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ON ALGEBRAS OF 2-CYCLIC REPRESENTATION TYPE

Dedicated to Professor K. Shoda on his sixtieth birthday

By

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§1. Let A be an associative algebra with a unit and of finite dimension over an algebraically closed field K and $A=\sum_i\sum_j Ae_{ij}$ be a decomposition of A into a direct sum of directly indecomposable left ideals where $Ae_{\kappa,i}\cong Ae_{\kappa,1}=Ae_\kappa$ and let N be its radical.

Now if an A-left module (or an A-right module) $\mathfrak{m}$ is a homomorphic image of one of Ae_i (or e_jA) we call $\mathfrak{m}$ a cyclic module and if an arbitary indecomposable A-left or right module is the sum of at most n cyclic modules we call A an algebra of n-cyclic representation type. It is known that A is generalized uniserial if and only if A is of 1-cyclic representation type[1].

In this paper we study the structure of an algebra of 2-cyclic representation type. In order to make the description short we give the next definitions and notations.

(i) If a module or an ideal has only one composition series then we call it *uniserial*.

(ii) If $\frac{Ne_1}{N^2e_1}$ and $\frac{Ne_2}{N^2e_2}$ $(e_1\neq e_2)$ have simple components isomorphic to each other then we call such a component *a vertice component* and $\left\{\frac{N^{j_1}e_1}{N^{j_1+1}e_1},\cdots,\frac{N^{j_r}e_r}{N^{j_r+1}e_r}\right\}$ is called a *chain* if, $\frac{N^{j_\nu}e_\nu}{N^{j_\nu+1}e_\nu}$ and $\frac{N^{j_{\nu+1}}e_{\nu+1}}{N^{j_{\nu+1}+1}e_{\nu+1}}$ $(\nu=1,\cdots,r-1)$ have simple components isomorphic to each other and $\overline{Ae_\nu}$ is not isomorphic to any composition factor of $\frac{Ae_{\nu+1}}{N^{j_{\nu+1}-j_\nu+1}e_{\nu+1}}$ $(j_{\nu+1}\geqq j_\nu)$.

(iii) The largest completely reducible part of an A-left (or A-right) module $\mathfrak{m}$ is denoted by $s(\mathfrak{m})$.

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1) See [I] and [II].

Moreover in this paper we shall assume that Au is a cyclic left ideal or a cyclic A-left module.

The main result is as follows:

An algebra A is of 2-cyclic representation type if and only if A satisfies the following conditions.

(1) Let $\mathfrak{p}$ be an arbitrary left ideal of Ne. Then $s\left(\frac{Ne}{\mathfrak{p}}\right)$ is a direct sum of at most two simple components and if it is a direct sum of two simple components then they are not isomorphic to each other except the case where $Ne=Au_1+Au_2$ and there is an integer λ such that $N^{\lambda}u_2=Au_1u_{\lambda}$ where $N^{\lambda-1}u_2=Au_{\lambda}\supsetneqq Au_1\cap Au_2$ and $\frac{Au_1}{Au_1\cap Au_2}$ has no composition factor isomorphic to $\overline{Ae}$.

(2) (i) Assume that $Ne_1=Au+Av$ (or $Ne_1=Au$) and $\frac{Au}{\mathfrak{p}_1}\cong\frac{Ne_2}{\mathfrak{p}_2}$ $(e_1\neq e_2)$ where $\mathfrak{p}_1$ is a left subideal in Au containing $Au\cap Av$ and $\mathfrak{p}_2$ a left subideal in Ne_2. Then there exists no composition factor of $\frac{Ne_2}{\mathfrak{p}_2}$ isomorphic to a vertice component except a simple component of $\frac{Ne_2}{N^2e_2}$.

(ii) If $Ne=Au_1+Au_2$ then at least one of $\frac{Au_i}{Au_1\cap Au_2}$ $(i=1,2)$ has no composition factor isomorphic to a vertice component.

(3) Assume that Aw is a cyclic subideal in Ne. If $Nw=Av_1+Av_2$ then $Av_1\cap Av_2=Nv_1=Nv_2$.

(4) Assume that $\left\{\frac{N^{\rho}e_1}{N^{\rho+1}e_1}, \frac{N^{\rho+\nu}e_2}{N^{\rho+\nu+1}e_2}\right\}$ $(\rho=1,\cdots,t-1,\ \nu\geqq 0)$ are chains.

(i) At least one of $\frac{Ae_1}{N^te_1}$ or $\frac{Ae_2}{N^{t+\nu}e_2}$ is uniserial.

(ii) If $\nu=0$ and $Ne_1=Au_1+Au_2$ where $\overline{Au_2}\cong\frac{Ne_2}{N^2e_2}$ then

(α) Au_i $(i=1,2)$ are uniserial and $Au_1\cap Au_2=Nu_2$

or (β) $Nu_2=Aw_1+Aw_2$, $Aw_2=Au_1\cap Au_2$ and $\frac{Ne_2}{N^3e_2}\cong\frac{Au_2}{Aw_1+Nw_2}$.

(5) The similar four conditions for right ideals as above are also satisfied.

§2. In this chapter we assume that A is of 2-cyclic representation type unless otherwise stated and we shall prove that A satisfies five conditions in §1.

[2.1] The followings are the consequences of the results in (IV).

Lemma 1. $\frac{Ne}{N^2e}$ *is the direct sum of at most two simple components and if it is the direct sum of two simple components then they are not isomorphic to each other.*

Lemma 2. *If* $\left\{\frac{Ne_1}{N^2e_1}, \frac{Ne_2}{N^2e_2}\right\}$ *is a chain then at least one of* $\frac{Ne_i}{N^2e_i}$ $(i=1, 2)$ *is simple.*[2]

Lemma 3. *If* $\frac{Ne_i}{N^2e_i} \supset \widetilde{Au_i}$ $(i=1, \cdots, r)$ *and* $\widetilde{Au_1} \simeq \widetilde{Au_i}$ *for all* i $(i=2, \cdots, r)$ *then* $r \leqq 2$.

This lemma is a consequence of the Lemma 1. Hence this is a consequence of the first half of the condition 1.

[2. 2] **Lemma 4.** *If* $s\left(\frac{Ne}{\mathfrak{p}}\right) = \widetilde{Au_1} \oplus \cdots \oplus \widetilde{Au_r}$ *for an arbitrary left ideal* $\mathfrak{p}$ *in* Ne *then* $r \leqq 2$.

(This is the first half of the condition 1.)

Proof. The dual module $\left(\frac{Ae}{\mathfrak{p}}\right)^*$ of $\frac{Ae}{\mathfrak{p}}$ is also directly indecomposable and $\left(\frac{Ae}{\mathfrak{p}}\right)^*$ is the sum of r cyclic modules. Hence if $r \gneqq 2$ then A is not of 2-cyclic representation type.

Corollary 1. *If the first half of the condition* 1 *is satisfied and* $Ne = Au_1 + Au_2$ *then* $\frac{Au_1}{Au_1 \cap Au_2}$ *and* $\frac{Au_2}{Au_1 \cap Au_2}$ *are uniserial.*

Proof. If there is a left ideal $\mathfrak{p}$ in Au_2 such that $\mathfrak{p} \supseteqq Au_1 \cap Au_2$ and $s\left(\frac{Au_i}{\mathfrak{p}}\right)$ is not simple then $s\left(\frac{Ne}{\mathfrak{p}}\right)$ is the direct sum of at least three simple components. Next since it is proved by Köthe[3] that $\frac{N^ie}{N^{i+1}e}$ is the direct sum of simple components not isomorphic to each other, we have

Corollary 2. $\frac{N^ie}{N^{i+1}e}$ *is the direct sum of at most two simple components not isomorphic to each other.*

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2) This is also the consequence of the first half of the condition 1.

3) See [III].

[2.3] Assume that $\frac{Ne}{\mathfrak{p}} \cong \frac{Au}{\mathfrak{p}_1}$ where Au is a subideal in Ne' $(e \neq e')$ which is not contained in N^2e' and $\mathfrak{p}$ and $\mathfrak{p}_1$ are subideals in Ne and Au. Now if $\frac{N^ie+\mathfrak{p}}{N^{i+1}e+\mathfrak{p}} = \widetilde{Au_1} \oplus \widetilde{Au_2}$ and $\mathfrak{p} \not\subset N^{i+1}e$ then $s\left(\frac{Ae}{N^{i+1}e}\right)$ is the direct sum of at least three simple components, but by the first half of the condition 1 this is a contradiction. Hence if $\frac{N^ie+\mathfrak{p}}{N^{i+1}e+\mathfrak{p}} = \widetilde{Au_1} \oplus \widetilde{Au_2}$ then $\mathfrak{p} \subseteq N^{i+1}e$ and $\frac{N^ie}{N^{i+1}e} = \widetilde{Au_1} \oplus \widetilde{Au_2}$. Similarly if $\frac{N^{i-1}u+\mathfrak{p}_1}{N^iu+\mathfrak{p}_1} = \widetilde{Av_1} \oplus \widetilde{Av_2}$ where $\widetilde{Au_i} \cong \widetilde{Av_i}$ $(i=1,2)$ then $\mathfrak{p}_1 \subset N^iu$ and $\frac{N^{i-1}u}{N^iu} = \widetilde{Av_1} \oplus \widetilde{Av_2}$. Hence by the following lemma 5 $\frac{Ne}{\mathfrak{p}}$ and $\frac{Au}{\mathfrak{p}_1}$ are uniserial.

Lemma 5. *Assume that* $Au \subseteq Ne'$, $\not\subseteq N^2e'$ $(e \neq e')$ *and there exists an integer* φ *such that* $\frac{N^\varphi e}{N^{\varphi+1}e} = \widetilde{Aw_1} \oplus \widetilde{Aw_2}$ *and* $\frac{N^{\varphi-1}u}{N^\varphi u} = \widetilde{Av_1} \oplus \widetilde{Av_2}$ *where* $\widetilde{Aw_1} \cong \widetilde{Av_1}$ *and* $\widetilde{Aw_2} \cong \widetilde{Av_2}$. *Then A is of unbounded representation type.*

For the proof of this lemma, see [V] or [VI].

From the lemma 5 we have

Corollary 3. *Assume that* Aw_i $(i=1,2)$ *are cyclic,* $\frac{Aw_i}{Aw_1 \cap Aw_2}$ $(i=1,2)$ *are simple and* $\frac{Aw_1}{Aw_1 \cap Aw_2} \not\cong \frac{Aw_2}{Aw_1 \cap Aw_2}$. *Then* $Aw_1 \cap Aw_2$ *is uniserial.*

Proof. Assume that $\overline{Aw_1} \cong \overline{Ae'}$ and $\overline{Aw_2} \cong \overline{Ae''}$ $(e' \neq e'')$. Then there exist $\mathfrak{p}_1$ and $\mathfrak{p}_2$ such that $Nw_1 \cong \frac{Ne'}{\mathfrak{p}_1}$ and $Nw_2 \cong \frac{Ne''}{\mathfrak{p}_2}$.

(i) If $Aw_1 \cap Aw_2$ is cyclic and there is an integer $\nu \geqq 1$ such that $\frac{N^\nu e'}{N^{\nu+1}e'} = \overline{A\xi_1} \oplus \overline{A\zeta_1}$ and $\frac{N^\nu e''}{N^{\nu+1}e''} = \overline{A\xi_2} \oplus \overline{A\zeta_2}$ where $N^{\nu+1}e' \supset \mathfrak{p}_1$, $N^{\nu+1}e'' \supset \mathfrak{p}_2$, $\overline{A\xi_1} \cong \overline{A\xi_2}$ and $\overline{A\zeta_1} \cong \overline{A\zeta_2}$ then by the lemma 5 A is not of 2-cyclic representation type.

(ii) If $Aw_1 \cap Aw_2$ is not cyclic then $\frac{Ne'}{N^2e'} = \overline{A\xi_1} \oplus \overline{A\zeta_1}$ and $\frac{Ne''}{N^2e''} = \overline{A\xi_2} \oplus \overline{A\zeta_2}$ where $\overline{A\xi_1} \cong \overline{A\xi_2}$ and $\overline{A\zeta_1} \cong \overline{A\zeta_2}$. Hence this contradicts the lemma 2.

The necessity of the condition 2 follows from the following lemmas.

Lemma 6. *Assume that* $Ne_1 = Au_1 + Au_2$. *Then at least one of*

$\dfrac{Au_1}{Au_1\cap Au_2}$ *and* $\dfrac{Au_2}{Au_1\cap Au_2}$ *have no composition factor isomorphic to a vertice component.*

In order to prove this lemma we shall prove the following lemma 7.

Lemma 7. *Assume that* $\mathfrak{m}=Ae_1m_1+Ae_2m_2+Ae_3m_3$ *is an A-left module such that* $e_1\neq e_2\neq e_3$, $s(Ae_1m_1)\cap s(Ae_2m_2)=Au_1m_1=Au_2m_2\neq 0$ *and* $s(Ae_2m_2)\cap s(Ae_3m_3)=Av_2m_2=Av_3m_3\neq 0$.

If $u_ir_jm_j=0$ *and* $v_ir_jm_j=0$ *for* $r_jm_j\in Ne_jm_j$ *then* $\mathfrak{m}$ *is directly indecomposable.*

Proof. We can put $u_1m_1=\alpha u_2m_2$ and $v_2m_2=\beta v_3m_3$ $(\alpha, \beta\in K)$. Now suppose that $\mathfrak{m}$ is directly decomposable. Then $\mathfrak{m}=Ae_1n_1+Ae_2n_2+Ae_3n_3$ and some Ae_in_i is a direct summand of $\mathfrak{m}$. Now let $n_i=\alpha_{i1}m_1+\alpha_{i2}m_2+\alpha_{i3}m_3$ $(i=1, 2, 3)$.
Then $\alpha_{ii}\in e_iAe_i$, $\notin e_iNe_i$ and $\alpha_{ij}\in e_iNe_j$ $(i\neq j)$.
Hence $u_1n_1=a_{11}u_1m_1$, $u_2n_2=a_{22}u_2m_2$, $v_2n_2=a_{22}v_2m_2$ and $v_3n_3=a_{33}v_3m_3$ where $\alpha_{ii}=a_{ii}+r_{ii}$, $a_{ii}\in K$ and $r_{ii}\in e_iNe_i$. Therefore $Au_1n_1=Au_1m_1$, $Au_2n_2=Au_2m_2$, $Av_2n_2=Av_2m_2$ and $Av_3n_3=Av_3m_3$. Thus $Ae_in_i\cap(Ae_jn_j+Ae_kn_k)\neq 0$ for $\{i, j, k\}=\{1, 2, 3\}$. But this is a contradiction.

The proof of the lemma 6.

By the corollary 1 $\dfrac{Au_i}{Au_1\cap Au_2}=\widetilde{Au_i}$ $(i=1, 2)$ are uniserial. Now we may assume that $\dfrac{N^\rho\tilde{u}_1}{N^{\rho+1}\tilde{u}_1}$ and $\dfrac{N^\mu\tilde{u}_2}{N^{\mu+1}\tilde{u}_2}$ are isomorphic to vertice components and $\dfrac{A\tilde{u}_1}{N^\rho\tilde{u}_1}$ and $\dfrac{A\tilde{u}_2}{N^\mu\tilde{u}_2}$ have no composition factor isomorphic to a vertice component. From now on we assume that $Au_1\cap Au_2=0$.

(i) Assume that $\rho=\mu=0$. Then there exist Ae_2 and Ae_3 $(e_1\neq e_2,\ e_3)$ such that $\dfrac{Au_1}{Nu^1}\cong\dfrac{Ne_2}{N^2e_2}$ and $\dfrac{Au_2}{Nu_2}\cong\dfrac{Ne_3}{N^2e_3}$ since $\dfrac{Au_i}{Nu_i}$ $(i=1, 2)$ are isomorphic to vertice components.

Now if $e_2=e_3$ then $\dfrac{Au_1}{Nu_1}\cong\dfrac{Au_2}{Nu_2}$. But this contradicts the lemma 1 or the corollary 2. Hence $e_1\neq e_2\neq e_3$. Then $\left\{\dfrac{Ne_2}{N^2e_2}, \dfrac{Ne_1}{N^2e_1}, \dfrac{Ne_3}{N^2e_3}\right\}$ is a chain and this contradicts the lemma 3.

(ii) Assume that $\rho>0$ or $\mu>0$.

If $\dfrac{N^{\rho-1}u_1}{N^\rho u_1}\cong\overline{Ae_2'}$ and $\dfrac{N^{\mu-1}u_2}{N^\mu u_2}\cong\overline{Ae_3'}$ then there exist Ae_2 and Ae_3 such that $\left\{\dfrac{Ne_2}{N^2e_2}, \dfrac{Ne_2'}{N^2e_2'}\right\}$ and $\left\{\dfrac{Ne_3}{N^2e_3}, \dfrac{Ne_3'}{N^2e_3'}\right\}$ are chains where $\dfrac{Ne_2'}{N^2e_2'}$ and $\dfrac{Ne_3'}{N^2e_3'}$

are assumed to be simple by the lemma 2. Now we construct an A-left module $\mathfrak{m}=Ae_1m_1+Ae_2m_2+Ae_3m_3$ in the following way;

(α) $N^{\rho+1}u_1m_1=N^{\mu+1}u_2m_1=0$,

(β) if $\frac{Ne_2}{N^2e_2}$ is simple then $N^2e_2m_2=0$ and if $Ne_2=Av_1+Av_2$ and $\overline{Av_1}\cong\frac{Ne_2'}{N^2e_2'}$ then $Nv_1m_2=Av_2m_2=0$.

(γ) if $\frac{Ne_3}{N^2e_3}$ is simple then $N^2e_3m_3=0$ and if $Ne_3=Aw_1+Aw_2$ and $\overline{Aw_1}\cong\frac{Ne_3'}{N^2e_3'}$ then $Nw_1m_3=Aw_2m_3=0$

and (δ) $Ne_2m_2=N^{\rho}u_1m_1$ and $Ne_3m_3=N^{\mu}u_2m_1$.

(From now on we assume that $Ne_2=Av_1$ and $Ne_3=Aw_1$.)

Then $N^{\rho}u_1r_2m_2\subset N^2e_2m_2=0$ and $N^{\mu}u_2r_2m_2\subset N^2e_2m_2=0$ for $r_2\in Ne_2$. Similary $N^{\rho}u_1r_3m_3\subset N^2e_3m_3=0$ and $N^{\mu}u_2r_3m_3\subset N^2e_3m_3=0$ for $r_3\in Ne_3$.

(1) Assume that $\rho=0$ and $\mu>0$. Then $e_2'=e_1$. If $e_3'\neq e_2$ then $e_3\neq e_1$, $v_1r'm_1=v_1r''m_1=0$ and $w_1r'm_1=w_1r''m_1=0$ for $r'\in Au_1$ and $r''\in Au_2$. Hence by the lemma 7 $\mathfrak{m}$ is directly indecomposable and this is a contradiction. If $e_3'=e_2$ then $e_3=e_1$. Hence $u_1=w_1$ and if we put $N^{\mu-1}u_2=Av'$ then $N^{\mu}u_2=Av_1v'$ and by the assumption Au_2 have no composition factor isomorphic to $\overline{Ae_1}$ and $\overline{Ae_2}$ except $\frac{N^{\mu-1}u_2}{N^{\mu}u_2}$.

Now suppose that $\mathfrak{m}$ is directly decomposable. Then $\mathfrak{m}=Ae_1n_1+Ae_1n_2+Ae_2n_3$ and some Ae_in_j is the direct summand of $\mathfrak{m}$. Now let $n_i=\alpha_{i1}m_1+\alpha_{i2}m_2+\alpha_{i3}m_3$ $(i=1, 2, 3)$. Then α_{11}, $\alpha_{22}\in e_1Ae_1$, $\notin e_1Ne_1$, $\alpha_{33}\in e_2Ae_2$, $\notin e_2Ne_2$, α_{13}, $\alpha_{23}\in e_2Ne_1$ and α_{31}, $\alpha_{32}\in e_2Ne_1$.

Hence $w_1n_1=a_{11}w_1m_1+a_{12}w_1m_2$ $(\alpha_{ij}=a_{ij}+r_{ij},\ a_{ij}\in K$ and $r_{ij}\in e_iNe_j)$, $v_1v'n_1=a_{11}v_1v'm_1$ (since $v_1v'm_2=0$), $w_1n_2=a_{21}w_1m_1+a_{22}w_1m_2$ and $v_1n_3=a_{31}v_1v'm_1+a_{33}v_1m_3$.

Therefore $a_{21}w_1n_1-a_{11}w_1n_2=(a_{12}a_{21}-a_{11}a_{22})w_1m_2=(a_{12}a_{21}-a_{11}a_{22})v_1v'm_1=\frac{a_{12}a_{21}-a_{11}a_{22}}{a_{11}}v_1v'n_1$ and $v_1n_3=\frac{a_{31}}{a_{11}}v_1v'n_1+a_{33}w_1m_1=\frac{a_{31}}{a_{11}}v_1v'n_1+\frac{a_{33}a_{22}w_1n_1-a_{33}a_{11}w_1n_2}{(a_{11}a_{22}-a_{21}a_{12})}$.

Thus $Ae_1n_1\cap(Ae_1n_2+Ae_2n_3)\neq0$, $Ae_1n_2\cap(Ae_1n_1+Ae_2n_3)\neq0$ or $Ae_2n_3\cap(Ae_1n_1+Ae_1n_2)\neq0$. But this is a contradiction.

If $\rho>0$ and $\mu=0$ then similarly as above we can show that this lemma is true.

(2) Assume that $\rho>0$ and $\mu>0$. Then we can assume that $e_1\neq e_2$, e_2', e_3, e_3'.

(2. 1) Assume that $e_2 \neq e'_3$ (accordingly $e'_2 \neq e_3$) and $e'_2 \neq e'_3$. If $v_1 r_1 m_1 \neq 0$ for $r_1 \in Au_1$ then there exists an integer ν such that $\frac{N^{\nu-1}u_1}{N^{\nu}u_1} \simeq \overline{Ae_2}$ and $\frac{N^{\nu}u_1}{N^{\nu+1}u_1} \simeq \overline{Av_1}$. But this contradicts the assumption since $e_2 \neq e'_2$ and $\nu \lneqq \rho$. Hence $v_1 r_1 m_1 = 0$. Next if $v_1 r_2 m_1 \neq 0$ for $r_2 \in Au_2$ then there exists an integer ν such that $\frac{N^{\nu-1}u_2}{N^{\nu}u_2} \simeq \overline{Ae_2}$ and $\frac{N^{\nu}u_2}{N^{\nu+1}u_2} \simeq \overline{Av_1}$. But if $\nu \lneqq \mu$ then this contradicts the assumption and if $\nu = \mu$ then $e'_3 = e_2$. But this contradicts the assumption. Thus $v_1 r_2 m_1 = 0$ for $r_2 \in Au_2$.

Similarly $w_1 r' m_1 = 0$ for $r' \in Ne_1$. Moreover $N^{\rho}u_1 r_1 m_1 \subset N^{\rho+1}u_1 m_1 = 0$ for $r_1 \in Au_1$ and if $N^{\rho}u_1 r_2 m_1 \neq 0$ for $r_2 \in Au_2$ then $N^{\rho}u_1 r_2 m_1 = N^{\mu}u_2 m_1$ and $e'_2 = e'_3$. But this is a contradiction. Thus $N^{\rho}u_1 r_2 m_1 = 0$. Similarly $N^{\mu}u_2 r' m_1 = 0$ for $r' \in Ne_1$. Therefore by the lemma 7 $\mathfrak{m}$ is directly indecomposable since $e_1 \neq e_2 \neq e_3$.

(2. 2) Assume that $e_2 = e'_3$ (accordingly $e_3 = e'_2$). Then there exist $r_1 \in Au_1$ and $r_2 \in Au_2$ such that $N^{\rho}u_1 m_1 = Aw_1 r_1 m_1$ and $N^{\mu}u_2 m_1 = Av_1 r_2 m_1$. In this case $N^{\rho}u_1 r m_1 = 0$ $(r \in Ne_1)$ and $N^{\mu}u_2 r' m_1 = 0$ $(r' \in Ne_1)$ since $e'_2 \neq e'_3$. Now suppose that $\mathfrak{m}$ is directly decomposable. Then $\mathfrak{m} = Ae_1 n_1 + Ae_2 n_2 + Ae_3 n_3$ and some $Ae_i n_i$ is the direct summand of $\mathfrak{m}$. Now let

$$n_i = \alpha_{i1} m_1 + \alpha_{i2} m_2 + \alpha_{i3} m_3 \qquad (i = 1, 2, 3) .$$

Then $\alpha_{ii} \in e_i A e_i$, $\notin e_i N e_i$ ann $\alpha_{ij} \in e_i N e_j$ $(i \neq j)$ since $e_1 \neq e_2 \neq e_3$.

Now $N^{\rho}u_1 n_1 = N^{\rho}u_1 m_1$ and $N^{\mu}u_2 n_1 = N^{\mu}u_2 m_1$. Next $v_1 n_2 = a_{22} v_1 m_2 + a_{21} v_1 r_2 m_1$ $(r_2 \in Au_2,\ a_{22},\ a_{21} \in K)$. Then $Ae_2 n_2 \cap Ae_1 n_1 \neq 0$ since $v_1 m_2 \in N^{\rho}u_1 m_1 = N^{\rho}u_1 n_1$ and $v_1 r_2 m_1 \in N^{\mu}u_2 m_1 = N^{\mu}u_2 n_1$. Similarly $Ae_3 n_3 \cap Ae_1 n_1 \neq 0$. But this is a contradiction and $\mathfrak{m}$ is directly indecomposable.

(2. 3) Assume that $e'_2 = e'_3$ (accordingly $e_2 = e_3$ add $v_1 = w_1$). Then we can assume that there exists $r \in Au_2$ such that $N^{\mu}u_2 m_1 = N^{\rho}u_1 r m_1$. Therefore $N^{\mu}u_2 r' m_1 \subset N^{\mu+1}u_1 m_1 = 0$ for $r' \in Au_1$ since $\mu \gneqq \rho$. Moreover $v_1 r m_1 = 0$ and $w_1 r' m_1 = 0$ for $r, r' \in Ne_1$ since $e_2 \neq e'_3$ and $e'_2 \neq e_3$.

Now suppose that $\mathfrak{m}$ is directly decomposable. Then $\mathfrak{m} = Ae_1 n_1 + Ae_2 n_2 + Ae_3 n_3$ and some $Ae_i n_i$ is the direct summand of $\mathfrak{m}$. Now let

$$n_i = \alpha_{i1} m_1 + \alpha_{i2} m_2 + \alpha_{i3} m_3 \qquad (i = 1, 2, 3) .$$

Then $\alpha_{i1} \in e_1 A e_1$, $\notin e_1 N e_1$, $\alpha_{22}, \alpha_{33} \in e_2 A e_2$, $\notin e_2 N e_2$, $\alpha_{1j} \in e_1 N e_j$ and $\alpha_{j1} \in e_j N e_1$ $(j \neq 1)$. Now $N^{\rho}u_1 n_1 \subset N^{\rho}u_1 m_1 + N^{\mu}u_2 m_1$ and $N^{\mu}u_2 n_1 = N^{\mu}u_2 m_1$. Next $v_1 n_2 = a_{22} v_1 m_2 + a_{22} v_1 m_3$ and $v_1 n_3 = a_{32} v_1 m_2 + a_{33} v_1 m_3$ $(a_{ij} \in K)$.

Hence $v_1 m_2 = \frac{a_{33} v_1 n_2 - a_{23} v_1 n_3}{(a_{22} a_{33} - a_{32} a_{23})}$ and $v_1 m_3 = \frac{a_{22} v_1 n_3 - a_{32} v_1 n_2}{(a_{22} a_{33} - a_{23} a_{32})}$. Thus $\frac{a_{22} v_1 n_3 - a_{32} v_1 n_2}{(a_{22} a_{33} - a_{23} a_{32})}$

$\in N^{\mu}u_2n_1$ since $v_1m_3 \in N^{\mu}u_2m_1$ and $\dfrac{a_{33}v_1n_2 - a_{23}v_1n_3}{(a_{22}a_{33} - a_{32}a_{23})} \in N^{\rho}u_1n_1 + N^{\mu}u_2n_1$ since $v_1m_2 \in N^{\rho}u_1m_1$ and $N^{\rho}u_1m_1 \subset N^{\rho}u_1n_1 + N^{\mu}u_2m_1 = N^{\rho}u_1n_1 + N^{\mu}u_2n_1$[4]. Therefore $Ae_in_i \cap (Ae_jn_j + Ae_kn_k) \neq 0$. But this is a contradiction.

By this lemma 6 we have

Corollary 4. *If* $\left\{\dfrac{Ne_2}{N^2e_2}, \cdots, \dfrac{Ne_r}{N^2e_r}\right\}$ *is a chain then* $r=2$.

Proof. Assume that $r=3$. If $\dfrac{Ne_i}{N^2e_i} \supset \overline{Au_i}$ $(i=1, 2, 3)$ and $\overline{Au_1} \simeq \overline{Au_2} \simeq \overline{Au_3}$ then this contradicts the lemma 3 and if $\dfrac{Ne_2}{N^2e_2} = \overline{Au_1} \oplus \overline{Au_2}$, $\overline{Au_1}$ is isomorphic to a simple component of $\dfrac{Ne_1}{N^2e_1}$ and $\overline{Au_2}$ is isomorphic to a simple component of $\dfrac{Ne_3}{N^2e_3}$ then this contradicts the lemma 6.

Lemma 8. *Assume that* $\dfrac{Ne_1}{\mathfrak{p}_1} \simeq \dfrac{Au}{\mathfrak{p}_2}$ *where* $Ne_2 = Au + Av$ (*or* $Ne_2 = Au$), $(e_1 \neq e_2)$, $\mathfrak{p}_1$ *is a left subideal in* Ne_1 *anc* $\mathfrak{p}_2$ *is a left subideal in* Au *which contains* $Au \cap Av$. *Then* $\dfrac{Ne_1}{\mathfrak{p}_1} = \widetilde{Ne_1}$ *has no composition factor isomorphic to a vertice component except* $\dfrac{\widetilde{Ne_1}}{\widetilde{N^2e_1}}$.

Proof. By the corollary 1, $\dfrac{Ne_1}{\mathfrak{p}_1}$ is uniserial. From now on we assume that $\mathfrak{p}_1 = 0$ and $\mathfrak{p}_2 = 0$. Now suppose that $\dfrac{N^{\rho+1}e_1}{N^{\rho+2}e_1}$ $(\rho \geqq 1)$ is isomorphic to a vertice component. Then there exist Ae_3 and Ae'_3 such that $\left\{\dfrac{Ne_3}{N^2e_3}, \dfrac{Ne'_3}{N^2e'_3}\right\}$ is a chain $\left(\dfrac{Ne'_3}{N^2e'_3}\right.$ is assumed to be simple$\left.\right)$, $\dfrac{N^{\rho}e_1}{N^{\rho+1}e_1} \simeq \overline{Ae'_3}$ $\left(\text{accordingly } \dfrac{N^{\rho+1}e_1}{N^{\rho+2}e_1} \simeq \dfrac{Ne'_3}{N^2e'_3}\right)$. Now we put $N^{\rho+1}e_1 = Au_1$, $N^{\rho}u = Au_2$, $Ne_3 = Aw$ (or $Ne_3 = Aw + Aw'$) and $Ne'_3 = Aw''$. Then $Au_1 = Aw''u'$ where $N^{\rho}e_1 = Au'$. Moreover we may assume that any composition factor of $\dfrac{N^2e_1}{N^{\rho+1}e_1}$ is not isomorphic to a vertice component.

Now we construct an A-left module $\mathfrak{m} = Ae_1m_1 + Ae_2m_2 + Ae_3m_3$ in the following way:

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4) We can get $N^{\rho}u_1m_1 \subset N^{\rho}u_1n_1 + N^{\mu}u_2m_1$ from $N^{\rho}u_1n_1 \subset N^{\rho}u_1m_1 + N^{\mu}u_2m_1$ since $N^{\rho}u_1m_1$ is simple.

(α) $N^{\rho+2}e_1m_1=N^{\rho+1}um_2=N^2e_3m_3=vm_2=0$.
(If $Ne_3=Aw+Aw'$ then $Nwm_3=Aw'm_3=0$.)
(β) $N^{\rho+1}e_1m_1=N^{\rho}um_2=Ne_3m_3$.

(1) Assume that $e_1\neq e_3$ and $e_2\neq e_3$. Then $N^{\rho+1}e_1rm_1\subset N^{\rho+2}e_1m_1=0$ for $r\in Ne_1$, $N^{\rho+1}e_1r'm_2\subset N^{\rho+1}um_2=0$ for $r'\in Ne_2$, $N^{\rho+1}e_1r''m_3\subset N^{\rho+1}e_3m_3=0$ for $r''\in Ne_3$, $N^{\rho}upm_1\subset N^{\rho+2}e_1m_1=0$ for $p\in Ne_1$, $N^{\rho}up'm_2\subset N^{\rho+1}um_2=0$ for $p'\in Ne_2$, and $N^{\rho}up''m_3\subset N^{\rho+1}e_3m_3=0$ for $p''\in Ne_3$. Next $Ne_3pm_1=0$ ($p\in Ne_1$) and $Ne_3p'm_2=0$ ($p'\in Ne_2$) since $e_1\neq e_3$, $e_2\neq e_3$ and $\frac{Ne_1m_1}{N^{\rho+1}e_1m_1}$ has no composition factor isomorphic to a vertice component. Then by the lemma 7 m is directly indccomposable.

(2) Assume that $e_1=e_3$. Then $Ne_1m_1=Awm_1$, $Awm_3=N^{\rho+1}e_1m_1=N^{\rho}um_2$ and we put $N^{\rho+1}e_1m_1=Au_1m_1$ and $N^{\rho}um_2=Au_2m_2$.

Now suppose that m is directly decomposable. Then $\mathfrak{m}=Ae_1n_1+Ae_2n_2+Ae_3n_3$ and some Ae_in_i is the direct summand of m. Now let $n_i=\alpha_{i1}m_1+\alpha_{i2}m_2+\alpha_{i3}m_3$ ($i=1,2,3$). Then α_{11}, $\alpha_{33}\in e_1Ae_1$, $\notin e_1Ne_1$, $\alpha_{22}\in e_2Ae_2$, $\notin e_2Ne_2$, α_{21}, $\alpha_{23}\in e_2Ne_1$ and α_{12}, $\alpha_{32}\in e_1Ne_2$. Now $u_1n_1=a_{11}u_1m_1$ and $u_2n_2=a_{22}u_2m_2$ ($a_{ii}\in K$) since $e_1\neq e_2$ and $\rho\geq 1$. Next $wn_3=a_{31}wm_1+a_{33}wm_3$ and $wn_1=a_{11}wm_1+a_{13}wm_3$.

Hence $wm_1=\frac{a_{13}wn_3-a_{23}wn_1}{a_{31}a_{13}-a_{11}a_{33}}$ and $wm_3=\frac{a_{11}wn_3-a_{31}wn_1}{a_{11}a_{33}-a_{13}a_{31}}$. Thus $\frac{a_{11}wn_3-a_{31}wn_1}{a_{11}a_{33}-a_{13}a_{31}}=\frac{u_1n_1}{a_{11}}=\frac{u_2n_2}{a_{22}}$. Therefore $Ae_1n_1\cap(Ae_1n_3+Ae_2n_2)\neq 0$, $Ae_1n_3\cap(Ae_1n_1+Ae_2n_2)\neq 0$ or $Ae_2n_2\cap(Ae_1n_1+Ae_1n_3)\neq 0$. But this is a contradiction. If $e_3=e_2$ then similarly as this we can show that this is true.

By the condition 1 and 2 we have the following corollary.

Corollary 5. *Assume that* $Ne=Au_1+Au_2$ *and* $Au_1\cap Au_2\neq 0$. *If* $Au_1\cap Au_2\subset N\xi_i$, $\not\subset N^2\xi_i$ *where* $A\xi_i\subset Au_i$ ($i=1,2$) *then* $\overline{A\xi_1}\not\cong\overline{A\xi_2}$.

Proof. By the condition 1 (accordingly by the corollary 1) $\frac{Au_i}{Au_1\cap Au_2}$ ($i=1,2$) are uniserial. Now suppose that $\overline{A\xi_1}\simeq\overline{A\xi_2}$. If we put $N^{\lambda}u_1=A\xi_1$ and $N^{\mu}u_2=A\xi_2$ and assume that $\frac{N^{\lambda-1}u_1}{N^{\lambda}u_1}\simeq\frac{N^{\mu-1}u_2}{N^{\mu}u_2}$ then $\overline{A\xi_1}$ ($\simeq\overline{A\xi_2}$) is isomorphic to a vertice component but this contradicts the condition 2. Thus we may assume that $\frac{Au_1}{N^{\lambda+1}u_1}\simeq\frac{N^{\mu+\lambda}u_2}{N^{\mu+1}u_2}$.

Next if $\frac{N^{\mu-\lambda-1}u_2}{N^{\mu-\lambda}u_2}\not\cong\overline{Ae}$ then $\frac{N^{\mu-\lambda}u_2}{N^{\mu-\lambda+1}u_2}$ is isomorphic to a vertice component but this contradicts the condition 2. Therefore $\frac{N^{\mu-\lambda-1}u_2}{N^{\mu-\lambda}u_2}\simeq\overline{Ae}$, Hence $\frac{Ae}{Au_2}$ is homomorphic onto $N^{\mu-\lambda-1}u_2$. If $u_2N^{\mu-\lambda-1}u_2 \neq 0$ then $N^{\mu-\lambda}u_2 = A\zeta_1 + A\zeta_2$ where $A\zeta_1 = Au_1 \cap Au_2$. Hence $N^{\mu-\lambda-1}u_2 = A\xi_2$ and $\overline{A\xi_1} \cong \overline{A\xi_2} \cong \overline{Ae}$. But in this case similarly as above we can see that this is a contradiction. Therefore $N^{\mu+1}u_2 = 0$ since $\frac{Au_1}{N^{\lambda+1}u_1} \cong \frac{N^{\mu-\lambda}u_2}{N^{\mu+1}u_2}$ and $N^{\lambda+1}u_1 = Au_1 \cap Au_2$.[5)]

[2.4] In order to prove that the rest conditions are satisfied we shall prove the following lemmas.

Lemma 9. *Assume that there exist* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *such that* $s\left(\frac{Ae_1}{\mathfrak{p}_1}\right) \supset \widetilde{Au_1}$, $s\left(\frac{Ae_2}{\mathfrak{p}_2}\right) \supset \widetilde{Au_2}$ *and* $\widetilde{Au_1} \cong \widetilde{Au_2}$ *where* $\mathfrak{p}_i$ $(i=1,2)$ *are left subideals in* Ne_i *and* $\frac{Ae_i}{\mathfrak{p}_i} = \widetilde{Ae_i}$. *If there exist* $\widetilde{Aw_i}$ $(i=1,2)$ *which are left subideals in* $\widetilde{Ne_i}$ $(i=1,2)$ *such that* $\widetilde{Au_i} \subset \widetilde{Nw_i}$, $\not\subset \widetilde{N^2w_i}$ $(i=1,2)$ *and the isomorphism* $\widetilde{Au_1} \cong \widetilde{Au_2}$ *cannot be extended to any homomorphism of* $\widetilde{Aw_1}$ *onto* $\widetilde{Aw_2}$ *and of* $\widetilde{Aw_2}$ *onto* $\widetilde{Aw_1}$ *then* $\frac{\widetilde{Aw_1}}{\widetilde{Nw_1}} \not\cong \frac{\widetilde{Aw_2}}{\widetilde{Nw_2}}$.

Proof. Suppose that $\frac{\widetilde{Aw_1}}{\widetilde{Nw_1}} \cong \frac{\widetilde{Aw_2}}{\widetilde{Nw_2}} \cong \overline{Ae'}$. If $\widetilde{Nw_1} = \widetilde{Au_1}$ and $\widetilde{Nw_2} = \widetilde{Au_2}$ then this is a contradiction since $\widetilde{Aw_1} \cong \widetilde{Aw_2}$. If $\widetilde{Nw_1} = \widetilde{Av_1} \oplus \widetilde{Au_1}$ and $\widetilde{Nw_2} = \widetilde{Au_2}$ then this is a contradiction since $\frac{\widetilde{Aw_1}}{\widetilde{Av_1}} \cong \widetilde{Aw_2}$. If $\widetilde{Nw_1} = \widetilde{Av_1} \oplus \widetilde{Au_1}$, $\widetilde{Nw_2} = \widetilde{Av_2} \oplus \widetilde{Au_2}$ and $\widetilde{Av_i}$ $(i=1,2)$ are simple then this is a contradiction since $\widetilde{Aw_1} \cong \widetilde{Aw_2}$. If $\widetilde{Nw_1} = \widetilde{Av_1} \oplus \widetilde{Au_1}$, $\widetilde{Nw_2} = \widetilde{Av_2} \oplus \widetilde{Au_2}$ and there exists $\mathfrak{p}' \subset \widetilde{Av_1}$ such that $\widetilde{Av_2} \cong \frac{\widetilde{Av_1}}{\mathfrak{p}'}$ then this is a contradiction since $\widetilde{Aw_2} \cong \frac{\widetilde{Aw_1}}{\mathfrak{p}'}$.

By this lemma we can see that these $\widetilde{Au_i}$ $(i=1,2)$ are isomorphic to a vertice component.

Next let A be an algebra (not necessarily of 2-cyclic representation type) satisfying the condition (1) and (2).

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5) In this corollary if $\overline{A\xi_1} \not\cong \overline{A\xi_2}$ and $Au_1 \cap Au_2 \neq 0$ then no composition factor of $\frac{Au_1}{Au_1 \cap Au_2}$ is not isomorphic to any composition factor of $\frac{Au_2}{Au_1 \cap Au_2}$. (The proof is as similarly as above.)

Corollary 6. *Assume that* $\left\{\frac{N^{\rho}e_1}{N^{\rho+1}e_1}, \frac{N^{\rho+\nu}e_2}{N^{\rho+\nu+1}e_2}\right\}$ $(\rho=1, \cdots, t-1,\ \nu\geqq 0)$ *are chains. If there exist* Au_i $(i=1,2)$ *such that* $Au_1\subset N^{t-1}e_1$, $\not\subset N^te_1$, $Au_2\subset N^{t+\nu-1}e_2$, $\not\subset N^{t+\nu}e_2$ *and* $\frac{Au_1}{Au_1\cap N^te_1}\cong\frac{Au_2}{Au_2\cap N^{t+\nu}e_2}$ *then there exist* Aw_i $(\subset Ae_i)$ $(i=1,2)$ *such that* $\frac{Ae_1}{N^te_1}\supset\widetilde{Aw_1}\supset\widetilde{Au_1}$, $\frac{Ae_2}{N^{t+\nu}e_2}\supset\widetilde{Aw_2}\supset\widetilde{Au_2}$, $Aw_1\subset Ne_1$, $\not\subset N^2e_1$, $Aw_2\subset N^{\nu+1}e_2$, $\not\subset N^{\nu+2}e_2$ *and the homomorphism of* $\widetilde{Aw_1}$ *onto* $\widetilde{Aw_2}$ *(or of* $\widetilde{Aw_2}$ *onto* $\widetilde{Aw_1}$*) is the extension of the isomorphism* $\widetilde{Au_1}\cong\widetilde{Au_2}$.

Proof. Let Aw_1' and Aw_2' be maximal subideals in Ae_1 and Ae_2 such that $\widetilde{Aw_1'}\supset\widetilde{Au_1}$, $\widetilde{Aw_2'}\supset\widetilde{Au_2}$ and $\widetilde{Aw_i'}$ $(i=1,2)$ are uniserial. If $\widetilde{Ne_1}=\widetilde{Aw_1'}$ and $\widetilde{N^{\nu+1}e_2}=\widetilde{Aw_2'}$ then this is trivial. Now assume that $\widetilde{Ne_1}=\widetilde{Aw_1''}\oplus\widetilde{Aw_1'}$, $\widetilde{Ne_2}=\widetilde{Aw_2'}$, $\widetilde{Au_1}=\widetilde{N^{\rho_1}w_1'}$ and $\widetilde{Au_2}=\widetilde{N^{\rho_2}w_2'}$. If $\frac{\widetilde{N^{\rho_2-1}w_2'}}{\widetilde{N^{\rho_2}w_2'}}$ is not isomorphic to $\frac{\widetilde{N^{\rho_1-1}w_1'}}{\widetilde{N^{\rho_1}w_1'}}$ then $\widetilde{Au_1}$ is isomorphic to a vertice component. Now assume that there exists an integer λ_1 such that $\frac{\widetilde{N^{\rho_2-1}w_2'}}{\widetilde{N^{\rho_2}w_2'}}\cong\frac{\widetilde{N^{\lambda_1-1}w_1''}}{\widetilde{N^{\lambda_1}w_1''}}$. Then there exists an integer μ such that $\frac{\widetilde{Aw_2''}}{\widetilde{N^{\lambda}w_1''}}\cong\frac{\widetilde{N^{\nu}w_1'}}{\widetilde{N^{\mu}w_2'}}$. Otherwise $\widetilde{Aw_2''}$ has a composition factor isomorphis to a vertice component but this contradicts the condition 2. Now from the assumption that $\left\{\frac{Ne_1}{N^2e_1}, \frac{N^{\nu+1}e_2}{N^{\nu+2}e_2}\right\}$ is a chain $\overline{Ae_1}$ is not isomorphic to any composition factor of $\frac{Ae_2}{N^{\nu+1}e_2}$. Hence $\frac{\widetilde{Aw_1''}}{\widetilde{Nw_1''}}$ is a vertice component. But this contradicts the condition 2. Thus $\widetilde{Aw_1'}\cong\widetilde{Aw_2'}$ and this isomorphism is the extension of $\widetilde{Au_1}\cong\widetilde{Au_2}$.

If $\widetilde{Ne_1}=\widetilde{Aw_1'}+\widetilde{Aw_1''}$, $\widetilde{Ne_2}=\widetilde{Aw_2'}$ and $\widetilde{Aw_1'}\cap\widetilde{Aw_1''}=\widetilde{Au_1}$, then by the same way as above $\widetilde{Aw_1'}\cong\widetilde{Aw_2'}$ and this isomorphism is the extension of $\widetilde{Au_1}\cong\widetilde{Au_2}$.

Next assume that $\widetilde{Ne_1}=\widetilde{Aw}$, $\widetilde{N^{\varphi}w}=\widetilde{Aw_1'}\oplus\widetilde{Aw_1''}$, $\widetilde{Ne_2}=\widetilde{Aw_2'}$ and $\widetilde{Aw_2'}\not\cong\widetilde{Aw_1'}$. If we put $\frac{\widetilde{N^{\varphi-1}w}}{\widetilde{N^{\varphi}w}}\cong\overline{Ae'}$ and assume that $\widetilde{Aw_1''}\cong\frac{\widetilde{Aw_2'}}{\widetilde{N^{\rho_2}w_2'}}$ and

$\widetilde{N^{\rho_2}w'_2} \simeq \widetilde{N^{\rho_1}w'_1}$ then there exists a subideal $\mathfrak{p}$ in Ne', such that $\frac{Ne'}{\mathfrak{p}} = \widetilde{Av_1} \oplus \widetilde{Av_2}$, $\widetilde{Av_1} \simeq \widetilde{A\tilde{w}'_1}$ and $\widetilde{Av_2} \simeq \widetilde{Aw'_2}$. In this case $\widetilde{N^{\rho_2}v_2} \simeq \widetilde{Au_2}$ and $\widetilde{N^{\rho_1}v_1} \simeq \widetilde{Au_1}$. But this contradicts the condition 2 since $\widetilde{Au_1}$ ($\simeq \widetilde{Au_2}$) is isomorphic to a vertice component.

If $\widetilde{Ne_1} = \widetilde{Aw_1} \supsetneqq \widetilde{Aw'_1} + \widetilde{Aw''_1}$ and $\widetilde{Ne_2} = \widetilde{Aw_2} \supsetneqq \widetilde{Aw'_2} + \widetilde{Aw''_2}$ where $\widetilde{Au_1} \subset \widetilde{Aw'_1}$ and $\widetilde{Au_2} \subset \widetilde{Aw'_2}$ then similarly as above we can see that the corollary holds.

As we can see from the proof of this corollary there does not exist any homomorphism of $\widetilde{Ae_1}$ into $\widetilde{Ae_2}$ which is the extension of the isomorphism $\widetilde{Au_1} \simeq \widetilde{Au_2}$.

Now let A be an algebra (not necessarily 2-cyclic representation type) satisfying the condition (1) and (2). Then we have

Lemma 10. *If* $\left\{\frac{N^{j_1}e_1}{N^{j_1+1}e_1}, \dots, \frac{N^{j_r}e_r}{N^{j_r+1}e_r}\right\}$ *is a chain then* $r=2$.

Proof. Suppose that $r=3$ and $\left\{\frac{N^{j_1-\nu}e_1}{N^{j_1-\nu+1}e_1}, \frac{N^{j_2-\nu}e_2}{N^{j_2-\nu+1}e_2}, \frac{N^{j_3-\nu}e_3}{N^{j_3-\nu+1}e_3}\right\}$ is not a chain for all $\nu \gneqq 0$.

(1) Assume that $\frac{N^{j_1}e_1}{N^{j_1+1}e_1} \supset \widetilde{Au_1}$, $\frac{N^{j_2}e_2}{N^{j_2+1}e_2} \supset \widetilde{Au_2}$, $\frac{N^{j_3}e_3}{N^{j_3+1}e_3} \supset \widetilde{Au_3}$, and $\widetilde{Au_1} \simeq \widetilde{Au_2} \simeq \widetilde{Au_3}$. Then $\widetilde{Au_1}$ is assumed to be isomorphic to a vertice component.
(Namely we assume that $\left\{\frac{N^{j_1-\nu}e_1}{N^{j_1-\nu+1}e_1}, \frac{N^{j_i-\nu}e_i}{N^{j_i-\nu+1}e_i}\right\}$ ($i=2, 3$) is not a chain for all ν.)

(1, 1) Assume that $\left\{\frac{N^{j_2-\nu}e_2}{N^{j_2-\nu+1}e_2}, \frac{N^{j_3-\nu}e_3}{N^{j_3-\nu+1}e_3}\right\}$ is not a chain for all ν. If there exist ξ_i ($i=1, 2, 3$) such that $\widetilde{Au_i} \subset \widetilde{N\xi_i}$ and $\not\subset \widetilde{N^2\xi_i}$ ($i=1, 2, 3$) then by the lemma 9 $\overline{A\xi_i}$ ($i=1, 2, 3$) are not isomorphic to each other. If we put $\overline{A\xi_i} \simeq \overline{Ae_{\xi_i}}$ then $\left\{\frac{Ne_{\xi_1}}{N^2e_{\xi_1}}, \frac{Ne_{\xi_2}}{N^2e_{\xi_2}}, \frac{Ne_{\xi_3}}{N^2e_{\xi_3}}\right\}$ is a chain but this contradicts the corollary 4.

(1. 2) Assume that $\widetilde{Aw_2}$ and $\widetilde{Aw_3}$ are the largest left subideals of $\widetilde{Ae_2} = \frac{Ae_2}{N^{j_2+1}e_2}$ and $\widetilde{Ae_3} = \frac{Ae_3}{N^{j_3+1}e_3}$ such that the homomorphism of $\widetilde{Aw_i}$ onto $\widetilde{Aw_j}$ ($i, j=2, 3$) is the extension of the isomorphism $\widetilde{Au_2} \simeq \widetilde{Au_3}$. Then

by the lemma 9 $\dfrac{\widetilde{Aw_1}}{\widetilde{Nw_2}}$ is isomorphic to a vertice component.

(1. 2. 1) If $\widetilde{Aw_2}$ is uniserial or $\widetilde{Aw_2}\supset\widetilde{A\eta_2}\oplus\widetilde{A\eta_2'}$ where $\widetilde{A\eta_2}\supset\widetilde{Au_2}$ then this contradicts the lemma 8.

(1. 2. 2) Assume that $\widetilde{Aw_2}\supset\widetilde{A\eta_2}+\widetilde{A\eta_2'}$ and $\widetilde{A\eta_2}\cap\widetilde{A\eta_2'}\supset\widetilde{Au_2}$. If we take $\widetilde{A\xi_2}$ and $\widetilde{A\xi_3}$ such that $\widetilde{Aw_2}\subset\widetilde{N\xi_2}$, $\not\subset\widetilde{N^2\xi_2}$ and $\widetilde{Aw_3}\subset\widetilde{N\xi_3}$, $\not\subset\widetilde{N^2\xi_3}$, then by the assumption the isomorphism of $\widetilde{Au_2}\simeq\widetilde{Au_3}$ cannot be extended to the homomorphism of $\widetilde{A\xi_2}$ onto $\widetilde{A\xi_3}$ (or of $\widetilde{A\xi_3}$ onto $\widetilde{A\xi_2}$) and by the lemma 9 $\dfrac{\widetilde{A\xi_2}}{\widetilde{N\xi_2}}\not\simeq\dfrac{\widetilde{A\xi_3}}{\widetilde{N\xi_3}}$.

Now from the assumption there exist $\widetilde{A\varphi_3}$ and $\widetilde{A\varphi_3'}$ such that $\widetilde{Aw_3}\subset\widetilde{A\varphi_3}+\widetilde{A\varphi_3'}$, $\widetilde{A\varphi_3}\cap\widetilde{A\varphi_3'}\supset\widetilde{Au_3}$ and the homomorphism of $\widetilde{A\varphi_3}$ onto $\widetilde{A\eta_2}$ (or of $\widetilde{A\eta_2}$ onto $\widetilde{A\varphi_3}$) and that of $\widetilde{A\varphi_3'}$ onto $\widetilde{A\eta_2'}$ (or of $\widetilde{A\eta_2'}$ onto $A\varphi_3'$) are the extension of the isomorphism $\widetilde{Au_2}\simeq\widetilde{Au_3}$. Then by the following lemma 11 this is a contradiction.

(2) Assume that $\dfrac{N^{j_1}e_1}{N^{j_1+1}e_1}\supset\widetilde{Au_1}$, $\dfrac{N^{j_2}e_2}{N^{j_2+1}e_2}=\widetilde{Au_2}\oplus\widetilde{Au_2'}$, $\dfrac{N^{j_3}e_3}{N^{j_3+1}e_3}\supset\widetilde{Au_3}$, $\widetilde{Au_1}\simeq\widetilde{Au_2}$ and $\widetilde{Au_2'}\simeq\widetilde{Au_3}$. Similarly as (1) we can assume that $\widetilde{Au_1}$ is isomorphic to a vertice component. If $\widetilde{Au_2}$ and $\widetilde{Au_2'}$ are isomorphic to vertice components then this contradicts the lemma 6. Hence we assume that $\widetilde{Aw_2}$ and $\widetilde{Aw_3}$ are the largest left subideals in $\widetilde{Ae_2}$ and $\widetilde{Ae_3}$ such that the homomorphism of $\widetilde{Aw_2}$ onto $\widetilde{Aw_3}$ (or of $\widetilde{Aw_3}$ onto $\widetilde{Aw_2}$) is the extension of $\widetilde{Au_2}\simeq\widetilde{Au_3}$. Hence $\dfrac{\widetilde{Aw_2}}{\widetilde{Nw_2}}$ is isomorphic to a vertice component. If $\widetilde{Aw_2}\subset\widetilde{Au_2}$ then by the same way as (1) this is a contradiction.

If $\widetilde{Aw_2}\not\supset\widetilde{Au_2}$ then $\widetilde{Aw_2}\cap\widetilde{Au_2}=0$ and this contradicts the lemma 6.

Lemma 11. *Assume that* $s\left(\dfrac{Ae_1}{\mathfrak{p}_1}\right)\supset\widetilde{Au_1}$, $s\left(\dfrac{Ae_2}{\mathfrak{p}_2}\right)\supset\widetilde{Au_2}$ *and* $\widetilde{Au_1}\simeq\widetilde{Au_2}$ *where each* $\mathfrak{p}_i$ $(i=1, 2)$ *is a left subideal in* Ae_i *and there is no homomorphism of* $\dfrac{Ae_1}{\mathfrak{p}_1}$ *into* $\dfrac{Ae_2}{\mathfrak{p}_2}$ $\left(\text{or of } \dfrac{Ae_2}{\mathfrak{p}_2} \text{ into } \dfrac{Ae_1}{\mathfrak{p}_1}\right)$ *which is the extension of the isomorphism* $\widetilde{Au_1}\simeq\widetilde{Au_2}$. *Then at least one of* $s\left(\dfrac{Ae_i}{\mathfrak{p}_i}\right)$ *is simple.*

Proof. Assume that $s\left(\frac{Ae_1}{\mathfrak{p}_1}\right)=\widetilde{Au_1}\oplus\widetilde{Av_1}$ and $s\left(\frac{Ae_2}{\mathfrak{p}_2}\right)=\widetilde{Au_2}\oplus\widetilde{Av_2}$. Now we construct an A-left module $\mathfrak{m}=Ae_1m_1+Ae_2m_2$ where $\mathfrak{p}_im_i=0$ and $u_1m_1=u_2m_2$ and suppose that $\mathfrak{m}$ is directly decomposable. Then $\mathfrak{m}=Ae_1n_1\oplus Ae_2n_2$ where $n_i=\alpha_{i1}m_1+\alpha_{i2}m_2$ $(i=1,2)$. Now we may assume that $e_in_i=n_i$, $\alpha_{ii}\in e_iAe_i$, $\notin e_iNe_i$ and $\alpha_{ij}\in e_iNe_j$ for $i\neq j$. Then $u_in_i\neq 0$ and $v_jn_j\neq 0$ since by the assumption that there does not exist any homomorphism of $\frac{Ae_1}{\mathfrak{p}_1}$ into $\frac{Ae_2}{\mathfrak{p}_2}$ and of $\frac{Ae_2}{\mathfrak{p}_2}$ into $\frac{Ae_1}{\mathfrak{p}_1}$ which is the extension of $\widetilde{Au_1}\cong\widetilde{Au_2}$, there does not exist $r\in Ne_1$ or $r'\in Ne_2$ such that $u_1=u_2r$ or $u_2=u_1r'$. Hence $Ae_1n_1\sim Ae_1m_1$ and $Ae_2n_2\sim Ae_2m_2$.[6)] Now if t_i is the length of the composition series of Ae_im_i then the length of $\mathfrak{m}$ is t_1+t_2-1. But from $\mathfrak{m}=Ae_1n_1\oplus Ae_2n_2$, the length of $\mathfrak{m}$ is t_1+t_2 and this is a contradiction.

Therefore $\mathfrak{m}$ is directly indecomposable and $s(\mathfrak{m})$ is the direct sum of at least three simple components. Thus the dual module $\mathfrak{m}^*$ of $\mathfrak{m}$ is directly indecomposable and is the sum of at least three cyclic right modules and A is not of 2-cyclic representation type. Hence this is a contradiction and at least one of $s\left(\frac{Ae_i}{\mathfrak{p}_i}\right)$ is simple.

From the lemma 10 we have the following lemma 12.

Lemma 12. *If* $\left\{\frac{N^{j_1}e_1}{N^{j_1+1}e_1}, \frac{N^{j_2}e_2}{N^{j_2+1}e_2}\right\}$ *is a chain for a pair of integers* (j_1, j_2) *then there does not exist* Ae_3 *such that* $\left\{\frac{N^{i_2}e_2}{N^{i_2+1}e_2}, \frac{N^{i_3}e_3}{N^{i_3+1}e_3}\right\}$ *is a chain for any integers* i_2 *and* i_3.

Proof. Suppose that $\left\{\frac{N^{i_2}e_2}{N^{i_2+1}e_2}, \frac{N^{i_3}e_3}{N^{i_3+1}e_3}\right\}$ is a chain. If $i_2=j_2$ then this contradicts the lemma 10. Hence we assume that $i_2\lneqq j_2$. Moreover similarly as the lemma 10 we can assume that the simple component $\widetilde{Au_1}$ of $\frac{N^{j_2}e_2}{N^{j_2+1}e_2}$ which is isomorphic to a simple component $\widetilde{Av_1}$ of $\frac{N^{j_1}e_1}{N^{j_1+1}e_1}$, is isomorphic to a vertice component.

Next we can assume that the simple component $\widetilde{Au_2}$ of $\frac{N^{i_2}e_2}{N^{i_2+1}e_2}$, which is isomorphic to a simple component $\widetilde{Aw_3}$ of $\frac{N^{i_3}e_3}{N^{i_3+1}e_3}$, is also isomorphic to a vertice component. If it is not isomorphic to a vertice component then we can extend this isomorphism to the homomorphism

6) By the condition (1) and (2) we can see that the kernel of the homomorphism $Ae_i\sim Ae_in_i$ is $N^\rho w_1+N^\mu w_2$ where $Ne_i=Aw_1+Aw_2$.

$A\xi_2$ onto $A\xi_3$ (or of $\widetilde{A\xi_3}$ onto $\widetilde{A\xi_2}$) such that $\frac{\widetilde{A\xi_2}}{\widetilde{N\xi_2}}$ is isomorphic to a vertice compount and we may only take it instead of $\widetilde{Au_2}$. Therefore by the same way as the lemma 10 this is a contradiction.

[2.5] Now assume that $\left\{\frac{N^\rho e_1}{N^{\rho+1}e_1}, \frac{N^{\rho+\nu}e_2}{N^{\rho+\nu+1}e_2}\right\}$ ($\nu \geqq 0$, $\rho=1, \cdots, t-1$) are chains.

(1) First we shall show that if $\nu=0$ then at least one of $\frac{Ae_i}{N^t e_i}$ ($i=1, 2$) is uniserial. By the lemma 2 we can assume that $\frac{Ne_1}{N^2e_1}$ is simple and $\frac{Ne_2}{N^2e_2}$ is not simple. If $\frac{N^2e_1}{N^3e_1}$ is not simple then $s\left(\frac{Ae_2}{N^3e_2}\right)$ is simple. Hence if $Ne_2=Aw_1+Aw_2$ then $N^2e_2=Nw_1=Nw_2$. Now we assume that $\overline{Aw_1} \cong \overline{Ae'}$ and $\overline{Aw_2} \cong \overline{Ae''} \cong \frac{Ne_1}{N^2e_1}$. Then $\frac{Ne''}{N^2e''}$ is not simple and $\left\{\frac{Ne'}{N^2e'}, \frac{Ne''}{N^2e''}\right\}$ is a chain. Hence by the following lemma 13 we can show that this is a contradiction. Thus $\frac{N^2e_1}{N^3e_1}$ is simple and in this way we can show that $\frac{Ne_1}{N^te_1}$ is uniserial.

Lemma 13. *Assume that* $\frac{N^\rho e_1}{N^{\rho+1}e_1}=\widetilde{Au_1}\oplus\widetilde{Au_2}$, $\frac{Ne_1}{N^\rho e_1}$ *is uniserial,* $\frac{\widetilde{N^\mu e_2}}{\widetilde{N^{\mu+1}e_2}} \cong \widetilde{Au_2}$ *and* $\frac{Ne_2}{\mathfrak{p}_2} \cong \frac{\widetilde{N^\mu e_1}}{\widetilde{Au_1}}$ $(e_1 \neq e_2)$. *Then* $\overline{Ae_2} \cong \frac{N^{\mu+1}e_1}{N^\mu e_1}$ *or* $\mu=1$.

Proof. Assume that $\mu \gneqq 1$ and $\overline{Ae_2} \ncong \frac{N^{\mu-1}e_1}{N^\mu e_1}$. Now if we put $\frac{N^{\mu-1}e_1}{N^\mu e_1} \cong Ae_1'$ and we take Ae_1' instead of Ae_1 then $\frac{Ne_2}{\mathfrak{p}_2} \cong \frac{\widetilde{N^2e_1'}}{\widetilde{Au_1}}$. Hence we may assume that $\mu=2$.

Next assume that $\frac{Ne_1}{N^2e_1} \cong \overline{Ae_3}$. Then there exists a subideal $\mathfrak{p}_3$ in Ne_3 such that $\frac{Ae_3}{\mathfrak{p}_3} \cong \frac{Ne_1}{\widetilde{Au_2}+\widetilde{Nu_1}}$.

Now if we put $\frac{\widetilde{N^{\rho-1}e_3}}{\widetilde{N^\rho e_3}}=\widetilde{Aw_1}\oplus\widetilde{Aw_2}$ then there exist $r \in Ne_1$, $\notin N^2e_1$ such that $\tilde{u}_1=\widetilde{w_1 r}$ and $\tilde{u}_2=\widetilde{w_2 r}$ and by the assumption $Aw_2 \subset \mathfrak{p}_3$.

In order to show that this is a contradiction we construct an A-left module $\mathfrak{m}=Ae_1m_1+Ae_2m_2+Ae_3m_3$ in the following way and show that this is directly indecomposable.

$$(1)\qquad N^{\rho+1}e_1m_1 = N^{\rho}e_2m_2 = \mathfrak{p}_3m_3 = 0$$
$$(\text{or } N^{\rho}vm_1 = N^{\rho}e_2m_2 = \mathfrak{p}_3m_3 = 0)\,.$$

$$(2)\qquad w_1m_3 = u_1m_1 \quad\text{and}\quad Au_2m_1 = N^{\rho-1}e_2m_2\,.$$

Now suppose that $\mathfrak{m}$ is directly decomposable. Then $\mathfrak{m}=Ae_1n_1+Ae_2n_2+Ae_3n_3$ and some Ae_in_i is a direct summand of $\mathfrak{m}$. Now let $n_i=\alpha_{i1}m_1+\alpha_{i2}m_2+\alpha_{i3}m_3$ $(i=1, 2, 3)$.

(i) Assume that $e_1\neq e_2\neq e_3$. Then $\alpha_{ii}\in e_iAe_i$, $\notin e_iNe_i$ and $\alpha_{ij}\in e_iNe_j$ $(i\neq j)$.
Now $u_1\alpha_{12}m_2\in N^{\rho}e_2m_2=0$ and $u_1\alpha_{13}m_3\in N^{\rho}e_3m_3\subset\mathfrak{p}_3m_3=0$ since Ae_1 is not isomorphic into Ae_2 and into Ae_3.
Next if $r_{11}\in e_1Ne_1$ then $u_1r_{11}m_1\in N^{\rho+1}e_1m_1=0$. Thus $u_1n_1=a_{11}u_1m_1$ $(a_{11}\in K)$. Similarly $u_2n_1=a_{11}u_2m_1$.
Next $N^{\rho}e_2r_1m_1\subset N^{\rho+1}e_1m_1=0$ for $r_1\in Ne_1$ since $\dfrac{Ne_1}{N^2e_1}\not\cong\overline{Ae_2}$ and $N^{\rho}e_2r_3m_3\in N^{\rho+1}e_3m_3=0$ for $r_3\in Ne_3$ and $N^{\rho}e_2r_2m_2\in N^{\rho+1}e_2m_2=0$ for $r_2\in Ne_2$. Hence $N^{\rho}e_2n_2=N^{\rho}e_2m_2$ and $s(Ae_2n_2)\frown s(Ae_1n_1)\neq 0$.

Lastly we shall show that if $w_1n_3=0$ then $w_2n_3\neq 0$.
Now suppose that $w_1n_3=0$ and $w_2n_3=0$. Then $w_1\alpha_{31}m_1+w_1\alpha_{33}m_3=0$ and $w_2\alpha_{31}m_1+w_2\alpha_{33}m_3=0$ since $w_i\alpha_{32}m_2\in N^{\rho}e_2m_2=0$ for $\alpha_{32}\in e_3Ne_2$. Now from the assumption $w_2\alpha_{33}m_3=0$. Hence $w_2\alpha_{31}m_1=0$. If $\alpha_{31}\in Ne_1$, $\notin N^2e_1$, then $w_2\alpha_{31}m_1\neq 0$. Thus $w_1\alpha_{33}m_3=0$. But this is a contradiction. Hence $w_1n_3\neq 0$ or $w_2n_3\neq 0$. Now assume that $w_2n_3\neq 0$. Then $w_2n_3=w_2\alpha_{31}m_1=u_2m_1\neq 0$ and $w_2\alpha_{31}n_1=a_{11}w_2\alpha_{31}m_1+w_2\alpha_{31}\alpha_{13}m_3\neq 0$. But $w_2\alpha_{31}\alpha_{13}m_3\in N^{\rho+1}e_3m_3=0$ since $w_2\alpha_{31}\in N^{\rho}e_1$. Thus $w_2\alpha_{31}m_1=u_2m_1=\dfrac{1}{a_{11}}w_2\alpha_{31}n_1$ and $w_2n_3=\dfrac{1}{a_{11}}w_2\alpha_{31}n_1$ $=\dfrac{1}{a_{11}}u_2n_1$. If $w_1n_3\neq 0$ then $w_1n_3=w_1\alpha_{31}m_1+w_1\alpha_{33}m_3=a_{31}w_1rm_1+a_{33}w_1m_3$ $=a_{31}u_1m_1+a_{33}u_1m_1=(a_{31}+a_{33})u_1m_1=\dfrac{a_{31}+a_{33}}{a_{11}}u_1n_1$ $(a_{ij}\in K)$. Therefore $s(Ae_1n_1)$ $\frown s(Ae_3n_3)\neq 0$. Thus $Ae_in_i\frown(Ae_jn_j+Ae_kn_k)\neq 0$ where $\{i, j, k\}=\{1, 2, 3\}$. But this is a contradiction and $\mathfrak{m}$ is directly indecomposable.

(ii) Assume that $e_1=e_2$. Then $\alpha_{ii}\in e_iAe_i$, $\in e_iNe_i$, $\alpha_{13}\in e_1Ne_3$ and $\alpha_{31}\in e_3Ne_1$. Now if we put $N^{\rho-1}e_1=Av$ then $\widetilde{Au_2}\simeq\widetilde{Av}$ and $u_2m_1=vm_2$. Similarly as (i) $u_in_1=a_{11}u_im_1$ $(i=1, 2)$. Next $vn_2=a_{21}vm_1+a_{22}vm_2$ $(a_{ij}\in K)$ since $\overline{Ae_3}\simeq\dfrac{Ne_1}{N^2e_1}\not\cong\overline{Ae_1}$. On the other hand $vn_1=a_{11}vm_1+a_{12}vm_2$. Hence

$vn_1 = a_{11}vm_1 + a_{12}u_2m_1 = a_{11}vm_1 + \dfrac{a_{12}}{a_{11}}u_2n_1$ and $vm_1 = \dfrac{a_{11}vn_1 - a_{12}u_2n_1}{a_{11}^2}$. Moreover $vn_2 = a_{21}vm_1 + a_{22}u_2m_1 = a_{21}vm_1 + \dfrac{a_{22}}{a_{11}}u_2n_1$ and $vm_1 = \dfrac{a_{11}vn_2 - a_{22}u_2n_1}{a_{11}a_{21}}$ $(a_{21} \neq 0)$.

(If $a_{21} = 0$ then $vn_2 = \dfrac{a_{22}}{a_{11}}u_2n_1$ and $Ae_1n_2 \cap Ae_1n_1 \neq 0$.)

Thus $\dfrac{a_{11}vn_1 - a_{12}u_2n_1}{a_{11}^2} = \dfrac{a_{11}vn_2 - a_{22}u_2n_1}{a_{11}a_{21}}$ and $(sAe_1n_1) \cap s(Ae_1n_2) \neq 0$. Similarly as (i) $s(Ae_1n_1) \cap s(Ae_3n_3) \neq 0$ and $Ae_in_i \cap (Ae_jn_j + Ae_kn_k) \neq 0$. But this is a contradiction and $\mathfrak{m}$ is directly indecomposable.

(iii) Assume that $e_1 = e_3$. Then Ne_1 is uniserial and this is a contradiction. This lemma is equivalent to the condition (4, ii, α).

The following lemma is necessary for the proof of the condition (4, ii, β).

Lemma 14. *Assume that* $Ne_1 = Av + Aw$, $\dfrac{N^{\rho-1}v}{N^{\rho}v} = \widetilde{Au_1} \oplus \widetilde{Au_2}$, $\widetilde{Au_2} = \widetilde{N^{\nu}w}$ *and* $\dfrac{N^{\rho-2}v}{N^{\rho-1}v} \not\cong \dfrac{\widetilde{N^{\nu-1}w}}{\widetilde{N^{\nu}w}}$. *Then there does not exist Ae_2 such that* $\dfrac{\widetilde{N^{\mu}e_2}}{\widetilde{N^{\mu+1}e_2}} \cong \widetilde{Au_2}$ *and* $\dfrac{N^{\rho-2}v}{N^{\rho-1}v} \not\cong \dfrac{\widetilde{N^{\mu-1}e_2}}{\widetilde{N^{\mu}e_2}}$ *where* $\widetilde{Ne_2} = \dfrac{Ne_2}{\mathfrak{p}}$ *is uniserial ($\mathfrak{p}$ is a subideal in Ne_2).*

Proof. (i) Assume that there exists Ae_2 such that $\dfrac{\widetilde{N^{\mu}e_2}}{\widetilde{N^{\mu+1}e_2}} \cong \widetilde{Au_2}$ and $\dfrac{N^{\rho-2}v}{N^{\rho-1}v} \not\cong \dfrac{\widetilde{N^{\mu-1}e_2}}{\widetilde{N^{\mu}e_2}} \not\cong \dfrac{\widetilde{N^{\nu-1}w}}{\widetilde{N^{\nu}w}}$ where $\widetilde{Ne_2} = \dfrac{Ne_2}{\mathfrak{p}}$ is uniserial. But this contradicts the lemma 3.

(ii) Assume that there exists Ae_2 such that $\dfrac{\widetilde{N^{\mu}e_2}}{\widetilde{N^{\mu+1}e_2}} \cong \widetilde{Au_2}$ and $\dfrac{\widetilde{N^{\mu-1}e_2}}{\widetilde{N^{\mu}e_2}} \cong \dfrac{\widetilde{N^{\nu-1}w}}{\widetilde{N^{\nu}w}}$ where $\widetilde{Ne_2} = \dfrac{Ne_2}{\mathfrak{p}}$ is uniserial. Now we put $\dfrac{N^{\rho-2}v}{N^{\rho-1}v} \cong \widetilde{Ae_3}$ and $Ne_3 = Aw_1 + Aw_2$. Then there exists $r \in N^{\rho-2}v$ such that $w_1r = u_1$ and $w_2r = u_2$. Now we construct an A-left module $\mathfrak{m} = Ae_1m_1 + Ae_2m_2 + Ae_3m_3$ in the following way:

$$(1) \qquad Nu_1m_1 = Nu_2m_1 = N^{\mu+1}e_2m_2 = Aw_2m_3 = Nw_1m_3 = 0\,.$$

$$(2) \qquad w_1m_3 = u_1m_1 \quad \text{and} \quad Au_2m_1 = N^{\mu}e_2m_2\,.$$

Then by the same way as the lemma 13 $\mathfrak{m}$ is directly indecomposable.

Next we shall show that if $\nu \neq 0$ then at least one of $\frac{Ae_1}{N^t e_1}$ and $\frac{Ae_2}{N^{t+\nu}e_2}$ is uniserial.

Assume that $\nu \neq 0$ and $Ne_1 = Aw_1 + Aw_2$. Then $s\left(\frac{Ae_2}{N^{\nu+2}e_2}\right)$ is simple by the lemma 11 where $s\left(\frac{Ae_2}{N^{\nu+2}e_2}\right) \cong \overline{Aw_2}$.

(α) We assume that $Ne_2 = A\xi_1 + A\xi_2$, $A\xi_1 \cap A\xi_2 = N^{\nu+1}e_2$ and $\frac{N^{\nu+1}e_2}{N^{\nu+2}e_2}$ is simple.

(i) Assume that $s\left(\frac{A\xi_1}{A\xi_1 \cap A\xi_2}\right) \not\cong s\left(\frac{A\xi_2}{A\xi_1 \cap A\xi_2}\right)$. If we put $s\left(\frac{A\xi_1}{A\xi_1 \cap A\xi_2}\right) \cong \overline{Ae'}$ and $s\left(\frac{A\xi_2}{A\xi_1 \cap A\xi_2}\right) \cong \overline{Ae''}$ then $e' \neq e'' \neq e_1$ and $\left\{\frac{Ne'}{N^2e'}, \frac{Ne''}{N^2e''}, \frac{Ne_1}{N^2e_1}\right\}$ is a chain. But this contradicts the lemma 11.

(ii) Assume that $s\left(\frac{A\xi_1}{A\xi_1 \cap A\xi_2}\right) \cong s\left(\frac{A\xi_2}{A\xi_1 \cap A\xi_2}\right)$. If $\frac{N^{\mu_1}\xi_1}{A\xi_1 \cap A\xi_2} \cong \frac{N^{\mu_2}\xi_2}{A\xi_1 \cap A\xi_2}$ and $\frac{N^{\mu_1-1}\xi_1}{A\xi_1 \cap A\xi_2} \not\cong \frac{N^{\mu_2-1}\xi_2}{A\xi_1 \cap A\xi_2}$ then $\frac{N^{\mu_i-1}\xi_i}{N^{\mu_i}\xi_i}$ $(i=1, 2)$ are isomorphic to a vertice component but this contradicts the lemma 6. Thus there exists an integer μ such that $\frac{A\xi_1}{A\xi_1 \cap A\xi_2} \cong \frac{N^{\mu}\xi_2}{A\xi_1 \cap A\xi_2}$.

Next assume that $\frac{N^{\mu-1}\xi_2}{N^{\mu}\xi_2} \not\cong \overline{Ae_2}$. Then $\frac{N^{\mu}\xi_2}{N^{\mu+1}\xi_2}$ and $\overline{A\xi_1}$ are isomorphic to a vertice component but this contradicts the lemma 6. Hence $\frac{N^{\mu-1}\xi_2}{N^{\mu}\xi_2} \cong \overline{Ae_2}$. Thus $N^{\mu}\xi_2 = A\xi_1\eta_{\mu}$ and $\xi_2\eta_{\mu} = 0$ since if $\xi_2\eta_{\mu} \neq 0$ then $A\xi_2\eta_{\mu} = A\xi_1 \cap A\xi_2$ and $A\xi_1\eta_{\mu} \cap A\xi_2\eta_{\mu} \subsetneqq A\xi_1 \cap A\xi_2$ but this contradicts the above assumption.

Moreover if $\frac{A\xi_1}{A\xi_1 \cap A\xi_2}$ has a composition factor isomorphic to $\overline{A\xi_1}$ then $\frac{A\xi_1}{A\xi_1 \cap A\xi_2}$ has a composition factor isomorphic to $\overline{Ae_2}$ but similarly as above this is a contradiction. Hence $A\xi_1 \cap A\xi_2 = 0$. But this contradicts the assumption.

Thus $\frac{Ne_1}{N^2e_1}$ is simple.

(β) Next assume that $Ne_2 = A\xi_1 + A\xi_2$, $A\xi_1 \cap A\xi_2 = A\eta = N^{\nu}e_2$ and $\frac{N\eta}{N^2\eta} \cong \overline{Aw_2}$ where $Ne_1 = Aw_1 + Aw_2$ and $\overline{A\eta} \not\cong \overline{Ae_1}$.

(i) If $s\left(\frac{A\xi_1}{A\xi_1 \cap A\xi_2}\right) \cong \overline{Ae'}$, $s\left(\frac{A\xi_2}{A\xi_1 \cap A\xi_2}\right) \cong \overline{Ae''}$ and $e' \neq e''$ then $\frac{Ne'}{N^2e'}$ $\left(\text{or } \frac{Ne''}{N^2e''}\right)$ is isomorphic to a vertice component since $\left\{\frac{Ne'}{N^2e'}, \frac{Ne''}{N^2e''}\right\}$ is a chain and $\frac{N^2e'}{N^3e'}$ $\left(\text{or } \frac{N^2e''}{N^3e''}\right)$ is isomorphic to a vertice component since $\left\{\frac{N^2e''}{N^3e''}, \frac{Ne_1}{N^2e_1}\right\}$ is a chain. But this contradicts the lemma 8.

(ii) Next if $s\left(\frac{A\xi_1}{A\xi_1 \frown A\xi_2}\right) \cong s\left(\frac{A\xi_2}{A\xi_1 \frown A\xi_2}\right)$ then similarly as above this contradicts the corollary 5. Hence we can see that the condition (4, i) is true.

Next we shall show that the condition (4, ii) is true. Now assume that $\left\{\frac{N^\rho e_1}{N^{\rho+1}e_1}, \frac{N^\rho e_2}{N^{\rho+1}e_2}\right\}$ $(\rho=1, \cdots, t-1)$ are chains and $\frac{Ne_2}{N^t e_2}$ is uniserial.

(i) Assume that $Ne_1 = Au_1 + Au_2$ where Au_i $(i=1, 2)$ are uniserial, $\overline{Au_2} \cong \frac{Ne_2}{N^2e_2}$ and $\frac{Au_2}{Au_1 \frown Au_2}$ is not simple. If we put $Nu_2 = Avu_2$ and $Ne_2 = A\xi$ then $N^2e_2 = Av\xi$ since $\overline{Au_2} \cong \overline{A\xi}$. Now we put $\overline{Au_1} \cong \overline{Ae'}$, $\overline{Au_2} \cong \overline{Ae''}$ and $Avu_2 \cong \overline{Ae'''}$. Then $e' \neq e''$ and $e'''N = vA$. If $e'''N = vA + v'A$ and $v'e = v'$ then $e \neq e''$ $(ve'' = v)$. Hence $Ne = Av' + A\alpha$ and $\overline{Av'} \cong Av\xi$. Therefore $\overline{A\xi}$ and $\overline{Av\xi}$ are isomorphic to vertice components. But this contradicts the lemma 8. Next $e'''N^2 = vu_2A + v\xi A$ and $e'N = u_1A$ where $\overline{e'A} \ncong \overline{vA}$ and $\overline{vu_2A} \cong \overline{u_1A}$. But this contradicts the lemma 14. Thus $Au_1 \frown Au_2 = Nu_2$.

REMARK. From this result we can see that the following two cases are equivalent.

(1) $Ne_1 = Au_1 + Au_2$, $Nu_2 \not\subset Au_1$ and $\frac{Ne_2}{N^3e_2} \cong \frac{Au_2}{N^2u_2}$ $(e_2 \neq e_1)$.

(2) $\frac{Ne_1}{N^2e_1}$ is simple, $N^2e_1 = Au_1 + Au_2$, $\frac{Ne_1}{N^2e_1} \ncong \overline{Ae_2}$ and $\overline{Au_2} \cong \frac{Ne_2}{N^2e_2}$.

(ii) Assume that $Ne_1 = Au_1 + Au_2$ where $N^\mu u_1 = Aw_1 + Aw_2$. Then $Aw_2 \subset Au_2$ (or $Aw_1 \subset Au_2$) and $s\left(\frac{Au_1}{Au_1 \frown Au_2}\right) \ncong s\left(\frac{Au_2}{Au_1 \frown Au_2}\right)$ since $Au_1 \frown Au_2 \neq 0$. Hence similarly as (i) each composition factor of $\frac{Au_1}{Au_1 \frown Au_2}$ is not isomorphic to any composition factor of $\frac{Au_2}{Au_1 \frown Au_2}$.

Now if $\frac{Ne_2}{N^t e_2} \cong \frac{Au_2}{N^s u_2}$ where $N^s u_2 \subsetneqq Au_1 \frown Au_2$ then there exists p such that $\overline{Aw_2} \cong \frac{N^p e_2}{N^{p+1}e_2}$ $\left(\text{or } \overline{Aw_1} \cong \frac{N^p e_2}{N^{p+1}e_2}\right)$. But this contradicts the lemma 13 since $\frac{Ne_2}{N^2e_2} \ncong \frac{Au_1}{Nu_1}$.

Next if $\frac{Ne_2}{N^t e_2} \cong \frac{Au_1}{Aw_2 + Nw_1}$ $\left(\text{or } \frac{Ne_2}{N^t e_2} \cong \frac{Au_1}{Aw_1 + Nw_2}\right)$ then $t \geqq 3$. But this contradicts the first half of the condition (4, ii). Since similarly as (i) if $\frac{Ne_2}{N^t e_2} \cong \frac{Au_1}{Aw_2 + Nw_1}$ and $t \geqq 3$ then there exist Ae' and Ae'' such that $\frac{Ne'}{N^2e'}$ is simple, $N^2e' = Au_1 + Au_2$, $\frac{Ne'}{N^2e'} \not\cong \overline{Ae''}$ and $\overline{Au_2} \cong \frac{Ne''}{N^2e''}$ and this contradicts the first half of the condition (4, ii). Hence $\frac{Ne_2}{N^3e_2} \cong \frac{Au_1}{Aw_1 + Nw_2}$ $\left(\text{or } \frac{Ne_2}{N^3e_2} \cong \frac{Au_1}{Nw_1 + Aw_2}\right)$ and $\mu = 1$. Thus the condition 4 is true.

[2. 6] Next we shall prove that the condition 3 holds. For that purpose we shall prove the following lemma 15.

(2. 6. 1) **Lemma 15.** *Assume that* $\left\{\frac{N^i e_1}{N^{i+1}e_1}, \frac{N^i e_2}{N^{i+1}e_2}\right\}$ $(i = 1, 2)$ *are chains,* $\frac{Ne_1}{N^3e_1}$ *is uniserial and if there exists* Ae_3 *such that* $Ne_3 = Aw + Aw'$ *then* $N^3w \supseteqq Aw \cap Aw'$. *If* Ae_1 *(or* Ae_2*) is homomorphic onto* Aw *where* $\frac{Nw}{N^2w} \cong \frac{Ne_1}{N^2e_1}$ *then* $N^2w = 0$.

Proof. Assume that $N^2w \neq 0$.

(i) Assume that $\frac{Ne_2}{N^3e_2}$ is uniserial and $Ae_2 \sim Aw$. Then $\frac{Ne_1}{N^3e_1} \cong \frac{Ne_2}{N^3e_2}$. Now we put $\frac{Ne_1}{N^2e_1} \cong \overline{Ae'}$, $\frac{N^2e_1}{N^3e_1} \cong \overline{Ae''}$, $Ne_1 = Au_1$, $Ne_2 = Au_2$, $Ne_3 = Aw$ and $Ne' = Av$. Then $N^2e_1 = Avu_1$ and $N^2e_2 = Avu_2$. If $\frac{Ae_2}{N^3e_2} \cong \frac{Ne_3}{N^4e_3}$ then $N^2e_3 = Au_2w$ and $N^3e_3 = Avu_2w$.

(If $Ne_3 = Aw + Aw'$ and $\frac{Ae_2}{N^3e_2} \cong \frac{Aw}{N^3w}$ then $Nw = Au_2w$ and $N^2w = Avu_2w$.)

Now by the condition 2 we can see that $e' \neq e''$, $e_2 \neq e'$, $e_1 \neq e'$, $e_3 \neq e_2$ and $e_3 \neq e_1$.

If $e_2 = e'$, $e_1 = e'$, $e_3 = e_2$ or $e_3 = e_1$ then $e' = e''$ and $\frac{Ne_1}{N^2e_1}$ and $\frac{N^2e_1}{N^3e_1}$ are isomorphic to a vertice component and this contradicts the condition 2. Now we construct an A-left module $\mathfrak{m} = Ae_1m_1 + Ae_2m_2 + Ae_3m_3$ where $N^3e_1m_1 = N^3e_2m_2 = N^4e_3m_3 = 0$, (if $Ne_3 = Aw + Aw'$ then $N^3wm_3 = w'm_3 = 0$) $u_1m_1 = u_2m_2$ and $vu_1m_1 = vu_2m_2 = vu_2wm_3$ and suppose that $\mathfrak{m}$ is directly decomposable. Then $\mathfrak{m} = Ae_1n_1 + Ae_2n_2 + Ae_3n_3$ and some Ae_in_i is a direct summand of $\mathfrak{m}$. Now let

$$n_i = \alpha_{i1}m_1 + \alpha_{i2}m_2 + \alpha_{i3}m_3 \qquad (i = 1, 2, 3),$$

where $e_in_i=n_i$. Then $\alpha_{ii}\in e_iAe_i$, $\notin e_iNe_i$ and $\alpha_{ij}\in e_iNe_j$ for $i\neq j$. First of all $u_2wn_3=u_2w\alpha_{31}m_1+u_2w\alpha_{32}m_2+u_2w\alpha_{33}m_3$. But $u_2w\alpha_{31}m_1\in N^3e_1m_1=0$ and $u_2w\alpha_{32}m_2\in N^3m_2=0$ and $u_2wxm_3=0$ for $x\ni Ne_3$ since $\overline{Aw}\not\cong\overline{Ae_3}$.

Hence $u_2wn_3=a_{33}u_2wm_3$ ($\alpha_{33}=a_{33}+r_{33}$, $a_{33}\in K$ and $r_{33}\in Ne_3$).

Next $u_1\alpha_{12}m_2=0$ and $u_1\alpha_{13}m_3=0$ since $e'\neq e_1$ and $e_1\neq e_2$ and $u_1xm_1=0$ for $x\in Ne_1$ since $e_1\neq e'\neq e''$. Therefore $u_1n_1=a_{11}u_1m_1$ ($a_{11}\in K$).

Lastly assume that $u_2n_2=0$. Then $u_2\alpha_{21}m_1+u_2\alpha_{22}m_2+u_2\alpha_{33}m_3=0$. Similarly as above $u_2\alpha_{21}m_1=0$. If $u_2\alpha_{23}m_3\neq 0$ then $u_2\alpha_{23}m_3=a_{23}u_2wm_3$ ($a_{23}\in K$) since $\overline{Ae_2}\cong\overline{Aw}$. Thus $a_{22}u_2m_2+a_{23}u_2u_3m_3=0$ since $u_2\alpha_{22}m_2=a_{22}u_2m_2$ ($a_{22}\in K$). But from the assumption that $Au_2m_2\neq Au_2wm_3$ this is a contradiction. Thus $u_2n_2\neq 0$ and $u_2n_2=a_{22}u_2m_2+a_{23}u_2wm_3$ and $Au_2n_2\subset Au_2m_2+Au_2wm_2$ $=Au_1m_1+Au_2wm_3=Au_1n_1+Au_2wn_3$. Hence $Ae_in_i\cap(Ae_jn_j+Ae_kn_k)\neq 0$ where $\{i, j, k\}=\{1, 2, 3\}$. This is a contradiction. Hence we can see that $\mathfrak{m}$ is directly indecomposable.

(ii) Assume that $Ne_2=Au_2$, $N^2e_2=Av_1u_2=Av_1u_2+Av_2u_2$, $Ne_1=Au_1$, $\overline{Au_1}\cong\overline{Au_2}$, $\frac{Nu_1}{N^2u_1}\cong\overline{Av_1u_2}$ and $\overline{Ae_2}\cong\overline{Aw}$. Now if we put $Ne_3=Aw$ and assume that $\overline{Ae_2}\cong\frac{Ne_3}{N^2e_3}$ then $N^2e_3=Au_2w$ and $N^3e_3=Av_1u_2w+Av_2u_2w$.

(If $Ne_3=Aw+Aw'$ then $\frac{Aw}{Nw}\cong\overline{Ae_2}$ and $Nw=Au_2w$ and $N^2w=Av_1u_2w$ $+Av_2u_2w$.)

If $\frac{Ne_1}{N^2e_1}\cong\overline{Ae'}$ and $\frac{N^2e_1}{N^3e_1}\cong\overline{Ae''}$ then similarly as (i) we can see that $e'\neq e''$, $e'\neq e_1$, $e'\neq e_2$, $e_2\neq e_3$ and $e_3\neq e_1$. Now we construct an A-left module $\mathfrak{m}=Ae_1m_1+Ae_2m_2+Ae_3m_3$ where $N^3e_1m_1=N^3e_2m_2=N^4e_3m_3=0$, $v_2u_2m_2$ $=v_2u_2wm=0$ (if $Ne_3=Aw+Aw'$ then $N^3wm_3=w'm_3=0$), $u_1m_1=u_2m_2$ and $v_1u_1m_1=v_1u_2m_2=v_1u_2wm_3$. Then similarly as (i) we can see that $\mathfrak{m}$ is directly indecomposable.

(iii) Assume that $Ne_2=Au_1+Au_2$ and $\overline{Au_1}\cong\frac{Ne_1}{N^2e_1}$. Then by the condition (4, ii) $Au_1\cap Au_2=Nu_1$. If $Ne_3=Aw+Aw'$ and $\overline{Ae_2}\cong\overline{Aw}$ then Nw $=Au_1w$. If $Au_2w\neq 0$ then $s\left(\frac{Ne_3}{N^2w}\right)$ is the direct sum of at least three simple components and this contradicts the condition 1 since $Au_1w\neq 0$ and $N^2w\supset Aw+Aw'$. Hence $u_2w=0$ and $N^2w=Nu_1w=0$. Thus if Ne_3 $=Aw+Aw'$ then we assume that $\overline{Ae_1}\cong\overline{Aw}$.

(iii. 1) Assume that $Ne_3=Aw$ and $\overline{Ae_2}\cong\overline{Aw}$. Now we put $Ne_1=Av_1$, $\overline{Av_1}\cong\overline{Au_1}\cong\overline{Ae'}$, $Ne'=Av$ and $\frac{N^2e_1}{N^3e_1}\cong\frac{Nu_1}{N^2u_1}\cong\overline{Ae''}$. Then $N^2e_1=Avv_1$, $Au_1\cap Au_2=Avu_1$, $N^2e_3=Au_1w+Au_2w$ and $N^3e_3=Avu_1w$. Now similarly as

(i) we can see that $e' \neq e''$, $e_1 \neq e'$, $e_2 \neq e'$, $e_1 \neq e_3$ and $e_2 \neq e_3$ and we construct an A-left module $\mathfrak{m} = Ae_1m_1 + Ae_2m_2 + Ae_3m_3$ where $N^3e_1m_1 = N^3e_2m_2 = N^4e_3m_3 = 0$, $v_1m_1 = u_1m_2$ and $vv_1m_1 = vu_1m_2 = vu_1u_3m_3$. Then similarly as (i) $\mathfrak{m}$ is directly indecomposable.

(iii. 2) Assume that $Ne_3 = Aw + Aw'$. Then $\overline{Ae_1} \cong \overline{Aw}$. Now we put $Ne_1 = Av_1$, $\overline{Av_1} \cong \overline{Au_1} \cong \overline{Ae'}$, $\frac{Nv_1}{N^2v_1} \cong \frac{Nu_1}{N^2u_1} \cong \overline{Ae''}$, $Ne' = Av$. Then $N^2e_1 = Avv_1$, $Au_1 \cap Au_2 = Avu_1$, $Nw = Av_1w$ and $N^2w = Avv_1w$. Now similarly as (i) we can see that $e' \neq e''$, $e_1 \neq e'$, $e_2 \neq e'$ $e_2 \neq e'$ $e_1 \neq e_3$ and $e_2 \neq e_3$ and we construct an A-left module $\mathfrak{m} = Ae_1m_1 + Ae_2m_2 + Ae_3m_3$ where $N^3e_1m_1 = N^3e_2m_2 = N^3wm_3 = w'm_3 = 0$, $v_1m_1 = u_1m_2$ and $vv_1m_1 = vu_1m_2 = vv_1wm_3$. Then $\mathfrak{m}$ is directly indecomposable.

(iv) Assume that $Ne_2 = Au_2$, $N^2e_2 = Av_1u_2 + Av_2u_2$, $\frac{N^2e_1}{N^3e_1} \cong \overline{Av_1u_2}$ and $Ae_1 \sim Aw$. Now we put $Ne_1 = Au_1$. Then $Nu_1 = Av_1u_1$, $Nw = Au_1w$ and $N^2w = Av_1u_1w$. Hence if we construct an A-left module $\mathfrak{m} = Ae_1m_1 + Ae_2m_2 + Ae_3m_3$ where $N^3e_1m_1 = Nv_1u_2m_2 = Av_2u_2m_2 = N^3wm_3 = 0$ (if $Ne_3 = Aw + Aw'$ then $N^3wm_3 = w'm_3 = 0$), $u_1m_1 = u_2m_2$ and $v_1u_1m_1 = v_1u_2m_2 = v_1u_1wm_3$ then similarly as (i) $\mathfrak{m}$ is directly indecomposable. Thus this is a contradiction. Therefore $N^2w = 0$.

(2. 6. 2) Now we shall show that if $\frac{Ne}{N^2e}$ is simple and $N^2e = Au_1 + Au_2$ then $Au_1 \cap Au_2 = Nu_1 = Nu_2$. For that purpose assume that $Nu_2 \supsetneqq Au_1 \cap Au_2$. First $\frac{Ne}{N^2e} \not\cong \overline{Ae}$. If $\frac{Ne}{N^2e} \cong \overline{Ae}$ then Ne is uniserial. Hence we put $\frac{Ne}{N^2e} \cong Ae'$ $(e \neq e')$.

(α) Assume that $\overline{Au_1} \cong \frac{Nu_2}{N^2u_2}$. Now we construct an A-left module $\mathfrak{m} = Aem_1 + Aem_2 + Aem_3$ where $Nu_1m_i = N^2u_im_i = 0$ $(i = 1, 2, 3)$, $Au_1m_1 = Nu_2m_2$ and $Au_1m_2 = Nu_2m_3$. Then $u_1rm_i \subset N^4em_i = 0$ for $r \in Ne$ since $\frac{Ne}{N^2e} \not\cong \overline{Ae}$ and $Nu_2r'm_i \subset N^4em_i = 0$ for $r' \in Ne$. Thus by the lemma 7 $\mathfrak{m}$ is directly indecomposable and this is a contradiction.

(β) Assume that $\overline{Au_1} \not\cong \frac{Nu_2}{N^2u_2}$. Now we put $Ne = Aw$, $Au_1 = Av_1w$, $Au_2 = Av_2w$ and $Nu_2 = Nv_2w = Avv_2w$ $(v \neq v_1)$ where $\frac{Ne}{N^2e} \cong \overline{Ae'}$, $Ne' = Av_1 + Av_2$, $\overline{Au_2} \cong \overline{Ae_1}$, $\frac{Nu_2}{N^2u.} \cong \overline{Ae_2}$ and $\overline{Au_1} \cong \overline{Ae_3}$.

(i) Assume that there does not exist Ae'' such that $\left\{ \frac{N^\rho e''}{N^{\rho+1}e''}, \right.$

$\left.\frac{N^{\rho+\nu}e}{N^{\rho+\nu+1}e}\right\}$ ($\nu\geqq 0$, $\rho=1, 2, 3$) are chains and Ae (or Ae'') is not homomorphic into Ae'' (or Ae). Then $e_3N=v_1A$. If $e_3N=v_1A+v_1'A$ and $v_1'f=v_1'$ then $\left\{\frac{Ne'}{N^2e'}, \frac{Nf}{N^2f}\right\}$ is a chain. Hence $\left\{\frac{N^2e}{N^3e}, \frac{Nf}{N^2f}\right\}$ is a chain. But this contradicts the above assumption. Similarly as this, $e_3N^2=v_1wA$, $e_1N=v_2A$, $e_1N^2=v_2wA$, $e_2N=vA$, $e_2N^2=vv_2A$ and $e_2N^3=vv_2wA$. Hence $\frac{e_3N}{e_3N^3}\cong\frac{e_1N}{e_1N^3}$ and $\frac{e_1A}{e_1N^3}\cong\frac{e_2N}{e_2N^4}$. But this contradicts the lemma 15.

(ii) Assume that there exists Ae'' such that $\left\{\frac{N^\rho e''}{N^{\rho+1}e''}, \frac{N^\rho e}{N^{\rho+1}e}\right\}$ ($\rho=1, 2, 3$). Then by the condition 4 $\frac{Ae''}{N^4e''}$ is uniserial.

(ii. 1) Assume that $\frac{Av_2w}{N^2v_2w}\cong\frac{N^2e''}{N^4e''}$. If we put $Ne''=Aw'$ then $N^2e''=Av_2w'$ and $N^3e''=Avv_2w'$. Then $e_3N=v_1A$, $e_3N^2=v_1wA$, $e_1N=v_2A$, $e_1N^2=v_2wA+v_2w'A$, $e_3N=vA$, $e_3N^2=vv_2A$ and $e_3N^3=vv_2wA+vv_2w'A$. Hence $\frac{e_3N}{e_3N^2}\cong\frac{e_1N}{e_1N^2}$, $\frac{e_3N^2}{e_3N^3}\cong\overline{v_2wA}$ and $\frac{e_2N}{e_2N^4}\cong\frac{e_1A}{e_1N^3}$. But this contradicts the lemma 15.

(ii. 2) Assume that $\frac{N^2e''}{N^3e''}\cong\overline{Av_1w}$. Now if we put $Ne''=Aw'$ then $N^2e''=Av_1w'$. Hence $e_3N=v_1A$, $e_3N^2=v_1wA+v_1w'A$, $e_1N=v_2A$, $e_1N^2=v_2wA$, $e_2N=vA$, $e_2N^2=vv_2A$ and $e_2N^3=vv_2wA$. Therefore $\overline{v_2A}\cong\overline{v_1A}$, $\overline{v_1wA}\cong\overline{v_2wA}$ and $\frac{e_1A}{e_1N^3}\cong\frac{e_2N}{e_2N^4}$. But this is a contradiction.

(iii) Assume that there exists Ae'' such that $\left\{\frac{N^\rho e''}{N^{\rho+1}e''}, \frac{N^{\rho+1}e}{N^{\rho+2}e}\right\}$ ($\rho=1, 2$) are chains. But this contradicts the lemma 14.

(iv) Assume that there exists Ae'' such that $\frac{Ne''}{N^2e''}\cong\overline{Avv_2w}.$ Now if we put $Ne''=Av'$ then $e_2N=v'A+vA$, $vN=vv_2A$, $vN^2=vv_2wA$ ($v'A\frown vA\subset vN^2$), $e_1N=v_2A$, $e_1N^2=v_2wA$, $e_3N=v_1A$ and $e_3N^2=v_1wA$. Hence $\frac{e_1N}{e_1N^3}\cong\frac{e_3N}{e_3N^3}$ and $\frac{e_1A}{e_1N^3}\cong\frac{vA}{vN^3}$. But this contradicts the lemma 15. Thus $Nu_2\subset Au_1\frown Au_2$. Similarly as this $Nu_1\subset Au_1\frown Au_2$. Therefore $Au_1\frown Au_2=Nu_1=Nu_2$. Generally if $\frac{Ne}{N^\rho e}$ is uniserial and $N^\rho e=Au_1+Au_2$ then $Au_1\frown Au_2=Nu_1=Nu_2$.

(2. 6. 3) Next we shall show that if $Ne=Au_1+Au_2$ and $N^\rho u_1=Aw_1$

$+Aw_2$ then $Aw_1 \cap Aw_2 = Nw_1 = Nw_2$. For that purpose we assume that $Nw_1 \not\subseteq Aw_1 \cap Aw_2$.

(i) Assume that $Au_1 \cap Au_2 = Aw_1 + Aw_2$. If $s\left(\frac{Au_1}{Au_1 \cap Au_2}\right) \not\cong s\left(\frac{Au_2}{Au_1 \cap Au_2}\right)$ then this contradicts the corollary 3.

If $N^\rho u_1 = Aw_1 + Aw_2$, $N^\mu u_2 = Aw_1 + Aw_2$ and $\rho = \mu = 1$ then similarly as above this contradicts the corollary 3 since $\overline{Au_1} \not\cong \overline{Au_2}$ by the lemma 1. Hence we assume that $\rho \gneqq 1$ or $\mu \gneqq 1$. Then if $\rho \gneqq 1$ and $\overline{Au_1} \cong \overline{Ae'}$ then $\widetilde{N^\rho e'} = \widetilde{Aw_1'} + \widetilde{Aw_2'}$ where $\widetilde{Ae'} = \frac{Ae'}{\mathfrak{p}'} \cong Au_1$ ($\mathfrak{p}'$ is a subideal in Ae') and $Aw_i \cong \widetilde{Aw_i'}$ $(i=1, 2)$ and $\frac{\widetilde{Ne'}}{\widetilde{N^\rho e'}}$ is uniserial. Hence by (2. 6. 2) $\widetilde{Aw_1'} \cap \widetilde{Aw_2'} = \widetilde{Nw_1'} = \widetilde{Nw_2'}$

Thus $Aw_1 \cap Aw_2 = Nw_1 = Nw_2$.

(ii) Assume that $N^\rho u_1 = Aw_1 + Aw_2$ and $Au_2 \supset Aw_2$. By the result of (i) we can see that $\rho = 1$ and Au_2 is uniserial. Hence $Aw_2 = N^\mu u_2$.

(ii. 1) Assume that $s\left(\frac{Au_1}{Au_1 \cap Au_2}\right) \cong s\left(\frac{Au_2}{Au_1 \cap Au_2}\right)$. Then similarly as (2. 5) we can see that $Au_1 \cap Au_2 = 0$.

(ii. 2) Assume that $s\left(\frac{Au_1}{Au_1 \cap Au_2}\right) \not\cong s\left(\frac{Au_2}{Au_1 \cap Au_2}\right)$. Now if we put $\overline{Au_1} \cong \overline{Ae'}$ and $\frac{N^{\mu-1}u_2}{N^\mu u_2} \cong \overline{Ae''}$ then $\widetilde{Ne'} = \widetilde{Av_1} + \widetilde{Av_2}$ where $\widetilde{Ae'} = \frac{Ae'}{\mathfrak{p}'} \cong Au_1$ ($\mathfrak{p}'$ is a subideal in Ae') and $\widetilde{Av_i} \cong Aw_i$ $(i=1, 2)$.

From now on we assume that $\mathfrak{p}' = 0$.

(α) Assume that there does not exist Af such that $\left\{\frac{N^\rho f}{N^{\rho+1} f}, \frac{N^{\rho+\nu} e}{N^{\rho+\nu+1} e}\right\}$ $(\nu \geqq 0, \rho = 1, 2, 3)$ are chains. Now we put $Nu_1 = Av_1u_1 + Av_2u_1$ where $v_1u_1 = w_1$ and $v_2u_1 = w_2$, $Nv_1u_1 = Awv_1u_1$, $N^\mu u_2 = Av'u_2$, $v_2u_1 = v'u_2$, $\overline{Av_1u_1} \cong \overline{Ae_1}$, $\overline{Avv_1u_1} \cong \overline{Ae_2}$ and $\overline{Av_2u_1} = \overline{Av'u_2} \cong \overline{Ae_3}$. Then similarly as (2. 6. 2) $e_2N = wA$, $e_2N^2 = wv_1A$, $e_2N^3 = wv_1u_1A$, $e_1N = v_1A$, $e_1N^2 = v_1u_1A$, $e_3N = v_2A + v'A$, $e_3N^2 = v_2u_1A = v'u_2A$. Thus $\frac{e_2N}{e_2N^4} \cong \frac{e_1A}{e_1N^3}$ and $\frac{e_1N}{e_1N^3} \cong \frac{v_2A}{v_2N^2}$. But this contradicts the lemma 15.

(β) Assume that there exists Ae' such that $\frac{Ne'}{N^3e'} \cong \frac{Au_1}{Aw_1 + Nw_2}$. Now we put $w_1 = v_1u_1$, $w_2 = v_2u_1$ and $Ne' = Aw'$. Then $N^2e' = Av_2w'$. Hence similarly as above $e_2N = wA$, $e_2N^2 = wv_1A$, $e_2N^3 = wv_1u_1A$, $e_1N = v_1A$, $e_1N^2 = v_1u_1A$, $e_3N = v_2A + v'A$ and $v_2N = v_2w_1A + v_2u_1A$ $(v_2u_1 = v'u_2)$. Thus $\frac{e_2N}{e_2N^4} \cong \frac{e_1A}{e_1N^3}$ and

$\frac{e_1N}{e_1N^3} \simeq \frac{v_2A}{v_2w_1A+v_2u_1N}$. But this contradicts the lemma 15.

(γ) Assume that there exists Ae' such that $\frac{Ne'}{N^{\mu+2}e'} \simeq \frac{Au_2}{N^{\mu+1}u_2}$. But this contradicts the condition (4. ii. β).

(δ) Assume that there exists Ae' such that $\frac{Ne'}{N^{\nu}e'} \simeq \frac{N^{\rho}u_2}{N^{\mu+1}u_2}$ $(\rho \geqq 1)$ and $\overline{Ae'} \not\cong \overline{Au_2}$. Now if we put $\overline{Au_2} \cong \overline{Ae''}$ and $\overline{Au_1} \simeq \overline{Ae'''}$ then $\left\{\frac{N^{j_1}e'}{N^{j_1+1}e'}, \frac{N^{j_2}e''}{N^{j_2+1}e''}, \frac{N^{j_3}e'''}{N^{j_3+1}e'''}\right\}$ is a chain. But this contradicts the lemma 11.

(ε) Assume that there exists Ae' such that $\frac{Ne'}{N^3e'} \simeq \frac{Au_1}{Av_2u_1+Nv_1u_1}$ $(e' \neq e)$. But this contradicts the condition (4. ii. α).

(φ) Assume that there exists Ae' such that $\frac{Ne'}{N^2e'} \simeq \overline{Av_1u_1}$ or $\frac{Ne'}{N^2e'} \simeq \overline{Awv_1u_1}$. Then $\overline{Av_1u_1}$ or $\overline{Awv_1u_1}$ and $\overline{Av_2u_1}$ are isomorphic to vertice components since $\overline{Ae'} \not\cong \overline{Au_1}$ (or $\overline{Ae'} \not\cong \overline{Av_1u_1}$) and $\overline{Au_1} \not\cong \frac{N^{\mu-1}u_2}{N^{\mu}u_2}$. But this contradicts the condition 2. Thus $Nw_1 \subset Aw_1 \cap Aw_2$. Similarly as this $Nw_2 \subset Aw_1 \cap Aw_2$. Generally if $\frac{Ne}{N^2e}$ is simple then it is clear by (2. 6. 2) and if $Ne = Au_1 + Au_2$ then Au_i $(i=1, 2)$ are uniserial or $N^{\rho}u_1 = Aw_1 + Aw_2$.

If $N^{\rho}u_1 = Aw_1 + Aw_2$ then $Aw_2 \subset Au_2$ and Au_2 is uniserial since the first half of the condition 1, the condition 2 and the condition 4 are true. Hence we can reduce to the above case (2. 6. 3).

Therefore we can see that the condition 3 is true.

[2. 7] Lastly we shall show that latter half of the condition (1) holds.

(2. 7. 1) Assume that $Ne = Au_1 + Au_2$. If $\overline{Au_1}$ is not isomorphic to any composition factor of $\frac{Au_2}{Au_1 \cap Au_2}$, $\overline{Au_2}$ is not isomorphic to any composition factor of $\frac{Au_1}{Au_1 \cap Au_2}$ and $\frac{N^{\rho}u_1}{N^{\rho+1}u_1} \simeq \frac{N^{\mu}u_2}{N^{\mu+1}u_2}$ $(N^{\rho+1}u_1 \supset Au_1 \cap Au_1$ and $N^{\mu+1}u_2 \supseteq Au_1 \cap Au_2)$ $(\rho \geqq 1,\ \mu \geqq 1)$ then $\frac{N^{\rho}u_1}{N^{\rho+1}u_1}$ is isomorphic to a vertice component since we may assume that $\frac{N^{\rho-1}u_1}{N^{\rho}u_1} \not\cong \frac{N^{\mu-1}u_2}{N^{\mu}u_2}$.

Next if $\overline{Au_1} \simeq \frac{N^{\mu}u_2}{N^{\mu+1}u_2}$ $(N^{\mu+1}u_2 \supset Au_1 \cap Au_2)$ and $\overline{Au_1}$ is not isomorphic to any composition factor of $\frac{Au_2}{N^{\mu}u_2}$ then $\frac{N^{\mu-1}u_2}{N^{\mu}u_2} \cong \overline{Ae}$. If $\frac{N^{\mu-1}u_2}{N^{\mu}u_2} \simeq \overline{Ae'}$

$(e \neq e')$ then $\frac{N^{\mu}u_2}{N^{\mu+1}u_2}$ is isomorphic to a vertice component since $\overline{Au_1} \cong \frac{N^{\mu}u_2}{N^{\mu+1}u_2}$. But this contradicts the condition 2. Thus $\frac{N^{\mu-1}u_2}{N^{\mu}u_2} \cong \overline{Ae}$. Now we put $N^{\mu-1}u_2 = Aw$. Then $Nw = Au_1w$ or Au_2w since $\frac{Au_2}{Au_1 \cap Au_2}$ is uniserial. If $Nw = Au_2w$ then $\overline{Au_2w} \cong \overline{Au_2}$ and $\overline{Au_1} \not\cong \overline{Au_2w}$. But this contradicts the assumption that $\overline{Au_1} \cong \frac{N^{\mu}u_2}{N^{\mu+1}u_2}$. Thus $Nw = Au_1w$.

Therefore if there does not exist an integer μ or ρ such that $N^{\mu}u_2 = Au_1w$ or $N^{\rho}u_1 = Au_2w'$ then $\frac{Au_1}{Au_1 \cap Au_2}$ and $\frac{Au_2}{Au_1 \cap Au_2}$ have no composition factor isomorphic to each other.

(2. 7. 2) If there exists an integer ρ or μ such that $N^{\rho}u_1 = Au_2w$ or $N^{\mu}u_2 = Au_1w'$ then there exists a left subideal $\mathfrak{p}$ of Ne such that $s\left(\frac{Ne}{\mathfrak{p}}\right)$ is the direct sum of two simple components isomorphic to each other.

Conversely if there does not exists ρ or μ such that $N^{\rho}u_1 = Au_2w$ or $N^{\mu}u_2 = Au_1w'$ then $\frac{Au_1}{Au_1 \cap Au_2}$ and $\frac{Au_2}{Au_1 \cap Au_2}$ have no composition factor isomorphic to each other.

(α) Assume that Au_1 and Au_2 are uniserial. Then an arbitrary left ideal $\mathfrak{p}$ of Ne is $N^{\nu}u_2 + N^{\mu}u_2$. Hence $s\left(\frac{Ne}{\mathfrak{p}}\right)$ is the direct sum of two simple component not isomorphic to each other.

(β) Assume that $Nu_1 = Aw_1 + Aw_2$ and $Aw_2 = N^{\mu}u_2$. Then by the condition 3 $Aw_1 \cap Aw_2 = Nw_1 = Nw_2$. Hence an arbitrary left ideal $\mathfrak{p}$ of Ne is $N^{\nu}u_1 + N^{\mu}u_2$. Hence $s\left(\frac{Ne}{\mathfrak{p}}\right)$ is the direct sum of two simple components not isomorphic to each other.

Thus we proved that if A is of 2-cyclic representation type then five conditions of §1 hold.

§3. In this chapter we shall show that if A satisfies five conditions in §1 then A is of 2-cyclic representation type.

First if A satisfies five conditions in §1 then the following results are proved to be true in the same way as in §2.

(a) If $Ne = Au_1 + Au_2$ then $\frac{Au_1}{Au_1 \cap Au_2}$ $(i = 1, 2)$ are uniserial. (This is the corollary 1 and a consequence of the condition 1.)

(b) $\frac{N^i e}{N^{i+1} e}$ is the direct sum of at most two simple components not isomorphic to each other.
(This is a consequence of the condition 1.)

(c) If $s\left(\frac{Aw_1}{Aw_1 \cap Aw_2}\right) \not\cong s\left(\frac{Aw_2}{Aw_1 \cap Aw_2}\right)$ where Aw_i $(i=1,2)$ are uniserial subideals in Ne then $Aw_1 \cap Aw_2$ is uniserial.
(This is a consequence of the condition (4. i).[7])

(d) Assume that $s\left(\frac{Ae_1}{\mathfrak{p}_1}\right) \supset \widetilde{Au_1}$, $s\left(\frac{Ae_2}{\mathfrak{p}_2}\right) \supset \widetilde{Au_2}$ and $\widetilde{Au_1} \cong \widetilde{Au_2}$. If $\widetilde{Au_i} \subset \widetilde{Nw_i}$, $\not\subset \widetilde{N^2 w_i}$ $(i=1,2)$ and this isomorphism $\widetilde{Au_1} \cong \widetilde{Au_2}$ cannot be extended to any homomorphism of $\widetilde{Aw_1}$ into $\widetilde{Aw_2}$ and of $\widetilde{Aw_2}$ into $\widetilde{Aw_1}$ then $\frac{\widetilde{Aw_1}}{\widetilde{Nw_1}} \not\cong \frac{\widetilde{Aw_2}}{\widetilde{Nw_2}}$ and $\widetilde{Au_i}$ is isomorphic to a vertice component.
(This is the lemma 10 and a consequence of the condition 1.)

(e) The condition 3 is equivalent to the lemma 15.
(The proof is as same as [2. 6].)

(f) The condition (4. ii. α) is equivalent to the first half of the lemma 14.
(The proof is as same as [2. 5.])

(g) If $\left\{\frac{N^{j_1} e_1}{N^{j_1+1} e_1}, \cdots, \frac{N^{j_r} e_r}{N^{j_r+1} e_r}\right\}$ is a chain then $r=2$.
(The proof is as same as the lemma 11 and this is the consequence of the condition 1 and 2.)

(h) If $\left\{\frac{N^{j_1} e_1}{N^{j_1+1} e_1}, \frac{N^{j_2} e_2}{N^{j_2+1} e_2}\right\}$ is a chain then there does not exist Ae_3 such that $\left\{\frac{N^{i_2} e_2}{N^{i_2+1} e_2}, \frac{N^{i_3} e_3}{N^{i_3+1} e_3}\right\}$ is a chain and at least one of $\frac{Ae_i}{N^{j_i+1} e_i}$ $(i=1,2)$ is uniserial.
(The proof is as same as the lemma 13 and this is the consequence of the condition 1 and 2).

(i) Assume that $\widetilde{Au} \subset s\left(\frac{Ae}{\mathfrak{p}}\right)$, $\widetilde{Au'} \subset s\left(\frac{Ae'}{\mathfrak{p}'}\right)$ and $\widetilde{Au} \cong \widetilde{Au'}$ where $s\left(\frac{Ae}{\mathfrak{p}}\right)$ and $s\left(\frac{Ae'}{\mathfrak{p}'}\right)$ are simple. If this isomorphism $\widetilde{Au} \cong \widetilde{Au'}$ cannot be extended to any homomorphism of $\widetilde{Aw}$ $(\supset \widetilde{Au})$ into $\widetilde{Aw'}$ $(\supset \widetilde{Au'})$ and of $\widetilde{Aw'}$ into $\widetilde{Aw}$ then it is not true that there exist Ax, $Ay \subseteq Ae$ and

7) cf. Lemma 6 or Corollary 3.

Ax', $Ay'\subseteq Ae'$ such that $Ax\cap Ay=Au$ and $Ax'\cap Ay'=Au'$. If $Au'\subset Nv'$ and $\not\subset N^2v'$ and there exist Ax and Ay such that $Ax\cap Ay=Au$ and $s\left(\frac{Ax}{Au}\right)\not\cong s\left(\frac{Ay}{Au}\right)$ then by (g) or (h) $\overline{Av'}\cong s\left(\frac{Ax}{Au}\right)$ or $\overline{Av'}\cong s\left(\frac{Ay}{Au}\right)$. But this contradicts the assumption that the isomorphism $Au\cong Au'$ is not extended to any homomorphism of $Aw(\supset Au)$ into $Aw'(\supset Au')$ and of Aw' into Aw.

Now assume that $s\left(\frac{Ax}{Au}\right)\cong s\left(\frac{Ay}{Au}\right)$. Then there exists Ae' such that $Ne'=Au_1+Au_2$, $Au_1\cap Au_2=As$, $s\left(\frac{Au_1}{As}\right)\cong s\left(\frac{Ax}{Au}\right)$ and $s\left(\frac{Au_2}{As}\right)\cong s\left(\frac{Ay}{Au}\right)$. Hence by the condition 1 $N^\rho u_2=Au_1v$ where $N^{\rho-1}u_2=Av$ and $Au_1\cap Au_2=As=Asv$ since $\frac{Au_1}{As}\cong\frac{N^\rho u_2}{As}$. If $vv\neq 0$ then $N^{\rho-1}u_2v\neq 0$ since $Av=N^{\rho-1}u_2$. But $u_2v=0$ since $N^\rho u_2=Au_1v$ and $Ae'\sim N^{\rho-1}u_2$. Hence $N^{\rho-1}u_2v=0$ and $v^2=0$. Thus $Asv=Asv^2=0$ and $As=0$. Therefore $Au_1\cap Au_2=0$.

Next we shall consider indecomposable modules which are the sum of at most two cyclic modules

[3. 1] First Aem has one of the following structures:

(3. 1. 1) Assume that $\frac{Nem}{N^2em}$ is simple.

(i) Nem is uniserial.

(ii) If $N^\rho em=Au_1m+Au_2m$ ($\rho\geqq 1$) then by the condition 3, $N^{\rho+1}em=Nu_1m=Nu_2m$ and by the condition 1 $\overline{Au_1m}\not\cong\overline{Au_2m}$. Hence by (c) $Au_1m\cap Au_2m$ is uniserial.

(3. 1. 2) Assume that $Nem=Au_1m+Au_2m$. Then similarly as above if $Au_1m\cap Au_2m\neq 0$ then $s\left(\frac{Au_1m}{Au_1m\cap Au_2m}\right)\not\cong s\left(\frac{Au_2m}{Au_1m\cap Au_2m}\right)$ and $Au_1m\cap Au_2m$ is uniserial.

(i) Au_im $(i=1, 2)$ are uniserial.

(ii) $Nu_1m=Av_1u_1m+Av_2u_1m$, Au_2m is uniserial, $Au_2m\supset Av_2u_1m$ and $N^2u_1m=Nv_1u_1m=Nv_2u_1m$. Hence we put $N^\mu u_2m=Av_2u_1m$. Now assume that $N^\rho u_1m=Av_1u_1m+Av_2u_1m$. If $Au_2m\supset N^\rho u_1m$ then this contradicts (c) (accordingly the condition (4. i)) since $s\left(\frac{Au_1m}{Au_1m\cap Au_2m}\right)\not\cong s\left(\frac{Au_2m}{Au_1m\cap Au_2m}\right)$. Hence we may assume that $Av_2u_1m\subset Au_2m$ and $Av_1u_1m\not\subset Au_2m$.

Next assume that $\overline{Au_1m}\cong\overline{Ae'}$ and $\overline{Au_2m}\cong\overline{Ae''}$. Then there exists an integer μ such that $\left\{\frac{N^\rho e'}{N^{\rho+1}e'}, \frac{N^\mu e''}{N^{\mu+1}e''}\right\}$ is a chain. Hence by (h) $\frac{Ne''}{N^{\mu+1}e''}$ is uniserial since $\frac{Ne'}{N^{\rho+1}e'}$ is not uniserial. Thus $Av_2u_1m=N^\mu u_2m$.

Moreover assume that $\rho \gneqq 1$ ($\rho=2$). Now if we put $\frac{N^{\mu-1}u_2m}{N^{\mu}u_2m} \cong \overline{Ae'''}$ then $\overline{Ae'''}$ is not isomorphic to any composition factor of Au_1m from the assumption and $\left\{\frac{N^2e'}{N^3e'}, \frac{Ne'''}{N^2e'''}\right\}$ is a chain. But by (f) this contradicts the condition (4. ii. α) since $\frac{N^2e'}{N^3e'}$ is not simple. Thus $\rho=1$.

Lastly by the condition 3 $N^2u_1m=Nv_1u_1m=Nv_2u_1m=N^{\mu+1}u_2m$.

[3. 2] Assume that $\mathfrak{m}=Ae_1m_1+Ae_2m_2$ is directly indecomposable and take m_1 and m_2 such that $l(Ae_1m_1)+l(Ae_2m_2)$ is minimal where $l(Ae_im_i)$ is the length of composition series of Ae_im_i. Then $Ae_1m_1 \cap Ae_2m_2 \neq 0$ and there exist Au_1m_1 and Au_2m_2 such that $s(Ae_1m_1) \supset Au_1m_1$, $s(Ae_2m_2) \supset Au_2m_2$ and $Au_1m_1=Au_2m_2$ where $u_1m_1=\alpha u_2m_2$ ($\alpha \in K$)).

(3. 2. 1) Assume that $s(Ae_im_i)$ $(i=1, 2)$ are simple. If there exists a homomorphism of Ae_1m_1 into Ae_2m_2 which is the extension of the isomorphism of $Au_1m_1 \cong Au_2m_2$ then there exists $v \in Ne_2$ such that $u_2m_2=\beta u_1vm_2$ ($\beta \in K$). Now if we take $n_1=m_1-\alpha\beta vm_2$ nstead of m_1 then $Au_1n_1=0$. But this contradicts the assumption on l. Similarly there does not exist a homomorphism of Ae_2m_2 into Ae_1m_1 which is the extension of the isomorphism $Au_1m_1 \cong Au_2m_2$. Hence by (d) Ne_1m_1 and Ne_2m_2 have composition factors isomorphic to vertice components and by (h) we may assume that Ae_2m_2 is uniserial.

(i) Assume that Ae_1m_1 is uniserial.

Then by the condition 3 (accordingly the lemma 15) $\frac{Ne_1m_1}{N^2e_1m_1} \left(\cong \frac{Ne_2m_2}{N^2e_2m_2}\right)$ is isomorphic to a vertice component or if $\frac{N^{\rho}e_1m_1}{N^{\rho+1}e_1m_1}$ ($\rho \gneqq 1$) $\left(\text{or } \frac{N^{\mu}e_2m_2}{N^{\mu+1}e_2m_2} (\mu \gneqq 1)\right)$ is isomorphic to a vertice component then $N^{\rho+1}e_1m_1=0$ (or $N^{\mu+1}e_2m_2=0$). Hence $Ae_1m_1 \cap Ae_2m_2=N^{\varphi}e_1m_1=N^{\varphi}e_2m_2$ where $Ne_1m_1 \cong Ne_2m_2$ or $Ae_1m_1 \cap Ae_2m_2=N^{\rho}e_1m_1=N^{\mu}e_2m_2$ where $\frac{N^{\rho-1}e_1m_1}{N^{\rho}e_1m_1} \not\cong \frac{N^{\mu-1}e_2m_2}{N^{\mu}e_2m_2}$ and $N^{\rho+1}e_1m_1 =N^{\mu+1}e_2m_2=0$. In the first case if we put $N^{\varphi-1}e_1m_1=Au_1'm_1$ and $N^{\varphi-1}e_2m_2 =Au_2'm_2$ then $N(u_1'm_1-u_2'm_2)=0$ since $Nu_1'm_1=Nu_2'm_2$.

(ii) Assume that $Ne_1m_1=Au_1'm_1+Au_2'm_1$ where $Au_i'm_1$ $(i=1, 2)$ are uniserial and $\frac{Ne_2m_2}{N^2e_2m_2} \cong \frac{N^{\nu}u_2'm_1}{N^{\nu+1}u_2'm_1}$ ($\nu \geqq 0$) or $\frac{N^{\varphi}e_2m_2}{N^{\varphi+1}e_2m_2} \cong \frac{Au_2'm_1}{Nu_2'm_1}$. Now $Au_1'm_1 \cap Au_2'm_1 \neq 0$ since $s(Ne_1m_1)$ is assumed to be simple. Moreover by the same way as (i) $s\left(\frac{Au_1'm_1}{Au_1'm_1 \cap Au_2'm_1}\right) \cong s\left(\frac{Au_2'm_1}{Au_1'm_1 \cap Au_2'm_1}\right)$. Hence if we put $Au_1'm_1 \cap Au_2'm_1 = N^\rho u_2'm_1$ then $\frac{N^\rho u_2'm_1}{N^{\rho+1}u_2'm_1}$ is isomorphic to a vertice component and similarly as (i) $N^{\rho+1}u_2'm_1=0$.

Next if $Ne_2m_2 \cong Au_2'm_1$ then by the condition (4. ii. α) $Au_1'm_1 \cap Au_2'm_1 = Nu_2'm_1$ since $e_1 \neq e_2$. If $N^\varphi e_2m_2 \cong Au_2'm_1$ ($\varphi \gneqq 1$) then $N^{\varphi+1}e_2m_2 = Nu_2'm_1 = 0$. If $Ne_2m_2 \cong N^{\rho'}u_2'm_1$ ($\rho' \gneqq 1$) and $N^{\rho'}u_2'm_1 \supsetneqq Au_1'm_1 \cap Au_2'm_1$ then this contradicts (h) since if we put $\frac{N^{\rho'-1}u_2'm_1}{N^{\rho'}u_2'm_1} \cong \overline{Ae'}$ then $e' \neq e_2$ and $\frac{Ne'}{\mathfrak{p}'} \cong Ne_2m_2$ ($\mathfrak{p}'$ is a subideal in Ne') and $\frac{Ne'}{\mathfrak{p}'}$ has two composition factor isomorphic to vertice components since $\frac{N^{\rho'}u_2'm_1}{N^{\rho'+1}u_2'm_1}$ and $\frac{N^\rho u_2'm_1}{N^{\rho+1}u_2'm_1}$ are isomorphic to vertice components.

If $Ne_2m_2 \cong N^\rho u_2'm_1$ and we assume that $s\left(\frac{Au_1'm_1}{Au_1'm_1 \cap Au_2'm_1}\right) \cong \overline{Ae''}$ and $s\left(\frac{Au_2'm_1}{Au_1'm_1 \cap Au_2'm_1}\right) \cong \overline{Ae'}$ then $\left\{\frac{Ne'}{N^2e'}, \frac{Ne''}{N^2e''}, \frac{Ne_2}{N^2e_2}\right\}$ is a chain but this contradicts (g). Thus in this case by the same way as (i) if $Au_2'm_1 \cong N^\varphi e_2m_2$ ($\varphi \gneqq 1$) then $Ae_1m_1 \cap Ae_2m_2 = Au_2'm_1$, $Nu_2'm_1 = N^{\varphi+1}e_2m_2 = 0$ and if $Au_2'm_1 \cong Ne_2m_2$ then $Ae_1m_1 \cap Ae_2m_2 = Au_2'm_1 = Ne_2m_2$ or $Ae_1m_1 \cap Ae_2m_2 = Nu_2'm_1 = N^2e_2m_2$ and $N^2u_2'm_1 = N^3e_2m_2 = 0$.

(iii) Assume that $Ne_1m_1 = Au_1'm_1 + Au_2'm_1$. If $N^\rho u_1'm_1 = Av_1u_1'm_1 + Av_2u_1'm_1$ then similarly as (3. 1. 2, ii) we can see that $\rho=1$, $Av_2u_1'm_1 = N^\mu u_2'm_1$ and $Av_1u_1'm_1 \cap Av_2u_1'm_1 = Nv_1u_1'm_1 = Nv_2u_1'm_1 = N^{\mu+1}u_2'm_1$. Hence by the condition (4. ii. β) $\frac{Ne_2m_2}{N^3e_2m_2} \cong \frac{Au_1'm_1}{Av_1u_1'm_1 + Nv_2u_1'm_1}$ and $N^3e_2m_2=0$ since $Av_1u_1'm_1 \cap Av_2u_1'm_1 = Nv_1u_1'm_1 = Nv_2u_1'm_1$.

(iv) Assume that $N^\rho e_1m_1 = Au_1'm_1 + Au_2'm_1$ ($\rho \gneqq 1$). Then by the condition 3 $Au_1'm_1 \cap Au_2'm_1 = Nu_1'm_1 = Nu_2'm_1$. In this case $Ne_2m_2 \cong Au_2'm_1$ or by the condition (4. ii. α) $\frac{Ne_1m_1}{Au_1'm_1} \cong Ne_2m_2$. If $Ne_2m_2 \cong Au_2'm_1$ then $N^3e_2m_2 = 0$ since $\frac{N^2e_2m_2}{N^3e_2m_2}$ is isomorphic to a vertice component. If $\frac{Ne_1m_1}{Au_1'm_1} \cong Ne_2m_2$ then $N^{\rho+1}e_2m_2=0$.

(3. 2. 2) Assume that $s(Ae_2m_2) = Au_2m_2$ and $s(Ae_1m_1) = Av_1m_1 \oplus Au_1m_1$.

(i) Assume that there exists a homomorphism of Ae_2m_2 into Ae_1m_1 which is the extension of the isomorphism $Au_1m_1 \cong Au_2m_2$.

If Ae_1m_1 is homomorphic onto Ae_2m_2 then $u_2=u_1$ and if we take $n_1 = m_1 - \alpha m_2$ instead of m_1 then $Au_1n_1=0$ and this contradicts the assumption on l. Similarly as this if there exists a homomorphism of Ae_1m_1 into Ae_2m_2 which is the extension of the isomorphism $Au_1m_1 \cong Au_2m_2$

then this is a contradiction. If $\frac{Ne_1m_1}{N^2e_1m_1}$ is simple then by the condition 3 $s(Ne_1m_1)=N^\rho e_1m_1=Av_1m_1\oplus Au_1m_1$, $\frac{Ne_1m_1}{N^\rho e_1m_1}$ is uniserial and $Ae_2m_2\cong\frac{N^\mu e_1m_1}{Av_1m_1}$ $(\mu\leqq\rho)$. Hence there exists a left subideal $\mathfrak{p}_2$ in Ne_2 such that $\frac{Ae_2}{\mathfrak{p}_2}$ $\cong N^\mu e_1m_1$. Then we can assume that $s\left(\widetilde{\frac{Ae_2}{\mathfrak{p}_2}}\right)=\widetilde{Av_2}\oplus\widetilde{Au_2}$ and $\frac{\widetilde{Ae_2}}{s(\widetilde{Ae_2})}$ is uniserial where $\widetilde{Ae_2}=\frac{Ae_2}{\mathfrak{p}_2}$.

Now by the assumption $v_2m_2=0$ and there exists $w\in Ne_1$ such that $v_1=\gamma v_2w$ and $u_1=\delta u_2w$ $(\gamma,\delta\in K)$. Thus in this case we can see that $\mathfrak{m}$ is directly indecomposable.

Now suppose that $\mathfrak{m}=Ae_1n_1\oplus Ae_2n_2$. Then $n_i=\alpha_{i1}m_1+\alpha_{i2}m_2$ where $\alpha_{ii}\in e_iAe_i$, $\notin e_iNe_i$ and $\alpha_{ij}\ni N$ $(i\neq j)$ since $e_1\neq e_2$ similarly as the lemma 14.

First if $u_1n_1=0$ then $u_1\alpha_{11}m_1+u_1\alpha_{12}m_2=0$. But $u_1r_{11}m_1\in N^{\rho+1}e_1m_1=0$ for $r_{11}\in e_1Ne_1$ and $u_1\alpha_{12}m_2\in N^\rho e_2m_2=0$ since $N^\rho e_2m_2=0$. Thus $a_{11}u_1m_1=0$ $(a_{11}\in K)$ but this is a contradiction. Therefore $u_1n_1\neq0$. Similarly as this $v_1n_1\neq0$.

Next we shall show that $u_2n_2\neq0$ or $v_2n_2\neq0$. Now suppose that $u_2n_2=0$ and $v_2n_2=0$. Then $v_2\alpha_{21}m_1+v_2\alpha_{22}m_2=0$ but $v_2\alpha_2m_2=0$. Hence $v_2\alpha_{21}m_1=0$. Thus $u_2\alpha_{21}m_1=0$ and $u_2\alpha_{22}m_2=0$ since $u_2n_2=0$. But this is a contradiction. Therefore $u_2n_2\neq0$ or $v_2n_2\neq0$ and if we consider about the length of the composition series it is a contradiction that $Ae_1n_1\cap Ae_2n_2$ $=0$. Thus $\mathfrak{m}$ is directly indecomposable.

Next assume that $Ne_1m_1=Aw_1m_1\oplus Aw_2m_1$ and $Au_1m_1\subsetneq Aw_1m_1$. If $Ae_2m_2\cong N^\mu w_1m_1$ then there exists $v\in Aw_1$ such that $u_1=u_2v$. Hence if we take $n_2=\alpha m_2-vm_1$ instead of m_2 then $u_2n_2=0$ and the length of Ae_2n_2 is smaller than that of Ae_2m_2 since Aw_1m_1 is uniserial, and this is a contradiction.

Lastly assume that $Ne_1m_1=Aw_1m_1+Aw_2m_1$ and $Aw_1m_1\cap Aw_2m\neq0$. Then by the same way as (3. 2. 1) $s(Aw_1m_1)=Nw_1m_1=Av_1m_1\oplus Au_1m_1$, Au_1m_1 $=Aw_1m_1\cap Aw_2m_1$ and Aw_2m_1 is uniserial.

If $Ae_2m_2\cong\frac{Aw_1m_1}{Av_1m_1}$ then by the same way as above $\mathfrak{m}=Ae_1m_1+$ Ae_2m_2 is directly indecomposable but if $Ae_2m_2\cong N^\nu w_2m_1$ then $\mathfrak{m}$ is directly decomposable similarly as above.

(ii) Assume that there does not exist any homomorphism of Ae_1m_1 into Ae_2m_2 and of Ae_2m_2 into Ae_1m_1 which is the extension of the isomorphism $Au_1m_1\cong Au_2m_2$. Then by the same way as (3. 2. 1) Ae_2m_2 is uniserial and Ae_1m_1 has one of the following types:

(a) $s(Ae_1m_1)=N^{\rho}e_1m_1=Av_1m_1 \oplus Au_1m_1$ and $\dfrac{Ae_1m_1}{N^{\rho}e_1m_1}$ is uniserial.

(b) $Ne_1m_1=Aw_1m_1 \oplus Aw_2m_1$ where $Aw_1m_1 \supset Au_1m_1$.

(c) $Ne_1m_1 = Aw_1m_1+Aw_2m_1$, $Nw_1m_1 = Au_1m_1+Av_1m_1$, $Aw_2m_1 \supset Au_1m_1$ and Aw_2m_1 is uniserial.

In the case (a) by the condition (4. ii. α) $\dfrac{Ne_1m_1}{Av_1m_1} \cong Ne_2m_2$ and $Ae_1m_1 \cap Ae_2m_2=Au_1m_1=Au_2m_2$ and $s(Ae_1m_1+Ae_2m_2)=Av_1m_1 \oplus Au_1m_1$.

In the case (b) if $\dfrac{N^{\mu}e_2m_2}{N^{\mu+1}e_2m_2}$ $(\mu \gneqq 1)$ or $\dfrac{N^{\nu}w_1m_1}{N^{\nu+1}w_1m_1}$ $\nu \gneqq 1)$ is isomorphic to a vertice component then $N^{\mu+1}e_2m_2=0$ or $N^{\nu+1}w_1m_1=0$. Thus unless $Aw_1m_1 \cong Ne_2m_2$ then $Ae_1m_1 \cap Ae_2m_2=Au_1m_1=Au_2m_2$ is isomorphic to a vertice component.
If $Aw_1m_1 \cong Ne_2m_2$ then $Ae_1m_1 \cap Ae_2m_2=N^{\varphi}w_1m_1=N^{\varphi+1}e_2m_2$ and if we put $N^{\varphi-1}w_1m_1=Au_1'm_1$ and $N^{\varphi}e_2m_2=Au_2'm_2$ $(\varphi \gneqq 1)$ then $N(u_1'm_1-\xi u_2'm_2)=0$.

In the case (c) $Ne_2m_2 \cong \dfrac{Aw_1m_1}{Av_1m_1}$ and $N^2w_1m_1=0$. Hence $Ae_1m_1 \cap Ae_2m_2 =Au_1m_1=Au_2m_2$ and $s(Ae_1m_1+Ae_2m_2)=Av_1m_1 \oplus Au_1m_1$.

(3. 2. 3) Assume that $s(Ae_1m_1)=Av_1m_1 \oplus Au_1m_1$ and $s(Ae_2m_2)=Av_2m_2 \oplus Au_2m_2$ and $Au_1m_1=Au_2m_2$. If there does not exist any homomorphism of Ae_1m_1 into Ae_2m_2 and of Ae_2m_2 into Ae_1m_1 which is the extension of the isomorphism $Au_1m_1 \cong Au_2m_2$ then this contradicts the condition (4. i). Hence there exists a homomorphism of Ae_1m_1 into Ae_2m_2 (or of Ae_2m_2 into Ae_1m_1) which is the extension of the isomorphism $Au_1m_1 \cong Au_2m_2$. Therefore there exists $v \in Ne_2$ (or $\in Ne_1$) such that $u_2=u_1v$ (or $u_1=u_2v$). Then if we take $n_1=m_1-\alpha v m_2$ instead of m_1 (or $n_2=\alpha m_2-vm_1$ instead of m_2) then $u_1n_1=0$ (or $u_2n_2=0$) and this contradicts the assumption on l.

[3. 3] Assume that $\mathfrak{m}=\sum_{i=1}^{\lambda}\sum_{j=1}^{s_i} Ae_im_{ij}$ is directly indecomposable and $\sum_{i=1}^{\lambda} s_i=s \geqq 3$.

Now if l_{ij} is the length of the composition series of Ae_im_{ij} then we assume that $\sum_{i,j} l_{ij}=l$ is minimal and we put $\mathfrak{m}=Ae_{\lambda}m_{\lambda,s_\lambda}+\mathfrak{m}'$ where $\mathfrak{m}'$ is the sum of $s-1$ cyclic A-left modules Ae_im_{ij} $(\neq Ae_{\lambda}m_{\lambda,s_\lambda})$ and it is the direct sum of p directly indecomposable modules which are shown in (3. 2) since $\sum l_{ij}=l$ is minimal.[8]

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8) If $\mathfrak{M}'=\sum Ae_im_{ij}$ is the direct sum of directly indecomposable modules shown in (3. 2) and we put $n_{ij}=m_{ij}+\sum r_{\xi\eta}m_{\xi\eta}$ then the length of Ae_in_{ij} is larger than that of Ae_im_{ij}.

(3. 3. 1) We assume that $s(Ae_\lambda m_{\lambda,s_\lambda})$ is simple and put $s(Ae_\lambda m_{\lambda,s_\lambda})$

$$= Au_{\lambda,s_\lambda,\alpha}m_{\lambda,s_\lambda} \text{ where } e_\alpha u_{ij\alpha} = u_{ij\alpha}.$$

Then
$$u_{\lambda,s_\lambda\alpha}m_{\lambda,s_\lambda} = \sum_{(i,j)\neq(\lambda,s_\lambda)} a_{ij}u_{ij\alpha}m_{ij} \qquad (a_{ij}\in k) \tag{I}$$

since $Ae_\lambda m_{\lambda,s_\lambda}\cap \mathfrak{m}'\neq 0$ and we may assume that the number of $u_{ij\alpha}m_{ij}$ of (I) is minimal. Now if $Ae_g m_{gh}+Ae_{g'}m_{g'h'}$ is a direct summand of $\mathfrak{m}'$ and $a_{gh}u_{gh\alpha}m_{gh}$ and $a_{g'h'}u_{g'h'\alpha}m_{g'h'}$ do not appear in (I) then this is a contradiction since $Ae_\lambda m_{\lambda,s_\lambda}\cap\mathfrak{m}'\neq 0$ and $Ae_g m_{gh}+Ae_{g'}m_{g'h'}$ is a direct summand of $\mathfrak{m}'$.

(a) If $Au_{ij\alpha}m_{ij}\not\subset s(Ae_i m_{ij})$ then there exists $Ae_{i'}m_{i'j'}$ such that $Ae_i m_{ij}+Ae_{i'}m_{i'j'}$ is directly indecomposable and $Ae_i m_{ij}\cap Ae_{i'}m_{i'j'}=Nu_{ij\alpha}m_{ij}=Nu_{i'j'\alpha}m_{i'j'}$ where $N(a_{ij}u_{ij\alpha}m_{ij}-a_{i'j'}u_{i'j'\alpha}m_{i'j'})=0$. Hence by (3. 2) $Ne_i m_{ij}\cong Ne_{i'}m_{i'j'}$ or $Au_{ij}m_{ij}\cong N_{i'}m_{i'j'}$ where $Ne_i m_{ij}=Au_{ij}m_{ij}+Av_{ij}m_{ij}$, $Au_{ij}m_{ij}$ is uniserial and $Au_{ij}m_{ij}\supset Au_{ij\alpha}m_{ij}$. If there exists $v_{ij\lambda}\in Ne_i$ such that $u_{ij\alpha}=u_{\lambda,s_\lambda,\alpha}v_{ij\lambda}$ then there exists $v_{i'j'\lambda}\in Ne_{i'}$ such that $u_{i'j'\alpha}=u_{\lambda,s_\lambda,\alpha}v_{i'j'\lambda}$. Hence if we take $n_{\lambda,s_\lambda}=m_{\lambda,s_\lambda}-a_{ij}v_{ij\lambda}m_{ij}-a_{i'j'}v_{i'j'\lambda}m_{i'j'}$ instead of m_{λ,s_λ} then $u_{\lambda,s_\lambda,\alpha}n_{\lambda,s_\lambda}=\sum_{(\xi,\eta)\neq}\sum_{\substack{(ij)\\(i'j')}} a_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}$ and $s(Ae_i m_{ij}+Ae_{i'}m_{i'j'})\cap$

$(Ae_\lambda n_{\lambda,s_\lambda}+\sum_{(\xi,\eta)\neq}\sum_{\substack{(ij)\\(i'j')}} Ae_\xi m_{\xi\eta})=0$. If $u_{ij\alpha}m_{ij}+\gamma u_{i'j'\alpha}m_{i'j'}=b_{\lambda,s_\lambda}u_{\lambda,s_\lambda\alpha}n_{\lambda,s_\lambda}$

$+\sum_{(\xi,\eta)\neq}\sum_{\substack{(ij)\\(i'j')}} b_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}$ then $u_{ij\alpha}m_{ij}+\gamma u_{i'j'\alpha}m_{i'j'}=b_{\lambda,s_\lambda}u_{\lambda,s_\lambda\alpha}a_{ij}u_{ij\alpha}m_{ij}$

$-b_{\lambda,s_\lambda}a_{ij}u_{ij\alpha}m_{ij}-b_{\lambda,s_\lambda}a_{i'j'}u_{i'j'\alpha}m_{i'j'}+\sum b_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}$ and $b_{\lambda,s_\lambda}u_{\lambda,s_\lambda,\alpha}m_{\lambda,s_\lambda}=(b_{\lambda,s_\lambda}a_{ij}+1)u_{ij\alpha}m_{ij}+(b_{\lambda,s_\lambda}a_{i'j'}+\gamma)u_{i'j'\alpha}m_{i'j'}-\sum b_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}$. Hence $u_{\lambda,s_\lambda,\alpha}m_{\lambda,s_\lambda}=\left(\frac{b_{\lambda,s_\lambda}a_{ij}+1}{b_{\lambda,s_\lambda}}\right)u_{ij\alpha}m_{ij}+\frac{b_{\lambda,s_\lambda}a_{i'j'}+\gamma}{b_{\lambda,s_\lambda}}u_{i'j'\alpha}m_{i'j'}-\sum\frac{b_{\xi,\eta}}{b_{\lambda,s_\lambda}}u_{\xi\eta\alpha}m_{\xi\eta}$ and $\frac{b_{\lambda,s_\lambda}a_{ij}+1}{b_{\lambda,s_\lambda}}=a_{ij}$. Thus $a_{ij}+\frac{1}{b_{\lambda,s_\lambda}}=a_{ij}$ and $\frac{1}{b_{\lambda,s_\lambda}}=0$ but this is a contradiction.

Next if there exists $v_{\lambda,s_\lambda,i}\in Ne_\lambda$ such that $u_{\lambda,s_\lambda,\alpha}=u_{ij\alpha}v_{\lambda,s_\lambda,i}$ or $Ne_i m_{ij}=Au_{ij}m_{ij}+Av_{ij}m_{ij}$ $(i=\lambda)$, and we take $n_{ij}=a_{ij}m_{ij}-v_{\lambda,s_\lambda,i}m_{\lambda,s_\lambda}$ instead of m_{ij} then $Nu_{ij\alpha}n_{ij}=N(u_{ij\alpha}m_{ij}-u_{ij\alpha}v_{\lambda,s_\lambda,i}m_{\lambda,s_\lambda})=N(u_{ij\alpha}m_{ij}-u_{\lambda,s_\lambda,\alpha}m_{\lambda,s_\lambda})=Nu_{ij\alpha}m_{ij}=Nu_{i'j'\alpha}m_{i'j'}$. Thus $u_{ij\alpha}n_{ij}+\sum_{(\xi,\eta)\neq}\sum_{\substack{(ij)\\(\lambda,s_\lambda)}} a_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}=0$ and

$s(Ae_\lambda m_{\lambda s_\lambda})\cap(Ae_i n_{ij}+\sum_{(\xi,\eta)\neq}\sum_{\substack{(i,j)\\(\lambda,s_\lambda)}} Ae_\xi m_{\xi\eta})=0$ by the same way as above.

But this is a contradiction. Therefore $\mathfrak{m}$ is assumed not to have such a direct summand and we may assume that $Au_{ij\alpha}m_{ij}\subset s(Ae_i m_{ij})$ for each (i, j). Hence we can assume that $Nu_{ij\alpha}m_{ij}=0$ for each (i, j).

(b) Assume that there exists $v_{ij\lambda}\in Ne_i$ such that $u_{ij\alpha}=u_{\lambda,s_\lambda,\alpha}v_{ij\lambda}$ and

$s(Ae_i m_{ij})$ is simple. If we take $N_{\lambda,s_\lambda}=m_{\lambda,s_\lambda}-a_{ij}v_{ij\lambda}m_{ij}$ instead of m_{λ,s_λ} then $u_{\lambda,s_\lambda,\alpha}n_{\lambda,s_\lambda}=\sum_{(\xi,\eta)\neq}\sum_{(i,j)} a_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}$. But this is a contradiction since $s(Ae_i m_{ij})\cap(Ae_\lambda n_{\lambda,s_\lambda}+\sum_{(\xi,\eta)\neq}\sum_{(i,j)} Ae_\xi m_{\xi\eta})=0$ similarly as above.

Moreover if $Ne_i m_{ij}=Au_{ij}m_{ij}\oplus Av_{ij}m_{ij}$ then similarly as above we can see that this is a contradition.

Next if there exists $v_{\lambda,s_\lambda,i}\in Ne_\lambda$ such that $u_{\lambda,s_\lambda,\alpha}=u_{ij\alpha}=u_{ij\alpha}v_{\lambda,s_\lambda,i}$ and $Ae_i m_{ij}$ is a direct summand of $\mathfrak{m}'$ then similarly as above this is a contradiction.[9]

Thus we can assume that $\mathfrak{m}'$ is the direct sum of the following directly indecomposable modules.

(1) $Ae_s m_{st}+Ae_{s'}m_{s't'}$ where $Ae_s m_{st}\cap Ae_{s'}m_{s't'}=Au_{st\alpha}m_{st}=Au_{s't'\alpha}m_{s't'}$ and there does not exist $v_{\lambda,s_\lambda,s}\in Ne_\lambda$ such that $u_{\lambda s_\lambda\alpha}=u_{st\alpha}v_{\lambda,s_\lambda,s}$ for each $u_{st\alpha}$.

(2) $Ae_p m_{pq}$ where $s(Ae_p m_{pq})=Nw$, $s(Ae_p m_{pq})=Au_{pq\alpha}m_{pq}\oplus Au_{pq\beta}m_{pq}$ $(\alpha\neq\beta)$ and there exists $v_{pq\lambda}\in Ne_p$ such that $u_{pq\alpha}=u_{\lambda,s_\lambda,\alpha}v_{pq\lambda}$.

(3) $Ae_p m_{pq}+Ae_r m_{rs}$ where $Ae_p m_{pq}$ has the type (2), $Ae_p m_{pq}\cap Ae_r m_{rs}=Au_{p\beta\alpha}m_{pq}=Au_{rs\alpha}m_{rs}$ and there exists a homomorphism of $Ae_r m_{rs}$ into $Ae_p m_{pq}$ which is the extension of $Au_{pq\alpha}m_{pq}\cong Au_{rs\alpha}m_{rs}$.

(4) $Ae_{p'}m_{p'q'}$ where there exists $v_{p'q's'}\in Ne_{p'}$ such that $u_{p'q'\alpha}=u_{s't'\alpha}v_{p'q's'}$ for each $u_{s't'\alpha}$.

(i) Assume that $\mathfrak{m}'$ has a direct summand $Ae_i m_{ij}+Ae_{i'}m_{i'j'}$ where $Ae_i m_{ij}\cap Ae_{i'}m_{i'j'}=Au_{ij\alpha}m_{ij}=Au_{i'j'\alpha}m_{i'j'}$ and $Au_{ij\alpha}m_{ij}$ is isomorphic to a vertice component.[10] In this case by the condition (4. ii. α) if $Au_{ij\alpha}m_{ij}\subset N^2e_i m_{ij}$ then $s(Ae_i m_{ij})$ is simple. Now we say that this module is of type (1_a).

First assume that $\mathfrak{m}'$ is the direct sum of directly indecomposable modules of type (1_a). Then there exists $Ae_i m_{ij}$ such that $u_{\xi\eta\alpha}=u_{ij\alpha}v_{\xi\eta i}$ for each (ξ,η) $(v_{\xi\eta i}\in Ne_\xi)$ since there exists Ae_i such that it is homomorphic into $Ae_\xi m_{\xi\eta}$ for each (ξ,η). Hence if we take $n_{ij}=a_{ij}m_{ij}+\sum_{(\xi,\eta)\neq}\sum_{\substack{(i,j)\\(\lambda,s_\lambda)}} a_{\xi\eta}v_{\xi\eta i}m_{\xi\eta}-v_{\lambda s,\lambda,i}m_{\lambda,s_\lambda}$ instead of m_{ij} then $u_{ij\alpha}n_{ij}=0$ and this contradicts the assumption on l. Therefore we assume that $\mathfrak{m}'$ is the direct sum of directly indecomposable modules of the type (1_a) and (4).

If $\mathfrak{m}'$ has at least two direct summands of type (4), $Ae_{p'}m_{p'q'}$ and $Ae_{r'}m_{r's'}$, then from the assumption $Ae_{s'}m_{s't'}$ of each direct summand $Ae_s m_{st}+Ae_{s'}m_{s't'}$ of $\mathfrak{M}'$ is homomorphic to a submodule of $Ae_{p'}m_{p'q'}$ and

9) We have only to take $n_{ij}=a_{ij}m_{ij}-v_{\lambda,s_\lambda i}m_{\lambda,s_\lambda}$ instead of m_{ij}.

10) From this result we have $s\left(\frac{Ae_i m_{ij}}{Au_{ij\alpha}m_{ij}}\right)\cong s\left(\frac{Ae_{i'}m_{i'j'}}{Au_{i'j'\alpha}m_{j'j'}}\right)$.

$n_{pq}=a_{pq}m_{pq}+\sum_{\substack{(\xi,\eta)\neq(p,q)\\(\lambda,s_\lambda)}} a_{\xi\eta}v_{\xi\eta p}m_{\xi\eta}$ instead of m_{pq} then $u_{\lambda,s_\lambda\alpha}m_{\lambda,s_\lambda}=u_{pq\alpha}n_{pq}$ and $a_{pq}u_{r's'\alpha}m_{r's'}=a_{pq}u_{pq}m_{pq}=u_{pq\alpha}n_{pq}-\sum_{(\xi,\eta)\neq(p,q)} a_{\xi\eta}u_{\xi\alpha}m_{\xi\eta}$. Hence if we take $n_{r's'}=a_{pq}m_{r's'}-v_{\lambda,s_\lambda r'}m_{\lambda,s_\lambda}$ instead of $m_{r's'}$ then $n_{r's'\alpha}n_{r's'}=-\sum_{(\xi,\eta)\neq(p,q)} a_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}$ and $s(Ae_pm_{pq}+Ae_{r'}m_{r's'})\frown(Ae_{r'}n_{r's'}+\sum\sum(Ae_\xi m_{\xi\eta}+Ae_{\xi'}m_{\xi'\eta'}))=0$. But this is a contradiction.

If $\mathfrak{m}'$ is the direct sum of modules of the type (α_1) and (α_2) then by the same way as this we can see that this is a contradiction.

(3. 3. 2) Assume that $s(Ae_\lambda m_{\lambda,s_\lambda})=Au_{\lambda,s_\lambda\alpha}m_{\lambda,s_\lambda}\oplus Au_{\lambda,s_\lambda\beta}m_{\lambda,s_\lambda}$. If $Ne_\lambda m_{\lambda,s_\lambda}=Aw_{\lambda,s_\lambda}m_{\lambda,s_\lambda}\oplus Aw'_{\lambda,s_\lambda}m_{\lambda,s_\lambda}$ then similarly as (3. 3. 1) we can see that this is a contradiction.

Next assume that Ne_λ has the type (3. 1. 1, ii) or (3. 1. 2, ii). If there exists Ae_im_{ij} in $\mathfrak{m}$ such that $s(Ae_im_{ij})$ is simple then we have only to take Ae_im_{ij} instead of $Ae_\lambda m_{\lambda,s_\lambda}$.

Otherwise by the same way as (3. 3. 1) we can see that this is a contradiction.

Thus we have the main theorem.

Theorem. *A is of 2-cyclic representation type if and only if A satisfies five conditions in* § 1.

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$Ae_{r'}m_{r's'}$. Hence by (g) and (h) $u_{p'q'\alpha}=u_{r's'\alpha}v_{p'q'r'}$ (or $u_{r's'\alpha}=u_{p'q'\alpha}v_{r's'p'}$) since $Au_{s't'\alpha}m_{s't'}$ is isomorphic to a vertice component. Thus if we take $n_{r's'}=a_{r's'}\mathfrak{m}_{r's'}+a_{p'q'}v_{p'q'r'}m_{p'q'}$ instead of $m_{r's'}$ (or $n_{p'q'}=a_{p'q'}m_{p'q'}+a_{r's'}v_{r's'p'}m_{r's'}$ instead of $m_{p'q'}$) then $u_{\lambda s_\lambda \alpha}m_{\lambda s_\lambda}=a_{r's'}u_{r's'\alpha}n_{r's'}+\sum_{(\xi,\eta)\neq}\sum_{(r',s')}a_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}$ (or $u_{\lambda s_\lambda \alpha}m_{\lambda s_\lambda}=a_{p'q'}u_{p'q'\alpha}n_{p'q'}+\sum_{(\xi,\eta)\neq}\sum_{(p',q')}a_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}$) and this is a contradiction. Therefore we assume that $\mathfrak{m}'$ is the direct sum of directly indecomposable modules of the type (1_a), $Ae_\xi m_{\xi\eta}+Ae_{\xi'}m_{\xi'\eta'}$, and a directly indecomposable modules of the type (4) $Ae_{p'}m_{p'q'}$.

Now similarly as above there exists Ae_im_{ij} such that $Ae_im_{ij}+Ae_{i'}m_{i'j'}$ is a direct summand of $\mathfrak{m}'$ and $u_{\xi\eta\alpha}=u_{ij\alpha}v_{\xi\eta i}$ for each (ξ, η) $(v_{\xi\eta i}\in Ne_\xi)$ and if we take $n_{ij}=a_{ij}m_{ij}+\sum_{(\xi,\eta)\neq}\sum_{(p',q')}a_{\xi\eta}v_{\xi\eta\alpha}m_{\xi\eta}-v_{\lambda,s_\lambda\alpha}m_{\lambda,s_\lambda}$ instead of m_{ij} then $u_{ij\alpha}n_{ij}=u_{p'q'\alpha}m_{p'q'}$. Hence $a_{ij}u_{i'j'\alpha}m_{i'j'}=a_{ij}u_{ij\alpha}m_{ij}=u_{ij\alpha}n_{ij}-\sum_{(\xi,\eta)\neq}\sum_{(p',q')}a_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}+u_{\lambda,s_\lambda\alpha}m_{\lambda,s_\lambda}$ and from the assumption $u_{p'q'\alpha}=u_{i'j'\alpha}v_{p'q'i'}$ $(v_{p'q'i'}\in Ne_{p'})$. Therefore if we take $n_{i'j'}=a_{ij}m_{i'j'}-v_{p'q'i'}m_{p'q'}$ instead of $m_{i'j'}$ then $u_{i'j'\alpha}n_{i'j'}=u_{\lambda,s_\lambda\alpha}m_{\lambda,s_\lambda}-\sum_{(\xi,\eta)\neq}\sum_{(p',q')}a_{\xi\eta}u_{\xi\eta\alpha}m_{\xi\eta}$ and this is a contradiction.

Next if $\mathfrak{m}'$ has a direct summand of the type (2), Ae_pm_{pq}, and of the type (1_a), $Ae_im_{ij}+Ae_{i'}m_{i'j'}$, then by the condition (4. ii. α) $Ne_pm_{pq}=Au_{pq\alpha}m_{pq}\oplus Au_{pq\beta}m_{pq}$. But in this case $u_{ij\alpha}=u_{pq\alpha}v_{ijp}$ and this contradicts the assumption.

(ii) Assume that $\mathfrak{m}'$ has a direct summand $Ae_im_{ij}+Ae_{i'}m_{i'j'}$ where $Ne_im_{ij}\simeq Ne_{i'}m_{i'j'}$ or $\dfrac{Ne_im_{ij}}{Au_{ij\beta}m_{ij}}\simeq Ne_{i'}m_{i'j'}$ $(s(Ne_im_{ij})=Au_{ij\alpha}m_{ij}\oplus Au_{ij\beta}m_{ij})$. Moreover we may assume that $Au_{ij}m_{ij}\subset N^2e_im_{ij}$. We say that this modnle is of type (1_b). Therefore if $\mathfrak{m}'$ has at least two direct summands of the type (1_b) $Ae_im_{ij}+Ae_{i'}m_{i'j'}$ and $Ae_km_{kl}+Ae_{k'}m_{k'l'}$ then $i=k$ and $i'=k'$. Hence similarly as (i) we may assume that $\mathfrak{m}'$ has at most one direct summand of the type (1_b). In this case if $\mathfrak{m}'$ has a direct summand of the type (4) Ae_pm_{pq} then by the condition (4. ii. α) we can see that $p=\lambda=i$ but this contradicts the assumption.

(iii) Assume that $\mathfrak{m}'$ has a direct summand of type (3), $Ae_pm_{pq}+Ae_rm_{rs}$. Then similarly as (i) and (ii) $\mathfrak{m}'$ has no direct summand of the type (1_a). Hence $\mathfrak{m}'$ has a direct summand of one of the following types.

(α_1) $Ae_pm_{pq}+Ae_rm_{r's'}$ where this is of the type (3), $u_{pq\alpha}=u_{\lambda,s_\lambda\alpha}v_{pq\lambda}$ and $u_{\lambda,s_\lambda\alpha}=u_{r's'\alpha}v_{\lambda,s_\lambda r'}$.

(α_2) Ae_km_{kl} where this is of the type (2) and $u_{kl\alpha}=u_{pq\alpha}u_{klp}$.

If $\mathfrak{m}'$ is the direct sum of modules of the type (α_1) then there exists $Ae_pm_{pq}+Ae_{r'}m_{r's'}$ such that $u_{\xi\eta\alpha}=u_{pq\alpha}v_{\xi\eta p}$ for all (ξ, η). Now if we take