textstyle\sum_{i=1}^{n}(-1)^{i-1}F(x_0)(x_1\otimes\cdots\otimes x_ix_{i+1}\otimes\cdots)+(-1)^{n-1}F(x_0)(x_1\otimes\cdots\otimes x_{n-1})x_n\\ &= (\delta_{n+1}^*f)(x_0\otimes x_1\otimes\cdots\otimes x_n) ;\end{aligned} \tag{68}$$

$$\begin{aligned}\partial_1^*(x_1\otimes(x_2\otimes\cdots\otimes x_n\otimes u)) &= x_2\otimes\cdots\otimes x_n\otimes ux_1^*-(\textstyle\sum_{i=1}^{n-1}(-1)^{i-1}x_1\otimes\cdots\\ &\otimes x_ix_{i+1}\otimes\cdots\otimes u+(-1)^{n-1}x_1\otimes\cdots\otimes x_{n-1}\otimes x_nu) = \partial_n^*(x_1\otimes\cdots\otimes x_n\otimes u) .\end{aligned} \tag{69}$$

(Now we really use the Frobenius algebra property of A, the property of our automorphism $*$ expressed in (34), (35), and the assumption $1M1=M$). The case $p=0$ (with $q\gtreqless 0$) has been settled in the preceding section ((51), (52), (60)). Further, for $q<0$ we have $H^{-1}(A, C^q(A, M))=\mathfrak{N}(C^{-1}(A, C^q(A, M)))=\mathfrak{N}(C^{q-1}(A, M))=H^{q-1}(A, M)$, which proves (67) for $p=-1$, $q<0$; observe that $C^q(A, M)1=C^q(A, M)$ because of $M1=M$.

Now we turn to the remaining cases. Our $C^1(A, M)$ is $\mathrm{Hom}_K(A, M)$ with the A-double-module structure defined by (10), (11). We now introduce another A-double-module stiucture into $\mathrm{Hom}_K(A, M)$, putting simply $F\in\mathrm{Hom}_K(A, K)$

$$(xF)(y) = xF(y) , \quad (Fx)(y) = F(xy) . \tag{70}$$

On doing this we shall denote $\mathrm{Hom}_K(A, M)$ by $L(A, M)$. We have $L(A, M)1=L(A, M)$ always, and $1L(A, M)=L(A, M)$ when $1M=M$.

Lemma 4. $H^p(A, L(A, M))=0$ for $p\leqq 0$.

To prove this lemma, let $F\in L(A, M)^A$. Then

$$0 = (xF-Fx)(y) = xF(y)-F(xy)$$

for all $x, y \in A$. Hence $x \to F(x)$ is a left A-homomornhism of A into M and indeed $F(x) = xF(1)$ $(x \in A)$. Let (ν_κ) and $(b'_1, \cdots, b'_k)$ be as in §4 ((57), (58)). Define $G \in L(A, M)$ by

$$G(b'_1) = F(1)\,, \quad G(b'_\iota) = 0 \quad (\iota \neq 1)\,.$$

Then

(71) $$\textstyle\sum_\iota (a_\iota G b_\iota)(x) = \sum_\iota a_\iota G(b_\iota x) = \sum_\iota x a_\iota G(b_\iota)$$
$$\textstyle = \sum_{\iota,\kappa} x a_\iota \nu_{\iota\lambda} G(b'_\kappa) = \sum_\iota x a_\iota \nu_{\iota 1} F(1) = x1F(1) = F(x)\,.$$

Thus $F \in \sigma L(A, M)$, and this shows

(72) $$\mathfrak{N}(L(A, M)) = 0\,.$$

Next we show that for $p \leqq 0$ $C^p(A, L(A, M))$ may be identified with $L(A, C^p(A, M))$. This is trivial for $p=0$. For $p=-n<0$, we identify $x_1 \otimes \cdots \otimes x_n \otimes F$ $(F \in L(A, M))$ with $(x_1 \otimes \cdots \otimes x_n \otimes F)_1 \in L(A, C^{-n}(A, M))$ such that

(73) $$x_1 \otimes \cdots \otimes x_n \otimes F(x) = (x_1 \otimes \cdots \otimes x_n \otimes F)_1(x)\,.$$

This is in accord with our A-double-module structures. For,

$$((x_1 \otimes \cdots \otimes F)y)_1(x) = (x_1 \otimes \cdots \otimes Fy)_1(x) = x_1 \otimes \cdots \otimes (Fy)(x)$$
$$= x_1 \otimes \cdots \otimes F(yx) = (x_1 \otimes \cdots \otimes F)_1(yx) = ((x \otimes \cdots \otimes F)_1 y)(x)\,,$$
$$\textstyle (y(x_1 \otimes \cdots \otimes F))_1(x) = (yx_1 \otimes \cdots \otimes F + \sum_{i=1}^{n-1}(-1)^i y \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes F$$
$$\textstyle +(-1)^n y \otimes x_1 \otimes \cdots \otimes x_n F)_1(x) = (yx_1 \otimes \cdots + \sum_{i=1}^{n-1}(-1)^i y \otimes \cdots$$
$$\otimes x_i x_{i+1} \otimes \cdots) \otimes F(x) + (-1)^n y \otimes x_1 \otimes \cdots \otimes (x_n F)(x) = yx_1 \otimes \cdots \otimes F(x)$$
$$\textstyle + \sum_{i=1}^{n-1}(-1)^i y \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes F(x) + (-1)^n y \otimes x_1 \otimes \cdots \otimes x_n F(x)$$
$$= y(x_1 \otimes \cdots \otimes F(x)) = y((x_1 \otimes \cdots \otimes F)_1(x)) = (y(x_1 \otimes \cdots \otimes F)_1)(x)\,.$$

Now we have, for $p \leqq 0$,

(74) $$H^p(A, L(A, M)) = \mathfrak{N}(C^p(A, L(A, M))) = \mathfrak{N}(L(A, C^p(A, M)) = 0\,,$$

the last equality being implied by (72) with $C^p(A, M)$ in place of M; for the first equality, which is implied by (60) or (51), observe that $L(A, M)1 = L(A, M)$ as was noted above. Lemma 4 is thus proved.

Lemma 5. *Let $M1 = M$. We have the exact sequence*

(75) $$0 \to M \to L(A, M) \to C^1(A, M)1 \to 0$$

of A-A-homomorphisms. (The assumption $M1 = M$ concerns only with the exactness at M).

To prove this lemma, let $F \in L(A, M) = \mathrm{Hom}_K(A, M)$. On considering

F as an element of $C^1(A, M)$, we denote it by F_0. The mapping $F \to F_0 1$ is clearly a K-homomorphism of $L(A, M)$ onto $C^1(A, M)1$. It is in fact an A-A-homomorphism. For,

$$(76) \quad ((xF)_0 1)(y) = (xF)_0(1y) - (xF)_0(1)y = (xF((y) - (xF)(1)y$$
$$= xF(y) - xF(1)y = x(F_0(1y) - F_0(1)y) = x(F_0 1)(y) = (x(F_0 1))(y)\,,$$
$$(77) \quad ((Fx)_0 1)(y) = (Fx)_0(1y) - (Fx)_0(1)y = (Fx)(y) - (Fx)(1)y$$
$$= F(xy) - F(x)y = F_0(xy) - F_0(x)y = (F_0 x)(y) = ((F_0 1)x)(y)\,.$$

Now, $(F_0 1)(x) = F_0(x) - F_0(1)x = F(x) - F(1)x$. So, $F_0 1 = 0$ if and only if $F(x) = F(1)x$ for all $x \in A$. Associating $u \in M$ with an element F of $L(A, M)$ such that $F(x) = ux$, we have thus a mapping of M upon the kernel $\{F \mid F_0 1 = 0\}$ of our homomorphism $L(A, M) \to C^1(A, M)1$. The elements of $L(A, M)$ associated with yu, $uy \in M$ map x onto yux, uyx respectively. They are nothing but yF, Fy. So our mapping $M \to \{F \mid F_0 1 = 0\}$ is an A-A-epimorphism. If $M1 = M$ then it is a monomorphism too. Thus we have the exact sequence of our lemma 5.

The exact sequence (74) entails, for any p, the exact sequence

$$(78) \qquad H^{p-1}(A, L(A, M)) \to H^{p-1}(A, M)1) \to H^p(A, M) \to H^p(A, L(A, M))\,.$$

If here $p \leqq 0$ then the first and the last terms are 0 by lemme 4. So we have

$$(79) \qquad H^p(A, M) = H^{p-1}(A, C^1(A, M)1) = H^p(A, C^1(A, M))$$

for $p \leqq 0$. The repeated application of this gives $H^p(A, M) = H^{p-n}(A, C_n(A, M))$ ($p \leqq 0$, $n > 0$), or the case $p < 0$, $q > 0$, $p + q \leqq 0$ of (67). As foı the case $p < 0$, $q > 0$, $p + q > 0$ we have, by what have just been proved, $H^p(A, C^q(A, M)) = H^p(A, C^{-p}(A, C^{p+q}(A, M))) = H^0(A, C^{p+q}(A, M)) = H^{p+q}(A, M)$. The case $p < 0$, $q > 0$ is thus settled

Next we consider $A \otimes M$ with an A-double-module structure different from the one in $C_1(A, M) = C^{-1}(A, M)$. We simply put

$$(80) \qquad x(y \otimes u) = xy \otimes u\,, \quad (y \otimes u)x = y \otimes ux\,.$$

On doing this, we denote $A \otimes M$ by $L'(A, M)$.

Lemma 6. $H^p(A, L'(A, M)) = 0$ *for* $p \geqq 0$.

To prove this, let $\sum_\iota a_\iota \otimes u_\iota$ ($u_\iota \in M$) be an element of $L'(A, M)^A$. We have

$$0 = x(\textstyle\sum_\iota a_\iota \otimes u_\iota) - (\sum_\iota a_\iota \otimes u_\iota)x = \sum_{\iota,\kappa} a_\kappa \lambda_{\kappa\iota}(x) \otimes u_\iota - \sum_\iota a_\iota \otimes u_\iota x_\iota$$
$$= \textstyle\sum_\iota a_\iota \otimes (\sum_\kappa \lambda_{\iota\kappa}(x) u_\kappa - u_\iota x)\,.$$

with $\lambda_{\kappa\iota}(x)$ in (28). Thus $u_\iota x = \sum_\kappa \lambda_{\iota\kappa}(x) u_\kappa$, for all $x \in A$, and this shows, in view of (29), that $u_\iota \to b_\iota$ gives a right A-homomorphism of A into M. Let u be the image of $1 \in A$ in this homomorphism, and consider the element $1 \otimes u$ of $L'(A, M)$. We have

(81) $$\sum_\iota a_\iota (1 \otimes u) b_\iota = \sum_\iota a_\iota \otimes u b_\iota = \sum_\iota a_\iota \otimes u_\iota .$$

This shows

(82) $$\mathfrak{N}(L'(A, M)) = 0 .$$

Next we show that for $p \geqq 0$ $C^p(A, L'(A, M))$ may be identified with $L'(A, C^p(A, M))$. This is again trivial for $p=0$. For $p=n>0$, we identify an element $\sum_\iota c_\iota \otimes f_\iota$ $(f_\iota \in C^n(A, M))$ of $L'(A, C^n(A, M))$ with the element f of $C^n(A, L'(A, M))$ such that

(84) $$f(x_1 \otimes \cdots \otimes x_n) = \sum_\iota c_\iota \otimes f_\iota (x_1 \otimes \cdots \otimes x_n) ,$$

where $(c_1, \cdots, c_k)$ is an arbitrary basis of A. Again this does not contradict the A-double-module structures. For, if $y c_\iota = \sum_\kappa c_\kappa \mu_{\kappa\iota}(y)$ $(y \in A)$, then

$$\begin{aligned}(yf)(x_1 \otimes \cdots \otimes x_n) &= y f(x_1 \otimes \cdots \otimes x_n) = y \sum_\iota c_\iota \otimes f_\iota (x_1 \otimes \cdots \otimes x_n) \\ &= \sum_{\iota, \kappa} c_\kappa \mu_\kappa (y) \otimes f(x_1 \otimes \cdots \otimes x_n) = \sum_\kappa c_\kappa \otimes (\sum_\iota \mu_{\kappa\iota}(y) f_\iota (x_1 \otimes \cdots \otimes x_n)) .\end{aligned}$$

Hence $\sum_\kappa c_\kappa \otimes (\sum_\iota \mu_{\kappa\iota}(y) f_\iota) = y (\sum_\iota c \otimes f_\iota)$ is the element of $L'(A, C^n(A, M))$ identified with yf.

Further

$$\begin{aligned}(fy)(x_1 \otimes \cdots \otimes x_n) &= f(yx_1 \otimes x_2 \otimes \cdots) + \sum_{i=1}^{n-1} (-1)^i f(y \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots) \\ &+ (-1)^i f(y \otimes x_1 \otimes \cdots) x_n = \sum_\iota c_\iota \otimes (f_\iota (yx_1 \otimes x_2 \otimes \cdots) + \sum_{i=1}^{n-1} (-1)^i f_\iota (y \otimes \cdots \\ &\otimes x_i x_{i+1} \otimes \cdots) + (-1)^i f_\iota (y \otimes x_1 \otimes \cdots) x_n) = \sum_\iota c_\iota \otimes (f_\iota y)(x_1 \otimes \cdots)\end{aligned}$$

So $\sum_\iota c_\iota \otimes f_\iota y = (\sum_\iota c_\iota \otimes f_\iota) y$ is identified with fy.

Now we have, for $p \geqq 0$,

(84) $$H^p(A, L'(A, M)) = \mathfrak{N}(C^p(A, L'(A, M)) = \mathfrak{N}(L'(A, C^p(A, M)) = 0 ,$$

since clearly $1L'(A, M) = L'(A, M)$. Lemma 6 is thus proved.

Lemma 7. *Let $1M = M$. We have the exact sequence*

(85) $$0 \to 1C^{-1}(A, M) \to L'(A, M) \to M \to 0$$

of A-A-homomorphisms. (The assumption $1M = M$ concerns only with the exactness at M).

To prove this, map each element $\sum_\iota c_\iota \otimes u_\iota$ of $L'(A, M)$ onto $\sum c_\iota u_\iota \in M$, where again (c_ι) is an arbitrary basis of A. The mapping is evidently A–A-homomorphic. Any element of form $\sum_\iota(c_\iota \otimes u_\iota - 1 \otimes c_\iota u_\iota)$ is mapped on 0, while an element of form $1 \otimes u$ is mapped on 0 clearly if and only if $u=0$. Thus the kernel of our mapping is the totality of elements $\sum_\iota(c_\iota \otimes u_\iota - 1 \otimes c_\iota u_\iota)$, which is nothing but $1C^{-1}(A, M)$. On $1C^{-1}(A, M)$ the A-double-module structures induced from $C^{-1}(A, M)$ and $L'(A, M)$ coincide. Further, if $1M=M$ then every element of M can be expressed in a form $\sum_\iota c_\iota u_\iota$. Lemma 7 is thus proved; cf. [11], § 4.

Our exact sequences (85) entails, for any p, the exact sequence

$$(86)\quad H^p(A, L'(A, M)) \to H^p(A, M) \to H^{p+1}(A, 1C^{-1}(A, M)) \to H^{p+1}(A, L'(A, M)).$$

If here $p \geqq 0$ then the first and the last terms are 0 by Lemma 6. So we have

$$(87)\qquad H^p(A, M) = H^{p+1}(A, 1C^{-1}(A, M)) = H^{p+1}(A, C^{-1}(A, M))$$

for $p \geqq 0$. Now the case $p>0$, $q<0$ of (67) is settled easily in the same manner as the case $p<0$, $q>0$. Theorem is thus completely proved.

§ 6. A second proof to the general reduction theorem.

The above proof to our general reduction theorem is rather constructive. Now we want to give a second proof which depends on the Cartan-Eilenberg axiomatic characterization of cohomology groups. To do so, we first observe that for a Frobenius algebra over a field the notion of injective left-(say) modules coincides with that of projective left modules (cf. [9]). Since an algebra inverse-isomorphic to a Frobenius algebra is a Frobenius algebra and since the tensor product of two Frobenius algebras is again a Frobenius algebra, the same remark holds also for projective and injective double-modules over a Frobenius algebra.

Lemma 8. *If M is a projective double-module over a Frobenius algebra A, then $\mathfrak{N}(M) = M^A/\sigma M = 0$, the notation being as in §§ 2, 4.*

Proof. We first consider the case $M = A \otimes A$, regarded as A-double-module under ordinary operation. With an element $u = \sum_\iota a_\iota \otimes x_\iota (x_\iota \in A)$ of M we have, for $y \in A$,

$$yu = \sum_\iota ya_\iota \otimes x_\iota = \sum_{\kappa,\iota} a_\iota \otimes \lambda_{\kappa\iota}(y) x_\iota$$

by (28). So, if $u \in M^A$, whence $yu = uy$, then $\sum_\kappa \lambda_{\kappa\iota}(y) x_\iota = x_\iota y$ for each ι, and $b \to x$ is, by (29), an A-right-homomorphism of A onto the module spanned by (x_ι). There exists therefore an element c in A such that $x_\iota = cb_\iota$ for every ι. Hence $u = \sum_\iota a_\iota \otimes x_\iota = \sum a_\iota (1 \otimes c) b_\iota \in \sigma M$, and this settles our lemma in case $M = A\otimes A$. As every projective A-double-module is a direct summand of a direct sum of modules isomorphic to $A\otimes A$ (and is indeed a direct sum of modules isomorphic to direct summands of $A\otimes A$ [9]), the lemma follows then generally.

Lemma 9. *If M is a projective double-module over an algebra A, with unit element* 1, *then for every $p\leqq 0$ the A-double-module $1C^p(A, M) = 1C^p(A, M)1$ is projective.*

Proof. Let $n=-p$, and consider again the case $M=A\otimes A$ first. We have $C^p(A, M) = C_n(A, A\otimes A) = C_n(A, A)\otimes A$ and $1C^p(A, M) = 1C_n(A, A)\otimes A$. Now, if we consider $C_m = A\otimes A\otimes \cdots \otimes A$ under ordinary operation as A-left-module,

$$0 \leftarrow A(=C_1) \leftarrow C_2 \leftarrow C_3 \leftarrow \cdots$$

gives an A-left projective resolution of the A-left-module A, where $x_1\otimes x_2\otimes\cdots\otimes x_m\in C_m$ is mapped upon $\sum_{i=1}^{m-1}(-1)^{i-1}(x_1\otimes\cdots\otimes x_i x_{i+1}\otimes\cdots\otimes x_m)\in C_{m-1}$. The kernel of the $(n+1)$-th mapping is $1(C_n\otimes A) = 1C_n(A, A)$, where the operation of 1 is now in the sense of our lemma, i.e. in the sense of (9). As A is naturally A-left-projective, the same must be the case for $1C_n(A, A)$. Hence $1C_n(A, A)\otimes A$ is A-two-sided projective, and our assertion is proved for $M=A\otimes A$. The general case follows again by the direct sum argument.

Lemma 10. *If M is an injective double-module over an algebra A, with unit element, then for every $p\geqq 0$ the A-double-module $C^p(A, M)1 = 1C^p(A, M)1$ is injective.*

Proof. We first consider $M=(A\otimes A)^0 = \mathrm{Hom}_K(A\otimes A, K)$ (K being the ground field) regarded as A-double-module by $(y\varphi z)(x_1\otimes x_2) = \varphi(zx_1\otimes x_2 y)$ ($\varphi\in(A\otimes A)^0$); $(A\otimes A)^0$ is an injective A-double-module and indeed every injective A-double-module is a direct sum of modules isomorphic to direct summands of $(A\otimes A)^0$ (cf. [9]). $C^p(A, M) = C^p(A, (A\otimes A)^0) = \mathrm{Hom}_K(C_p, (A\otimes A)^0)$ and this may be identified with $(C_p\otimes A\otimes A)^0 = \mathrm{Hom}_K(C_p(A, A\otimes A), K)$ if we associate ψ in $\mathrm{Hom}_K(C_p, (A\otimes A)^0)$ to φ in the latter module with $\varphi(x_1\otimes\cdots\otimes x_p\otimes y\otimes z) = \psi(x_1\otimes\cdots\otimes x_p)(y\otimes z)$. We then verify that the A-double-module structure of $\mathrm{Hom}_K(C_p(A, A\otimes A), K)$ induced by that of $C_p(A, A\otimes A)$ defined in (8), (9) corresponds to the A-double-module structure of $\mathrm{Hom}_K(C_p, (A\otimes A)^0) = C^p(A, (A\otimes A)^0)$ given in (10), (11). Now, $1C_p(A, A\otimes A)$ is a projective A-double-module, as was seen in Lemma 9. It follows that $\mathrm{Hom}_K(C_p(A, A\otimes A), K)$ is a direct sum of an injective A-double-module and an A-double-module annihilated by 1 on the right-hand side. Thus $\mathrm{Hom}_K(C_p(A, A\otimes A), K)1 = C_p(A, (A\otimes A)^0)1$ is an injective A-double-module. This settles the case $M=(A\otimes A)^0$ and the general case follows by the direct sum argument.

Lemma 11. *Let A be a Frobenius algebra. If M is a projective (or injective) A-double-module, then $H^p(A, M)=0$ for every $p\gtreqless 0$.*

Proof. Since $H^p(A, M)=\mathfrak{N}(C^p(A, M))=\mathfrak{N}(1C^p(A, M)1)$ by § 4, we have the assertion by virtue of Lemmas 8, 9, 10 and the remark at the beginning of the present section. (As a matter of fact, the parts $p>0$, $p<0$ of the lemma are clear from the general theory).

With the same remark and the lemma just proved in mind, we see readily that cohomology groups (the 0- and negative-dimensional ones being those defined in the present paper) of a Frobenius algebra A may be characterized by the following axioms:

(I) To every A-double-module M and to every integer $p\gtreqless 0$ there is associated a module $H^p(M)$;

(II) If $\varphi: M\to N$ is an A-two-sided homomorphism of M into N, then there is, for each p, a homomorphism $\tilde{\varphi}: H^p(M)\to H^p(N)$;

(III) If $0\to M\xrightarrow{\iota} N\xrightarrow{\varphi} Q\to 0$ is an exact sequence of A-double-modules, then there is, for each p, a homomorphism $\tilde{\delta}: H^p(Q)\to H^{p+1}(M)$; here the sequence

$$\cdots\to H^{p-1}(Q)\xrightarrow{\tilde{\delta}} H^p(M)\xrightarrow{\tilde{\iota}} H^p(N)\xrightarrow{\tilde{\varphi}} H^p(Q)\xrightarrow{\tilde{\delta}} H^{p+1}(M)\to\cdots$$

is exact; if the diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & N & \to & Q & \to & 0 \qquad \text{(exact)} \\ & & \varphi\downarrow & & \downarrow & & \downarrow\psi & & \\ 0 & \to & M_1 & \to & N_1 & \to & Q_1 & \to & 0 \qquad \text{(exact)} \end{array}$$

is commutative, then the diagram

$$\begin{array}{ccc} H^p(Q) & \xrightarrow{\tilde{\delta}} & H^{p+1}(M) \\ \tilde{\psi}\downarrow & & \downarrow\tilde{\varphi} \\ H^p(Q_1) & \xrightarrow{\tilde{\delta}} & H^{p+1}(M_1) \end{array}$$

is commutative;

(IV) If M is A-two-sided projective (or injective) then $H^p(M)=0$ for any $p\gtreqless 0$;

(V) $H^0(M)=\mathfrak{N}(M)$.

The proof runs similarly as in the first chapters by Eilenberg in [2], the above observations being kept in our mind. As a matter of fact, if we take a specific integer s arbitrarily, then (V) may be replaced by

(V′) $H^s(M) = H^s(A, M)$.

Here the right-hand side is our cohomology group which is now assumed to be known (for the specific dimension s).

Now we want to show that these considerations lead to a second proof to our general reduction theorem; observe that we have used, in our above considerations, only the results of § 4 (i.e. the case $p=0$ of the reduction theorem). Thus, let t be any fixed integer and consider the modules $H^t(A, C^q(A, M))$. We put

$$H^p(M) = H^t(A, C^{p-t}(A, M)).$$

As an exact sequence $0 \to M \to N \to Q \to 0$ of A-double-modules entails the exact sequence $0 \to C^q(A, M) \to C^q(A, N) \to C^q(A, Q) \to 0$ for every q, we see readily that our modules $H^p(M)$ satisfy the axiom (I)–(III). They satisfy (IV) by lemma 11. Now, take $s=t$ in (V′). Then (V′) is satisfied too. So we have $H^p(M) = H^p(A, M)$ for any p, or $H^t(A, C^{p-t}(A, M)) = H^p(A, M)$ for any pair p and t, which proves the general reduction theorem in § 5. (Considering the reduction theorem for non-mixed dimensions rather obvious and the case $t=0$ settled in § 4 (the case having been used above also), we could also take any s with $s \geqq t$ or $s \leqq t$ according as $t > 0$ or $t < 0$.)

§ 7. Case of Frobenius algebras over a commutative ring and supplementary remarks.

Let now K be a commutative ring with unit element, and let A be an algebra over K. Let us call A a Frobenius algebra when the following conditions are satisfied:

i) A possesses a linearly independent finite basis $(c_1, c_2, \cdots, c_k)$ over K such that c_1 is the unit element 1 of A;

ii) there exist a second (necessarily linearly independent) basis $(d_1, d_2, \cdots, d_k)$ and a K-linear mapping μ of A into K such that

$$\mu(d_\iota c_\kappa) = \delta_{\iota\kappa}.$$

Under this definition of Frobenius algebras over a general commutative ring K (with unit element), we see readily that our results remain valid for them; for the proof of the general reduction theorem, in particular, we employ the method in § 5; the argument in § 6, as it stands, fails to be transferred directly to the present general case, though a certain modification of it probably would.

It is evident that a group algebra of any finite group over any commutative ring (with unit element), and in particular a such over the ring of rational integers, is a Frobenius algebra in our present sense. Thus our treatment includes the case of complete cohomology of finite groups.

In closing we want to make the following remarks. As was observed in a remark in §1, $H^{-n}(A, M) = H^*_{n-1}(A, M)$ $(n>1)$ is nothing but $H_{n-1}(A, M^*)$, where M^* is obtained from M by a modification of its structure as A-right-module. It is naturally also possible to develop a theory similar as above by considering homology groups on the A-double-module ${}^\dagger M$ obtained from M by a modification of its A-left-module structure, indeed with the automorphism $\dagger$ inverse to $*$.

In order to get a complete sequence of cohomology groups, it is also possible to interprete the homology groups themselves as negative-dimensional cohomology groups, indeed $H_n(A, M)$ as $-(n+1)$-dimensional one, but to modify (positive-dimensional) cohomology groups by our automorphism, as was indicated in §1. This amounts, however, to consider the A-double-module ${}^*M$ (in the sense similar as above) instead of M and then to adopt the above described system of modifying homology groups by $\dagger$ on the left. Needless to say that the system adopted in the present paper is equivalent to considering $M^\dagger$ instead of M and modifying cohomology groups, rather than homology groups, by $*$ on the right.

Finally we want to note that, though we have dealt with Frobenius algebras only, the consideration in §6 already indicates that the proper setting for complete cohomology theory must be quasi-Frobenius algebras rather than Frobenius algebras (as are defined in [10] in case of algebras over a field and as should be defined suitably in case of algebras over a general commutative ring). Indeed, it is rather easy, for a quasi-Frobenius algebra A, to see the existence of an acyclic augmented complete complex:

$$\begin{tikzcd} \cdots \arrow[r] & X_1 \arrow[r, "d_1"] & X_0 \arrow[r, "d_0"] \arrow[d, "\varepsilon"'] & X_{-1} \arrow[r, "d_{-1}"] & X_{-2} \arrow[r] & \cdots \\ & & A \arrow[ur, "\mu"'] & & & \end{tikzcd}$$

similar to that of a Frobenius algebra in §3 (but lacking A^0) in which X_p are A-A-projective, all arrows are A-A-homomorphisms, ε, μ are respectively onto and into, and which will provide us a complete cohomology theory for A. What we have to do is, perhaps, to derive a concrete description of $H^0(A, M)$, for instance, which is a generalization of our module $\mathfrak{N}(M)$ in §§ 2, 4, and to this we shall come back elsewhere.

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References

[1] R. Brauer: On hyperkomplex arithmetic and a theorem of Speiser, A. Speiser Festschrift 233–245, Zürich, 1945.

[2] H. Cartan: Séminaire de topologie algébrique, 1950–51.

[3] H. Cartan–S. Eilenberg: Homological Algebra, Princeton 1956.

[4] C. Chevalley: Class Field Theory, Lecture notes, Nagoya Univ. 1954–55.

[5] S. Eilenberg–T. Nakayama: On the dimension of modules and algebras, II (Frobenius algebras and quasi-Frobenius rings), Nagoya Math. J. 9 (1955), 1–16.

[6] G. Hochschild: On the cohomology groups of an associative algebra, Ann. Math. 46 (1945), 58–67.

[7] G. Hochschild: On the cohomology theory for associative algebras, Ann. Math. 47 (1946), 568–579.

[8] M. Ikeda–H. Nagao–T. Nakayama: Algebras with vanishing n-cohomology groups, Nagoya Math. J. 7 (1954), 115–131.

[9] H. Nagao–T. Nakayama: On the structure of (M_0)- and (M_u)-modules, Math. Zeitschr. 59 (1953), 164–170.

[10] T. Nakayana: On Frobeniusean algebras, I, Ann. Math. 40 (1939), 611–633; II, ibid. 42 (1941), 1–21.

[11] T. Nakayama: Derivation and cohomology in simple and other rings, I, Duke Math. J. 19 (1951), 51–63.

[12] C. J. Nesbitt–R. M. Thrall: Some ring theorems with applications to modular representations, Ann. Math. 47 (1946), 551–567.

[13] G. Shimura: On a certain ideal of Frobenius algebra, Sci. Papers Coll. Gen. Ed. Univ. Tokyo 2 (1952), 117–124.

[14] J. Tate: Higher dimensional cohomology groups of class field theory, Ann. Math. 56 (1952), 224–297.