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<thead>
<tr>
<th><strong>Title</strong></th>
<th>Generalizations of Nakayama ring. VI. (Right US-n rings; n=3,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Harada, Manabu</td>
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<td><strong>Note</strong></td>
<td></td>
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Osaka University
GENERALIZATIONS OF NAKAYAMA RING VI

(RIGHT US-\(n\) RINGS; \(n=3, 4\))

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

MANABU HARADA

(Received June 4, 1986)

We have studied artinian right US-3 rings in [5] and right US-4 algebras over an algebraically closed field in [7]. We shall continue, in this paper, to study a right US-3 (resp. US-4) ring \(R\) when \(R\) is either hereditary or left serial.

In the first two sections, we shall give the characterization of a right US-3 (resp. US-4) ring \(R\), when \(R\) satisfies a weaker condition (*) (see § 1) than \(R\) being either hereditary or left serial. In the next two sections, we shall specify the characterizations given in the previous sections to hereditary rings and left serial rings. We shall exhibit several examples in the final section to illustrate the above characterizations.

1. US-3 rings

Throughout this paper we deal with an artinian ring \(R\) and every \(R\)-module is a unitary right \(R\)-module. We shall use the same terminologies and definitions given in [2]~[8].

As a generalization of right serial rings, we considered

\[ (**, n) \quad \text{Every maximal submodule in a direct sum } D \text{ of } n \text{ hollow modules contains a non-zero direct summand of } D \text{ [5].} \]

It is clear that if \(D/J(D)\) is not homogeneous, \(D\) satisfies (\(**, n\)). Hence we may restrict ourselves to hollow modules of a form \(eR/E\), where \(e\) is a primitive idempotent and \(E\) is a submodule of \(eR\). If (\(**, n\)) holds for any direct sum of \(n\) hollow modules, we call \(R\) a right US-\(n\) ring [5]. Since the concept of US-\(n\) rings is Morita equivalent, we study always a basic ring.

We studied right US-\(n\) algebras over an algebraically closed field for \(n=3\) and 4 in [5] and [7], respectively. In this and next sections we shall give a complete list of the structure of right US-3 (resp. US-4) rings with (*, 1') below. We can give theoretically the complete structure, however as we know a few properties of division rings, we can not give the complete examples for each structure.
We quote here a particular property of a semisimple module (cf. [8] and [9]).

Let $e$ be a primitive idempotent in $R$ and $D$ a semisimple $R$-module and $(\# , m)$ a left $eR$-module. For any two $R$-submodules $V_1$ and $V_2$ with $|V_1| = |V_2| = m$, there exists a unit $x$ in $eRe$ such that $xV_1 = V_2$.

Further we consider one more property:

\( (\ast, 1') \) \hspace{1cm} $eJ^i$ is a direct sum of hollow modules for each primitive idempotent $e$ and each $i$.

If $R$ satisfies $(\ast, 1)$, then $(\ast, 1')$ holds. Moreover, if $R$ is hereditary or left serial, $(\ast, 1')$ holds by [11], Corollary 4.2. Under the assumption $(\ast, 1')$, we obtain the following diagram (cf. [8]):

$$
\begin{array}{cccccc}
A_1 & A_2 & \cdots & A_m & eJ^i \\
\downarrow & \downarrow & & \downarrow & \\
A_{11} & A_{12} & \cdots & A_{1m} & & eJ^{i+1}
\end{array}
$$

where the $A$ are hollow.

Let $A_1$, $A_2$ be submodules in $eR$. If there exists a unit $x$ in $eRe$ such that $xA_1 \subset A_2$ or $xA_1 \supset A_2$, we indicate this situation by $A_1 \sim A_2$ [4]. We put $\Delta = eRe/eJ^i$ and $\Delta(A_i) = \{x \in \Delta, xA_i \subset A_i\}$ [2].

Let $D = A_1 \oplus A_2$; the $A_i$ are uniserial. A submodule $B = B_1 \oplus B_2 (A_1 \supset B_i)$ is called a standard submodule in $D$ [3].

**Lemma 1.** Let $A_1$ and $A_2$ be as in (1). If $A_1 \sim A_2$, $A_1 = xA_2$ for some unit element $x$ in $eRe$, and hence $A_1 \approx A_2$.

Proof. Since $A_1 \sim A_2$, there exists a unit $x$ in $eRe$ such that $xA_1 \subset A_2$ or $xA_1 \supset A_2$. We may assume that $xA_1 \subset A_2$. If $xA_1 \supset A_2$, $xA_1 \supset J(A_2) \subset eJ^{i+1}$, since $A_2$ is hollow. Hence $A_1 \subset x^{-1}eJ^{i+1} = eJ^{i+1}$, a contradiction. Therefore $xA_1 \subseteq A_2$.

**Lemma 2.** Let $A_1$ and $A_2$ be as in (1). Let $B$ be a hollow submodule in $A_2$, which appears on the level $eJ^{k+i}$ ($k \geq 0$) in (1). If $\Delta(A_1) = \Delta, A_1 \sim B$.

Proof. First assume $k \geq 1$ and $A_1 \sim B$, i.e., there exists a unit $x$ in $eRe$ such that $xA_1 \supset B$ or $xA_1 \subset B$. In the latter case $A_1 \subset eJ^{i+1}$. Hence $xA_1 \supset B$. Since $\Delta(A_i) = \Delta$, there exists an element $j$ in $eRe$ with $(x+j)A_i = A_i$. Let $b$ be a generator of $B$. Then we obtain $a$ in $A_1$ with $xa = b$. $b = (x+j)a = (x+j)a - ja$. Let $p_1$ be the projection of $eJ^i$ to $A_i$. $0 = p_1(b) = (x+j)a - p_1(ja)$. Assume $a \in eJ^i - eJ^{i+1}$, and $p(ja) \in eJ^{i+1}$, which is a contradiction, since $x+j$ is a unit in $eRe$. Finally assume $B = A_2$. Then $A_2 = x'A_1$ for some unit $x'$ in $eR$. Hence we obtain the same situation as above, which is a contradiction.
From [2], Theorem 2 we have

**Lemma 3.** If \( R \) is a right US-\( n \) ring, then \([\Delta : \Delta(A)] \leq n - 1\) for any sub-module \( A \) in \( eR \).

Put \( \bar{R} = R/J^{t+k} \). Then \( eRe/eJ\cong eRe/eJ=\Delta \). Let \( A_1 \) be as in (1). Then we can define \( \Delta(\bar{A}_1) = \Delta((A_1+J^{t+k})/J^{t+k}) = \{x \in \Delta, x(A_1+J^{t+k}) \subseteq (A_1+J^{t+k})\} \).

It is clear that \( \Delta(A_1) \) is a division subring of \( \Delta(\bar{A}_1) \).

**Lemma 4.** Let \( A_1 \) and \( A_2 \) be as in (1). If \( \Delta(A_1) = \Delta, \Delta(\bar{A}_1) = \Delta \).

Next assume that \( A_2 = xA_1 \) for some unit \( x \) in \( eRe \). If \([\Delta : \Delta(A_1)] = 2 \) (resp. 3), \([\Delta : \Delta(\bar{A}_1)] = 2 \) (resp. 3), where \( \bar{A}_1 = (A_1+J^{t+k})/J^{t+k} \subseteq \bar{R} = R/J^{t+k} \).

**Proof.** The first part is clear from the remark above. Assume \((x+j)\bar{A}_1 \subseteq \bar{A}_1\) for some \( j \) in \( eJe \). Since \((x+j)\bar{A}_1 \subseteq \bar{A}_2 + j\bar{A}_1 \subseteq \bar{A}_2 + eJ^{t+1}, (x+j)\bar{A}_1 \subseteq \bar{A}_1 + eJ^{t+1}) \cap (A_2+eJ^{t+1}) = eJ^{t+1}, \) a contradiction. Hence \( x \notin \Delta(\bar{A}_1) \). Further \([\Delta : \Delta(A_1)] \) is prime, and so \([\Delta : \Delta(\bar{A}_1)] = [\Delta : \Delta(\bar{A}_1)] \).

**Remark 5.** We shall study a right US-\( n \) ring and observe \([\Delta : \Delta(A_1)] \).

Since \([\Delta : \Delta(A_1)] \leq 3\), we may assume \( J^{t+1} = 0 \) by Lemma 4, [3], Lemma 1 and its proof, when we observe \([\Delta : \Delta(A_1)] \) (the \( x \) in Lemma 4 exists, provided \([\Delta : \Delta(A_1)] \geq 2\)).

**Theorem 1.** \( R \) is a right (basic) US-3 ring with \((\ast, 1')\) if and only if \( eR \) has one of the following structures for each primitive idempotent \( e \).

1) \( eRe/f^t \) is uniserial for some \( t \) and
2) \( f^t = 0 \) or \( f^t = A \oplus B \), where \( A \) is simple and \( B \) is uniserial, such that
   a) \([\Delta : \Delta(A)] = 2\) or b) \( \Delta = \Delta(A) = \Delta(B) \).

In case a) \( B \) is simple and \( A \oplus B \) satisfies \( (\#, 1) \).

In case b)
   i) \( B \) is simple and \( A \cong B \) or
   ii) \( B \) is not simple, and if \( A \) is isomorphic to a simple subfactor module \( B_i/B_{i+1} \) of \( B, B_{i+1} = 0 \) (i.e., \( B_i \) is the socle of \( B \)) and this isomorphism is given by \( j_i \): the left-sided multiplication of \( j \) in \( eJe \).

**Proof.** We assume that \( R \) is a right US-3 ring. From \((\ast, 1')\) and [5], Proposition 1,3) \( eJ = A \oplus B \), where \( A \) and \( B \) are hollow. We may assume \( |A| \leq |B| \). \([\Delta : \Delta(C)] \leq 2\) for any submodule \( C \) in \( eR \) by Lemma 3. Hence we divide ourselves into two cases: I) \([\Delta : \Delta(A)] = 2\) and II) \( \Delta = \Delta(A) \).

Case I). Since \([\Delta : \Delta(A)] = 2\), by [5], Proposition 1,2) there exists a unit element \( x \) in \( eRe \) such that \( xA \subseteq J(A) \oplus J(B) \) or \( xA \supset J(A) \oplus J(B) \). However \( A \subseteq f^{t+1} \) and so \( xA \supset J(A) \oplus J(B) \). On the other hand, \(|A| = |J(A)+1| \) and \( xA \supset J(A) \oplus J(B) \). Hence \( f(B) = 0 \). Further \( A \cong B \) by Lemma 1 and [5], Proposition 1,2). Therefore \( A \) and \( B \) are simple and \( f^{t+1} = 0 \). Which means that every
(simple) submodule \(C\) in \(eJ^i\) is characteristic if and only if \(\Delta(C)=\Delta\). Hence \([\Delta: \Delta(C)]=2\) and \(eJ^i\) satisfies \((\#, 1)\) by [5], Proposition 1,2).

Case II). We know from the above argument that \(\Delta=\Delta(A)=\Delta(B)\) (note that we did not use the assumption \(|A| \leq |B|\)). Let \(y\) be any unit element in \(eRe\).

Since \(\Delta=\Delta(A)\), there exists an element \(j\) in \(eFe\) such that \((y+j)A=A\). Then \((y+j)(A\oplus J(B))\subset A \oplus (y+j)J(B)\subset A \oplus eJ^{i+1}=A \oplus J(B)\). Hence \(\Delta(A \oplus J(B))=\Delta\). Assume that \(B\) is not simple. \(A \oplus J(B)\) or \(J(A) \oplus B\) is hollow by [5], Proposition 1,4)-iv).

\[J(A)=0,\quad \text{i.e., } A \text{ is simple.}\]

We shall show that \(B\) is uniserial. Assume \(eJ^{i+k}=BJ^k=C_1 \oplus C_2 \oplus \cdots\); the \(C_i\) are hollow. If \(\Delta(C_1) \neq \Delta, C_1 \sim A_i\) by [5], Proposition 1,2), which is a contradiction from Lemma 2. Hence \(\Delta=\Delta(C_1)=\Delta(C_2)\). However \(\{A, C_1, C_2\}\) derives a contradiction by Lemma 2 and [4], Corollary 2 of Theorem 2, provided \(C_2 \neq 0\).

Therefore \(B\) is uniserial.

Next assume \(g: A \Rightarrow B_i \Rightarrow B_{i+1}\). Take \(\{A, B_i, B_i(g^{-1})\}\); the graph of \(B_i\) with respect to \(g^{-1}\)). Since \(A\) is simple (and hence \(eFeB \subset B\) and \(\Delta(B)=\Delta, B\) is characteristic. Hence \(A \sim B_i(g^{-1})\), and so there exists a unit \(x_1\) in \(eRe\) such that \(x_1A \subset B_i(g^{-1})\). If \(B_{i+1} \neq 0, x_1A \subset B_{i+1} \subset eJ^{i+1}\), a contradiction. Hence \(B_{i+1}=0\) and \(g: A \sim B_{i+1}\), the socle of \(B\). Let \(j\) be an element in \(eFe\) such that \((x_1+j)A=A\), and put \(x_2=x_1+j\). Then \(A(g)=x_1A=(x_2-j)A\). Put \(A=aR\).

Then \(a+g(a)=(x_2-j)ar\) for some \(r\) in \(R\). \(eFeA \subset eJ^{i+1}\) and \(eJ^{i+1}=BJ\) imply \(eFeA \subset B_i\). Hence

\[a=xzar \quad \text{and} \quad g(a)=-jar,\]

and so \(g(a)=-(jx_2a)\). Therefore \(g=(-jx_2^{-1})a\) and \(-x_2^{-1} \in eFe\) (b-ii)). Finally assume that \(B\) is simple. If \(f: A \Rightarrow B_i\), \(\{A, B, A(f)\}\) derives a contradiction from [5], Lemma 1, (note \(eJ^{i+1}=0\) and use Lemma 8 below). Hence \(A \Rightarrow B\) (b-i)).

Conversely, assume that \(eR\) has one of the structures given in the theorem. Clearly \((\#, 1')\) holds. Let \(\{E_i\}_{i=1}^n\) be any set of submodules in \(eRe\). Case a): If \(E_i \supset eJ^i\) and \(E_2 \supset eJ^i, \Delta(E_i)=\Delta\) for \(i=1, 2\) and \(E_1 \subset E_2\) or \(E_1 \subset E_2\). Hence \(D=\bigoplus_{i=1}^n E_i\) contains a non-zero direct summand of \(D\) by [4], Corollary 1 of Theorem 2. If \(E_i \supset eJ^i\) and \(E_2=\supset eJ^i, E_2=xE_i(=A)\) for some \(x\) in \(eRe\) by \((\#, 1)\). Hence \(D\) satisfies \((\ast, 3)\) again by [4], Corollary 1 of Theorem 2. Case b-ii): If \(E_i \supset eJ^i, xE_i\) is a standard submodule in \(eJ^i\) for a unit \(x_i=(e+j)\) in \(eRe\) by assumption. Hence \(E_i \sim E_j\) for some pair \(i, j\). Further \(\Delta(\Delta(E))\) by assumption. Therefore \(D\) satisfies \((\ast, 3)\) by [4], Corollary 1 of Theorem 2. Case b-i): This is much simpler than the above. Thus \(R\) is right US-3.
In the last paragraph of the proof of "only if part" in Theorem 1, we have shown

**Lemma 6.** Assume that $eJ'=A \oplus A' \oplus B$ and 1) $A$ and $A'$ are simple modules with $\Delta(A)=\Delta$, and 2) $B$ is non-simple and uniserial. If $g: A \approx B_i \oplus B_{i+1}$ and $A \sim B_i(g^{-1}), B_{i+1}=0$ and $g$ is given by $j_1; j \in eJ, \text{ and hence } i \geq 1$ (cf. [7], Lemma 16).

We shall illustrate the structure in Theorem 1 as the following diagram:

1) $\begin{array}{ccc}
eR & eJ & eJ' \\
 & \cdot & \cdot \\
 & \cdot & \cdot & 0
\end{array}$

2) $\begin{array}{ccc}
eR & eJ & eJ' & eJ' \\
 & \cdot & \cdot & A & 0 \\
 & \cdot & B & B_p & 0
\end{array}$

where the straight line means uniserial.

It is clear that if $R$ has the structure above, $(\ast, 1)$ (and hence $(\ast, 1')$) holds. We note that if $(\ast, 1')$ does not hold, Theorem 1 is not true (see [6]). We shall give examples of a) and b) in § 5.

**2. US-4 rings**

Next we shall characterize a right US-4 ring with $(\ast, 1')$.

**Lemma 7.** Let $R$ be a right US-4 ring and $\{A_i\}_{i=1}^t$ a set of submodules in $eJ$. Then 1) if $\Delta(A_i)=\Delta$ or all $i \leq 3$ and $A_k \sim A_k$, for $k \neq k' \leq 3$, then $A_k \sim (\text{some } A_i)$. 2) $A_i \sim A_j$ for some pair $i, j$. 3) If $[\Delta: \Delta(A_i)]=2$ for $i=1, 2, A_i \sim A_2$. 4) If $[\Delta: \Delta(A_i)]=3$, $A_i \sim A_j$ for all $j$. 5) If $[\Delta: \Delta(A_i)]=3$, $A_i \sim A_j$ for some $i, j \leq 3$.

Proof. This is clear from [4], Corollary 2 of Theorem 2.

**Lemma 8.** Let $A_1$ and $A_2$ be as in (1). Assume $J^{i+1}=0$. If $\Delta(A_i)=\Delta$, $A_i$ is characteristic.

Proof. This is clear.

**Lemma 9.** Let $R$ be a right US-4 (basic) ring, and $\{A_i\}_{i=1}^t$ a set of hollow submodules on the level $eJ'$ in (1). If $\Delta(A_i)=\Delta$ for all $i, t \leq 3$.

Proof. This is clear from Lemmas 7, 8 and Remark 5.

From now on we assume that $R$ is a right US-4 (basic) ring satisfying $(\ast, 1')$. Let $D=(eJ'=A_1 \oplus A_2 \oplus \cdots \oplus A_t)$, where the $A_i$ are hollow. In the
following lemmas, we mainly assume that $D$ is characteristic. We note $[\Delta: \Delta(A_i)] \leq 3$ for all $i$ by Lemma 3.

**Lemma 10.** Assume $[\Delta: \Delta(A_i)] = 2$ for all $i$. Then i) $t=2$. ii) There exists a unit $x$ in $eRe$ such that $xA_1 = A_2$. iii) $A_i$ is a uniserial module with $|A_i| \leq 2$. iv) If there are characteristic submodules in $A_1 \oplus A_2$, they are linear with respect to the inclusion. v) If $B$ is not a characteristic submodule in $A_1 \oplus A_2$, $[\Delta: \Delta(B)] = 2$ and those submodules are related by $\sim$.

**Proof.** We may assume $|A_i| \leq |A_2| \leq \cdots \leq |A_t|$ (note $t \geq 2$). By Lemmas 1 and 7, $A_k = x_k A_1$ for all $k$. Hence

(a) if $[\Delta: \Delta(A_i)] \geq 2$ for all $i$, there exists a unit $x_i$ in $eRe$ such that $x_i A_1 = A_i$ for all $i$.

On the other hand, since $[\Delta: \Delta(A_1)] = 2$, $D = \Delta(A_1) A_1 + x_2 \Delta(A_1) A_1 = A_1 \oplus A_2$, $t = 2$. We note that from the above argument and Lemma 3 we obtain

(\beta) If $[\Delta: \Delta(A_i)] \geq 2$ for all $i$, $t \leq 3$.

Assume that $A_1/A_1 J^s$ is uniserial and $A_1 J^s = B_1 \oplus B_2 \oplus \cdots \oplus B_s$, where the $B_j$ are hollow and $s \geq 2$. In order to show $s \leq 1$, we may assume $ef^{i+1} = 0$ by Remark 5. Since $D = \Delta(A_1) A_1 + x_2 \Delta(A_1) A_1 = A_1 \oplus A_2$, $t = 2$. We note that there exists a unit $x$ in $eRe$ such that $x A_1 = A_2$. Hence $\Delta(B_p) \neq \Delta$ for all $p$. On the other hand, $DJ^s = A_1 J^s \oplus A_2 J^s = \bigoplus_{p=1}^s B_p + \bigoplus_{p=1}^s x B_p$, which is a contradiction to (\beta). Therefore $A_1$ and $A_2$ are uniserial. Next assume $A_1 J^s \neq 0$. $\Delta(A_1 J \oplus A_2 J^s) \neq \Delta$ by existence of $x_2$. Hence $\{A_1, A_1 J \oplus A_2 J^s\}$ derives a contradiction by Lemma 7. Therefore $|A_1| \leq 2$. Since $\Delta(A_1 J \oplus (A_2)) \subset \Delta(A_1), \Delta(A_1 \oplus J(A_2)) = \Delta(A_2)$ for $\Delta(A_2 \oplus J(A_2)) = \Delta$. Similarly $[\Delta: \Delta(J(A_1))] = 2$. Let $E$ be a submodule with $[\Delta: \Delta(E)] = 3$. Then there exists a unit element $x$ in $eRe$ such that $xE \subset A_1$ or $xE \supset A_1$ by Lemma 7. In the former case $[\Delta: \Delta(E)] = [\Delta: \Delta(xE)] = 2$. If $xE \supset A_1$, $x E = A_1 + E'$; $E' \subset A_2$. Hence $[\Delta: \Delta(xE)] = 2$ from the above. Therefore there are no submodules $E$ with $[\Delta: \Delta(E)] = 3$. Finally assume that $A_1 \oplus A_2$ contains two characteristic submodules $C_1, C_2$ such that $C_1 \sim C_2$. Consider $\{A_1, A_1, C_1, C_2\}$, and $A_1 \sim C_1$ or $A_1 \sim C_2$ by Lemma 7. If $A_1 \supset C_1, C_1 = 0$ and if $A_1 \subset C_1, C_1 = A_1 \oplus F'$; $F \subset A_1$, and so $C_1 = A_1 \oplus A_2$. Hence $C_1 \supset C_2$ or $C_1 \subset C_2$. Let $\Delta(E) = \Delta$. If $|A_1| = 1$, $E$ is characteristic. Assume $|A_1| = 2$. Put $C_1 = A_1 \oplus B_2$. Then $E \sim C_1$ from the above. Hence $E \subset C_1$ or $E \supset C_1$, and so $E$ is characteristic.

**Lemma 11.** Assume $[\Delta: \Delta(A_i)] = 3$ for all $i$. Then $t \leq 3$, and the $A_i$ are simple and there exists a unit $x_i$ in $eRe$ such that $x_i A_i = A_i$ for each $i$. If $t = 3$, $D$ satisfies (\#, 1) and (\#, 2) and $[\Delta: \Delta(C)] \leq 3$ for every submodule $C$ in $D$. If $t = 2$, $D$ satisfies (\#, 1).

**Proof.** Since $[\Delta: \Delta(A_i)] = 3$, there exists a unit $x_i$ in $eRe$ such that $x_i A_i = A_i$
from (α) and \( t \leq 3 \) by (β). Assume \( t = 3 \). Taking \( \{ A_i, J(D) \} \), we know from Lemma 7 that \( A_i \) is simple and hence \( e f^{i+1} = 0 \). It is clear from Lemmas 7 and 8 that there are no simple submodules \( B \) in \( D \) with \( \Delta(B) = \Delta \). Hence \( D \) satisfies (\#, 1). Let \( C \) be a submodule of \( D \) with \( |C| = 2 \). Then \( D = C \oplus A_i \) for some \( i \). Hence \( \Delta(C) = \Delta \) by Lemma 7, and so \( D \) satisfies (\#, 2). We obtain the similar result for \( t = 2 \).

**Lemma 12.** Assume \( [\Delta : \Delta(A_i)] = 1 \) and \( \Delta(A_i) \neq \Delta \) for \( i \geq 2 \). Then \( A_i \) is uniserial and \( t \leq 3 \).

i) \( t = 3 \):

Then all \( A_i \) are simple, \( [\Delta : \Delta(A_i)] = 2 \) for \( i = 2, 3 \), \( A_1 \cong A_2 \) and \( A_2 \oplus A_3 \) satisfies (\#, 1).

ii) \( t = 2 \):

a) \( A_i \) is not simple.

Then \( [\Delta : \Delta(A_2)] = 2 \), and \( A_2 \) is a simple submodule isomorphic to \( B \), the socle of \( A_1 \). If \( A_2 \cong E_1/E_{i+1}(A_1 \supseteq E_i \supseteq E_{i+1}) \), \( E_i = B \) and \( E_{i+1} = 0 \). Further \( B \oplus A_2 \) satisfies (\#, 1) except \( B \).

b) \( A_i \) is simple.

Then

1) \( [\Delta : \Delta(A_2)] = 2 \), \( A_i \oplus A_2/J(A_2) \) satisfies (\#, 1) except \( A_1 \).

2) \( A_2/A_2J' \) is uniserial for some \( t \) and

2-i). \( A_2J' = 0 \) or

2-ii). \( A_2J' = B_1 \oplus B_2; B_1 \) is simple and \( B_2 \) is uniserial.

2-ii-1) \( \Delta(B_1) = \Delta(B_2) = \Delta \).

2-ii-1-1) \( B_1 \cong B_2/J(B_2) \).

2-ii-1-2) \( A_1 \cong F_i/F_{i+1}(A_2 \supseteq F_i \supseteq F_{i+1} \supseteq B_1 \oplus B_2) \).

2-ii-1-3) If \( f: A_1 \cong G_j/G_{j+1}(f' : B_1 \cong G_j/G_{j+1}(B_2 \supseteq G_j \supseteq G_{j+1}), \text{then } G_{j+1} = 0 \) and \( f(f') \) is given by \( j; j' \in eJ' \).

2-ii-1-4) If \( f: A_1 \cong B_1 \), we have the same result as 2-ii-1-3).

2-ii-2). \( [\Delta : \Delta(B_1)] = [\Delta : \Delta(B_2)] = 2 \).

2-ii-2-1) \( B_1 \) and \( B_2 \) are simple and \( B_1 \oplus B_2 \) satisfies (\#, 1).

2-ii-2-2) \( A_1 \cong F_i/F_{i+1}(A_2 \supseteq F_i \supseteq F_{i+1} \supseteq B_1 \oplus B_2) \).

2-ii-2-3) If \( A_1 \cong B_1 \), then \( f \) is given by \( j; j' \in eJ' \).

Proof. It is clear, from the assumption and Lemmas 1 and 7, that \( [\Delta : \Delta(A_i)] = 2 \) for all \( i \geq 2 \). Assume that \( A_i \) contains two independent submodules \( B_1, B_2 \). If \( \Delta(B_1) = \Delta(B_2) = \Delta \), \( \{ B_1, B_2, A_2, A_3 \} \) derives a contradiction by Lemmas 7, 8 and Remark 5. On the other hand, if \( \Delta(B_1) \neq \Delta \), \( \{ B_1, B_2, A_2, A_3 \} \) derives again a contradiction. Hence

\[ A_i \text{ is uniserial} \]

by (\#, 1').

a) \( J(A_i) = 0 \): Consider \( \{ A_i, J(D) \} \). Then \( A_i \ncong A_2 \) by Lemma 2. Hence \( J(D) \sim A_i \) or \( J(D) \sim A_1 \) by Lemma 7. However \( J(D) \ncong A_2 \), since \( J(D) \) is
characteristic and \( J(A_i) \neq 0 \). Hence
\[
\text{the } A_i \text{ are simple for all } i \geq 2.
\]
Since \([\Delta: \Delta(A_i)] = 2\), there exists \( x_i \) in \( eRe \) such that \( x_i A_2 = A_i \) for \( i > 2 \) by Lemmas 1 and 7. Hence in order to show \( t \leq 3 \), we may assume \( J^{t+1} = 0 \) by Remark 5. Noting \( A_i \in \Delta(A_i) \), \( \Delta = \Delta(A_2) \oplus x_2 \Delta(A_2) \), which implies that \( A_2 \oplus A_3 = \Delta A_2 \supseteq \sum_{i=1}^{t+1} \Delta A_i \). Hence \( t \leq 3 \). Now we resume to the original situation. We note \( eReA_1 \subset J(A_i) \), and hence \( A_1 \) is characteristic. Since \( \Delta(J(A_i)) = \Delta, \Delta(J(A_2) \oplus A_2) \neq \Delta \). Consider \( \{ A_i, J(A_i) \oplus A_2, A_2 \oplus A_2 \} \). \( \Delta(A_i) = \Delta \)
and \( \Delta(J(A_i) \oplus A_2) \neq \Delta \) imply \( (J(A_i) \oplus A_2) \cong (A_2 \oplus A_2) \) by Lemma 7. Hence there exists a unit \( x \) in \( eRe \) such that \( x(J(A_i) \oplus A_2) \subset (A_2 \oplus A_2) \) or \( x(J(A_i) \oplus A_2) \supset (A_2 \oplus A_2) \). However, \( A_2 \) is characteristic. Hence \( \Delta(J(A_i) \oplus A_2) \neq \Delta \). Assume \( \Delta(J(A_i) \oplus A_2) \neq \Delta \) for any \( i \). Therefore \( t = 2 \) provided \( J(A_i) = 0 \), i.e.,
\[
D = A_1 \oplus A_2 \quad (J(A_i) = 0).
\]
Now we take the similar manner to Lemma 6. Assume \( f: A_2 \cong E_i | E_{i+1}; A_i \supseteq E_i \supseteq E_{i+1} \). We note that \( A_i \) is characteristic. \( \{ A_i, J(A_i) \oplus A_2, A_2 \oplus A_2 \} \). \( \Delta(A_i) = \Delta \)
and \( \Delta(J(A_i) \oplus A_2) \neq \Delta \) imply \( (J(A_i) \oplus A_2) \cong (A_2 \oplus A_2) \) by Lemma 7. Hence \( E_{i+1} = 0 \) as the proof of Lemma 6. Further since \( \Delta(A_2) \neq \Delta, A_2 \cong E_{i+1} \); the socle of \( A_1 \). Let \( C(\neq E_{i+1}) \) be a simple submodule in \( E_{i+1} \oplus A_2 \). Consider \( \{ A_i, C, A_2, A_3 \} \). It is clear that if \( C \cong A_i, C \subset A_1 \). Hence \( C \cong A_2 \) by Lemmas 2 and 7, and so \( E_{i+1} \oplus A_2 \) satisfies \((\#), (1)\) except \( E_{i+1} \).

b) \( J(A_i) = 0, t \geq 3 \). Assume \( J(A_i) = 0 \). Since \( t \geq 3 \), there exists a unit \( x \) in \( eRe \) with \( xA_2 = A_3 \) by Lemmas 1 and 7, and so \( \Delta(A_1 \oplus J(A_2)) \neq \Delta \). Then \( A_2 \supseteq A_1 \oplus J(A_2) \) by Lemma 7. Assume \( A_2 \supseteq y(A_1 \oplus J(A_2)) \) for some unit \( y \). Since \( A_1 \) is simple and \( \Delta(A_1) = \Delta, p_1(yA_1) = A_1 \), where \( p_1: eJ \rightarrow A_1 \) the projection, which is a contradiction. Similarly, since \( A_1 \) is simple and \( A_2 \) is not, \( p_2(yA_1) \subset J(A_3) \) for any unit \( y \) in \( eRe \). Hence \( A_2 \subset J(A_3) \) satisfies \((\#), (1)\). Therefore \( A_2 \) (and so \( A_i (i \geq 2) \)) is simple.

Accordingly \( t = 3 \) from the initial paragraph of a). If \( f: A_1 \cong A_2, \{ A_1, A_1(f), A_2, A_3 \} \) derives a contradiction, since \( \Delta(A_2) = \Delta = A_2 \oplus A_3 \) as before (note \( eJ^{t+1} = 0 \)).

Hence \( A_1 \not\cong A_2 \). Further if \( A_2 \oplus A_3 \) contains a characteristic submodule \( B \neq 0, \{ A_1, B, A_3 \} \) derives a contradiction. Therefore \( A_2 \oplus A_3 \) satisfies \((\#), (1)\).

Case \( t = 2 \) and \( J(A_i) = 0 (D = A_1 \oplus A_2) \). First we shall show that \( A_1 \oplus A_2 \mid J(A_i) \) satisfies \((\#), (1)\) except \( A_1 \). Since \( \Delta(A_3) \neq \Delta \), there exists a unit \( x \) in \( eRe \) such that \( p_1(xA_2) = A_1 \), where \( p_1: eJ \rightarrow A_1 \) is the projection. Further \( eReA_2 \subset A_2 \), since \( A_1 \) is simple. Hence \( (x+j)(A_2 + J^{t+1}) = A_2 + J^{t+1} \) for any \( j \) in \( eRe \), and so \( \Delta(A_2) = \Delta((A_2 + J^{t+1}))/J^{t+1} \). Therefore we may assume \( J^{t+1} = 0 \) (cf. Remark 5). Then
$A_2 \oplus A_2$ satisfies $(\#_1,1)$ except $A_1$ from Lemma 7. Now we resume the original situation. Since $A_1$ is simple, $eJ^{i+1}=A_2J$. Assume that $A_2/A_2J$ is uniserial and $eJ^{i+1}=B_1 \oplus B_2 \oplus \cdots \oplus B_n$ where the $B_i$ are hollow. Then from Lemmas 10~16 below, $s \leq 3$. Further $[A: \Delta(B_j)] \leq 2$ by Lemmas 2 and 7. Assume $s=3$. Then $\Delta(B_i) \neq \Delta$ (resp. $[A: \Delta(B_j)] \neq 2$) for some $i$ (resp. $j$) by Lemmas 7 and 10. Hence we remain two cases $\Delta(B_i)=\Delta$, $[A: \Delta(B_j)] = 2$ for $j=2$, 3 and $\Delta(B_i)=\Delta$ for $i=1,2$. $[A: \Delta(B_j)]=2$. On the other hand, since $\Delta(A_1)=\Delta$, we do not have such cases by Lemmas 2 and 7. Therefore $s \leq 2$. Similarly we do not have a case $\Delta(B_i)=\Delta$ and $[A: \Delta(B_j)] = 2$. Thus we obtain two cases; 2-ii-1): $\Delta(B_i)=\Delta$ for $i=1,2$ and 2-ii-2): $[A: \Delta(B_j)]=2$ for $i=1,2$.

2-ii-1) We assume $|B_1| \leq |B_2|$. \{A_1, B_1, B_2, J(B_1) \oplus J(B_2)\} gives $J(B_1)=0$ from Lemmas 2 and 7. Assume $B_2J= C_1 \oplus C_2 \oplus \cdots \oplus C_s$, $s \geq 2$ and the $C_i$ are hollow. If $[A: \Delta(C_i)] \geq 2$, \{A_1, B_1, C_1, C_2\} derives a contradiction from Lemmas 2 and 7. Hence $\Delta(C_1)=\Delta(C_2)=\Delta$. Taking $R/J^{i+1}+1$, we obtain again a contradiction from \{A_1, B_1, C_1, C_2\} and Lemmas 2, 7 and 8. Accordingly $B_2$ is uniserial. If $f: B_1 \approx B_2/J(B_2)$, \{A_1, B_1, B_2, B_2(f-1)\} derives a contradiction. Hence $B_1 \approx B_2/J(B_2)$ (2-ii-1-1)). Further if $g: B_1 \approx G_i/g_i+1(B_2 \supset G_i \supset G_i+1)$, \{A_1, B_1, B_2, G_i(g^-1)\} gives $B_1 \sim G_i(g^+)$, since $\Delta(A_1)=\Delta(B_2)=\Delta$. Hence $G_i+1=0$ and $g$ is given by $j_k$, $j_k \in \mathfrak{e}f_{j_k}$ from Lemma 6. Similarly if $f_1: A_2 \approx G_i/g_{i+1}$, \{A_1, G_i(f^-1), A_2, A_2\} gives $A_2 \sim G_i(f^+)$, since $\Delta(A_2)=\Delta$. Then we can show similarly to Lemma 6 that $G_{i+1}=0$ and $f_1$ is given by $j'_{i'}$, $j' \in \mathfrak{e}f_{j'_k}$ (2-ii-1-3)). Next if $h: A_1 \approx F_j/F_{j+1}(A_2 \supset F_j \supset F_{j+1} \supset B_1 \oplus B_2)$, consider \{A_1, F_j(f^-1), A_2, A_2\}. Then $xA_1 \subset F_j(h^-1)$ for some unit $x$ in $eRe$, since $A_2 \neq F_j$ and $\Delta(A_1)=\Delta$. Hence $xA_1 \subset F_{j+1} \subset J(A_2)$, a contradiction. Accordingly $A_1 \approx F_j/F_{j+1}(F_j \neq A_2)$ (2-ii-1-2)). In the same manner given in the proof of 2-ii-1-3), we have 2-ii-1-4).

2-ii-2) Since $[A: \Delta(B_j)]=\Delta$ and $[A: \Delta(B_2)] = 2$, $B_1 \approx B_2$ by Lemmas 1 and 7. \{A_1, B_1, B_2, J(B_1) \oplus J(B_2)\} gives $J(B_1)=J(B_2)=0$. Accordingly $B_1$ and $B_2$ are simple. Let $C$ be any simple submodule in $B_1 \oplus B_2$. Then \{A_1, B_1, B_2, C\} shows $C=xB$ for some unit $x$ in $eRe$ by Lemmas 2 and 7. Hence $B_1 \oplus B_2$ satisfies $(\#_1, 1)$ (2-ii-2-1).

2-ii-2-2) is same to 2-ii-1-2). If $f: A_1 \approx B_1$, \{A_1, A_1(f), B_1, B_1\} gives $A_1 \sim A_1(f)$. Hence $f$ is given by $j_k$, $j \in \mathfrak{e}f_{j_k}$ (2-ii-2-3)).

REMARK 13. We shall consider the situation of ii-b) of Lemma 12. Taking $R=J^{i+1}$, we may assume that $eJ^i(=V)=A_1+ A_2$: the $A_i$ are simple, $\Delta(A_i)=\Delta$, and $[A: \Delta(A_2)] = 2$. Then $A_1 \approx A_2 (\approx g_1/Rg_1=\Delta')$. We shall express $\text{End}_{\Delta'}(V)$ as elements of matrices $(\Delta')$. Since $A_1$ is characteristic, for any element $x$ in $\Delta$, $x = \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}$ : $x_1 \in \Delta'$. Hence we
obtain two monomorphisms as rings $f_1, f_2$ of $\Delta$ to $\Delta'$ such that $f_i(x) = x_i$ and a homomorphism $g$ as additive groups of $\Delta$ to $\Delta'$ such that

\[ g(x'x) = f_1(x)g(x') + g(x)f_2(x'). \]

Then $\Delta(A_2) = g^{-1}(0)$ (note, from i), that $g^{-1}(0)$ is a division subring of $\Delta$.

Hence $[\Delta : \Delta(A_2)] = 2$ is equivalent to

ii) $[\Delta : g^{-1}(0)] = 2$.

Further (ii, 1) holds if and only if, for any $\alpha$ in $\Delta$, there exists $x \neq 0$ in $\Delta$ such that

\[ \alpha = -f_1(x^{-1})g(x) \quad (= g(x^{-1})f_2(x)), \] i.e., $F : \Delta \to \Delta'$ ($F(x) = f_1(x)^{-1}g(x)$) is surjective.

If $\alpha \neq 0$, $x \in g^{-1}(0)$. Hence if either $|\Delta|$, cardinal of $\Delta$, ($|\Delta| \leq |\Delta'|$) is finite or $|\Delta| < |\Delta'|$, iii) does not hold. Hence we assume that $|\Delta|$ is infinite. Further, since $f_i$ is a monomorphism, we may assume that $\Delta \subseteq \Delta'$ and $f_i$ is the inclusion. Now assume that $\Delta'$ is commutative. Then $g$ is a $K$-linear mapping from i), where $K = g^{-1}(0)$. Using those facts and $|\Delta| \geq \infty$, for any $g$ we can show by computation that there exists $\alpha$ in $\Delta'$ not satisfying iii) for any $x \in \Delta$. Therefore if $\Delta'$ is commutative, we do not have the case of ii) of Lemma 12.

**Remark 14.** Next we consider the case $t = 2$ in Lemma 11. Let $K$ be a field and $R$ a $K$-algebra. If $[\Delta' : K]$ is not divided by 3, this case does not occur. Because, since $V = A_1 \oplus A_2$ and $A_i \approx A_2$, $\operatorname{End}_A(V) = (\Delta')_2$ and $\Delta \subseteq (\Delta')_2$.

$[\Delta : \Delta(A_1)] = 3$ implies that $4[\Delta' : K]$ is divided by 3.

Finally we take division rings given by [10]. Let $D \supset D_1$ be division rings such that $[D : D_i] = 3$ and $[D : D_1] = 2$. Put $D = D_1 + uD_1$ and $D^* = \operatorname{Hom}_{D_1}(D, D_1)$. Then $[D^* : D_1] = 2$ and $D^*$ is a left $D$-vector space. Define $1^* \in D^*$ by setting $1^*(1) = 1$, $1^*(u) = 0$, and put $A_1 = 1^* D_1$. Then $D(A_1) = \{d \in D, dA_1 \subseteq A_1\} = u^{-1}D_1u$, and so $[D : D(A_1)] = 3$. For any $h$ in $D^*$ and $h^{-1}(0) = D_1u$, we have $D = D_1u \oplus D_1v_1$. Put $d = h(v_1)$. Then $(u_1^{-1}u)1^*(u_1) = 0$ and $(u_1^{-1}u)1^*(v_1) = d' \neq 0$. Hence $h = (u_1^{-1}u)1^*d' = 0$, and so $hD_1 = (u_1^{-1}u)A_1$. Therefore $D^*$ satisfies (i, 1), $[D : D(A_1)] = 3$ and $[D^* : D_1] = 2$. We shall use $D^*$ in §5, Example 3.

Now we resume to study the structure of right US-4 rings.

**Lemma 15.** If $R$ is a US-4 ring with $(*, 1')$. $D$ has one of the structures in Lemmas 10, 11, 12 and 16 below.

**Proof.** Assume 1) $\Delta(A_1) = \Delta(A_2) = \Delta(A_3) = \Delta$. Then $t = 3$ by Lemmas 7 and 8 (the case of Lemma 16 below). 2) $\Delta(A_1) = \Delta(A_2) = \Delta$ and $\Delta(A_i) \neq \Delta$ for $i \geq 3$. Then $\{A_1, A_2, A_i, A_3\}$ derives a contradiction from Lemmas 2 and 7. 3) $\Delta(A_i) = \Delta$ and $\Delta(A_i) \neq \Delta$ for $i \geq 2$. This is a case of Lemma 12. 4) $[\Delta : \Delta(A_i)] = 2$ for $i \leq$ some $l$, $[\Delta : \Delta(A_j)] = 3$ for $j > l$. Since $[\Delta : \Delta(A_k)] \geq 2$ for all $k$, from (a) there exists a unit $x_i$ in $eRe$ such that $x_iA_i = A_i$ for all $i$. Hence $\Delta(A_i) = x_i \Delta_i(A) x_i^{-1}$, and so we obtain the cases of Lemmas 10 and 11.
Lemma 16. Assume $\Delta(A_i) = \Delta$ for all $i$. Then $t \leq 3$, and

1) $t = 3$:
   \begin{itemize}
   \item $A_3$ is uniserial and $A_1, A_2$ are simple, $A_3 \ncong A_2$. If $A_3$ is simple, $A_3 \ncong A_1$ and $A_3 \ncong A_2$.
   \item If $A_3$ is not simple and $f_1: A_3 \ncong F_1/F_{i+1}$ (with $F_i$ given by $j_i$; $j_i \in \mathcal{E}e$), then $F_{i+1} = 0$ and $f$ is given by $j_i$; $j_i \in \mathcal{E}e$, and hence $i > 1$.
   \end{itemize}

2) $t = 2$:
   \begin{itemize}
   \item $A_1 \ncong A_2$ (with $g_i : g_i$, $j_i$, $f_i$).
   \item $A_3$ is uniserial; $A_3 \ncong A_2$ and $A_3 \ncong A_1$.
   \item If $A_3$ is simple, $A_3 \ncong A_1$ and $A_3 \ncong A_2$. If $A_3$ is not simple and $f_i : A_3 \ncong A_i$, then $A_{i+1} = 0$ and $f$ is given by $j_i$; $j_i \in \mathcal{E}e$, and hence $i > 1$.
   \item $f_1 = \mathcal{E}e$, $f_2 = \mathcal{E}e$, $f_3 = \mathcal{E}e$.
   \item $A_3$ is given by $j_i$; $j_i \in \mathcal{E}e$.
   \item $A_3 \ncong A_1$ and $A_3 \ncong A_2$.
   \end{itemize}
Then $B_{1} \cong B_{2}$ and $B_{3}, B_{2}$ are simple and $V = B_{1} \oplus B_{2}$ satisfies $(\#, 1)$.

b2-1) $A_{1} \cong F_{i}/F_{i+1}(A_{2} \supset F_{i} \supset F_{i+1} \supset V)$.

b2-2) If $f: A_{1} \cong B_{1}, f$ is given by $j_{i} ; j \in e_{j} f_{e}$. (cf. [7], Theorem 17.)

Proof. We know $t \leq 3$ by Lemma 9. Assume that $|A_{1}| \leq |A_{2}| \leq |A_{3}|$.

i) $t = 3$. Consider $\{ A_{1}, A_{2}, A_{3}, J(D) = J(A_{1}) \oplus J(A_{2}) \oplus J(A_{3}) \}$. Since $\Delta(D_{i}) = \Delta, J(A_{1}) \oplus J(A_{2}) \oplus J(A_{3})$ is contained in some $A_{i}$ by Lemmas 2 and 7. Hence $J(A_{1}) = J(A_{2}) = 0$ (note $|A_{3}| \geq |A_{1}|$). Assume that $A_{3}$ contains two independent submodules $B_{1}$ and $B_{2}$ in $e_{j} f_{e}$. Then both $\Delta: \Delta(\cup) \neq \Delta(\cup B_{1}) \neq \Delta(\cup B_{2})$. Hence $A_{3}$ is characteristic. Assume $A_{1} \cong A_{2}$. Then $\{ A_{1}, A_{2}, A_{3}, A_{1}(f) \}$ implies $A_{1}(f) \cong A_{2}(f) \sim (A_{3})$. Since $A_{1}$ is characteristic (we may assume $J^{i+1} = 0$ by Remark 5), $A_{1}(f) \subset A_{3}$, which is a contradiction. Finally assume $g: A_{1} \cong F_{i}/F_{i+1}$. Since $\Delta(A_{3}) = \Delta$ and $A_{1}, A_{2}$ are simple, $A_{3}$ is characteristic. Hence $\{ A_{1}, A_{2}, F_{i}(g^{-1}), A_{3} \}$ derives from Lemmas 2 and 7 that $A_{1} \sim F_{i}(g^{-1})$. Therefore $g$ is given by $j_{i}$ from Lemma 6.

ii) $t = 2$.

i) $f: A_{1} \cong A_{2}$. Assume $\Delta \neq \Delta(A_{1}(f))$. Then $\{ A_{1}, A_{2}, A_{1}(f), A_{2}(f) \}$ implies $A_{1}(f) \sim A_{2}$ for some $i$, say 1 from Lemma 7. Since $A_{2} \cong A_{1} \cong A_{1}(f), A_{2}(f) = xA_{1}$ for some unit $x$ in $e_{j} f_{e}$. Hence $\Delta(A_{1}(f)) = s \Delta(A_{1}(f)) x^{-1} = \Delta$, a contradiction. Accordingly $\Delta(A_{1}(f)) = \Delta$. Consider $\{ A_{1}, A_{2}, A_{1}(f), J(A_{1}) \oplus J(A_{2}) \}$, and $J(A_{1}) = 0$ by Lemma 7 (note $\Delta(A_{1}) = \Delta(A_{1}(f)) = \Delta$). Hence $A_{1}$ and $A_{2}$ are simple, and so $e_{j} f_{e} = 0$. Let $f$ and $f'$ be two isomorphisms of $A_{1}$ to $A_{2}$ and consider $\{ A_{1}, A_{2}, A_{1}(f), A_{2}(f) \}$. Since $e_{j} f_{e} \neq 0$, they are characteristic, and so $A_{1}(f) \cong A_{2}(f')$ by Lemmas 7 and 8. Hence $f = f'$. Considering an isomorphism $\delta f$ for $\delta \in \Delta$, $\Delta = \{ 0, 1 \}$, $\Delta' = \delta \Delta = \delta \Delta = 0, 1 \}$, where $A \cong 0, 1 \}$, and $\Delta = \delta \Delta = \delta \Delta = 0, 1 \}$. Then Hom$_{R}(A_{1}, A_{1}) = \{ 0, 1 \}$. Assume $A_{1} J = 0$ and $A_{1} J^{s} = C_{1} \cap C_{2} \oplus \cdots \oplus C_{s}$ ($s \geq 1$), where the $C_{i}$ are hollow. Consider $\{ A_{1}, A_{2}, A_{1} J \cap C_{1}, A_{1} J \cap C_{2}, \}$ ($s \geq 2$).

Then $A_{1} J \oplus C_{1} \cong A_{1} J \oplus C_{2}$ by Lemmas 2 and 7, provided $A_{1} J \neq 0$. Assume $\Delta(A_{1} J \cap C_{1}) = \Delta$ for $i = 1, 2$ and $x(A_{1} J \cap C_{1}) \subset A_{1} J \oplus C_{2}$ for some unit $x$. We may assume $J^{i+1} = 0$. There exists $j$ in $e_{j} f_{e}$ such that $(x+j)(A_{1} J \cap C_{1}) = A_{1} J \oplus C_{1}$. Then $x C_{1} \subset (x+j) C_{1} + j C_{1} \subset A_{1} J + C_{1}$. Hence $x C_{1} \subset (A_{1} J \cap C_{1}) \cap (A_{1} J \cap C_{2}) = A_{1} J$, and so $C_{1} \sim A_{1}$, a contradiction by Lemma 2. Hence $\Delta(A_{1} J \cap C_{1}) \neq \Delta$ for some $i$, say 1. $\{ A_{1}, A_{2}, A_{1} J \cap C_{1}, A_{1} J \cap C_{2} \}$ implies either $A_{1} \sim A_{1} J \cap C_{1}$ or $A_{2} \sim A_{1} J \cap C_{2}$ by Lemma 7. Which is again a contradiction by Lemma 2. Hence $s = 1$, and so

$A_{2}$ is uniserial, provided $A_{1} J \neq 0$.

Similarly $A_{1}$ is also uniserial, provided $A_{2} J \neq 0$. Now assume that $A_{2}$ is uniserial ($|A_{2}| \geq 2$ and hence so is $A_{1}$). We shall show $|A_{1}| \leq 2$. Assume $A_{1} J^{s} = 0$ and
hence $A_iJ^i \neq 0$. Consider $\{A_1, A_1J \oplus A_2J, A_1J^2 \oplus A_2J, A_3\}$. Since $A_i \sim A_2$ by Lemma 2, 1) $A_i \sim A_1J \oplus A_2J^i$ or 2) $A_i \sim A_1J^2 \oplus A_2J$ ($A_1$ and $A_2$ are symmetry) or 3) $A_1J \oplus A_2J^i \sim A_1J^2 \oplus A_2J$.

1) It is clear that $xA_i \supset A_iJ \oplus A_2J$ for a unit $x$. However $A_i$ is uniserial, and so $A_2J^i = 0$ (note $|A_1| \leq |A_2|$). 2) This is similar. 3) Assume $x(A_1J \oplus A_2J^i) \supset A_iJ^2 \oplus A_2J$. Since $\Delta(A_i) = \Delta$, there exists $j$ in $eRe$ such that $(x+j)A_i = A_i$. Let $a_{i,j}$ be an element in $A_2J$ ($a_{i,j} \in A_2$, $j \in J$). Then $x(a_{i,j} + a_{j,i}) = a_{i,j}$ for some $a_{i,j} \in A_1$, $a_{j,i} \in A_2$, $j \in J$ and $j \in J^i$. Hence $(x+j)a_{i,j} - ja_{j,i} + xa_{i,j} = a_{i,j}2$. On the other hand, $ja_{j,i}, xa_{i,j}$ are contained in $J^{i+2}$. Take the projection of $eJ^i$ onto $A_2$, and $a_{i,j} \in A_2 \cap J^{i+2} = A_2J^i$. Hence $A_2J^i = 0$. Similarly if $x(A_1J \oplus A_2J^i) \supset A_1J^2 \oplus A_2J$, $A_1J = 0$. Therefore $|A_i| \leq 2$.

We observe isomorphisms between sub-factor modules of $A_1$ and $A_2$, and then investigate submodules $X$ in $eJ^i$. It is well known that there exist sub-modules $A_1 \supset C \supset C'$ and $A_2 \supset D \supset D'$ such that $h: C/C' \cong D/D'$ and $X = \{c+d | \in C \oplus D, h(c+c') = d+D'\}$ (cf. [3]). We denote $X$ by $C(h)D$.

a-1) Let $|A_1| = 2$.

a-1-1) $f: A_1E_2 \approx A_2F_2$ ($= gRg$).

Then $\Delta' = gRg = Z$ from 2-i) and $f$ is a unique isomorphism.

a-1-2) ([7], Theorem 17) Assume $f: A_1E_2 \approx F_1F_{i+1}$ ($i > 1$). Consider $\{A_1, A_2, E_2 \oplus F_2, A_1(f)F_i\}$. Since $\Delta(A_1) = \Delta(A_2) = \Delta$, $A_2 \approx A_1(f)F_i$. Further $E_2 \oplus F_2$ being characteristic, from Lemma 7 there exists a unit $x'$ in $eRe$ such that $x'A_1 \subset A_1(f)F_i$. Let $p_i: eJ^i \rightarrow A_j$ be the projection and $x' = x+j$; $xA_1 = A_1$, $j \in eRe$ as usual. Then for a generator $a$ in $A_1$

\[(x+j)a = ar + f(ar) + z_1 + z_2, r \in R, z_1 \in E_2 \text{ and } z_2 \in F_{i+1}^i.
\]

Hence $xa + p_i(ja) = ar + z_1$ and $p_i(ja) = f(ar) + z_2$.

Since $p_i(ja) \in E_2, xa \equiv ar \pmod{E_2}$. Assume $i < p - 1$. Since $ja \in F_{p-1}^i \subset F_{i+1}$, $f(ar) \equiv f(xa) \equiv 0 \pmod{F_{i+1}}$. However $xa$ is a generator of $A_1$, and hence $f = 0$. Therefore $i \geq p - 1$.

a-1-3) See a-2-1) below.

a-1-4) $E_2 \approx F_2$ ($p = 2$). We have the situation of 2-i).

Assume further $f: A_1E_2 \approx F_2$ ($A_3F_2 \approx E_2$), and consider $\{A_1, A_2, A_1(f), E_2 \oplus F_2\}$. Then $A_1 \sim A_1(f)$ by Lemma 7 and so $A_1(f) = xA_1$ for some unit $x$ in $eRe$, since $A_1 \approx A_1(f)$. If $p > 2$, see a-2-2) below.

a-1-5) Let $X$ be a submodule in $eJ^i$.

i) $X = A_1(f_1)F_1 = F_1(f_1^{-1}) (f_1: A_1 \approx F_1/F_1^{i+2})$. If $i = 1$, consider $R/J^{i+3}$. Then this contradicts 2-i). Hence $i \neq 1$, $F_1 = F_{p-1}$ and $F_{i+2} = 0$ from a-1-2). $\{A_1, A_2, E_2 \oplus F_2, A_1(f_1)\}$ shows $A_1(f_1) = xA_1$ for some unit $x$ in $eRe$.

ii) $X = A_1(f_2)A_1^2 (f_2: A_1E_2 \approx A_1F_2)$. Then $X = B_1$ from a-1).

iii) $X = A_1(f_3)F_1 (f_3: A_1E_2 \approx F_1/F_{i+1}, i > 1)$ and hence $i = p - 1$ or $p$ by a-1-2). Then $\{A_1 \oplus F_{i+1}, A_2, E_2 \oplus F_2, A_1(f_3)F_1\}$ shows $A_1(f_3)F_1 = x(A_1 \oplus F_{i+1})$. 
iv) \( X = A_2(f_2^i) \) \( (f_1: E_1 \cong A_2|F_2) \). \( \{ A_1, A_2, E_2 \oplus F_2, A_2(f_i^j) \} \) shows \( A_2 = xA_2(f_i^j) \).

v) \( X = F_i(f_2^j) \) \( (f_3: E_2 \cong F_i|F_{i+1}, i \geq 2) \). In this case \( eI^j = E_2 \oplus F_2 \). Hence this is the case of a-2). Accordingly \( X = B_1, B_1' \) or \( B'' \), provided \( X \) is not isomorphic to a standard submodule in \( eI^j \) via \( x_i \).

Thus we have shown that \( X \) is isomorphic to a standard submodule in \( eI^j \) via \( x_i \) except \( B_1 \) or \( B_1' \) and \( B'' \).

a-2) \( |A_1| = 1 \).

a-2-1) Let \( f: A_1 \approx F_i|F_{i+1} (i < p) \). If \( F_i(f_i^{-1}) \supset xA_1 \) for some unit \( x \) in \( eRe \), \( xA_1 \subset F_{i+1} \subset A_2 \), since \( (F_i(f_i^{-1})) = F_{i+1} \), which is a contradiction from Lemma 2. We note further that \( A_2 \) is characteristic, since \( \Delta(A_2) = \Delta \) and \( A_1 \) is simple. Assume \( \Delta(F_i(f_i^{-1})) \neq \Delta \). Then \( \{ A_2, A_1, F_i(f_i^{-1}) \} \) gives \( F_i(f_i^{-1}) = F_{i+1} \) since they are characteristic. Therefore \( f = f' \). Accordingly, \( \Delta \approx g_4 \approx \bar{Z} \) as given in the proof of 2-i). Further assume \( g: A_1 \approx F_{i+1}/F_{i+1} (j < p) \). Again consider \( \{ A_2, A_1, F_i(f_i^{-1}), F_j(g^{-1}) \} \). Then \( F_i(f_i^{-1}) \supset F_j(g^{-1}) \) if \( i < j \), and so \( F_j(g^{-1}) \subset F_{i+1} \), a contradiction.

a-2-2) Assume that \( f_1, f_2: A_1 \approx F_p \) and they are not given by \( j \) in \( eRe \). Then \( \{ A_2, A_1, A_1(f_1), A_1(f_2) \} \) gives, from Lemmas 6 and 7, that \( A_1(f_1) = xA_1(f_2) \) for some unit \( x \) in \( eRe \). Since \( \Delta(A_1) = \Delta \), there exists \( j \) in \( eRe \) such that \( (x' + j)A_1 = A_1 \). Put \( x = x' + j \). Then for a generator \( a \) in \( A_1 \)

\[(x-j)(a_2 + f_2(a)) = ar + f_1(ar), r \in R.\]

Hence

\[xa = ar, xf_2(a) - ja = f_1(ar).\]

Next assume further that \( g: A_1 \approx F_{i+1}/F_{i+1} (j < p) \). Consider \( \{ A_2, A_1, F_i(q^{-1}), A_1(f) \} \), and \( F_i(q^{-1}) \sim A_i(f) \) since \( F_i(g^{-1}) \) is characteristic. Which is a contradiction. In particular, if \( eFeA_i = 0, A_i(f) \) is characteristic, since \( \Delta(A_i(f)) = \Delta \) (if \( \Delta \neq \Delta(A_i(f)) \)), \( \{ A_1, A_2, A_1(f) \} \) gives \( A_1 \sim A_i(f) \). Then \( f \) is given by \( j \) from Lemma 6). Hence \( f_1 = f_2 \) from the first paragraph, and so \( \Delta \approx g \approx \bar{Z} \) as in the proof of 2-2).

b) \( A_2/A_2J^k \) \( (k \geq 1) \) is uniserial and \( A_2J^k = \sum_{i=1}^{s} B_i \) \( (s \geq 2) \). Then \( A_1 \) is simple from the initial paragraph of ii). Then \( DJ^k = eI^{i+k} = A_2J^k \). Since \( eI^{i+k} = B_1 \oplus \cdots \oplus B_n \), \( s \leq 3 \) from Lemma 15, and \( [\Delta: \Delta(B_i)] \leq 2 \) for all \( i \) by Lemmas 2 and 7. If \( [\Delta: \Delta(B_i)] = 1 \) and \( [\Delta: \Delta(B_2)] = 2 \), \( \{ A_1, B_1, B_2, B_3 \} \) derives a contradiction. Hence either \( [\Delta: \Delta(B_i)] = 1 \) for all \( i \) \( (b_i) \) or \( [\Delta: \Delta(B_i)] = 2 \) for all \( i \) \( (b_i) \). In the former case \( s = 2 \) by Lemma 7 and in the latter case \( s = 2 \) and \( B_2 = xB_1 \) for some unit \( x \) in \( eRe \) by Lemma 10.
Then \( \{A_1, B_1, B_2, J(B_1) \oplus J(B_2) \} \) implies \( J(B_1) = 0 \) \((|B_1| = |B_2|)\). If \( f: B_1 \approx B_2 \), \( \{A_1, B_1, B_2, B_i(f) \} \) derives a contradiction. Hence \( B_1 \approx B_2 \). We can show as before that \( B_2 \) is uniserial.

\( b_1 \) This is the case of \( a \cdot 2 \cdot 1 \).

\( b_1 \) \(2\) Assume \( f: A_1 \approx B_1 \). \( \{A_1, A_i(f), B_1, B_2 \} \) derives \( A_1 \approx A_i(f) \), i.e., \((x+j)A_1 = A_i(f): xA_1 = A_i \) and \( j \in e_f e \). \((x+j)a = ar + f(ar); A_i = aR, r \in R. \) Hence \( xa = ar \) and \( ja = f(ar) \). Put \( xa = b \), and \( A_i = bR. f(b) = jx^{-1}b. \)

\( b_1 \) \(3\) Assume \( f: A_1 \approx H_i \bigcup H_{i+1} \). \( \{A_1, B_1, B_2, H_i(f^{-1}) \} \) shows \( A_1 \approx H_i(f^{-1}) \). Hence \( H_{i+1} = 0 \) and \( f \) is given by \( j_i \) as above (cf. Lemma 6).

\( b_1 \) \(4\) Assume \( f: B_i \approx H_i \bigcup H_{i+1} \). \( \{A_1, B_1, B_2, H_i(f^{-1}) \} \) derives \( B_i \approx H_i(f^{-1}) \), since \( \Delta(B_i) = \Delta \). Hence \( H_{i+1} = 0 \) and \( f \) is given by \( j_i \) from Lemma 6.

\( b_2 \) \([\Delta: \Delta(B_i)] = 2 \) for \( i = 1, 2, (B_2 = xB_1) \).
\( \{A_1, B_1, B_2, J(B_1) \oplus J(B_2) \} \) shows, from Lemma 2, that \( J(B_2) = 0 \), i.e., \( B_2 \) is simple. Further since \( \Delta(A_1) = \Delta \) and \([\Delta: \Delta(B_2)] = 2, [\Delta: \Delta(E)] = 2 \) for all simple submodules \( E \) in \( V = B_1 \bigcup B_2 \) by Lemmas 2 and 8. Hence \( V \) satisfies \((\#), 1) \) by Lemma 7.

\( b_2 \) \(1\) If \( A_1 \approx F_i / F_{i+1}, \Delta = \bar{Z} \) by \( a \cdot 1 \cdot 3 \). Hence \( \Delta(B_i) = \Delta. \)

\( b_2 \) \(2\) Assume \( f: A_1 \approx B_1 \). \( \{A_1, B_1, B_i(f) \} \) derives \( A_1 \approx A_i(f) \). Hence \( f \) is given by \( j_i \) as \( b_1 \) \(2\).

**Remark 17.** If \( R \) is an algebra over an algebraically closed field \( K, \Delta = \bar{Z} \) and the first part of \( a \cdot 2 \cdot 2 \) does not occur (take \( f_2 = kf_1, k \neq 1; k \in K \)). We can express \( f \) in \( a \cdot 1 \cdot 2 \) as an element in \( e_f e \), however it is little complicated (cf. [7], Theorem 17).

In order to make the converse version clear, we illustrate the structure of Lemmas 10~16 as follows:

1) (Lemma 10)

\[
\begin{array}{c|c|c}
  eR & eJ^i & eJ^{i+1} \\
  \hline
  A_1 & B_1 & 0 \\
  A_2 = xA_1 - B_2 = xB_1 - 0 \\
\end{array}
\]

\([\Delta: \Delta(A_1)] = [\Delta: \Delta(A_2)] = 2, \) every characteristic submodule in \( eJ^i \) is linear with respect to the inclusion and \( [\Delta: \Delta(C)] = 2 \) for any non-characteristic submodule \( C \) in \( eJ^i \). Further those \( C \) are related to one another with respect to \( \sim \).

2) (Lemma 11)

\[
\begin{array}{c|c}
  eR & eJ^i \\
  \hline
  A_1 & 0 \\
  A_2 = x_2 A_1 - 0 \\
  A_3 = x_3 A_1 - 0 \\
\end{array}
\]

(3)
$[\Delta: \Delta(A_1)] = [\Delta: \Delta(A_2)] = [\Delta: \Delta(A_3)] = \Delta$ and $A_1 \oplus A_2 \oplus A_3$ satisfies $(\#_i, 1)$ and $(\#_j, 2)$. Further $[\Delta: \Delta(C)] \leq 3$ for every submodule $C$ in $A_1 \oplus A_2 \oplus A_3$. ($A_3$ may be zero.)

3) (Lemma 12, i))

$$eR \begin{array}{ccc} & e & f^i \\ A_1 & - & 0 \\ A_2 & 0 & - \\ A_3 & = x_A A_2 - 0 \\ \end{array}$$

$[\Delta: \Delta(A_1)] = 1$ and $[\Delta: \Delta(A_2)] = [\Delta: \Delta(A_3)] = 2$. Further $A_2 \oplus A_3$ satisfies $(\#_1, 1)$.

4) (Lemma 12, ii-a))

$$eR \begin{array}{ccc} & e & f^i & e^*^i+/1 \\ A_1 & - & E_n - 0 \\ A_2 & 0 & - \\ \end{array}$$

$[\Delta: \Delta(A_1)] = 1$, $[\Delta: \Delta(A_2)] = 2$ and $A_2 \oplus E_n$ satisfies $(\#_1, 1)$ except $E_n$.

5) (Lemma 12, ii-b ii-b-2-ii-1))

$$eR \begin{array}{ccc} & e & f^i & e^i^+/ & e^i^p \\ A_1 & - & 0 & B_1 & - 0 \\ A_2 & 0 & - & B_2 & 0 \\ \end{array}$$

$[\Delta: \Delta(A_1)] = 1$, $[\Delta: \Delta(A_2)] = 2$ and $[\Delta: \Delta(B_1)] = [\Delta: \Delta(B_2)] = 1$. $A_1 \oplus A_2 \oplus (A_2)$ satisfies $(\#_1, 1)$ except $A_1$. ($B_2$ may be zero.)

5')

$$eR \begin{array}{ccc} & e & f^i & e^i^+/ \\ A_1 & - & 0 & B_1 & - 0 \\ A_2 & 0 & - & B_2 & 0 \\ \end{array}$$
[\Delta: \Delta(B_2)] = [\Delta: \Delta(B_1)] = 2 and \( B_2 \approx B_1 \). \( B_1 \oplus B_2 \) satisfies (\#, 1).

6) (Lemma 16, 1))

\[
\begin{array}{ccc}
\Delta & \Delta(\Lambda) & \Delta(\Lambda) \\
\Delta(\Lambda) & \Delta(\Lambda) & \Delta(\Lambda) \\
\hline
0 & E_1 & 0 \\
A_2 & 0 & A_0 \\
A_1 & 0 & A_0 \\
\end{array}
\]

[\Delta: \Delta(A_1)] = [\Delta: \Delta(A_2)] = [\Delta: \Delta(A_3)] = 1. If \( f: A_2 \approx E_2 \), \( f \) is given by \( j; j \in eF_e \). (\( A_3, A_4 \) and \( E_4 \) may be zero.)

7) (Lemma 16, 2-i))

\[
\begin{array}{ccc}
\Delta & \Delta(\Lambda) \\
\Delta(\Lambda) & \Delta(\Lambda) \\
\hline
A_1 & 0 \\
A_2 & 0 \\
\end{array}
\]

[\Delta: \Delta(A_1)] = [\Delta: \Delta(A_2)] = 1 and \( \Delta \approx \Delta' \approx \overline{Z} \).

8) (Lemma 16, 2-ii-a-1))

\[
\begin{array}{ccc}
\Delta & \Delta(\Lambda) & \Delta(\Lambda) \\
\Delta(\Lambda) & \Delta(\Lambda) & \Delta(\Lambda) \\
\hline
A_1 & E_2 & 0 \\
A_2 & F_2 & F_0 \\
\end{array}
\]

[\Delta: \Delta(A_1)] = [\Delta: \Delta(A_2)] = 1. If \( f: A_1[B_2 \approx A_3/F_2, \Delta \approx \Delta' \approx \overline{Z} \). Every submodule except \( B_1, B_1' \) and \( B'' \) is isomorphic to a standard submodule via \( x \). (If \( n=2 \) and \( E_2 \approx F_2, \Delta \approx \Delta' \approx \overline{Z} \).) If \( E_2=0 \), the conditions in a-2) of Lemma 16 are fulfilled.

9) (Lemma 16, 2-ii-b))

\[
\begin{array}{ccc}
\Delta & \Delta(\Lambda) & \Delta(\Lambda) \\
\Delta(\Lambda) & \Delta(\Lambda) & \Delta(\Lambda) \\
\hline
A_1 & 0 \\
B_1 & 0 \\
A_2 & 0 \\
B_2 & F_0 \\
\end{array}
\]

[\Delta: \Delta(A_1)] = [\Delta: \Delta(A_2)] = 1, [\Delta: \Delta(B_1)] = [\Delta: \Delta(B_2)] = 1. If \( f: A_1 \approx B_1 \), \( f \) is given by \( j; j \in eF_e \). Similar facts hold for other cases.

10) (Lemma 16, 2-ii-b_2))
We shall show that if $eR$ has one of the structures of the above diagrams 1)~10), then $R$ is a right US-4 ring with $(\ast, 1')$. It is clear from the diagrams that $(\ast, 1')$ holds. Let $\{U_i\}_{i=1}^4$ be a set of submodules in $eR$.

Diagram 1). If $U_1$ and $U_2$ are characteristic, $U_1 \supset U_2$ or $U_1 \subset U_2$. Hence $U_1 + U_2$ satisfies $(\ast, 2)$ by [4], Corollary 1 of Theorem 2. Hence $D = \sum_{i=1}^4 U_i$ satisfies $(\ast, 4)$ by [2], Lemma 1. Assume that $U_1 \cap U_2 \supset eJ^i$. Then $U_i$ for $i=1, 2$ is characteristic, and hence $D$ satisfies $(\ast, 4)$ from the above. Next assume that $U_1 \supset eJ^i$ and $eJ^i \supset U_j$ for $j > 1$. Since $\Delta(U_1) = \Delta, U_1 \oplus U_2$ satisfies $(\ast, 2)$ by [4], Corollary 1 of Theorem 2. Finally assume $eJ^i \supset U_j$ for all $j$. If $\{U_j\}_{j=1}^4$ is a set of non-characteristic submodules, then we may assume $U_i \supset x_j U_2 \supset x_j U_3$ for some units $x_j$ in $eR$ by assumption. Since $[\Delta: \Delta(U_i)] = 2$, $U_1 \oplus U_2 \oplus U_3$ satisfies $(\ast, 3)$ by [4], Corollary 3 of Theorem 2. Therefore $D$ satisfies $(\ast, 4)$.

2) As is shown in 1), we may assume that $eJ^i \supset U_j$ for all $j$. Then $U_i \supset x_j U_2 \supset x_j U_3 \supset x_j U_4$ by assumption, where the $x_j$ are units in $eR$. Then from the assumption $[\Delta: \Delta(C)] \leq 3$ and the argument of the proof of [4], Corollary 3 of Theorem 2, $D$ satisfies $(\ast, 4)$.

3) Let $eJ^i \supset U_j$ for all $j$. Then $U_i = A_1 \oplus B_i$ or $U_i \subset A_2 \oplus A_3$ by assumption, where $B_i \subset A_2 \oplus A_3$. First assume $U_i \subset A_2 \oplus A_3$ or $U_j = A_1 \oplus B_j$ ($B_j = 0$) for all $j \leq 3$. Then $D$ satisfies $(\ast, 4)$ by [4], Corollary 3 of Theorem 2 (note $A_1$ and $A_2 \oplus A_3$ are characteristic and see the remark above). If $U_i = A_1$ and $U_2 = A_1 \oplus B_2$, $U_1 \oplus U_2$ satisfies $(\ast, 2)$ by [4], Corollary 1 of Theorem 2. Thus $D$ satisfies $(\ast, 4)$.

4) Every submodule in $eJ^i$ is isomorphic to a standard submodule in $eJ^i$ via $x_1$. Hence we may assume that all $U_j$ are standard. Then $D$ satisfies $(\ast, 4)$ by [4], Corollaries 1~3 of Theorem 2.

5) and 5′) Let $eJ^i \supset U_i \supset A_2 J$ and $U_i = A_1 \oplus A_2$. Then $U_i/A_2 J = x(A_2/A_2 J)$, and so $x A_2 = U_i$. Further $A_1 \oplus A_2 J$ is characteristic. If $U_i = A_1 \oplus A_2 J$ and $U_i \subset A_2 J$, $U_i \oplus U_2$ satisfies $(\ast, 2)$. Accordingly we may assume that $U_i$ is $A_1$ or a submodule of $A_2$. Therefore $D$ satisfies $(\ast, 4)$.

6) and 7) These are clear.
8) First we note $B_1 \supseteq E_2 \oplus F_2 \supseteq B'$ ($E_2 \oplus F_2 \supseteq B''$) and $B', B''$ do not appear simultaneously. If the $U_i$ are standard for all $i$, $U_i \sim U_j$ for some pair $i, j$. Hence $D$ satisfies (**, 4) by [4], Corollary 2 of Theorem 2. The conditions given in Lemma 16 show that $A_i \sim A_i(f)$, $F_i \sim F_i'(g^{-1})$, ... etc.. Hence we obtain the desired result.

9) and 10) These are simpler than 8), (if $A_i \approx F_i/F_{i+1} \supseteq B_1 \oplus B_2$), $D \approx \bar{Z}$. Hence $D(C) = \Delta$ for any submodule $C$ in $eR$).

Thus we obtain

**Theorem 2.** $R$ is a right US-4 (basic) ring with $(*, 1')$ if and only if $eR$ has one of the structures given in Lemmas 10~16 (cf. Diagrams 1)~10)) for each primitive idempotent $e$.

3. Hereditary rings

In this section, we shall study a hereditary and right US-3 (resp. US-4) ring $R$. If $R$ is hereditary, $(*, 1')$ holds, and hence we can make use of the results in the previous sections.

**Lemma 18.** Assume that $R$ is basic and hereditary. Then a submodule $A$ in $eR$ is characteristic if and only if $\Delta(A) = \Delta$. Every non-zero element in $\text{Hom}_R(eR, fR)$ is a monomorphism, where $e$ and $f$ are primitive idempotents.

Proof. The second half is clear (see [9], Lemma 2). Hence, since $eje = 0$, the first one is clear

From now on we assume that $R$ is a hereditary and basic ring. First we assume further that $R$ is right US-3.

**Theorem 3.** Let $R$ be a hereditary (and basic) ring. Then $R$ is a right US-3 ring if and only if $eR$ has the following structure for each primitive idempotent $e$:

i) $eR\lceil eJ'$ is uniserial for some $t$ and

ii) $eJ' = 0$ or $eJ' = A \oplus B$ such that either

a) $A$ and $B$ are simple and $A \oplus B$ satisfies ($\emptyset$, 1), and $[\Delta : \Delta(A)] = 2$, or

b) $A$ is simple, $B$ is uniserial and $A$ is not isomorphic to any sub-factor modules of $B$ (and hence $\Delta(A) = \Delta(B) = \Delta$).

Proof. If $R$ is right US-3, $eR$ has the structure in Theorem 1. We consider the case b) of Theorem 1. Assume that $f: A \approx (the \ socle \ of \ B)$. Then $\{A, A(f), B\}$ derives a contradiction, since $A$ and $B$ are characteristic by Lemma 18. Thus we obtain the theorem from Theorem 1.

Let $R$ be a basic hereditary ring. Then
where the $\Delta_i$ are division rings and the $M_{ij}$ are left $\Delta_i$- and right $\Delta_j$-modules [1].

We shall express explicitly the content of Theorem 3 for $M_{ij}$ in a row of the above ring.

1) 

\[
\begin{pmatrix}
\cdots & \Delta_i 0 & \cdots & 0 & \Delta_i 0 & \cdots & \Delta_i 0 & \cdots & 0 & \Delta_i 0 \\
0 & \cdots & 0 & \Delta_i 0 & \cdots & 0 & \Delta_i 0 & \cdots & 0 & \Delta_i 0 \\
\vdots & & \ddots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\Delta_i & \cdots & M_{1n} & \cdots & M_{2n} & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

\[R = \]

where the $\Delta_i$ are division rings and the $M_{ij}$ are left $\Delta_i$- and right $\Delta_j$-modules [1].

We shall express explicitly the content of Theorem 3 for $M_{ij}$ in a row of the above ring.

1) 

\[
(0 \cdots \Delta_i 0 \cdots 0) \Delta_i 0 \cdots \Delta_i 0 \cdots 0)
\]

2) 

\[
(0 \cdots \Delta_i 0 \cdots \Delta_i 0 \cdots \Delta_i 0 \cdots 0)
\]

(4) 

\[
\begin{pmatrix}
\cdots & \Delta_i 0 & \cdots & 0 & \Delta_i 0 & \cdots & \Delta_i 0 & \cdots & 0 & \Delta_i 0 \\
0 & \cdots & 0 & \Delta_i 0 & \cdots & 0 & \Delta_i 0 & \cdots & 0 & \Delta_i 0 \\
\vdots & & \ddots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\Delta_i & \cdots & M_{1n} & \cdots & M_{2n} & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

\[R = \]

As is given in the proof of [9], Theorem 1, we can show a ring monomorphisms $\rho_{rs}: \Delta_s \rightarrow \Delta_r$ for $r<s<k$ such that $xu_r = u_r \rho_{rs}(x)$ for $x \in \Delta_r$ and $\rho_{rs} \rho_{sw} = \rho_{sw}$.

Next we shall characterize a hereditary (basic) and right US-4 ring. If $R$ is hereditary, some results in the previous sections may not occur as shown in Theorem 3. We shall observe them.

In the case b) of Lemma 12, $A_2$ is simple.

Because, since $A_1$ is simple and $[\Delta: \Delta(A_2)] = 2$, $A_1 \cong A_2 / J(A_2)$. Hence $A_1 \cong A_2$ by Lemma 18.

We shall observe the conditions in Lemma 16 for a hereditary ring. a-1-1), a-1-2), a-1-3), any of b-1-1)(~4) and b-2-2) do not occur from Lemma 18. For instance, if $f': A_i / E_2 \cong F_{p-1} / F_p$ (a-1-2), $f: A_i \cong F_{p-1}$ by Lemma 18. Then $A_i \sim A_i(f)$ by a-1-5). However, $A_i$ is characteristic, and so $A_1 = A_1(f)$. Therefore $f=0$.

We shall use the notations after Theorem 3.

**Lemma 19.** In case 2-i) in Lemma 16, $e_i R$ is of the form $(0, \cdots, Z, \cdots)$.
0 \cdots \overline{Z} \cdots 0). In case of 2-a-1-4) in Lemma 16, \(A_1\) (resp. \(A_2\)) is of the form 
\((0, \cdots, \overline{Z}, 0, \overline{Z}, 0 \cdots)\) (resp. 
\((0, \cdots, \overline{Z}, 0, \overline{Z}, 0 \cdots)\)).

Proof. Let \( E_2 \approx F_\varphi \cong \overline{e_k R}\) by Lemma 16. Let \(A_2 \approx e_s R\). Then \(e_s R\) is uniserial and \( M_{s \overline{k}} = u_{s \overline{z}} \overline{Z} \) \((\approx F_\varphi)\). Since \( M_{s \overline{k}} \) is a left \(\Delta_s\)-module, \(\Delta_s \subset \overline{Z}\). Hence \(\Delta_s = \overline{Z}\). We have the same for 2-i).

Thus we have

**Theorem 4.** Let \(R\) be a hereditary (basic) ring. Then \(R\) is right US-4 if and only if for each \(e = e_i, e \in R\) has one of the following structures: 1~11

1) 
\((0 \cdots 0 \Delta, 0 \Delta, 0 \Delta, 0 \cdots \Delta, 0 \cdots 0)\)

2) (Lemma 10) 
\((0 \cdots 0 \Delta, 0 \Delta, 0 \cdots \Delta, 0 \cdots 0) \begin{pmatrix} u_{t+1} \Delta i_{t+1} \\ v_{t+1} \Delta i_{t+1} \end{pmatrix} \begin{pmatrix} u_{t+2} \Delta i_{t+2} \\ v_{t+2} \Delta i_{t+2} \end{pmatrix} \cdots 0 \cdots 0) \begin{pmatrix} \Delta: \Delta, \Delta \end{pmatrix} = 2 \) \((i = 1, 2)\) and \(u_{t+s+1}, v_{t+s+1}\) may be zero. The conditions in Lemma 10 are satisfied.

3) (Lemma 11) 
\((0 \cdots 0 \Delta, 0 \Delta, 0 \cdots \Delta, 0 \cdots 0) \begin{pmatrix} u_{t+1} \Delta i_{t+1} \\ v_{t+1} \Delta i_{t+1} \end{pmatrix} \begin{pmatrix} u_{t+2} \Delta i_{t+2} \\ v_{t+2} \Delta i_{t+2} \end{pmatrix} \cdots 0 \cdots 0) \begin{pmatrix} \Delta: \Delta, \Delta \end{pmatrix} = 3 \) \((i = 1, 2)\) and \(A_i \oplus A_j \oplus A_l\) satisfies \((\#, 1)\) and \((\#, 2)\). \(u_{t+1}\) may be zero.

4) (Lemma 12-i) 
\((0 \cdots 0 \Delta, 0 \Delta, 0 \cdots \Delta, 0 \cdots 0) \begin{pmatrix} u_{t+1} \Delta i_{t+1} \\ v_{t+1} \Delta i_{t+1} \end{pmatrix} \begin{pmatrix} u_{t+2} \Delta i_{t+2} \\ v_{t+2} \Delta i_{t+2} \end{pmatrix} \cdots 0 \cdots 0) \begin{pmatrix} \Delta: \Delta, \Delta \end{pmatrix} = 2 \) \((i = 1, 2)\) and \(A_i \oplus A_j \oplus A_l\) satisfies \((\#, 1)\).

5) (Lemma 12-ii-a and b) 
\((0 \cdots 0 \Delta, 0 \Delta, 0 \cdots \Delta, 0 \cdots 0) \begin{pmatrix} u_{t+1} \Delta i_{t+1} \\ v_{t+1} \Delta i_{t+1} \end{pmatrix} \begin{pmatrix} u_{t+2} \Delta i_{t+2} \\ v_{t+2} \Delta i_{t+2} \end{pmatrix} \cdots 0 \cdots 0) \begin{pmatrix} \Delta: \Delta, \Delta \end{pmatrix} = 2, and \(u_{t+1}, u_{t+2}, u_{t+3}\) satisfies \((\#, 1)\), except \(u_{t+1}, u_{t+2}, u_{t+3}\) may be zero.

6) (Lemma 16, 1)
\[
\begin{align*}
\left(\begin{array}{c}
\Delta_0 \cdots \Delta_{i_0} \cdots \Delta_{i_0} \cdots 0 \\
u + 1 \Delta_{i_0 + 1} \\
0 \\
v_{i_0 + 2} \Delta_{i_0 + 2} \\
0 \\
0 \\
v_{i_0 + 3} \Delta_{i_0 + 3} \\
0 \\
0
\end{array}\right) & \quad \text{...... } A_1 \\
\left(\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right) & \quad \text{...... } A_2 \\
\left(\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right) & \quad \text{...... } A_3 \\
\Delta(A_i) &= \Delta \; (i = 1, 2, 3) \text{ and } u_{i+1} \text{ may be zero.}
\end{align*}
\]

\[
\begin{align*}
\Delta_1 & \subseteq \Delta_{i_1} \subseteq \cdots \subseteq \Delta_{i_2} \subseteq \Delta_{i_2 + 1} \\
\Delta_{i_2 + 1} & \subseteq \Delta_{i_2} \subseteq \cdots \subseteq \Delta_{i_3} \\
\end{align*}
\]

7) (Lemma 16, 2-i))

\[
\begin{align*}
\left(\begin{array}{c}
0 \\
Z_0 \\
Z \\
0 \\
0 \\
\left(\begin{array}{c}
u_{i+1} \bar{Z} \\
v_{i_0 + 1} \bar{Z}
\end{array}\right) \\
0
\end{array}\right) & \quad \text{...... } A_2 \\
\left(\begin{array}{c}
\left(\begin{array}{c}
u_{i+1} \bar{Z} \\
v_{i_0 + 1} \bar{Z}
\end{array}\right) \\
0
\end{array}\right) & \quad \text{...... } A_1 \\
\end{align*}
\]

8) (Lemma 16, 2-ii-a))

\[
\begin{align*}
\left(\begin{array}{c}
u_{i+1} \Delta_{i_1 + 1} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right) & \quad \text{...... } A_1 \\
\left(\begin{array}{c}
u_{i+2} \Delta_{i_1 + 2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right) & \quad \text{...... } A_2 \\
\Delta(A_i) &= \Delta \; (i = 1, 2), \; u_{i+2} \text{ or } \{v_{i+4}, \ldots, v_p\} \text{ may be zero.}
\end{align*}
\]

\[
\begin{align*}
\Delta_1 & \subseteq \Delta_{i_1} \subseteq \cdots \subseteq \Delta_{i_2} \subseteq \Delta_{i_2 + 1} \\
\Delta_{i_2 + 1} & \subseteq \Delta_{i_2} \subseteq \cdots \subseteq \Delta_{i_3} \\
\end{align*}
\]

9) (Lemma 16, 2-ii-a'))

\[
\begin{align*}
\left(\begin{array}{c}
0 \\
\bar{Z} \\
\bar{Z} \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right) & \quad \text{...... } A_1 \\
\left(\begin{array}{c}
\left(\begin{array}{c}
u_{i+1} \bar{Z} \\
v_{i+2} \bar{Z}
\end{array}\right) \\
0
\end{array}\right) & \quad \text{...... } A_2 \\
\end{align*}
\]

\[
\begin{align*}
u_{i+2} \text{ may be zero.}
\end{align*}
\]

10) (Lemma 16, 2-ii-b1))

\[
\begin{align*}
\left(\begin{array}{c}
\nu_{i+1} \Delta_{i_1 + 1} \\
v_{i+2} \Delta_{i_1 + 2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right) & \quad \text{...... } A_1 \\
\left(\begin{array}{c}
u_{i+2} \Delta_{i_1 + 1} \\
v_{i+3} \Delta_{i_2 + 1} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right) & \quad \text{...... } B_1 \\
\left(\begin{array}{c}
u_{i+3} \Delta_{i_2 + 1} \\
v_{i+3} \Delta_{i_3 + 2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right) & \quad \text{...... } A_2 \\
\left(\begin{array}{c}
u_{i+3} \Delta_{i_3 + 2} \\
v_{i+3} \Delta_{i_3 + 2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right) & \quad \text{...... } B_2
\end{align*}
\]
\( \Delta(A_i) = \Delta(B_i) = \Delta \quad (i = 1, 2). \)

\[
\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_{t+1} \supset \Delta_{t+2} \supset \cdots \supset \Delta_t \supset \Delta_{t+1} \]

11) (Lemma 16, 2-ii-b))

\[
0 \cdots \Delta_0 \cdots \Delta_1 \cdots \Delta_i 0 \cdots
\]

\[
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & \omega_{t+1} & \omega_{t+2} & \cdots & \omega_{t+1}
\end{pmatrix}
\]

\[
\Delta(A_i) = \Delta \quad \text{and} \quad [\Delta : \Delta : (B_i)] = 2 \quad (i = 1, 2). \quad \omega_{t+1} \Delta_{t+1} \oplus \omega_{t+1} \Delta_{t+1} \text{ satisfies (1, 1),}
\]

where \( \mathbb{Z} = \mathbb{Z}/2 \), the \( \Delta \)'s are division rings and \( \Delta_i \subset \Delta_i \subset \Delta_{i+1} \) except 6, 8, 10), and 11). The series: \( 0 \cdots \Delta_0 \cdots \Delta_i \cdots \) on the same level means a uniserial module.

4. Left serial rings

We shall investigate the same problem for a left serial ring \( R \). In this case \((*, 1')\) holds, too by [11], Corollary 4.2. Therefore we can make use of the results in §§ 1 and 2.

From now on we always assume that \( R \) is a left serial ring.

**Lemma 20.** If \( eJ^i = A_1 \oplus A_2 \) and the \( A_i \) are uniserial, every submodule \( E \) in \( eJ^i \) is isomorphic to a standard submodule \( B_1 \oplus B_2 \) via \( x_i : x \) is a unit in \( eRe \), where \( B_i \subset A_i \).

See the proof of [3], Theorem 1.

**Lemma 21.** Let \( eJ^i = A_1 \oplus A_2 \) and the \( A_i \) hollow. If \( \Delta(A_i) \neq \Delta \), there exists a unit \( x \) in \( eRe \) such that \( xA_1 = A_2 \).

Proof. Since \( \Delta(A_i) \neq \Delta \), there exists a unit \( y \) in \( eRe \) such that \((y + j')A_i \subset A_i \) for all \( j' \) in \( eJ \). Let \( p \) be the projection of \( eJ^i \) onto \( A_2 \). Then \( f = py_{ij} \) is an element in \( \text{Hom}(A_1, A_2) \). If \( f \) is not an epimorphism, \( f = j_i \) for some \( j \) in \( eJ \), since \( A_2 \) is a hollow module \((\subset eJ^{i+1}) \) and \( R \) is left serial. Then \((y - j)A_1 \subset A_1 \),
a contradiction. Hence there exists a unit $x$ in $eRe$ such that $x_1 = f$, and so $x_2 A = A_1$.

**Lemma 22.** Let $eJ' = A_1 \oplus A_2$ be as in Lemma 21. If $\Delta(A_1) = \Delta$, $\Delta(A_1J^* \oplus A_2J') = \Delta$.

**Proof.** From Lemma 21, $\Delta(A_2) = \Delta$. Hence we may assume $k \leq k'$. Let $x$ be any unit element in $eRe$. Since $\Delta(A_1) = \Delta$, there exists $j$ in $eRe$ such that $(x+j)A_1 = A_1$. Hence $(x+j)(A_1J^* \oplus A_2J') \subseteq A_1J^* + (x+j)A_2J' \subseteq A_1J^* \oplus A_2J'$, and so $x = x+j \in (A_1J^* \oplus A_2J')$.

From Theorem 1, Lemmas 21, 22 and [8], Proposition 2, we obtain

**Theorem 5.** Let $R$ be a left serial ring. Then $R$ is a right US-3 ring if and only if $eR$ has the following structure for each primitive idempotent $e$:

- There exists an integer $t$ such that
  - $eR/eJ^*$ is uniserial and
  - $eJ^* = 0$ or $eJ^*$ is a direct sum of a simple module and a uniserial module.

Finally we shall give a characterization of a left serial and right US-4 ring. As was shown in the previous section, we shall refine the results in § 2.

In Lemma 10, every submodule in $eJ^*$ is standard up to $x_i$ ($x$ is a unit in $eRe$) by Lemma 20. Further since $\Delta(A_1 \oplus A_2J) = \Delta$,

$$A_1 \oplus A_2 \supseteq (A_1 \oplus J)(A_2) \supseteq 0$$

is the set of all characteristic submodules in $eJ^*$.

From the above proof we have

**Remark 23.** Let $R$ be left serial and assume $eJ^* = A_1 \oplus A_2$; the $A_i$ are uniserial. If $[\Delta: \Delta(A_i)] = 2$, $[\Delta: \Delta(C)] \leq 2$ for every submodule $C$ in $eJ^*$ and \{eJ^*\}$ is the set of characteristic submodules in $eJ^*$. Hence, if $R$ is left serial, i), ii) and iii) in Lemma 10 imply iv) and v). However hereditariness does not as is shown from the following example:

Let $K \subseteq L$ be fields such that $[L: K] = 2$. Put

$$R = \begin{pmatrix}
L & L & L \otimes L & L \otimes L \\
0 & K & L & L \\
0 & 0 & L & L \\
0 & 0 & 0 & L
\end{pmatrix}$$

Then $R$ is hereditary. Put $L = 1K + uK$, $e_{12} = e$, and $eJ = A_1 \oplus A_2$; $A_1 = 1e_{12}R$, $A_2 = ue_{12}R$ satisfy i), ii) and iii) in Lemma 10. Further $[\Delta: \Delta(B)] = 2$ for any submodule $B$ in $eJ^*$ if $\Delta \not= \Delta(B)$, since $[L: K] = 2$. \{eJ, eJ^*, eJ^3, (1 \otimes u \pm u \otimes 1)e_{12}R,$
(1 ⊗ u ± u ⊗ 1)e_i R} is the set of characteristic submodules provided \( u \in K \), and 
(1 ⊗ 1)e_i R \sim (1 ⊗ 1 + u \otimes x)e_i R \), provided \( x \notin K \).

**Lemma 24.** Let \( B_1 \) and \( B_2 \) be simple submodules in \( e^i R \) and \( V = B_1 \oplus B_2 \). 
If \( B_1 \cong B_2 \), \( V \) always satisfies (\( \# \), 1).

**Proof.** Since \( R \) is left serial, every simple submodule in \( V \) is isomorphic to \( B_1 \) via \( x_i \); \( x \) is a unit in \( e Re \). Hence \( V \) satisfies (\( \# \), 1).

In Lemma 12, we do not have the case \( t = 2 \) by Lemma 21.
In Lemma 16, we have always \( A_1 \cong A_2 \), since \( \Delta(A_1) = \Delta(A_2) = \Delta \). Hence 2-i), 2-a-1-1), 2-a-2-3) and \( p = 2 \) in 2-a-2-4) do not occur. Similarly 2-a-2-1) does not occur.

Thus we obtain

**Theorem 6.** Let \( R \) be a left serial ring. Then \( R \) is right US-4 if and only if, for each primitive idempotent \( e \), \( eR \) has one of the following structures:

1) \( eR \) is uniserial: \( eR \ eJ \)

2) \( eR \ eJ^{i-1} \ eJ^i \ eJ^{i+1} \)

3) \( eR \ eJ^{i-1} \ eJ^i \)

4) \( eR \ eJ^{i-1} \ eJ^i \)

\( [\Delta: \Delta(A_i)] = 2 \). In this case \( A_1 \cong A_2 \) and \( B_i \) may be zero.

\( [\Delta: \Delta(A_i)] = 3 \) and \( A_1 \oplus A_2 \oplus A_3 \) satisfies (\( \# \), 2). In this case \( A_1 \cong A_2 \cong A_3 \).

\( \Delta(A_1) = \Delta, [\Delta: \Delta(A_i)] = 2 \) (i=2, 3). In this case \( A_2 \cong A_3 \).
5) $e, e^{-1} J^i, J^{i+1}, J^p$

$\Delta(A_i) = \Delta$ $(i = 1, 2, 3)$. In this case $A_1 \cong A_2$ and $A_2$ may be zero.

6) $e, e^{-1} J^i, J^{i+1}, J^p$

$\Delta(A_i) = \Delta$ $(i = 1, 2)$.

7) $e, e^{-1} J^i, J^{i+1}, J^k, J^{k+1}, J^p$

$\Delta(A_i) = \Delta$ $(i = 1, 2, 3)$ and $\Delta(B_j) = \Delta$ $(j = 1, 2)$.

8) $e, e^{-1} J^i, J^{i+1}, J^{p-1}, J^p$

$\Delta(A_1) = \Delta(A_2) = \Delta$ and $[\Delta : \Delta(B_1)] = 2$. In this case $B_1 \cong B_2$, where each straight line means "uniserial".

5. Examples

We shall give examples of hereditary (resp. left serial) and right US-3 (resp. US-4) rings. Let $K$ be a field. By $L$ and $L'$ we denote extension fields of $K$ with $[L : K] = 2$ and $[L' : K] = 3$, respectively, and $\mathbb{Z} = \mathbb{Z}/2$, where $\mathbb{Z}$ is the ring of integers.

The following two rings are hereditary, left serial and right US-3 rings.
is the second type b) of Theorem 1 and is the first type a) of Theorem 1.

On the other hand

\[
\begin{pmatrix}
K & K & L \\
K & K & K \\
0 & 0 & K
\end{pmatrix}
\]
is a hereditary, non-left serial and right US-3 ring, and

\[
\begin{pmatrix}
L & L & 0 \\
0 & K & K \\
0 & 0 & K
\end{pmatrix}
\]
with \(e_{12}e_{23}=0\) is a left serial, non-hereditary and right US-3 ring.

Next we shall give hereditary and right US-4 rings for each structure in Theorem 4. However, we can not construct an example of the case 5) from the reason given in Remark 13.

\[
\begin{align*}
1 & \begin{pmatrix} K & K & K \\ K & K & K \\ 0 & 0 & K \end{pmatrix} & 2 & \begin{pmatrix} L & L & L \\ L & L & L \\ 0 & 0 & K \end{pmatrix} & 3 & \begin{pmatrix} L' & L' & L' \\ L' & L' & L' \\ 0 & 0 & K \end{pmatrix} \\
3' & \begin{pmatrix} D & D^* \\ 0 & D_1 \end{pmatrix} & 4 & \begin{pmatrix} L & L & L \\ L & 0 & 0 \\ 0 & K & K \end{pmatrix} & 6 & \begin{pmatrix} K & K & K & K \\ K & 0 & 0 & 0 \\ K & 0 & 0 & K \\ 0 & K & K & K \end{pmatrix} \\
7 & \begin{pmatrix} \bar{Z} & \bar{Z} & \bar{Z} \\ 0 & \bar{Z} & \bar{Z} \\ 0 & \bar{Z} & \bar{Z} \end{pmatrix} & 8 & \begin{pmatrix} K & K & K & K \\ K & K & 0 & 0 \\ K & 0 & 0 & K \\ 0 & K & K & K \end{pmatrix}
\end{align*}
\]
where $L$ is an extension of $\mathbb{Z}$ with $[L: \mathbb{Z}]=2$. $e_\text{tr}R$ is of the form $2-1$) in Lemma 16 and $e_\text{sr}R$ is of the form in Lemma 10.

The rings of $1)\sim 6), 8), 10)$ and $11)$ are left serial.

If $R$ is either hereditary or left serial, $A_1/E_2 \cong A_2/F_2$ implies $A_1 \cong A_2$ in Lemma 16. In general this is not true for US-4 rings.

We shall give rings of the type $a)$ in Lemma 16. Let $R=\sum \oplus e_i R$ and $e_i e_j = \delta_{ij} e_i$ (the $e_i$ are primitive idempotents).

1) $A_1/E_2 \cong A_2/F_2$ and $E_2 \cong F_2$

\[
e_i R = e_i \mathbb{Z} + e_i J
\]

\[
A_1 = (1, 2)\mathbb{Z} + (1, 2)(2, 3)\mathbb{Z}
\]

\[
E_2 = (1, 2)(2, 3)\mathbb{Z}
\]

\[
e_2 R = e_2 \mathbb{Z} + e_2 J
\]

\[
(2, 3)\mathbb{Z} + (2, 3)\mathbb{Z} + e_2 J
\]

\[
(2, 3)\mathbb{Z}
\]

and $(1, 2)(2, 3)'=(1, 2)'(2, 3)'=0$. This is a type of $a-1-1$) and $a-1-4)$. ($R$ is a finite ring.)

2) $A_1/E_2 \cong A_2/F_2$, $E_2 \cong F_2$

\[
\begin{pmatrix}
\mathbb{Z} & 0 & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z}
\end{pmatrix}
\]
GENERALIZATIONS OF NAKAYAMA RING VI

\[ e_1 R = e_1 \mathbb{Z} + e_1 J \]

\[ A_1 = (1, 2)\mathbb{Z} + (1, 2)(2, 3)K \quad A_2 = (1, 2)' \mathbb{Z} + (1, 2)'(2, 4)K \]

\[ E_2 = (1, 2)(2, 3)K \quad F_2 = (1, 2)'(2, 4)K \]

\[ e_2 R = e_2 K + e_2 J \quad e_3 R = e_3 K \quad e_4 R = e_4 K \]

\[ (2, 3)K \quad (2, 4)K \]

\[ 0 \quad 0 \]

and \((1, 2)(2, 4) = (1, 2)'(2, 3) = 0\), where \(K\) is a finite field of characteristic 2. This is a type of a-1-1).

3) \[ e_1 R = e_1 \mathbb{Z} + e_1 J \]

\[ A_1 = (1, 2)\mathbb{Z} + E_2 \quad A_2 = (1, 1)\mathbb{Z} + F_2 \]

\[ E_2 = (1, 2)(2, 3)K \quad F_2 = (1, 1)(1, 2)\mathbb{Z} + F_3 \]

\[ 0 \quad F_3 = (1, 1)(1, 2)(2, 3)K \]

\[ e_2 R = e_2 \mathbb{Z} + e_2 J \quad e_3 R = e_3 K \]

\[ (2, 3)K \]

\[ 0 \]

This is a type of a-1-2). If \(K=\mathbb{Z}, R\) is a left serial and finite ring.

4) \[ e_1 R = e_1 \mathbb{Z} + e_1 J \]

\[ A_1 = (1, 2)\mathbb{Z} + E_2 \quad A_2 = (1, 3)\mathbb{Z} + F_2 \]

\[ E_2 = (1, 2)(2, 4)\mathbb{Z} \quad F_2 = (1, 3)(3, 5)\mathbb{Z} + F_3 \]

\[ 0 \quad F_3 = (1, 3)(3, 5)(5, 4)\mathbb{Z} + F_4 \]

\[ F_4 = (1, 3)(3, 5)(5, 4)(4, 6)K \]

\[ 0 \]
\[ e_2 R = e_2 \bar{Z} + e_2 J \]  
\[ e_3 R = e_3 \bar{Z} + e_3 J \]  
\[ e_4 R = e_4 \bar{Z} \]

This is a type of a-1-3).

Other products among \((i, j)\) are zero (e.g. \((1, 1)(1, 1) = 0\)). In the above \(e_i(k, l)e_j = (k, l)\delta_{ih}\delta_{ij}\), \((\delta_{ij}\) is Kronecker delta).

Similarly we can construct a US-4 ring of a-2-1) in Lemma 16. Finally we shall give an example concerning \(ii)\) of Lemma 12.

Let \(K\) be a field of characteristic 2 and \(L\) an extension of \(K\); \(L = K(a)\) and \(a^2 \in K\). Put \(g(a) = b \neq 0\) in \(L\) and \(g(1) = 0\). Then \(g\) is a derivation of \(L\) over \(K\). Put

\[ R = \begin{pmatrix} L & L \\ L & L \end{pmatrix}, \]

where \(l (l_1) = \begin{pmatrix} L,L \end{pmatrix} (l_1) \begin{pmatrix} L \\ L \end{pmatrix} \) (\(l_1, l_2 \in L\)) as in Remark 13. Then \(e_{1i} J = A_1 \oplus A_2\) and \(\Delta(A_1) = \Delta, [\Delta : \Delta(A_2)] = 2\). However, \(e_{1i} R\) does not satisfy \((\# 1)\) as an \(L-L\)-module. Hence \(e_{1i} R\) has the similar form to \(ii)\) of Lemma 12, but \(R\) is not right US-4.

References


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