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GENERALIZATIONS OF NAKAYAMA RING VI

(RIHGT US-n RINGS; $n=3$, 4)

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

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We have studied artinian right US-3 rings in [5] and right US-4 algebras over an algebraically closed field in [7]. We shall continue, in this paper, to study a right US-3 (resp. US-4) ring R when R is either hereditary or left serial.

In the first two sections, we shall give the characterization of a right US-3 (resp. US-4) ring R, when R satisfies a weaker condition $(*, 1')$ (see § 1) than R being either hereditary or left serial. In the next two sections, we shall specify the characterizations given in the previous sections to hereditary rings and left serial rings. We shall exhibit several examples in the final section to illutsrate the above characterizations.

1. US-3 rings

Throughout this paper we deal with an artinian ring R and every R-module is a unitary right R-module. We shall use the same terminologies and definitions given in [2]~[8].

As a generalization of right serial rings, we considered

$(**, n)$ *Every maximal submodule in a direct sum D of n hollow modules contains a non-zero direct summand of D* [5].

It is clear that if $D/J(D)$ is not homogeneous, D satisfies $(**, n)$. Hence we may restrict ourselves to hollow modules of a form eR/E, where e is a primitive idempotent and E is a submodule of eR. If $(**, n)$ holds for any direct sum of n hollow modules, we call R a *right US-n ring* [5]. Since the concept of US-n rings is Morita equivalent, we study always a basic ring.

We studied right US-n algebras over an algebraically closed field for $n=3$ and 4 in [5] and [7], respectively. In this and next sections we shall give a complete list of the structure of right US-3 (resp. US-4) rings with $(*, 1')$ below. We can give theoretically the complete structure, however as we know a few properties of division rings, we can not give the complete examples for each structure.

We quote here a particular property of a semisimple module (cf. [8] and [9]).

($\sharp$, m) *Let e be a primitive idempotent in R and D a semisimple R-module and a left eRe-module. For any two R-submodules V_1 and V_2 with $|V_1| = |V_2| = m$, there exists a unit x in eRe such that $xV_1 = V_2$.*

Further we consider one more property:

($*$, 1$'$) *eJ^i is a direct sum of hollow modules for each primitive idempotent e and each i.*

If R satisfies ($*$, 1), then ($*$, 1$'$) holds. Moreover, if R is hereditary or left serial, ($*$, 1$'$) holds by [11], Corollary 4.2. Under the assumption ($*$, 1$'$), we obtain the following diagram (cf. [8]):

$$
(1) \qquad
\begin{array}{ccccccccc}
 & \overline{A_1} & & & \overline{A_2 \quad \cdots} & & \overline{A_m} & & eJ^t \\
 & | & & & | & & & & \\
\overline{A_{11}} & \overline{\cdots} & \overline{A_{1n_1}} & \overline{A_{21}} & \overline{\cdots} & \overline{A_{2n_2}} & \cdots & & eJ^{t+1}
\end{array}
$$

where the A are hollow.

Let A_1, A_2 be submodules in eR. If there exists a unit x in eRe such that $xA_1 \subset A_2$ or $xA_1 \supset A_2$, we indicate this situation by $A_1 \sim A_2$ [4]. We put $\Delta = eRe/eJe$ $(= \overline{eRe})$ and $\Delta(A_1) = \{\bar{x} \mid \in \Delta,\ xA_1 \subset A_1\}$ [2].

Let $D = A_1 \oplus A_2$; the A_1 are uniserial. A submodule $B = B_1 \oplus B_2$ $(A_i \supset B_i)$ is called a *standard submodule* in D [3].

Lemma 1. *Let A_1 and A_2 be as in* (1). *If $A_1 \sim A_2$, $A_1 = xA_2$ for some unit element x in eRe, and hence $A_1 \approx A_2$.*

Proof. Since $A_1 \sim A_2$, there exists a unit x in eRe such that $xA_1 \supset A_2$ or $xA_1 \subset A_2$. We may assume that $xA_1 \subset A_2$. If $xA_1 \neq A_2$, $xA_1 \subset J(A_2) \subset eJ^{t+1}$, since A_2 is hollow. Hence $A_1 \subset x^{-1}eJ^{t+1} = eJ^{t+1}$, a contradiction. Therefore $xA_1 = A_2$.

Lemma 2. *Let A_1 and A_2 be as in* (1). *Let B be a hollow submodule in A_2, which appears on the level eJ^{k+t} $(k \geqslant 0)$ in* (1). *If $\Delta(A_1) = \Delta$, $A_1 \not\sim B$.*

Proof. First assume $k \geqslant 1$ and $A_1 \sim B$, i.e., there exists a unit x in eRe such that $xA_1 \supset B$ or $xA_1 \subset B$. In the latter case $A_1 \subset eJ^{t+1}$. Hence $xA_1 \supset B$. Since $\Delta(A_1) = \Delta$, there exists an element j in eJe with $(x+j)A_1 = A_1$. Let b be a generator of B. Then we obtain a in A_1 with $xa = b$. $b = (x+j-j)a = (x+j)a - ja$. Let p_1 be the projection of eJ^t to A_1. $0 = p_1(b) = (x+j)a - p_1(ja)$. Assume $a \in eJ^p - eJ^{p+1}$, and $p(ja) \in eJ^{p+1}$, which is a contradiction, since $x+j$ is a unit in eRe. Finally assume $B = A_2$. Then $A_2 = x'A_1$ for some unit x' in eR. Hence we obtain the same situation as above, which is a contradiction.

From [2], Theorem 2 we have

Lemma 3. *If R is a right US-n ring, then $[\Delta\colon \Delta(A)]\leqslant n-1$ for any submodule A in eR.*

Put $\tilde{R}=R/J^{t+k}$. Then $\widetilde{eRe}/\widetilde{eJe}\approx eRe/eJe=\Delta$. Let A_1 be as in (1). Then we can define $\Delta(\tilde{A}_1)=\Delta((A_1+J^{t+k})/J^{t+k})=\{\bar{x}\mid\in\Delta,\ x(A_1+J^{t+k})\subset(A_1+J^{t+k})\}$. It is clear that $\Delta(A_1)$ is a division subring of $\Delta(\tilde{A}_1)$.

Lemma 4. *Let A_1 and A_2 be as in* (1). *If $\Delta(A_1)=\Delta$, $\Delta(\tilde{A}_1)=\Delta$. Next assume that $A_2=xA_1$ for some unit x in eRe. If $[\Delta\colon \Delta(A_1)]=2$* (*resp.* 3), *$[\Delta\colon \Delta(\tilde{A}_1)]=2$* (*resp.* 3), *where $\tilde{A}_1=(A_1+J^{t+k})/J^{t+k}\subset\tilde{R}=R/J^{t+k}$.*

Proof. The first part is clear from the remark above. Assume $(x+j)\tilde{A}_1\subset\tilde{A}_1$ for some j in eJe. Since $(x+j)A_1\subset A_2+jA_1\subset A_2+eJ^{t+1}$, $(x+j)A_1\subset(A_1+eJ^{t+1})\cap(A_2+eJ^{t+1})=eJ^{t+1}$, a contradiction. Hence $x\notin\Delta(A_1)$. Further $[\Delta\colon \Delta(A_1)]$ is prime, and so $[\Delta\colon \Delta(A_1)]=[\Delta\colon \Delta(\tilde{A}_1)]$.

REMARK 5. We shall study a right US-n ring and observe $[\Delta\colon \Delta(A_1)]$. Since $[\Delta\colon \Delta(A_1)]\leqslant 3$, we may assume $J^{t+1}=0$ by Lemma 4, [3], Lemma 1 and its proof, when we observe $[\Delta\colon \Delta(A_1)]$ (the x in Lemma 4 exists, provided $[\Delta\colon \Delta(A_1)]\geqslant 2$).

Theorem 1. *R is a right* (*basic*) *US-3 ring with* $(*, 1')$ *if and only if eR has one of the following structures for each primitive idempotent e.*

1) *eR/eJ^t is uniserial for some t and*

2) *$eJ^t=0$ or $eJ^t=A\oplus B$, where A is simple and B is uniserial, such that* a) $[\Delta\colon\Delta(A)]=2$ *or* b) $\Delta=\Delta(A)=\Delta(B)$.

In case a) *B is simple and $A\oplus B$ satisfies* ($\sharp$, 1).

In case b)

i) *B is simple and $A\not\approx B$ or*

ii) *B is not simple, and if A is isomorphic to a simple subfactor module B_i/B_{i+1} of B, $B_{i+1}=0$* (*i.e., B_i is the socle of B*) *and this isomorphism is given by j_l: the left-sided multiplication of j in eJe.*

Proof. We assume that R is a right US-3 ring. From $(*, 1')$ and [5], Proposition 1,3) $eJ=A\oplus B$, where A and B are hollow. We may assume $|A|\leqslant|B|$. $[\Delta\colon \Delta(C)]\leqslant 2$ for any submodule C in eR by Lemma 3. Hence we divide ourselves into two cases: I) $[\Delta\colon \Delta(A)]=2$ and II) $\Delta=\Delta(A)$.

Case I). Since $[\Delta\colon \Delta(A)]=2$, by [5], Proposition 1,2) there exists a unit element x in eRe such that $xA\subset J(A)\oplus J(B)$ or $xA\supset J(A)\oplus J(B)$. However $A\not\subset eJ^{t+1}$ and so $xA\supset J(A)\oplus J(B)$. On the other hand, $|A|=|J(A)+1|$ and $xA\neq J(A)\oplus J(B)$. Hence $J(B)=0$. Further $A\cong B$ by Lemma 1 and [5], Proposition 1.2). Therefore A and B are simple and $eJ^{t+1}=0$. Which means that every

(simple) submodule C in eJ^t is characteristic if and only if $\Delta(C)=\Delta$. Hence $[\Delta\colon \Delta(C)]=2$ and eJ^t satisfies ($\sharp$, 1) by [5], Proposition 1,2).
Case II). We know from the above argument that $\Delta=\Delta(A)=\Delta(B)$ (note that we did not use the assumption $|A|\leqslant|B|$). Let y be any unit element in eRe. Since $\Delta=\Delta(A)$, there exists an element j in eJe such that $(y+j)A=A$. Then $(y+j)(A\oplus J(B))\subset A\oplus(y+j)J(B)\subset A\oplus eJ^{t+1}=A\oplus J(B)$. Hence $\Delta(A\oplus J(B))=\Delta$. Assume that B is not simple. $A\oplus J(B)$ or $J(A)\oplus B$ is hollow by [5], Proposition 1,4)-iv). Hence

$$J(A)=0\,, \quad \text{i.e., } A \text{ is simple.}$$

We shall show that B is uniserial. Assume $eJ^{t+k}=BJ^k=C_1\oplus C_2\oplus\cdots$; the C_i are hollow. If $\Delta(C_1)\neq\Delta$, $C_1\sim A_1$ by [5], Proposition 1,2), which is a contradiction from Lemma 2. Hence $\Delta=\Delta(C_1)=\Delta(C_2)$. However $\{A, C_1, C_2\}$ derives a contradiction by Lemma 2 and [4], Corollary 2 of Theorem 2, provided $C_2\neq 0$. Therefore

$$B \text{ is uniserial}\,.$$

Next assume $g\colon A\approx B_i/B_{i+1}$; $B\supset B_i\supset B_{i+1}$. Take $\{A, B_i, B_i(g^{-1})$; the graph of B_i with respect ot $g^{-1}\}$. Since A is simple (and hence $eJeB\subset B$) and $\Delta(B)=\Delta$, B is characteristic. Hence $A\sim B_i(g^{-1})$, and so there exists a unit x_1 in eRe such that $x_1A\subset B_i(g^{-1})$. If $B_{i+1}\neq 0$, $x_1A\subset B_{i+1}\subset eJ^{k+1}$, a contradiction. Hence $B_{i+1}=0$ and $g\colon A\approx B_n$, the socle of B. Let j be an element in eJe such that $(x_1+j)A=A$, and put $x_2=x_1+j$. Then $A(g)=x_1A=(x_2-j)A$. Put $A=aR$. Then $a+g(a)=(x_2-j)ar$ for some r in R. $eJeA\subset eJ^{t+1}$ and $eJ^{t+1}=BJ$ imply $eJeA\subset B_n$. Hence

$$a=x_2ar \quad \text{and} \quad g(a)=-jar\,,$$

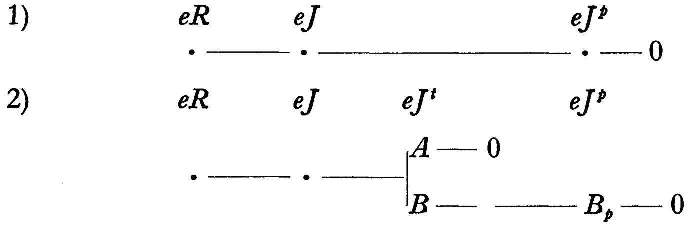
and so $g(a)=-jx_2^{-1}a$. Therefore $g=(-jx_2^{-1})_l$ and $-x_2j^{-1}\in eJe$ (b-ii)). Finally assume that B is simple. If $f\colon A\approx B$, $\{A, B, A(f)\}$ derives a contradiction from [5], Lemma 1, (note $eJ^{t+1}=0$ and use Lemma 8 below). Hence $A\not\approx B$ (b-i)). Conversely, assume that eR has one of the structures given in the theorem. Clearly ($*$, 1$'$) holds. Let $\{E_i\}_{i=1}^3$ be any set of submodules in eR. Case a): If $E_1\supset eJ^t$ and $E_2\supset eJ^t$, $\Delta(E_i)=\Delta$ for $i=1, 2$ and $E_1\supset E_2$ or $E_1\subset E_2$. Hence $D=\sum_{i=1}^{3}\oplus E_i$ contains a non-zero direct summand of D by [4], Corollary 1 of Theorem 2. If $E_1\subsetneqq eJ^t$ and $E_2\subsetneqq eJ^t$, $E_2=xE_1$ ($\approx A$) for some x in eRe by ($\sharp$, 1). Hence D satisfies ($**$, 3) again by [4], Corollary 1 of Theorem 2. Case b-ii): If $E_i\subset eJ^t$, x_1E_i is a standard submodule in eJ^t for a unit $x_1=(e+j)$ in eRe by assumption. Hence $E_i\sim E_j$ for some pair i, j. Further $\Delta=\Delta(E)$ by assumption. Therefore D satisfies ($**$, 3) by [4], Corollary 1 of Theorem 2. Case b-i): This is much simpler than the above. Thus R is right US-3.

In the last paragraph of the proof of "only if part" in Theorem 1, we have shown

Lemma 6. *Assume that* $eJ^t=A\oplus A'\oplus B$ *and* 1) A *and* A' *are simple modules with* $\Delta(A)=\Delta$, *and* 2) B *is non-simple and uniserial. If* $g\colon A\approx B_i/B_{i+1}$ *and* $A\sim B_i(g^{-1})$, $B_{i+1}=0$ *and* g *is given by* j_l; $j\in eJe$, *and hence* $i>1$ (cf. [7], Lemma 16).

We shall illustrate the structure in Theorem 1 as the following diagram:

(2)



where the straight line means uniserial.

It is clear that if R has the structure above, (*, 1) (and hence (*, 1′)) holds. We note that if (*, 1′) does not hold, Theorem 1 is not true (see [6]). We shall give examples of a) and b) in § 5.

2. US-4 rings

Next we shall characterize a right US-4 ring with (*, 1′).

Lemma 7. *Let* R *be a right* US-4 *ring and* $\{A_i\}_{i=1}^4$ *a set of submodules in* eJ. *Then* 1) *if* $\Delta(A_i)=\Delta$ *or all* $i\leqslant 3$ *and* $A_k\not\sim A_k$, *for* $k\neq k'\leqslant 3$, *then* $A_4\sim$ (*some* A_i). 2) $A_i\sim A_j$ *for some pair* i, j. 3) *If* $[\Delta\colon\Delta(A_i)]=2$ *for* $i=1, 2$, $A_1\sim A_2$. 4) *If* $[\Delta\colon\Delta(A_1)]=3$, $A_1\sim A_j$ *for all* j. 5) *If* $[\Delta\colon\Delta(A_1)]=2$, $A_i\sim A_j$ *for some* $i, j\leqslant 3$.

Proof. This is clear from [4], Corollary 2 of Theorem 2.

Lemma 8. *Let* A_1 *and* A_2 *be as in* (1). *Assume* $J^{i+1}=0$. *If* $\Delta(A_1)=\Delta$, A_1 *is characteristic.*

Proof. This is clear.

Lemma 9. *Let* R *be a right* US-4 (*basic*) *ring, and* $\{A_i\}_{i=1}^t$ *a set of hollow submodules on the level* eJ^t *in* (1). *If* $\Delta(A_i)=\Delta$ *for all* i, $t\leqslant 3$.

Proof. This is clear from Lemmas 7, 8 and Remark 5.

From now on we assume that R is a right US-4 (basic) ring satisfying (*, 1′). Let $D=(eJ^t=)A_1\oplus A_2\oplus\cdots\oplus A_t$, where the A_i are hollow. In the following lemmas, we mainly assume that D is characteristic. We note $[\Delta: \Delta(A_i)]\leqslant 3$ for all i by Lemma 3.

Lemma 10. *Assume* $[\Delta: \Delta(A_i)]=2$ *for all* i. *Then* i) $t=2$. ii) *There exists a unit* x *in eRe such that* $xA_1=A_2$. iii) A_1 *is a uniserial module with* $|A_1|\leqslant 2$. iv) *If there are characteristic submodules in* $A_1\oplus A_2$, *they are linear with respect to the inclusion.* v) *If* B *is not a characteristic submodule in* $A_1\oplus A_2$, $[\Delta: \Delta(B)]=2$ *and those submodules are related by* $\sim$.

Proof. We may assume $|A_1|\leqslant|A_2|\leqslant\cdots\leqslant|A_t|$ (note $t\geqslant 2$). By Lemmas 1 and 7, $A_k=x_kA_1$ for all k. Hence

(α) if $[\Delta: \Delta(A_i)]\geqslant 2$ for all i, there exists a unit x_i in eRe such that $x_iA_1=A_i$ for all i.

On the other hand, since $[\Delta: \Delta(A_1)]=2$, $\Delta=\Delta(A_1)+x_2\Delta(A_1)$. Assume $eJ^{i+1}=0$ from Remark 5. Since $D=\Delta A_1=\Delta(A_1)A_1+x_2\Delta(A_1)A_1=A_1\oplus A_2$, $t=2$. We note that from the above argument and Lemma 3 we obtain

(β) If $[\Delta: \Delta(A_i)]\geqslant 2$ for all i, $t\leqslant 3$.

Assume that A_1/A_1J^k is uniserial and $A_1J^k=B_1\oplus B_2\oplus\cdots\oplus B_s$, where the B_j are hollow and $s\geqslant 2$. In order to show $s\leqslant 1$, we may assume $eJ^{i+k+1}=0$ by Remark 5. First we note that there exists a unit x in eRe such that $xA_1=A_2$. Hence $\Delta(B_p)\neq\Delta$ for all p. On the other hand, $DJ^k=A_1J^k\oplus A_2J^k=\sum_{p=1}^{s}\oplus B_p\oplus\sum_{p=1}^{s}\oplus xB_p$, which is a contradiction to (β). Therefore A_1 and A_2 are uniserial. Next assume $A_1J^2\neq 0$. $\Delta(A_1J\oplus A_2J^2)\neq\Delta$ by existence of x_2. Hence $\{A_1, A_1J\oplus A_2J^2\}$ derives a contradiction by Lemma 7. Therefore $|A_1|\leqslant 2$. Since $\Delta(A_1J\oplus(A_2))\subset\Delta(A_1)$, $\Delta(A_1\oplus J(A_2))=\Delta(A_1)$ for $\Delta(A_1\oplus J(A_2))\neq\Delta$. Similarly $[\Delta: \Delta(J(A_1))]=2$. Let E be a submodule with $[\Delta: \Delta(E)]=3$. Then there exists a unit element x in eRe such that $xE\subset A_1$ or $xE\supset A_1$ by Lemma 7. In the former case $[\Delta: \Delta(E)]=[\Delta: \Delta(xE)]=2$. If $xE\supset A_1$, $xE=A_1\oplus E'$; $E'\subset A_2$. Hence $[\Delta: \Delta(xE)]=2$ from the above. Therefore there are no submodules E with $[\Delta: \Delta(E)]=3$. Finally assume that $A_1\oplus A_2$ contains two characteristic submodules C_1, C_2 such that $C_1\not\sim C_2$. Consider $\{A_1, A_1, C_1, C_2\}$, and $A_1\sim C_1$ or $A_1\sim C_2$ by Lemma 7. If $A_1\supset C_1$, $C_1=0$ and if $A_1\subset C_1$, $C_1=A_1\oplus F$; $F\subset A_2$, and so $C_1=A_1\oplus A_2$. Hence $C_1\supset C_2$ or $C_1\subset C_2$. Let $\Delta(E)=\Delta$. If $|A_1|=1$, E is characteristic. Assume $|A_1|=2$. Put $C_1=A_2\oplus B_2$. Then $E\sim C_1$ from the above. Hence $E\subset C_1$ or $E\supset C_1$, and so E is characteristic.

Lemma 11. *Assume* $[\Delta: \Delta(A_i)]=3$ *for all* i. *Then* $t\leqslant 3$, *and the* A_i *are simple and there exists a unit* x_i *in eRe such that* $x_iA_1=A_i$ *for each* i. *If* $t=3$, D *satisfies* (#, 1) *and* (#, 2) *and* $[\Delta: \Delta(C)]\leqslant 3$ *for every submodule* C *in* D. *If* $t=2$, D *satisfies* (#, 1).

Proof. Since $[\Delta: \Delta(A_1)]=3$, there exists a unit x_i in eRe such that $x_iA_1=A_i$

from (α) and $t \leqslant 3$ by (β). Assume $t=3$. Taking $\{A_1, J(D)\}$, we know from Lemma 7 that A_1 is simple and hence $eJ^{i+1}=0$. It is clear from Lemmas 7 and 8 that there are no simple submodules B in D with $\Delta(B)=\Delta$. Hence D satisfies (#, 1). Let C be a submodule of D with $|C|=2$. Then $D=C\oplus A_i$ for some i. Hence $\Delta(C)\neq\Delta$ by Lemma 7, and so D satisfies (#, 2). We obtain the similar result for $t=2$.

Lemma 12. *Assume* $[\Delta\colon \Delta(A_1)]=1$ *and* $\Delta(A_i)\neq\Delta$ *for* $i\geqslant 2$. *Then* A_1 *is uniserial and* $t\leqslant 3$.

i) $t=3$:

Then all A_i *are simple,* $[\Delta\colon \Delta(A_i)]=2$ *for* $i=2, 3$, $A_1 \not\approx A_2$ *and* $A_2\oplus A_3$ *satisfies* (#, 1).

ii) $t=2$: a) A_1 *is not simple.*

Then $[\Delta\colon \Delta(A_2)]=2$, *and* A_2 *is a simple submodule isomorphic to* B, *the socle of* A_1. *If* $A_2\approx E_i/E_{i+1}$ $(A_1\supset E_i\supset E_{i+1})$, $E_i=B$ *and* $E_{i+1}=0$. *Further* $B\oplus A_2$ *satisfies* (#, 1) *except* B.

b) A_1 *is simple.*

Then 1) $[\Delta\colon \Delta(A_2)]=2$, $A_1\oplus A_2/J(A_2)$ *satisfies* (#, 1) *except* A_1.

2) A_2/A_2J^t *is uniserial for some t and*

2-i). $A_2J^t=0$ *or*

2-ii) $A_2J^t=B_1\oplus B_2$; B_1 *is simple and* B_2 *is uniserial.*

2-ii-1) $\Delta(B_1)=\Delta(B_2)=\Delta$.

2-ii-1-1) $B_1\not\approx B_2/J(B_2)$.

2-ii-1-2) $A_1\not\approx F_i/F_{i+1}$ $(A_2\supsetneqq F_i\supset F_{i+1}\supset B_1\oplus B_2)$.

2-ii-1-3) *If* $f\colon A_1\approx G_j/G_{j+1}$ $(f'\colon B_1\approx G_j/G_{j+1})$ $(B_2\supset G_j\supset G_{j+1})$, *then* $G_{j+1}=0$ *and* f (f') *is given by* j_l; $j\in eJe$.

2-ii-1-4) *If* $f\colon A_1\approx B_1$, *we have the same result as* 2-ii-1-3).

2-ii-2). $[\Delta\colon \Delta(B_1)]=[\Delta\colon \Delta(B_2)]=2$.

2-ii-2-1) B_1 *and* B_2 *are simple and* $B_1\oplus B_2$ *satisfies* (#, 1).

2-ii-2-2) $A_1\not\approx F_i/F_{i+1}$ $(A_2\supsetneqq F_i\supset F_{i+1}\supset B_1\oplus B_2)$.

2-ii-2-3) *If* $A_1\approx B_1$, *then* f *is given by* j'_l; $j'\in eJe$.

Proof. It is clear, from the assumption and Lemmas 1 and 7, that $[\Delta\colon \Delta(A_i)]=2$ for all $i\geqslant 2$. Assume that A_1 contains two independent submodules B_1, B_2. If $\Delta(B_1)=\Delta(B_2)=\Delta$, $\{B_1, B_2, A_2, A_2\}$ derives a contradiction by Lemmas 7, 8 and Remark 5. On the other hand, if $\Delta(B_1)\neq\Delta$, $\{B_1, B_1, A_2, A_2\}$ derives again a contradiction. Hence

A_1 is uniserial

by (*, 1′).

a) $J(A_1)\neq 0$: Consider $\{A_1, A_2, J(D)\}$. Then $A_1 \not\sim A_2$ by Lemma 2. Hence $J(D)\sim A_1$ or $J(D)\sim A_1$ by Lemma 7. However $J(D)\not\sim A_2$, since $J(D)$ is characteristic and $J(A_1)\neq 0$. Hence

$$\text{the } A_i \text{ are simple for all } i\geqslant 2\,.$$

Since $[\Delta\colon \Delta(A_i)]=2$, there exists x_i in eRe such that $x_iA_2=A_i$ for $i>2$ by Lemmas 1 and 7. Hence in order to show $t\leqslant 3$, we may assume $J^{i+1}=0$ by Remark 5. Noting $\bar{x}_3\notin\Delta(A_2)$, $\Delta=\Delta(A_2)\oplus\bar{x}_3\Delta(A_2)$, which implies that $A_2\oplus A_3=\Delta A_2\supset\sum_{i=2}^{t}\oplus A_i$. Hence $t\leqslant 3$. Assume $t=3$. Now we resume to the original situation. We note $eJeA_1\subset J(A_1)$, and hence A_1 is characteristic. Since $\Delta(\mathrm{J}(A_1))=\Delta$, $\Delta(\mathrm{J}(A_1)\oplus A_2)\neq\Delta$. Consider $\{A_1, \mathrm{J}(A_1)\oplus A_2, A_2\oplus A_3\}$. $\Delta(A_1)=\Delta$ and $\Delta(\mathrm{J}(A_1)\oplus A_2)\neq\Delta$ imply $(\mathrm{J}(A_1)\oplus A_2)\sim(A_2\oplus A_3)$ by Lemma 7. Hence there exists a unit x in eRe such that $x(\mathrm{J}(A_1)\oplus A_2)\subset(A_2\oplus A_3)$ or $x(\mathrm{J}(A_1)\oplus A_2)\supset(A_2\oplus A_3)$. However, $\Delta(\mathrm{J}(A_1))=\Delta$ implies $x\mathrm{J}(A_1)\not\subset A_2\oplus A_3$. Hence $x(\mathrm{J}(A_1)\oplus A_2)\supset A_2\oplus A_3$. Taking $\tilde{R}=R/J^{i+1}$, we know that it is impossible. Therefore $t=2$ provided $\mathrm{J}(A_1)\neq 0$, i.e.,

$$D=A_1\oplus A_2\quad(\mathrm{J}(A_1)\neq 0)\,.$$

Now we take the similar manner to Lemma 6. Assume $f\colon A_2\approx E_i/E_{i+1}$; $A_1\supset E_i\supset E_{i+1}$. We note that A_1 is characteristic. $\{A_1, A_2, E_i(f^{-1})\}$ implies $A_2\sim E_i(f^{-1})$ from the above remark and Lemma 7. Hence $E_{i+1}=0$ as the proof of Lemma 6. Further since $\Delta(A_2)\neq\Delta$, $A_2\approx E_n$; the socle of A_1. Let $C(\neq E_n)$ be a simple submodule in $E_n\oplus A_2$. Consider $\{A_1, C, A_2, A_2\}$. It is clear that if $C\sim A_1$, $C\subset A_1$. Hence $C\sim A_2$ by Lemmas 2 and 7, and so $E_n\oplus A_2$ satisfies ($\sharp$, 1) except E_n.

b) $\mathrm{J}(A_1)=0$, $t\geqslant 3$. Assume $\mathrm{J}(A_2)\neq 0$. Since $t\geqslant 3$, there exists a unit x in eRe with $xA_2=A_3$ by Lemmas 1 and 7, and so $\Delta(A_1\oplus\mathrm{J}(A_2))\neq\Delta$. Then $A_2\sim A_1\oplus\mathrm{J}(A_2)$ by Lemma 7. Assume $A_2\supset y(A_1\oplus\mathrm{J}(A_2))$ for some unit y. Since A_1 is simple and $\Delta(A_1)=\Delta$, $p_1(yA_1)=A_1$, where $p_1\colon eJ^i\to A_1$ the projection, which is a contradiction. Similarly, since A_1 is simple and A_2 is not, $p_2(y'A_1)\subset\mathrm{J}(A_2)$ for any unit y' in eRe. Hence $A_2\not\subset y'(A_1\oplus\mathrm{J}(A_2))$. Therefore

$$A_2 \text{ (and so } A_i\ (i\geqslant 2)) \text{ is simple.}$$

Accordingly $t=3$ from the initial paragraph of a). If $f\colon A_1\approx A_2$, $\{A_1, A_1(f), A_2, A_2\}$ derives a contradiction, since $\Delta A_2=A_2\oplus A_3$ as before (note $eJ^{i+1}=0$). Hence $A_1\not\approx A_2$. Further if $A_2\oplus A_3$ contains a characteristic submodule $B\neq 0$, $\{A_1, B, A_2, A_2\}$ derives a contradiction. Therefore $A_2\oplus A_3$ satisfies ($\sharp$, 1).

Case $t=2$ and $\mathrm{J}(A_1)=0$ $(D=A_1\oplus A_2)$. First we shall show that $A_1\oplus A_2/\mathrm{J}(A_2)$ satisfies ($\sharp$, 1) except A_1. Since $\Delta(A_2)\neq\Delta$, there exists a unit x in eRe such that $p_1(xA_2)=A_1$, where $p_1\colon eJ^i\to A_1$ is the projection. Further $eJeA_2\subset A_2$, since A_1 is simple. Hence $(x+j)(A_2+J^{i+1})\neq A_2+J^{i+1}$ for any j in eJe, and so $\Delta(A_2)=\Delta((A_2+J^{i+1})/J^{i+1})$. Therefore we may assume $J^{i+1}=0$ (cf. Remark 5). Then

$A_1 \oplus A_2$ satisfies (#, 1) except A_1 from Lemma 7. Now we resume the original situation. Since A_1 is simple, $eJ^{i+1} = A_2J$. Assume that A_2/A_2J^t is uniserial and $eJ^{i+t} = B_1 \oplus B_2 \oplus \cdots \oplus B_s$, where the B_i are hollow. Then from Lemmas 10$\sim$16 below, $s \leqslant 3$. Further $[\Delta : \Delta(B_i)] \leqslant 2$ by Lemmas 2 and 7. Assume $s=3$. Then $\Delta(B_i) \neq \Delta$ (resp. $[\Delta : \Delta(B_j)] \neq 2$) for some i (resp. j) by Lemmas 7 and 10. Hence we remain two cases $\Delta(B_1) = \Delta$, $[\Delta : \Delta(B_j)] = 2$ for $j=2, 3$ and $\Delta(B_i) = \Delta$ for $i=1, 2$, $[\Delta : \Delta(B_3)] = 2$. On the other hand, since $\Delta(A_1) = \Delta$, we do not have such cases by Lemmas 2 and 7. Therefore $s \leqslant 2$. Similarly we do not have a case $\Delta(B_1) = \Delta$ and $[\Delta : \Delta(B_2)] = 2$. Thus we obtain two cases; 2-ii-1): $\Delta(B_i) = \Delta$ for $i=1,2$ and 2-ii-2): $[\Delta : \Delta(B_j)] = 2$ for $j=1, 2$.

2-ii-1) We assume $|B_1| \leqslant |B_2|$. $\{A_1, B_1, B_2, \mathrm{J}(B_1) \oplus \mathrm{J}(B_2)\}$ gives $\mathrm{J}(B_1) = 0$ from Lemmas 2 and 7. Assume $B_2J^k = C_1 \oplus C_2 \oplus \cdots \oplus C_s$; $s \geqslant 2$ and the C_i are hollow. If $[\Delta : \Delta(C_1)] \geqslant 2$, $\{A_1, B_1, C_1, C_1\}$ derives a contradiction from Lemmas 2 and 7. Hence $\Delta(C_1) = \Delta(C_2) = \Delta$. Taking $R/J^{i+t+k+1}$, we obtain again a contradiction from $\{A_1, B_1, C_1, C_2\}$ and Lemmas 2, 7 and 8. Accordingly B_2 is uniserial. If $f\colon B_1 \approx B_2/\mathrm{J}(B_2)$, $\{A_1, B_1, B_2, B_2(f^{-1})\}$ derives a contradiction. Hence $B_1 \not\approx B_2/\mathrm{J}(B_2)$ (2-ii-1-1)). Further if $g\colon B_1 \approx G_i/G_{i+1} (B_2 \supsetneqq G_i \supset G_{i+1})$, $\{A_1, B_1, B_2, G_i(g^{-1})\}$ gives $B_1 \sim G_i(g^{-1})$, since $\Delta(A_1) = \Delta(B_2) = \Delta$. Hence $G_{i+1} = 0$ and g is given by $j_l\colon j \in eJe$ from Lemma 6. Similarly if $f_1\colon A_1 \approx G_i/G_{i+1}$, $\{A_1, G_i(f_1^{-1}), A_2, A_2\}$ gives $A_1 \sim G_i(f_1^{-1})$, since $\Delta(A_1) = \Delta$. Then we can show similarly to Lemma 6 that $G_{i+1} = 0$ and f_1 is given by j'_l; $j' \in eJe$ (2-ii-1-3)). Next if $h\colon A_1 \approx F_j/F_{j+1} (A_2 \supsetneqq F_j \supset F_{j+1} \supset B_1 \oplus B_2)$, consider $\{A_1, F_j(h^{-1}), A_2, A_2\}$. Then $xA_1 \subset F_j(h^{-1})$ for some unit x in eRe, since $A_2 \neq F_j$ and $\Delta(A_1) = \Delta$. Hence $xA_1 \subset F_{j+1} \subset \mathrm{J}(A_2)$, a contradiction. Accordingly $A_1 \not\approx F_j/F_{j+1}$ $(F_j \neq A_2)$ (2-ii-1-2)). In the same manner given in the proof of 2-ii-1-3), we have 2-ii-1-4).

2-ii-2) Since $[\Delta : \Delta(B_1)] = [\Delta : \Delta(B_2)] = 2$, $B_1 \approx B_2$ by Lemmas 1 and 7. $\{A_1, B_1, B_2, \mathrm{J}(B_1) \oplus \mathrm{J}(B_2)\}$ gives $\mathrm{J}(B_1) = \mathrm{J}(B_2) = 0$. Accordingly B_1 and B_2 are simple. Let C be any simple submodule in $B_1 \oplus B_2$. Then $\{A_1, B_1, B_1, C\}$ shows $C = xB_1$ for some unit x in eRe by Lemmas 2 and 7. Hence $B_1 \oplus B_2$ satisfies (#, 1) (2-ii-2-1).

2-ii-2-2) is same to 2-ii-1-2). If $f\colon A_1 \approx B_1$, $\{A_1, A_1(f), B_1, B_1\}$ gives $A_1 \sim A_1(f)$. Hence f is given by j_l; $j \in eJe$ (2-ii-2-3)).

REMARK 13. We shall consider the situation of ii-b) of Lemma 12. Taking $\tilde{R} = R/J^{i+1}$, we may assume that $eJ^i (=V) = A_1 \oplus A_2$: the A_i are simple, $\Delta(A_1) = \Delta$, and $[\Delta : \Delta(A_2)] = 2$. Then $A_1 \approx A_2$ $(\approx \overline{g_1Rg_1} = \Delta')$. We shall express $\mathrm{End}_{\Delta'}(V)$ as elements of matrices $(\Delta')_2$. Since A_1 is characteristic, for any element x in Δ,

$$x = \begin{pmatrix} x_1 & x_3 \\ 0 & x_2 \end{pmatrix} \qquad : x_1 \in \Delta' .$$

Δ being a division ring, x_2 and x_3 are uniquely determined by x_1. Hence we obtain two monomorphisms as rings f_1, f_2 of Δ to Δ' such that $f_i(x)=x_i$ and a homomorphism g as additive groups of Δ to Δ' such that

i) $g(xx')=f_1(x)g(x')+g(x)f_2(x')$.

Then $\Delta(A_2)=g^{-1}(0)$ (note, from i), that $g^{-1}(0)$ is a division subring of Δ). Hence $[\Delta\colon \Delta(A_2)]=2$ is equivalent to

ii) $[\Delta\colon g^{-1}(0)]=2$.

Further (♯, 1) holds if and only if, for any α in Δ, there exists $x\neq 0$ in Δ such that

iii) $\alpha=-f_1(x^{-1})g(x)$ $(=g(x^{-1})f_2(x))$, i.e., $F\colon \Delta\to\Delta'$ $(F(x)=f_1(x)^{-1}g(x))$ is surjective.

If $\alpha\neq 0$, $x\notin g^{-1}(0)$. Hence if either $|\Delta|$, cardinal of Δ, $(|\Delta|\leqslant|\Delta'|)$ is finite or $|\Delta|<|\Delta'|$, iii) does not hold. Hence we assume that $|\Delta|$ is infinite. Further, since f_1 is a monomorphism, we may assume that $\Delta\subset\Delta'$ and f_1 is the inclusion. Now assume that Δ' is commutative. Then g is a K-linear mapping from i), where $K=g^{-1}(0)$. Using those facts and $|\Delta|\geqslant\infty$, for any g we can show by computation that there exists α in Δ' not satisfying iii) for any $x\in\Delta$. Therefore if Δ' is commutative, we do not have the case of ii) of Lemma 12.

REMARK 14. Next we consider the case $t=2$ in Lemma 11. Let K be a field and R a K-algebra. If $[\Delta'\colon K]$ is not divided by 3, this case does not occur. Because, since $V=A_1\oplus A_2$ and $A_1\approx A_2$, $\mathrm{End}_\Delta(V)=(\Delta')_2$ and $\Delta\subset(\Delta')_2$. $[\Delta\colon \Delta(A_1)]=3$ implies that $4[\Delta'\colon K]$ is divided by 3.

Finally we take division rings given by [10]. Let $D\supset D_1$ be division rings such that $[D\colon D_1]_r=3$ and $[D\colon D_1]_l=2$. Put $D=D_11+D_1u$, and $D^*=\mathrm{Hom}_{D_1}({}_{D_1}D, {}_{D_1}D_1)$. Then $[D^*\colon D_1]_r=2$ and D^* is a left D-vector space. Define $1^*\in D^*$ by setting $1^*(1)=1$, $1^*(u)=0$, and put $A_1=1^*D_1$. Then $D(A_1)=\{d\,|\in D,\ dA_1\subset A_1\}$ $=u^{-1}D_1u$, and so $[D\colon D(A_1)]_r=3$. For any h in D^* and $h^{-1}(0)=D_1u_1$, we have $D=D_1u_1\oplus D_1v_1$. Put $d=h(v_1)$. Then $(u_1^{-1}u)1^*(u_1)=0$ and $(u_1^{-1}u)1^*(v_1)=d'\neq 0$. Hence $h=(u_1^{-1}u)1^*d'^{-1}d$, and so $hD_1=(u_1^{-1}u)A_1$. Therefore D^* satisfies (♯, 1), $[D\colon D(A_1)]=3$ and $[D^*\colon D_1]=2$. We shall use D^* in § 5, Example 3'.

Now we resume to study the structure of right US-4 rings.

Lemma 15. *If R is a US-4 ring with $(*, 1')$. D has one of the structures in Lemmas* 10, 11, 12 *and* 16 *below.*

Proof. Assume 1) $\Delta(A_1)=\Delta(A_2)=\Delta(A_3)=\Delta$. Then $t=3$ by Lemmas 7 and 8 (the case of Lemma 16 below). 2) $\Delta(A_1)=\Delta_2(A)=\Delta$ and $\Delta(A_i)\neq\Delta$ for $i\geqslant 3$. Then $\{A_1, A_2, A_3, A_3\}$ derives a contradiction from Lemmas 2 and 7. 3) $\Delta(A_1)=\Delta$ and $\Delta(A)_i\neq\Delta$ for $i\geqslant 2$. This is a case of Lemma 12. 4) $[\Delta\colon \Delta(A_i)]=2$ for $i\leqslant$ some l, $[\Delta\colon \Delta(A_j)]=3$ for $j>l$. Since $[\Delta\colon \Delta(A_k)]\geqslant 2$ for all k, from (α) there exists a unit x_i in eRe such that $x_iA_1=A_i$ for all i. Hence $\Delta(A_j)=x_j\Delta_1(A)x_j^{-1}$, and so we obtain the cases of Lemmas 10 and 11.

Lemma 16. *Assume $\Delta(A_i)=\Delta$ for all i. Then $t\leqslant 3$, and*

1) $t=3$:

A_3 is uniserial and A_1, A_2 are simple, $A_1\not\approx A_2$. If A_3 is simple, $A_3\not\approx A_1$ and $A_3\not\approx A_2$. If A_3 is not simple and f: $A_1\approx F_i/F_{i+1}$ ($A_3\supset F_i\supset F_{i+1}$), then $F_{i+1}=0$ and f is given by j_l; $j\in eJe$, and hence $i>1$.

2) $t=2$: i) $A_1\approx A_2$ ($\approx g_1R/g_1J$).

Then A_1 and A_2 are simple and $\Delta=\overline{g_1Rg_1}\approx\bar{Z}=Z/2$.

ii) $A_1\not\approx A_2$ ($|A_1|\leqslant|A_2|$)

a) *A_2 is uniserial; $A_2=F_1\supset F_2\supset\cdots\supset F_p\supset F_{p+1}=0$.*

Then A_1 is a uniserial module with $|A_1|\leqslant 2$; $A_1=E_1\supset E_2\supset E_3=0$.

a-1) $|A_1|=2$.

a-1-1) *If f: $A_1/E_2\approx A_2/F_2$ ($\approx g_2R/g_2J$), $\Delta\approx\overline{g_2Rg_2}\approx\bar{Z}$. f is a unique isomorphism. In this case put $B_1=\{x+y\mid\in A_1\oplus A_2, f(\bar{x})=\bar{y}\}$.*

a-1-2) *If $A_1/E_2\approx F_i/F_{i+1}$, $i>1$, then $i\geqslant p-1$.*

a-1-3) *If f: $E_2\approx F_i/F_{i+1}$ ($\approx g_3R/g_3J$) ($p>i\geqslant 2$), $\Delta\approx\overline{g_3Rg_3}\approx\bar{Z}$. We have the same result as* a-2-1) *below, replacing A_1 with E_2. In this case put $B_i'=\{x+y\mid\in E_2\oplus F_i, f(x)=\bar{y}\}$.*

a-1-4) *f: $E_2\approx F_p$. If $p=2$, $\Delta\approx\overline{g_4Rg_4}\approx\bar{Z}$, where $E_2\approx F_2\approx g_4R/g_4R$. Further if f': $A_1/E_2\approx F_2$ ($A_2/F_2\approx E_2$), $A_1(f)=xA_1$ for some unit x in eRe.*

If $p>2$, we have the same result as a-2-2) *below, replacing A_1 with E_2. If f is not given by j_l, put $B''=E_2(f)$.*

a-1-5) *Further every submodule in eJ^i except B_1, B_i' and B'' is isomorphic to a standard submodule in eJ^i via x_l; x is a unit in eRe.*

a-2). $|A_1|=1$:

a-2-1) *If $A_1\approx F_i/F_{i+1}$ ($\approx g_5R/g_5J$) for some $i<p$, $\Delta\approx\overline{gR_5g_5}\approx\bar{Z}$. Further $A_1\not\approx F_j/F_{j+1}$ for any ($i\neq$) $j<p$.*

a-2-2) *Assume f_1, f_2: $A_1\approx F_p$ ($\approx g_6R/g_6J$).*

If the f_i are not given by j_l' in eJe, there exists a unit x in eRe such that $xA_1=A_1$ and $xf_1-f_2x_1=j_l$ ($j\in eJe$). In this case $A_1\not\approx F_i/F_{i+1}$ ($i<p$). In particular if $eJeA_1=0$, $\Delta\approx\overline{g_6Rg_6}\approx\bar{Z}$.

b) *A_2/A_2J^k is uniserial and A_2J^k is not unisrial, i.e., $A_2J^k=B_1\oplus B_2\oplus\cdots\oplus B_s$, where the B_i are hollow. Then A_1 is simple and $s=2$. Further*

b_1) $\Delta(B_2)=\Delta(B_1)=\Delta$.

Then $B_1\not\approx B_2$ ($|B_1|\leqslant|B_2|$), and B_1 is simple, B_2 is uniserial.

b_1-1) *If f: $A_1\approx F_i/F_{i+1}$ ($A_2\supset F_i\supset F_{i+1}\supset B_1\oplus B_2$), then we obtain the same result modulo $B_1\oplus B_2$ as given in* a-2-1).

b_1-2) *If f: $A_1\approx B_1$, f is givn given by j_l; $j\in eJe$.*

b_1-3) *If f: $A_1\approx H_i/H_{i+1}$ ($B_2\supset H_i\supset H_{i+1}$), then $H_{i+1}=0$ and f is given by j_l; $j\in eJe$.*

b_1-4) *If f: $B_1\approx H_i/H_{i+1}$, then $H_{i+1}=0$ and f is given by j_l; $j\in eJe$.*

b_2) $[\Delta\colon\Delta(B_i)]=2$ *for $i=1, 2$.*

Then $B_1 \approx B_2$ and B_1, B_2 are simple and $V = B_1 \oplus B_2$ satisfies ($\sharp$, 1).

b$_2$-1) $A_1 \not\approx F_i/F_{i+1}(A_2 \supset F_i \supset F_{i+1} \supset V)$.

b$_2$-2) *If f: $A_1 \approx B_1$, f is given by j_l; $j \in eJe$.* (cf. [7], Theorem 17.)

Proof. We know $t \leqslant 3$ by Lemma 9. Assume that $|A_1| \leqslant |A_2| \leqslant |A_3|$.

i) $t=3$. Consider $\{A_1, A_2, A_3, \mathrm{J}(D) = \mathrm{J}(A_1) \oplus \mathrm{J}(A_2) \oplus \mathrm{J}(A_3)\}$. Since $\Delta(A_i) = \Delta$, $\mathrm{J}(A_1) \oplus \mathrm{J}(A_2) \oplus \mathrm{J}_3(A)$ is contained in some A_i by Lemmas 2 and 7. Hence $\mathrm{J}(A_1) = \mathrm{J}(A_2) = 0$ (note $|A_3| \geqslant |A_i|$). Assume that A_3 contains two independent submodules B_1 and B_2 in eJ^{i+k} on the same level in (1). Take $\bar{R} = R/J^{i+k+1}$. Then both $[\Delta\colon \Delta(B_1)]$ and $[\Delta\colon \Delta(B_2)]$ are not equal to 1 and $[\Delta\colon \Delta(B_i)] \neq 3$ for any i by Lemmas 2 and 7, and hence $[\Delta\colon \Delta(B_i)] = 2$ for $i=1$ or 2 by Lemma 3, (say $i=1$). Then $\{B_1, B_1, A_1, A_2\}$ contradicts Lemmas 2 and 7, since $B_i \subset A_3$. Hence A_3 is uniserial. Assume f: $A_1 \approx A_2$. Then $\{A_1, A_2, A_3, A_1(f)\}$ implies $A_1(f) \sim$(some A_i). Since A_i is characteristic (we may assume $J^{i+1} = 0$ by Remark 5), $A_1(f) \subset A_i$, which is a contradiction. Finally assume g: $A_1 \approx F_i/F_{i+1}$. Since $\Delta(A_3) = \Delta$ and A_1, A_2 are simple, A_3 is characteristic. Hence $\{A_1, A_2, F_i(g^{-1}), A_3\}$ derives from Lemmas 2 and 7 that $A_1 \sim F_i(g^{-1})$. Therefore g is given by j_l from Lemma 6.

2) $t=2$.

i) f: $A_1 \approx A_2$. Assume $\Delta \neq \Delta(A_1(f))$. Then $\{A_1, A_2, A_1(f), A(f)\}$ implies $A_1(f) \sim A_i$ for some i, say 1 from Lemma 7. Since $A_2 \approx A_1 \approx A_1(f)$, $A_1(f) = xA_1$ for some unit x in eRe. Hence $\Delta(A_1(f)) = \bar{x}\Delta(A_1(f))\bar{x}^{-1} = \Delta$, a contradiction. Accordingly $\Delta(A_1(f)) = \Delta$. Consider $\{A_1, A_2, A_1(f), \mathrm{J}(A_1) \oplus \mathrm{J}(A_2)\}$, and $\mathrm{J}(A_1) = 0$ by Lemma 7 (note $\Delta(A_i) = \Delta(A_1(f)) = \Delta$). Hence A_1 and A_2 are simple, and so $eJ^{i+1} = 0$. Let f and f' be two isomorphisms of A_1 to A_2 and consider $\{A_1, A_2, A_1(f), A_1(f')\}$. Since $eJ^{i+1} = 0$, they are characteristic, and so $A_1(f) = A_1(f')$ by Lemmas 7 and 8. Hence $f = f'$. Considering an isomorphism δf for $\delta \in \Delta$, $\Delta = \{0, 1\}$. Since $\mathrm{Hom}_R(A_1, A_1) = \{0, 1\}$, $\Delta' = \overline{gRg} = \{0, 1\}$, where $A \approx gR/gJ$.

ii) $A_1 \not\approx A_2$ ($|A_1| \leqslant |A_2|$). Assume $A_1 J \neq 0$ and $A_2 J^k = C_1 \oplus C_2 \oplus \cdots \oplus C_s$ ($s \geqslant 1$), where the C_i are hollow. Consider $\{A_1, A_2, A_1 J \oplus C_1, A_1 J \oplus C_2\}$ ($s \geqslant 2$). Then $A_1 J \oplus C_1 \sim A_1 J \oplus C_2$ by Lemmas 2 and 7, provided $A_1 J \neq 0$, Assume $\Delta(A_1 J \oplus C_i) = \Delta$ for $i=1$, 2 and $x(A_1 J \oplus C_1) \subset A_1 J \oplus C_2$ for some unit x. We may assume $J^{i+k+1} = 0$. There exists j in eJe such that $(x+j)(A_1 J \oplus C_1) = A_1 J \oplus C_1$. Then $xC_1 \subset (x+j)C_1 + jC_1 \subset A_1 J + C_1$. Hence $xC_1 \subset (A_1 J \oplus C_1) \cap (A_1 J \oplus C_2) = A_1 J$, and so $C_1 \sim A_1$, a contradiction by Lemma 2. Hence $\Delta(A_1 J \oplus C_1) \neq \Delta$ for some i, say 1. $\{A_1, A_2, A_1 J \oplus C_1, A_1 J \oplus C_1\}$ implies either $A_1 \sim A_1 J \oplus C_1$ or $A_2 \sim A_1 J \oplus C_2$ by Lemma 7. Which is again a contradiction by Lemma 2. Hence $s=1$, and so

$$A_2 \text{ is uniserial, provided } A_1 J \neq 0\,.$$

Similarly A_1 is also uniserial, provided $A_2 J \neq 0$. Now assume that A_2 is uniserial ($|A_2| \geqslant 2$ and hence so is A_1). We shall show $|A_1| \leqslant 2$. Assume $A_1 J^2 \neq 0$ and hence $A_2J^2\neq 0$. Consider $\{A_1, A_1J\oplus A_2J^2, A_1J^2\oplus A_2J, A_2\}$. Since $A_1\nsim A_2$ by Lemma 2, 1) $A_1\sim A_1J\oplus A_2J^2$ or 2) $A_1\sim A_1J^2\oplus A_2J$ (A_1 and A_2 are symmetry) or 3) $A_1J\oplus A_2J^2\sim A_1J^2\oplus A_2J$.

1) It is clear that $xA_1\supset A_1J\oplus A_2J_1$ for a unit x. However A_1 is uniserial, and so $A_2J^2=0$ (note $|A_1|\leqslant|A_2|$). 2) This is similar. 3) Aassume $x(A_1J\oplus A_2J^2)\supset A_1J^2\oplus A_2J$. Since $\Delta(A_1)=\Delta$, there exists j in eJe such that $(x+j)A_1=A_1$. Let a_2j_2 be an element in A_2J ($a_2\in A_2$, $j_2\in J$). Then $x(a_1j_3+a_2'j_4)=a_2j_2$ for some $a_1\in A_1$, $a_2'\in A_3$, $j_3\in J$ and $j_4\in J^2$. Hence $(x+j)a_1j_3-ja_1j_3+xa_2'j_4=a_2j_2$. On the other hand, ja_1j_3, $xa_2'j_4$ are contained in J^{i+2}. Take the projection of eJ^i onto A_2, and $a_2j_2\in A_2\cap J^{i+2}=A_2J^2$. Hence $A_2J=0$. Similarly if $x(A_1J\oplus A_2J^2)\supset A_1J^2\oplus A_2J$, $A_1J=0$. Therefore $|A_1|\leqslant 2$.

We observe isomorphisms between sub-factor modules of A_1 and A_2, and then investigate submodules X in eJ^i. It is well known that there exist submodules $A_1\supset C\supset C'$ and $A_2\supset D\supset D'$ such that $h\colon C/C'\approx D/D'$ and $X=\{c+d\mid\in C\oplus D, h(c+C')=d+D'\}$ (cf. [3]). We denote X by $C(h)D$.

a-1) Let $|A_1|=2$.

a-1-1) $f\colon A_1/E_2\approx A_2/F_2$ $(\approx gR/gJ)$.

Then $\Delta'=\overline{gRg}=\bar{Z}$ from 2-i) and f is a unique isomorphism.

a-1-2) ([7], Theorem 17) Assume $f\colon A_1/E_2\approx F_i/F_{i+1}$ $(i>1)$. Consider $\{A_1, A_2, E_2\oplus F_2, A_1(f)F_i\}$. Since $\Delta(A_1)=\Delta(A_2)=\Delta$, $A_2\not\approx A_1(f)F_i$. Further $E_2\oplus F_2$ being characteristic, from Lemma 7 there exists a unit x' in eRe such that $x'A_1\subset A_1(f)F_i$. Let $p_j\colon eJ^i\to A_j$ be the projection and $x'=x+j$; $xA_1=A_1$, $j\in eJe$ as usual. Then for a generator a in A_1

$$(x+j)a = ar+f(ar)+z_1+z_2;\ r\in R,\ z_1\in E_2 \text{ and } z_2\in F_{i+1}\,.$$

Hence $$xa+p_1(ja) = ar+z_1 \text{ and } p_2(ja) = f(ar)+z_2\,.$$

Since $p_1(ja)\in E_2$, $xa\equiv ar$ (mod E_2). Assume $i<p-1$. Since $ja\in F_{p-1}\subset F_{i+1}$, $f(ar)\equiv f(xa)\equiv 0$ (mod F_{i+1}). However xa is a generator of A_1, and hence $f=0$. Therefore $i\geqslant p-1$.

a-1-3). See a-2-1) below.

a-1-4). $E_2\approx F_2$ $(p=2)$. We have the situation of 2-i).

Assume further $f\colon A_1/E_2\approx F_2$ $(A_2/F_2\approx E_2)$, and consider $\{A_1, A_2, A_1(f), E_2\oplus F_2\}$. Then $A_1\sim A_1(f)$ by Lemma 7 and so $A_1(f)=xA_1$ for some unit x in eRe, since $A_1\approx A_1(f)$. If $p>2$, see a-2-2) below.

a-1-5) Let X be a submodule in eJ^i.

i) $X=A_1(f_1)F_i=F_i(f_1^{-1})$ $(f_1\colon A_1\approx F_i/F_{i+2})$. If $i=1$, consider R/J^{i+3}. Then this contradicts 2-i). Hence $i\neq 1$, $F_i=F_{p-1}$ and $F_{i+2}=0$ from a-1-2). $\{A_1, A_2, E_2\oplus F_2, A_1(f_1)\}$ shows $A_1(f_1)=xA_1$ for some unit x in eRe.

ii) $X=A_1(f_2)A_2$ $(f_2\colon A_1/E_2\approx A_2/F_2)$. Then $X=B_1$ from a-1-1).

iii) $X=A_1(f_3)F_i$ $(f_3\colon A_1/E_2\approx F_i/F_{i+1}, i>1)$ and hence $i=p-1$ or p by a-1-2). Then $\{A_1\oplus F_{i+1}, A_2, E_2\oplus F_2, A_1(f_3)F_i\}$ shows $A_1(f_3)F_i=x(A_1\oplus F_{i+1})$.

iv) $X=A_2(f_4^{-1})$ (f_4: $E_2\approx A_2/F_2$). $\{A_1, A_2, E_2\oplus F_2, A_2(f_4^{-1})\}$ shows $A_2=xA_2(f_4^{-1})$.

v) $X=F_i(f_5^{-1})$ (f_5: $E_2\approx F_i/F_{i+1}$, $i\geqslant 2$). In this case $eJ^{i+1}=E_2\oplus F_2$. Hence this is the case of a-2). Accordingly $X=B_1$, B_i' or B'', provided X is not isomorphic to a standard submodule in eJ^{i+1} via x_l.

Thus we have shown that X is isomorphic to a standard submodule in eJ^i via x_l except B_1, B_i' and B''.

a-2) $|A_1|=1$.

a-2-1) *Let* f: $A_1\approx F_i/F_{i+1}$ $(i<p)$. *If* $F_i(f^{-1})\supset xA_1$ for some unit x in eRe, $xA_1\subset F_{i+1}\subset A_2$, since $\mathrm{J}(F_i(f^{-1}))=F_{i+1}$, which is a contradiction from lemma 2. We note further that A_2 is characteristic, since $\Delta(A_2)=\Delta$ and A_1 is simple. Assume $\Delta(F_i(f^{-1}))\neq\Delta$. Then $\{A_2, A_1, F_i(f^{-1})\}$ derives a contradiction from the above remarks and Lemma 7. It is clear that $eJe(F_i(f^{-1}))\subset eJe(F_i\oplus A_1)\subset F_{i+1}$. Hence $F_i(f^{-1})$ is also characteristic. Let f': $A_1\approx F_i/F_{i+1}$ be another isomorphism. $\{A_2, A_1, F_i(f^{-1}), F_i(f'^{-1})\}$ gives $F_i(f^{-1})=F_i(f'^{-1})$ since they are characteristic. Therefore $f=f'$. Accordingly, $\Delta\approx\overline{g_4R_4g}\approx\bar{Z}$ as given in the proof of 2-i). Further assume g: $A_1\approx F_j/F_{j+1}$ $(j<p)$. Again consider $\{A_2, A_1, F_i(f^{-1}), F_j(g^{-1})\}$. Then $F_i(f^{-1})\supset F_j(g^{-1})$ if $i<j$, and so $F_j(g^{-1})\subset F_{i+1}$, a contradiction.

a-2-2) Assume that f_1, f_2: $A_1\approx F_p$ and they are not given by j_l' in eJe. Then $\{A_2, A_1, A_1(f_1), A_1(f_2)\}$ gives, from Lemmas 6 and 7, that $A_1(f_1)=x'A_1(f_2)$ for some unit x' in eRe. Since $\Delta(A_1)=\Delta$, there exists j in eJe such that $(x'+j)A_1=A_1$. Put $x=x'+j$. Then for a generator a in A_1

$$(x-j)(a_2+f_2(a)) = ar+f_1(ar);\ r\in R\,.$$

Hence

$$xa = ar,\ xf_2(a)-ja = f_1(ar)\,.$$

Next assume further that q: $A_1\approx F_i/F_{i+1}$ $(i<p)$. Consider $\{A_2, A_1, F_i(q^{-1}), A_1(f)\}$, and $F_i(q^{-1})\sim A_1(f_1)$ since $F_i(q^{-1})$ is characteristic. Which is a contradiction. In particular, if $eJeA_1=0$, $A_1(f)$ is characteristic, since $\Delta(A_1(f))=\Delta$ (if $\Delta\neq\Delta(A_1(f))$, $\{A_1, A_2, A_1(f)\}$ gives $A_1\sim A_1(f)$). Then f is given by j_l from Lemma 6). Hence $f_1=f_2$ from the first paragraph, and so $\Delta\approx\overline{g_5R_5g}\approx\bar{Z}$ as in the proof of 2-i).

b) A_2/A_2J^k $(k\geqslant 1)$ is uniserial and $A_2J^k=\sum_{i=1}^{s}\oplus B_i$ $(s\geqslant 2)$. Then A_1 is simple from the initial paragraph of ii). Then $DJ^k=eJ^{i+k}=A_2J^k$. Since $eJ^{i+k}=B_1\oplus\cdots\oplus B_s$, $s\leqslant 3$ from Lemma 15, and $[\Delta\colon\Delta(B_i)]\leqslant 2$ for all i by Lemmas 2 and 7. If $[\Delta\colon\Delta(B_1)]=1$ and $[\Delta\colon\Delta(B_2)]=2$, $\{A_1, B_1, B_2, B_2\}$ derives a contradiction. Hence either $[\Delta\colon\Delta(B_i)]=1$ for all i (b_1)) or $[\Delta\colon\Delta(B_i)]=2$ for all i (b_2)). In the former case $s=2$ by Lemma 7 and in the latter case also $s=2$ and $B_2=xB_1$ for some unit x in eRe by Lemma 10.

b$_1$) $\Delta(B_1)=\Delta(B_2)=\Delta$.

Then $\{A_1, B_1, B_2, \mathrm{J}(B_1)\oplus\mathrm{J}(B_2)\}$ implies $\mathrm{J}(B_1)=0$ ($|B_1|\leqslant|B_2|$). If $f\colon B_1\approx B_2$, $\{A_1, B_1, B_2, B_1(f)\}$ derives a contradiction. Hence $B_1\not\approx B_2$. We can show as before that B_2 is uniserial.

b$_1$-1) This is the case of a-2-1).

b$_1$-2) Assume $f\colon A_1\approx B_1$. $\{A_1, A_1(f), B_1, B_2\}$ derives $A_1\sim A_1(f)$, i.e., $(x+j)A_1=A_1(f)$: $xA_1=A_1$ and $j\in eJe$. $(x+j)a=ar+f(ar)$; $A_1=aR$, $r\in R$. Hence $xa=ar$ and $ja=f(ar)$. Put $xa=b$, and $A_1=bR$. $f(b)=jx^{-1}b$.

b$_1$-3) Assume $f\colon A_1\approx H_i/H_{i+1}$. $\{A_1, B_1, B_2, H_i(f^{-1})\}$ shows $A_1\sim H_i(f^{-1})$. Hence $H_{i+1}=0$ and f is given by j_l as above (cf. Lemma 6).

b$_1$-4) Assume $f\colon B_1\approx H_i/H_{i+1}$. $\{A_1, B_1, B_2, H_i(f^{-1})\}$ derives $B_1\sim H_i(f^{-1})$, since $\Delta(B_2)=\Delta$. Hence $H_{i+1}=0$ and f is given by j_l from Lemma 6.

b$_2$) $[\Delta\colon\Delta(B_i)]=2$ for $i=1, 2$, ($B_2=xB_1$).

$\{A_1, B_1, B_2, \mathrm{J}(B_1)\oplus\mathrm{J}(B_2)\}$ shows, from Lemma 2, that $\mathrm{J}(B_2)=0$, i.e., B_2 is simple. Further since $\Delta(A_1)=\Delta$ and $[\Delta\colon\Delta(B_1)]=2$, $[\Delta\colon\Delta(E)]=2$ for all simple submodules E in $V=B_1\oplus B_2$ by Lemmas 2 and 8. Hence V satisfies ($\sharp$, 1) by Lemma 7.

b$_2$-1) If $A_1\approx F_i/F_{i+1}$, $\Delta=\bar{Z}$ by a-1-3). Hence $\Delta(B_1)=\Delta$.

b$_2$-2). Assume $f\colon A_1\approx B_1$. $\{A_1, B_1, B_1, A_1(f)\}$ derives $A_1\sim A_1(f)$. Hence f is given by j_l as b$_1$-2).

REMARK 17. If R is an algebra over an algebraically closed field K, $\Delta\neq\bar{Z}$ and the first part of a-2-2) does not occur (take $f_2=kf_1$, $k\neq 1$; $k\in K$). We can express f in a-1-2) as an element in eJe, however it is little complicated (cf. [7], Theorem 17).

In order to make the converse version clear, we illustrate the structure of Lemmas 10$\sim$16 as follows:

1) (Lemma 10)

$$
\begin{array}{c|l}
eR & eJ^i \qquad\quad eJ^{i+1} \\
 & A_1 \;\text{——}\; B_1 \;\text{——}\; 0 \\
\text{————} & \\
 & A_2 = xA_1 - B_2 = xB_1 - 0
\end{array}
$$

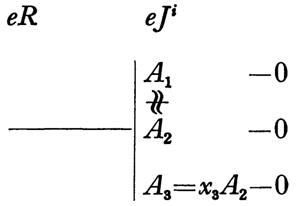
$[\Delta\colon\Delta(A_1)]=[\Delta\colon\Delta(A_2)]=2$, every characteristic submodule in eJ^i is linear with respect to the inclusion and $[\Delta\colon\Delta(C)]=2$ for any non-characteristic submodule C in eJ^i. Further those C are related to one another with respect to $\sim$.

2) (Lemma 11)

$$
\begin{array}{c|l}
eR & eJ^i \\
 & A_1 \qquad -0 \\
\text{————} & A_2=x_2A_1-0 \\
 & A_3=x_3A_1-0
\end{array}
\tag{3}
$$

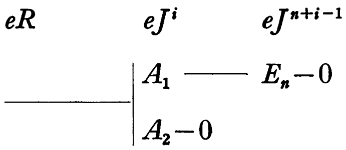
$[\Delta\colon \Delta(A_1)]=[\Delta\colon \Delta(A_2)]=[\Delta\colon \Delta(A_3)]=3$ and $A_1\oplus A_2\oplus A_3$ satisfies (♯, 1) and (♯, 2). Further $[\Delta\colon \Delta(C)]\leqslant 3$ for every submodule C in $A_1\oplus A_2\oplus A_3$. (A_3 may be zero.)

3) (Lemma 12, i))



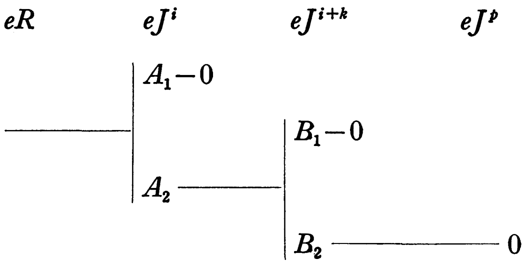
$[\Delta\colon \Delta(A_1)]=1$ and $[\Delta\colon \Delta(A_2)]=[\Delta\colon \Delta(A_3)]=2$. Further $A_2\oplus A_3$ satisfies (♯, 1).

4) (Lemma 12, ii-a))



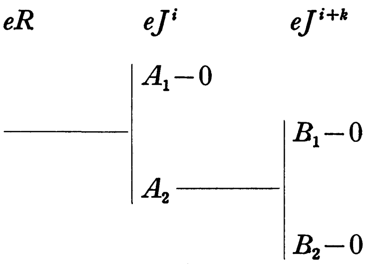
$[\Delta\colon \Delta(A_1)]=1$, $[\Delta\colon \Delta(A_2)]=2$ and $A_2\oplus E_n$ satisfies (♯, 1) except E_n.

5) (Lemma 12, ii-b ii-b-2-ii-1))



$[\Delta\colon \Delta(A_1)]=1$, $[\Delta\colon \Delta(A_2)]=2$ and $[\Delta\colon \Delta(B_1)]=[\Delta\colon \Delta(B_2)]=1$. $A_1\oplus A_2/J(A_2)$ satisfies (♯, 1) except A_1. (B_2 may be zero.)

5′)



$[\Delta\colon \Delta(B_1)]=[\Delta\colon \Delta(B_2)]=2$ and $B_1\approx B_2$. $B_1\oplus B_2$ satisfies ($\sharp$, 1).

6) (Lemma 16, 1))

$$
\begin{array}{lll}
eR & eJ^{i} & eJ^{i+n-1} \\
 & \left|\, A_1 \text{———————} \right. & E_n-0 \\
\text{——————} & \left|\, A_2-0 \right. & \\
 & \left|\, \not\approx \right. & \\
 & \left|\, A_3-0 \right. &
\end{array}
$$

$[\Delta\colon \Delta(A_1)]=[\Delta\colon \Delta(A_2)]=[\Delta\colon \Delta(A_3)]=1$. If $f\colon A_2\approx E_n$, f is given by j_l; $j\in eJe$. (A_2, A_3 and E_2 may be zero.)

7) (Lemma 16, 2-i))

$$
\begin{array}{ll}
eR & eJ^{i} \\
 & \left|\, A_1-0 \right. \\
\text{——————} & \left| \right. \\
 & \left|\, A_2-0 \right.
\end{array}
$$

$[\Delta\colon \Delta(A_1)]=[\Delta\colon \Delta(A_2)]=1$ and $\Delta\approx\Delta'\approx\bar{\bar{Z}}$.

8) (Lemma 16, 2-ii-a-1))

$$
\begin{array}{llll}
eR & eJ^{i} & eJ^{i+1} & eJ^{i+n-1} \\
 & \left|\, A_1- \right. & E_2-0 & \\
\text{——————} & \left| \right. & & \\
 & \left|\, A_2- \right. & F_2 \text{————} & F_n-0
\end{array}
$$

$[\Delta\colon \Delta(A_1)]=[\Delta\colon \Delta(A_2)]=1$. If $f\colon A_1/E_2\approx A_2/F_2$, $\Delta\approx\Delta'\approx\bar{\bar{Z}}$. Every submodule except B_1, B_i' and B'' is isomorphic to a standard submodule via x_l. (If $n=2$ and $E_2\approx F_2$, $\Delta\approx\Delta'\approx\bar{\bar{Z}}$.) If $E_2=0$, the conditions in a-2) of Lemma 16 are fulfiled.

9) (Lemma 16, 2-ii-b$_1$))

$$
\begin{array}{llll}
eR & eJ^{i} & eJ^{i+k} & eJ^{i+n-1} \\
 & \left|\, A_1-0 \right. & & \\
\text{——————} & \left| \right. & \left|\, B_1-0 \right. & \\
 & \left|\, A_2 \text{————} \right. & \left| \right. & \\
 & & \left|\, B_2 \text{——————} \right. & F_n-0
\end{array}
$$

$[\Delta\colon \Delta(A_1)]=[\Delta\colon \Delta(A_2)]=1$, $[\Delta\colon \Delta(B_1)]=[\Delta\colon \Delta(B_2)]=1$. If $f\colon A_1\approx B_1$, f is given by j_l; $j\in eJe$. Similar facts hold for other cases.

10) (Lemma 16, 2-ii-b$_2$))

$$
\begin{array}{lll}
eR & eJ^{i} & eJ^{i+k} \\
 & \vert\, A_1 - 0 & \\
\overline{\qquad\qquad} & \vert & \vert\, B_1 \qquad -0 \\
 & \vert\, A_2\ \overline{\qquad\quad} & \vert \\
 & & \vert\, B_2 = xB_1 - 0
\end{array}
$$

$[\Delta: \Delta(A_1)]=[\Delta: \Delta(A_2)]=1$, $[\Delta: \Delta(B_2)]=2$ and $B_1 \oplus B_2$ satisfies (#, 1).

We shall show that if eR has one of the structures of the above diagrams 1)$\sim$10), then R is a right US-4 ring with (*, 1′). It is clear from the diagrams that (*, 1′) holds. Let $\{U_i\}_{i=1}^{4}$ be a set of submodules in eR.

Diagram 1). If U_1 and U_2 are characteristic, $U_1 \supset U_2$ or $U_1 \subset U_2$. Hence $U_1 \oplus U_2$ satisfies (**, 2) by [4], Corollary 1 of Theorem 2. Hence $D=\sum_{i=1}^{4} \oplus U_i$ satisfies (**, 4) by [2], Lemma 1. Assume that $U_1 \cap U_2 \supset eJ^i$. Then U_i for $i=1, 2$ is characteristic, and hence D satisfies (**, 4) from the above. Next assume that $U_1 \supset eJ^i$ and $eJ^i \supset U_j$ for $j>1$. Since $\Delta(U_1)=\Delta$, $U_1 \oplus U_2$ satisfies (**, 2) by [4], Corollary 1 of Theorem 2. Finally assume $eJ^i \supset U_j$ for all j. If $\{U_j\}_{j=1}^{3}$ is a set of non-characteristic submodules, then we may assume $U_1 \supset x_2U_2 \supset x_3U_3$ for some units x_i in eRe by assumption. Since $[\Delta: \Delta(U_i)]=2$, $U_1 \oplus U_2 \oplus U_3$ satisfies (**, 3) by [4], Corollary 3 of Theorem 2. Therefore D satisfies (**, 4).

2) As is shown in 1), we may assume that $eJ^i \supset U_j$ for all j. Then $U_1 \supset x_2U_2 \supset x_3U_3 \supset x_4U_4$ by assumption, where the x_i are units in eRe. Then from the assumption $[\Delta: \Delta(C)] \leqslant 3$ and the argument of the proof of [4], Corollary 3 of Theorem 2, D satisfies (**, 4).

3) Let $eJ^i \supset U_j$ for all j. Then $U_i = A_1 \oplus B_i$ or $U_i \subset A_2 \oplus A_3$ by assumption, where $B_i \subset A_2 \oplus A_3$. First assume $U_j \subset A_2 \oplus A_3$ or $U_j = A_1 \oplus B_j$ ($B_j \neq 0$) for all $j \leqslant 3$. Then D satisfies (**, 4) by [4], Corollary 3 of Theorem 2 (note A_1 and $A_2 \oplus A_3$ are characteristic and see the remark above). If $U_1 = A_1$ and $U_2 = A_1 \oplus B_2$, $U_1 \oplus U_2$ satisfies (**, 2) by [4], Corollary 1 of Theorem 2. Thus D satisfies (**, 4)

4) Every submodule in eJ^i is isomorphic to a standard submodule in eJ^i via x_l. Hence we may assume that all U_j are standard. Then D satisfies (*, 4) by [4], Corollaries 1$\sim$3 of Theorem 2.

5) and 5′) Let $eJ^i \supset U_1 \supset A_2J$ and $U_1 \neq A_1 \oplus A_2$. Then $U_1/A_2J = x(A_2/A_2J)$, and so $xA_2 = U_1$. Further $A_1 \oplus A_2J$ is characteristic. If $U_1 = A_1 \oplus A_2J$ and $U_2 \subset A_2J$, $U_1 \oplus U_2$ satisfies (*, 2). Accordingly we may assume that U_i is A_1 or a submodule of A_2 Therefore D satisfies (**, 4).

6) and 7) These are clear.

8) First we note $B_1 \supset E_2 \oplus F_2 \supset B_i'$ ($E_2 \oplus F_2 \supset B''$) and B_i', B'' do not appear simultaneously. If the U_i are standard for all i, $U_i \sim U_j$ for some pair i, j. Hence D satisfies (**, 4) by [4], Corollary 2 of Theorem 2. The conditions given in Lemma 16 show that $A_1 \sim A_1(f)$, $F_p(f^{-1}) \sim F_p(g^{-1})$, $\cdots$ etc.. Hence we obtain the desired result.

9) and 10) These are simpler than 8), (if $A_1 \approx F_i/F_{i+1}$ ($F_{i+1} \supset B_1 \oplus B_2$), $\Delta \approx \bar{Z}$. Hence $\Delta(C)=\Delta$ for any submodule C in eR).

Thus we obtain

Theorem 2. *R is a right US-4 (basic) ring with (*, 1′) if and only if eR has one of the structures given in Lemmas 10~16 (cf. Diagrams 1)~10)) for each primitive idempotent e.*

3. Hereditary rings

In this section, we shall study a hereditary and right US-3 (resp. US-4) ring R. If R is hereditary, (*, 1′) holds, and hence we can make use of the results in the previous sections.

Lemma 18. *Assume that R is basic and hereditary. Then a submodule A in eR is characteristic if and only if $\Delta(A)=\Delta$. Every non-zero element in* Hom_R *(eR, fR) is a monomorphism, where e and f are primitive idempotents.*

Proof. The second half is clear (see [9], Lemma 2). Hence, since $eJe=0$, the first one is clear

From now on we assume that R is a hereditary and basic ring. First we assume further that R is right US-3.

Theorem 3. *Let R be a hereditary (and basic) ring. Then R is a right US-3 ring if and onyl if eR has the following structure for each primitive idempotent e:*

i) *eR/eJ^t is uniserial for some t and*

ii) *$eJ^t=0$ or $eJ^t=A \oplus B$ such that either*

a) *A and B are simple and $A \oplus B$ satisfies (#, 1), and $[\Delta: \Delta(A)]=2$, or*

b) *A is simple, B is uniserial and A is not isomorphic to any sub-factor modules of B (and hence $\Delta(A)=\Delta(B)=\Delta$).*

Proof. If R is right US-3, eR has the structure in Theorem 1. We consider the case b) of Theorem 1. Assume that f: $A \approx$(the socle of B). Then $\{A, A(f), B\}$ derives a contradiction, since A and B are characteristic by Lemma 18. Thus we obtain the theorem from Theorem 1.

Let R be a basic herediatry ring. Then

$$R=\begin{pmatrix} \Delta_1 & M_{12} \cdots\cdots\cdots M_{1n} \\ & \Delta_2 \quad M_{23} \cdots M_{2n} \\ & 0 \qquad \ddots \\ & \qquad\qquad M_{n-1n} \\ & \qquad\qquad \Delta_n \end{pmatrix}.$$

where the Δ_i are division rings and the M_{ij} are left Δ_i- and right Δ_j-modules [1].

We shall express explicitly the content of Theorem 3 for M_{ij} in a row of the above ring.

1)

$(0 \cdots \Delta_i 0 \Delta_{i_1} 0 \cdots 0 \Delta_{i_t} 0 \cdots \Delta_{i_p} 0)$

2)

$$(0 \cdots \Delta_i 0 \cdots \Delta_{i_1} 0 \cdots \Delta_{i_t} 0 \cdots \begin{pmatrix} u_p \Delta_{i_p} \\ v_p \Delta_{i_p} \end{pmatrix} \cdots 0) \qquad \begin{matrix} \cdots\cdots A \\ \cdots\cdots B \end{matrix}$$

(4)

where $\begin{pmatrix} u_p \Delta_{i_p} \\ v_p \Delta_{i_p} \end{pmatrix} = u_p \Delta_{i_p} \oplus v_p \Delta_{i_p}$ satisafies (#, 1).

3)

$$(0\,0 \Delta_i 0 \cdots \Delta_{i_1} 0 \cdots \Delta_{i_t} 0 \cdots$$

$$\cdots \begin{pmatrix} u_{t+1} \Delta_{i_{t+1}} \\ \\ 0 \end{pmatrix} 0 \begin{pmatrix} 0 \\ \\ v_{t+2} \Delta_{i_{t+2}} \end{pmatrix} 0 \cdots \begin{pmatrix} 0 \\ \\ v_p \Delta_{i_p} \end{pmatrix} 0 \cdots 0) \qquad \begin{matrix} \cdots\cdots A \\ \\ \cdots\cdots B \end{matrix}$$

As is given in the proof of [9], Theorem 1, we can show a ring monomorphisms $\rho_{rs}\colon \Delta_r \rightarrow \Delta_s$ for $r<s<k$ such that $xu_r = u_s \rho_{rs}(x)$ for $x \in \Delta_r$ and $\rho_{rs} \rho_{sv} = \rho_{xv}$.

Next we shall characterize a hereditary (basic) and right US-4 ring. If R is hereditary, some results in the previous sections may not occur as shown in Theorem 3. We shall observe them.

In the case b) of Lemma 12, A_2 is simple.
Because, since A_1 is simple and $[\Delta\colon \Delta(A_2)]=2$, $A_1 \approx A_2/J(A_2)$. Hence $A_1 \approx A_2$ by Lemma 18.

We shall observe the conditions in Lemma 16 for a hereditary ring. a-1-1), a-1-2), a-1-3), any of b-1-1)$\sim$4) and b_2-2) do not occur from Lemma 18. For instance, if $f'\colon A_1/E_2 \approx F_{p-1}/F_p$ (a-1-2)), $f\colon A_1 \approx F_{p-1}$ by Lemma 18. Then $A_1 \sim A_1(f)$ by a-1-5). However, A_1 is characteristic, and so $A_1 = A_1(f)$. Therefore $f=0$.

We shall use the notations after Theorem 3.

Lemma 19. *In case* 2-i) *in Lemma* 16, $e_{ii}R$ *is of the form* $(0, \cdots, \bar{Z}, \cdots$

$0 \cdots \bar{Z} \cdots 0)$. *In case of* 2-a-1-4) *in Lemma* 16, A_1 (*resp.* A_2)) *is of the form* $(0, \cdots, \bar{Z}, 0, \bar{Z}, 0 \cdots)$ (*resp.* $(0, \cdots \bar{Z}, 0, \bar{Z}, 0 \cdots)$).

Proof. Let $E_2 \approx F_p \approx e_{kk}R$. Then $\overline{e_{kk}R} \approx \bar{Z}$ by Lemma 16. Let $A_2 \approx e_{ss}R$. Then $e_{ss}R$ is uniserial and $M_{sk} = u_{sk}\bar{Z}$ $(\approx F_p)$. Since M_{sk} is a left Δ_s-module, $\Delta_s \subset \bar{Z}$. Hence $\Delta_s = \bar{Z}$. We have the same for 2-i).

Thus we have

Theorem 4. *Let R be a hereditary (basic) ring. Then R is right US-4 if and only if for each* $e = e_{ii}$, *eR has one of the following structures:* $1 \sim 11$

1)

$$(0 \cdots 0\Delta_i 0\Delta_{i_1} 0\Delta_{i_t} 0 \cdots \Delta_{i_p} \cdots 0)$$

2) (Lemma 10)

$$(0 \cdots 0\Delta_i 0\Delta_{i_1} 0 \cdots \Delta_{i_t} 0 \cdots \begin{pmatrix} u_{t+1}\Delta_{i_{t+1}} \\ v_{t+1}\Delta_{i_{t+1}} \end{pmatrix} 0 \begin{pmatrix} u_{t+2}\Delta_{i_{t+2}} \\ v_{t+2}\Delta_{i_{t+2}} \end{pmatrix} \cdots 0 \cdots 0) \qquad \begin{matrix} \cdots\cdots A_1 \\ \cdots\cdots A_2 \end{matrix}$$

$[\Delta\colon \Delta(A_i)] = 2$ $(i=1, 2)$ *and* u_{t+2}, v_{t+2} *may be zero. The conditions in Lemma* 10 *are satisfied.*

3) (Lemma 11)

$$(0 \cdots \Delta_i 0 \cdots \Delta_{i_1} 0 \cdots \Delta_{i_t} 0 \cdots \begin{pmatrix} u_{t+1}\Delta_{i_{t+1}} \\ v_{t+1}\Delta_{i_{t+1}} \\ w_{t+1}\Delta_{i_{t+1}} \end{pmatrix} \cdots 0) \qquad \begin{matrix} \cdots\cdots A_1 \\ \cdots\cdots A_2 \\ \cdots\cdots A_3 \end{matrix}$$

$[\Delta\colon \Delta(A_i)] = 3$ *for each i and* $A_1 \oplus A_2 \oplus A_3$ *satisfies* (♯, 1) *and* (♯, 2). w_{t+1} *may be zero.*

4) (Lemma 12-i))

$$(0 \cdots \Delta_i 0 \cdots \Delta_{i_1} 0 \cdots \Delta_{i_t} 0 \cdots 0 \cdots \begin{pmatrix} u_{t+1}\Delta_{i_{t+1}} \\ v_{t+1}\Delta_{i_{t+1}} \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ w_{t+2}\Delta_{i_{t+2}} \end{pmatrix} \cdots 0) \qquad \begin{matrix} \cdots\cdots A_1 \\ \cdots\cdots A_2 \\ \cdots\cdots A_3 \end{matrix}$$

$\Delta(A_3) = \Delta$, $[\Delta\colon \Delta(A_i)] = 2$ $(i=1, 2)$ *and* $A_1 \oplus A_2$ *satisfies* (♯, 1).

(5)

5) (Lemma 12-ii-a) and b))

$$(0 \cdots \Delta_i 0 \cdots \Delta_{i_1} 0 \cdots \Delta_{i_t} 0 \cdots$$

$$\begin{pmatrix} u_{t+1}\Delta_{i_{t+1}} \\ 0 \end{pmatrix} 0 \cdots 0 \begin{pmatrix} u_{p-1}\Delta_{i_{p-1}} \\ 0 \end{pmatrix} 0 \cdots \begin{pmatrix} u_p\Delta_{i_p} \\ v_p\Delta_{i_p} \end{pmatrix} \cdots 0) \qquad \begin{matrix} \cdots\cdots A_1 \\ \cdots\cdots A_2 \end{matrix}$$

$\Delta(A_1) = \Delta$, $[\Delta\colon \Delta(A_2)] = 2$, *and* $u_p\Delta_{i_p} \oplus v_p\Delta_{i_p}$ *satisfies* (♯, 1), *except* $u_p\Delta_{i_p}$ ($\{u_{t+1}, \cdots, u_{p-1}\}$ *may be zero.*)

6) (Lemma 16, 1))

$$(0 \cdots \Delta_i 0 \cdots \Delta_{i_1} 0 \cdots \Delta_{i_t} 0 \cdots 0$$

$$\begin{pmatrix} u_{t+1}\Delta_{i_{t+1}} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ v_{t+2}\Delta_{i_{t+2}} \\ 0 \end{pmatrix} 0 \cdots \begin{pmatrix} 0 \\ 0 \\ w_{t+3}\Delta_{i_{t+3}} \end{pmatrix} 0 \cdots \begin{pmatrix} 0 \\ 0 \\ w_p\Delta_{i_p} \end{pmatrix} \cdots 0 \qquad \begin{matrix} \cdots\cdots A_1 \\ \cdots\cdots A_2 \\ \cdots\cdots A_3 \end{matrix}$$

$\Delta(A_i)=\Delta$ $(i=1, 2, 3)$ *and* u_{t+1} *may be zero.*

$$\Delta_1 \subset \Delta_{i_1} \subset \cdots \subset \Delta_{i_t} \begin{matrix} \subset \Delta_{i_{t+1}} \\ \subset \Delta_{i_{t+2}} \\ \subset \Delta_{i_{t+3}} \subset \cdots \subset \Delta_{i_p} \end{matrix}$$

7) (Lemma 16, 2-i))

$$(0 \cdots \bar{Z} 0 \cdots \bar{Z} \cdots 0 \bar{Z} \cdots 0 \cdots \begin{pmatrix} u_{t+1}\bar{Z} \\ v_{t+1}\bar{Z} \end{pmatrix} \cdots 0) \qquad \begin{matrix} \cdots\cdots A_2 \\ \cdots\cdots A_1 \end{matrix}$$

8) (Lemma 16, 2-ii-a))

$$(0 \cdots \Delta_i 0 \cdots \Delta_{i_1} 0 \cdots \Delta_{i_t} 0 \cdots$$

$$\begin{pmatrix} u_{t+1}\Delta_{i_{t+1}} \\ 0 \end{pmatrix} \cdots 0 \cdots \begin{pmatrix} 0 \\ v_{t+2}\Delta_{i_{t+2}} \end{pmatrix} 0 \cdots \begin{pmatrix} u_{t+3}\Delta_{i_{t+3}} \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ v_p\Delta_{i_p} \end{pmatrix} \cdots 0) \qquad \begin{matrix} \cdots\cdots A_1 \\ \cdots\cdots A_2 \end{matrix}$$

$\Delta(A_i)=\Delta$ $(i=1, 2)$, u_{t+3} *or* $\{v_{t+4}, \cdots, v_p\}$ *may be zero.*

$$\Delta_i \subset \Delta_{i_1} \subset \cdots \subset \Delta_{i_t} \begin{matrix} \subset \Delta_{i_{t+1}} \subset \Delta_{i_{t+3}} \\ \\ \subset \Delta_{i_{t+2}} \subset \Delta_{i_{t+4}} \subset \cdots \subset \Delta_{i_p} \end{matrix}$$

9) (Lemma 16, 2-ii-a')

$$(0 \cdots \bar{Z} 0 \cdots \Delta_{i_1} 0 \cdots \Delta_{i_t} 0 \cdots$$

$$\begin{pmatrix} 0 \\ v_{t+1}\bar{Z} \end{pmatrix} 0 \begin{pmatrix} u_{t+2}\bar{Z} \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ v_{p-1}\bar{Z} \end{pmatrix} \cdots 0 \begin{pmatrix} u_p\bar{Z} \\ v_p\bar{Z} \end{pmatrix} 0) \qquad \begin{matrix} \cdots\cdots A_1 \\ \cdots\cdots A_2 \end{matrix}$$

u_{t+2} *may be zero.*

10) (Lemma 16, 2-ii-b_1))

$$(0 \cdots \Delta_i 0 \cdots \Delta_{i_1} 0 \cdots \Delta_{i_t} 0 \cdots$$

$$\begin{pmatrix} u_{t+1}\Delta_{i_{t+1}} \\ \\ 0 \\ \\ \end{pmatrix}\begin{pmatrix} 0 \\ \\ v_{t+2}\Delta_{i_{t+2}} \\ \\ \end{pmatrix} 0 \begin{pmatrix} 0 \\ \\ v_{t+s}\Delta_{i_{t+s}} \\ \\ \end{pmatrix} 0 \begin{pmatrix} 0 \\ w_{t+s+1}\Delta_{i_{t+s+1}} \\ 0 \\ \\ \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \\ z_{t+s+2}\Delta_{i_{t+s+2}} \end{pmatrix} 0) \qquad \begin{matrix} \cdots\cdots A_1 \\ \cdots\cdots B_1 \\ \cdots\cdots A_2 \\ \cdots\cdots B_2 \end{matrix}$$

$\Delta(A_i)=\Delta(B_i)=\Delta$ $(i=1, 2)$.

$$\begin{array}{llll} & \subset\Delta_{i_{t+1}} & & \\ \Delta_1\subset\Delta_{i_1}\subset\cdots\subset\Delta_{i_t} & & \subset\Delta_{i_{t+s+1}} & \\ & \subset\Delta_{i_{t+2}}\subset\cdots\subset\Delta_{i_{t+s}} & & \\ & & \subset\Delta_{i_{t+s+2}}\subset\cdots\subset\Delta_{i_p} & \end{array}$$

11) (Lemma 16, 2-ii-b))

$$(0\cdots\Delta_i 0\cdots\Delta_{i_1}0\cdots\Delta_{i_t}0\cdots$$

$$\begin{array}{c} \begin{pmatrix} u_{t+1}\Delta_{i_{t+1}} \\ \\ 0 \end{pmatrix} 0 \begin{pmatrix} 0 \\ \\ v_{t+2}\Delta_{i_{t+2}} \end{pmatrix} 0 \begin{pmatrix} 0 \\ \\ v_{t+s}\Delta_{i_{t+s}} \end{pmatrix} 0 \begin{pmatrix} 0 \\ w_{t+s+1}\Delta_{i_{t+s+1}} \\ \\ z_{t+s+1}\Delta_{i_{t+s+1}} \end{pmatrix} 0) \end{array} \qquad \begin{array}{l} \cdots\cdots A_1 \\ \cdots\cdots B_1 \\ \cdots\cdots A_2 \\ \cdots\cdots B_2 \end{array}$$

$$\begin{array}{ll} & \subset\Delta_{i_{t+1}} \\ \Delta_i\subset\Delta_{i_1}\subset\cdots\subset\Delta_{i_t} & \subset\Delta_{i_{t+2}}\subset\cdots\subset\Delta_{i_{t-s}}\subset\Delta_{i_{t+s+1}} \end{array}$$

$\Delta(A_i)=\Delta$ *and* $[\Delta\colon\Delta\colon(B_i)]=2$ $(i=1, 2)$. $w_{t+s+1}\Delta_{i_{t+s+1}}\oplus z_{t+s+1}\Delta_{i_{t+s+1}}$ *satisfies* (#, 1),
where $\bar{Z}=Z/2$, *the* Δ*'s are division rings and* $\Delta_{i_1}\subset\Delta_{i_1}\cdots\subset\Delta_{i_p}$ *except* 6), 8), 10) and 11). *The series*: $(0\cdots\Delta_i 0\cdots\Delta_{i_1}0\cdots)$ *on the same level means a uniserial module.*

4. Left serial rings

We shall investigate the same problem for a left serial ring R. In this case (*, 1′) holds, too by [11], Corollary 4,2. Therefore we can make use of the results in §§ 1 and 2.

From now on we always assume that R is a left serial ring.

Lemma 20. *If* $eJ^i=A_1\oplus A_2$ *and the* A_i *are uniserial, every submodule* E *in* eJ^i *is isomorphic to a standard submodule* $B_1\oplus B_2$ *via* x_l*:*x *is a unit in eRe, where* $B_i\subset A_i$.

See the proof of [3], Theorem 1.

Lemma 21. *Let* $eJ^i=A_1\oplus A_2$ *and the* A_i *hollow. If* $\Delta(A_1)\neq\Delta$, *there exists a unit* x *in eRe such that* $xA_1=A_2$.

Proof. Since $\Delta(A_1)\neq\Delta$, there exists a unit y in eRe such that $(y+j')A_1\not\subset A_1$ for all j' in eJe. Let p be the projection of eJ^i onto A_2. Then $f=py_l\,|\,A_1$ is an element in $\mathrm{Hom}_R(A_1, A_2)$. If f is not an epimorphism, $f=j_l$ for some j in eJe, since A_2 is a hollow module ($\not\subset eJ^{i+1}$) and R is left serial. Then $(y-j)A_1\subset A_1$, a contradiction. Hence there exists a unit x in eRe such that $x_l=f$, and so $x_2A=A_1$.

Lemma 22. *Let* $eJ^i=A_1\oplus A_2$ *be as in Lemma* 21. *If* $\Delta(A_1)=\Delta$, $\Delta(A_1J^k\oplus A_2J^{k'})=\Delta$.

Proof. From Lemma 21, $\Delta(A_2)=\Delta$. Hence we may assume $k\leqslant k'$. Let x be any unit element in eRe. Since $\Delta(A_1)=\Delta$, there exists j in eJe such that $(x+j)A_1=A_1$. Hence $(x+j)(A_1J^k\oplus A_2J^{k'})\subset A_1J^k+(x+j)A_2J^{k'}\subset A_1J^k\oplus A_2J^{k'}$, and so $\bar{x}=\overline{x+j}\in\Delta(A_1J^k\oplus A_2J^{k'})$.

From Theorem 1, Lemmas 21, 22 and [8], Proposition 2, we obtain

Theorem 5. *Let R be a left serial ring. Then R is a right US-3 ring if and only if eR has the following structure for each primitive idempotent e:*
There exists an integer t such that
i) eR/eJ^t *is uniserial and*
ii) $eJ^t=0$ *or* eJ^t *is a direct sum of a simple module and a uniserial module.*

Finally we shall give a characterization of a left serial and right US-4 ring. As was shown in the previous section, we shall refine the results in § 2.

In Lemma 10, every submodule in eJ^i is standard up to x_l (x is a unit in eRe) by Lemma 20. Further since $\Delta(A_1\oplus A_2J)\neq\Delta$,

$$A_1\oplus A_2\supset \mathrm{J}(A_1)\oplus \mathrm{J}(A_2)\supset 0$$

is the set of all characteristic submodules in eJ^i.
From the above proof we have

REMARK 23. Let R be left serial and assume $eJ^i=A_1\oplus A_2$; the A_i are uniserial. If $[\Delta\colon \Delta(A_1)]=2$, $[\Delta\colon \Delta(C)]\leq 2$ for every submodule C in eJ^i and $\{eJ^{i+t}\}$ is the set of characteristic submodules in eJ^i. Hence, if R is left serial, i), ii) and iii) in Lemma 10 imply iv) and v). However hereditarity does not as is shown from the following example:
Let $K\subset L$ be fields such that $[L\colon K]=2$. Put

$$R=\begin{pmatrix} L & L & L\otimes_K L & L\otimes_K L \\ 0 & K & L & L \\ 0 & 0 & L & L \\ 0 & 0 & 0 & L \end{pmatrix}$$

Then R is hereditary. Put $L=1K+uK$, $e_{11}=e$, and $eJ=A_1\oplus A_2$; $A_1=1e_{12}R$, $A_2=ue_{12}R$ satisfy i), ii) and iii) in Lemma 10. Further $[\Delta\colon \Delta(B)]=2$ for any submodule B in eJ^2 if $\Delta\neq\Delta(B)$, since $[L\colon K]=2$. $\{eJ, eJ^2, eJ^3, (1\otimes u\pm u\otimes 1)e_{33}R,$

$(1\otimes u \pm u\otimes 1)e_{44}R\}$ is the set of characteristic submodules provided $u^2 \in K$, and $(1\otimes 1)e_{44}R \not\approx (1\otimes 1 + u\otimes x)e_{44}R$, provided $x \notin K$.

Lemma 24. *Let B_1 and B_2 be simple submodules in eJ^i and $V=B_1\oplus B_2$. If $B_1 \approx B_2$, V always satisfies ($\sharp$, 1).*

Proof. Since R is left serial, every simple submodule in V is isomorphic to B_1 via x_l; x is a unit in eRe. Hence V satisfies ($\sharp$, 1).

In Lemma 12, we do not have the case $t=2$ by Lemma 21.

In Lemma 16, we have always $A_1 \not\approx A_2$, since $\Delta(A_1)=\Delta(A_2)=\Delta$. Hence 2-i), 2-a-1-1), 2-a-2-3) and $p=2$ in 2-a-1-4) do not occur. Similarly 2-a-2-1) does not occur.

Thus we obtain

Theorem 6. *Let R be a left serial ring. Then R is right US-4 if and only if, for each primitive idempotent e, eR has one of the following structures:*

1) *eR is uniserial:*

$$\begin{array}{lll} eR & eJ & \qquad\qquad eJ^p \\ \bullet \!-\!\!-\! & \bullet \!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\! & 0 \end{array}$$

2)

$$\begin{array}{cc|l} eR & eJ^{i-1} & eJ^i \quad eJ^{i+1} \\ & & A_1 - B_1 - 0 \\ \bullet \!-\!\!-\! & \bullet \!-\!\!-\! & \\ & & A_2 - B_2 - 0 \end{array}$$

$[\Delta\colon \Delta(A_1)]=2$. *In this case $A_1 \approx A_2$ and B_1 may be zero.*

3)

$$\begin{array}{cc|l} eR & eJ^{i-1} & eJ^i \\ & & A_1 - 0 \\ \bullet \!-\!\!-\! & \bullet \!-\!\!-\! & A_2 - 0 \\ & & A_3 - 0 \end{array} \tag{6}$$

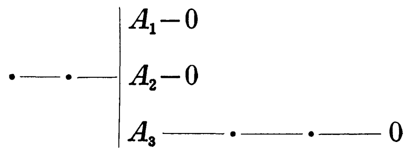
$[\Delta\colon \Delta(A_i)]=3$ *and $A_1\oplus A_2\oplus A_3$ satisfies ($\sharp$, 2). In this case $A_1 \approx A_2 \approx A_3$.*

4)

$$\begin{array}{cc|l} eR & eJ^{i-1} & eJ^i \\ & & A_1 - 0 \\ \bullet \!-\!\!-\! & \bullet \!-\!\!-\! & A_2 - 0 \\ & & A_3 - 0 \end{array}$$

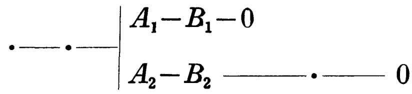
$\Delta(A_1)=\Delta$, $[\Delta\colon \Delta(A_i)]=2$ $(i=2, 3)$. *In this case $A_2 \approx A_3$.*

5) $eR \quad eJ^{i-1} \quad eJ^{i} \quad eJ^{i+1} \quad eJ^{p}$



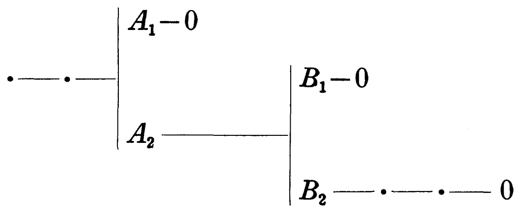
$\Delta(A_i)=\Delta$ $(i=1, 2, 3)$. *In this case* $A_1 \not\approx A_2$ *and* A_2 *may be zero.*

6) $eR \quad eJ^{i-1} \quad eJ^{i} \quad eJ^{i+1} \quad eJ^{p}$



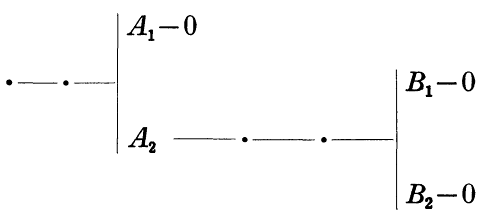
$\Delta(A_i)=\Delta$ $(i=1, 2)$.

7) $eR \quad eJ^{i-1} \quad eJ^{i} \quad eJ^{i+1} \quad eJ^{k} \quad eJ^{k+1} \quad eJ^{p}$



$\Delta(A_i)=\Delta$ $(i=1, 2, 3)$ *and* $\Delta(B_j)=\Delta$ $(j=1, 2)$.

8) $eR \quad eJ^{i-1} \quad eJ^{i} \quad eJ^{i+1} \quad eJ^{p-1} \quad eJ^{p}$



$\Delta(A_1)=\Delta(A_2)=\Delta$ *and* $[\Delta\colon \Delta(B_1)]=2$. *In this case* $B_1 \approx B_2$, *where each straight line means "uniserial".*

5. Examples

We shall give examples of hereditary (resp. left serial) and right US-3 (resp. US-4) rings. Let K be a field. By L and L' we denote extension fields of K with $[L\colon K]=2$ and $[L'\colon K]=3$, respectively, and $\bar{Z}=Z/2$, where Z is the ring of integers.

The following two rings are hereditary, left serial and right US-3 rings.

$\begin{pmatrix} K & K & K & K \\ & K & K & K \\ & & K & 0 \\ 0 & & & K \end{pmatrix}$ is the second type b) of Theorem 1 and $\begin{pmatrix} L & L & L \\ & L & L \\ 0 & & K \end{pmatrix}$ is the first type a) of Theorem 1.

On the other hand

$$\begin{pmatrix} K & L & L \\ 0 & L & L \\ 0 & 0 & K \end{pmatrix} \text{ is a hereditary, non-left serial and}$$

right US-3 ring, and

$$\begin{pmatrix} L & L & 0 \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix} \text{ with } e_{12}e_{23}=0 \text{ is a left}$$

serial, non-hereditary and right US-3 ring.

Next we shall give hereditary and right US-4 rings for each structure in Theorem 4. However, we can not construct an example of the case 5) from the reason given in Remark 13.

1 $\begin{pmatrix} K & K & K \\ & K & K \\ 0 & & K \end{pmatrix}$ 2 $\begin{pmatrix} L & L & L & L \\ & L & L & L \\ & & K & K \\ 0 & & & K \end{pmatrix}$ 3 $\begin{pmatrix} L' & L' & L' \\ & L' & L' \\ 0 & & K \end{pmatrix}$

3′ $\begin{pmatrix} D & D^* \\ 0 & D_1 \end{pmatrix}$, where D, D_1 and D^* are given in Remark 14.

4 $\begin{pmatrix} L & L & L \\ & L & 0 \\ 0 & & K \end{pmatrix}$ 6 $\begin{pmatrix} K & K & K & K & K \\ & K & 0 & 0 & 0 \\ & & K & 0 & 0 \\ & & & K & K \\ 0 & & & & K \end{pmatrix}$

7 $\begin{pmatrix} \bar{Z} & \begin{pmatrix} \bar{Z} \\ 0 \end{pmatrix} & \begin{pmatrix} \bar{Z} \\ \bar{Z} \end{pmatrix} \\ & \bar{Z} & \begin{pmatrix} \bar{Z} \\ \bar{Z} \end{pmatrix} \\ 0 & & \bar{Z} \end{pmatrix}$ 8 $\begin{pmatrix} K & K & K & K & K \\ & K & K & 0 & 0 \\ & & K & 0 & 0 \\ & & & K & K \\ 0 & & & & K \end{pmatrix}$

$$9\quad \begin{pmatrix} \bar{Z} & \begin{pmatrix}\bar{Z}\\ 0\end{pmatrix} & \begin{pmatrix}0\\ \bar{Z}\end{pmatrix} & \begin{pmatrix}\bar{Z}\\ \bar{Z}\end{pmatrix} \\ & \bar{Z} & 0 & \bar{Z} \\ 0 & & \bar{Z} & \bar{Z} \\ & & & \bar{Z} \end{pmatrix} \qquad 9'\quad \begin{pmatrix} \bar{Z} & \bar{L} & \bar{L} \\ & \bar{L} & \bar{L} \\ 0 & & \bar{Z} \end{pmatrix}$$

where $\bar{L}$ is an extension of $\bar{Z}$ with $[\bar{L}: \bar{Z}]=2$. $e_{11}R$ is of the form 2–1) in Lemma 16 and $e_{22}R$ is of the form in Lemma 10.

$$10\quad \begin{pmatrix} K & K & K & K & K & K \\ & K & 0 & 0 & 0 & 0 \\ & & K & K & K & K \\ & & & K & 0 & 0 \\ & & & & K & K \\ 0 & & & & & K \end{pmatrix} \qquad 11\quad \begin{pmatrix} L & L & L & L \\ & L & 0 & 0 \\ & & L & L \\ 0 & & & K \end{pmatrix}$$

The rings of 1)$\sim$6), 8), 10) and 11) are left serial.

If R is either hereditary or left serial, $A_1/E_2 \approx A_2/F_2$ implies $A_1 \approx A_2$ in Lemma 16. In general this is not true for US-4 rings.

We shall give rings of the type a) in Lemma 16. Let $R=\sum \oplus e_i R$ and $e_i e_j = \delta_{ij} e_i$ (the e_i are primitive idempotents).

1) $A_1/E_2 \approx A_2/F_2$ and $E_2 \approx F_2$

$$\begin{tikzcd}[column sep=small] & e_1R = e_1\bar{Z}+e_1J \arrow[dl, dash] \arrow[dr, dash] & \\ A_1 = (1, 2)\bar{Z}+(1, 2)(2, 3)\bar{Z} \arrow[d, dash] & & A_2 = (1, 2)'\bar{Z}+(1, 2)'(2, 3)'\bar{Z} \arrow[d, dash] \\ E_2 = (1, 2)(2, 3)\bar{Z} \arrow[d, dash] & & F_2 = (1, 2)'(2, 3)'\bar{Z} \arrow[d, dash] \\ 0 & & 0 \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & e_2R = e_2\bar{Z}+e_2J \arrow[dl, dash] \arrow[dr, dash] & & e_3R = e_3\bar{Z} \arrow[d, dash] \\ (2, 3)\bar{Z} \arrow[d, dash] & & (2, 3)'\bar{Z} \arrow[d, dash] & 0 \\ 0 & & 0 & \end{tikzcd}$$

and $(1, 2)\,(2, 3)'=(1, 2)'(2, 3)=0$. This is a type of a-1-1) and a-1-4). (R is a finite ring.)

2) $A_1/E_2 \approx A_2/F_2$, $E_2 \not\approx F_2$

$$\begin{tikzcd}[arrows=dash]
 & e_1R = e_1\bar{Z}+e_1J \arrow[dl] \arrow[dr] & \\
A_1 = (1, 2)\bar{Z}+(1, 2)(2, 3)K \arrow[d] & & A_2 = (1, 2)'\bar{Z}+(1, 2)'(2, 4)K \arrow[d] \\
E_2 = (1, 2)(2, 3)K \arrow[d] & & F_2 = (1, 2)'(2, 4)K \arrow[d] \\
0 & & 0
\end{tikzcd}$$

$$\begin{tikzcd}[arrows=dash]
 & e_2R = e_2K+e_2J \arrow[dl] \arrow[dr] & \\
(2, 3)K \arrow[d] & & (2, 4)K \arrow[d] \\
0 & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}[arrows=dash]
e_3R = e_3K \arrow[d] \\
0
\end{tikzcd}
\qquad
\begin{tikzcd}[arrows=dash]
e_4R = e_4K \arrow[d] \\
0
\end{tikzcd}$$

and $(1, 2)(2, 4)=(1, 2)'(2, 3)=0$, where K is a finite field of characteristic 2. This is a type of a-1-1).

3)

$$\begin{tikzcd}[arrows=dash]
 & e_1R = e_1\bar{Z}+e_1J \arrow[dl] \arrow[dr] & \\
A_1 = (1, 2)\bar{Z}+E_2 \arrow[d] & & A_2 = (1, 1)\bar{Z}+F_2 \arrow[d] \\
E_2 = (1, 2)(2, 3)K \arrow[d] & & F_2 = (1, 1)(1, 2)\bar{Z}+F_3 \arrow[d] \\
0 & & F_3 = (1, 1)(1, 2)(2, 3)K \arrow[d] \\
 & & 0
\end{tikzcd}$$

$$\begin{tikzcd}[arrows=dash]
e_2R = e_2\bar{Z}+e_2J \arrow[d] \\
(2, 3)K \arrow[d] \\
0
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[arrows=dash]
e_3R = e_3K \arrow[d] \\
0
\end{tikzcd}$$

This is a type of a-1-2). If $K=\bar{Z}$, R is a left serial and finite ring.

4)

$$\begin{tikzcd}[arrows=dash]
 & e_1R = e_1\bar{Z}+e_1J \arrow[dl] \arrow[dr] & \\
A_1 = (1, 2)\bar{Z}+E_2 \arrow[d] & & A_2 = (1, 3)\bar{Z}+F_2 \arrow[d] \\
E_2 = (1, 2)(2, 4)\bar{Z} \arrow[d] & & F_2 = (1, 3)(3, 5)\bar{Z}+F_3 \arrow[d] \\
0 & & F_3 = (1, 3)(3, 5)(5, 4)\bar{Z}+F_4 \arrow[d] \\
 & & F_4 = (1, 3)(3, 5)(5, 4)(4, 6)K \arrow[d] \\
 & & 0
\end{tikzcd}$$

$$\begin{array}{ccccc} e_2R = e_2\bar{Z}+e_2J & \qquad & e_3R = e_2\bar{Z}+e_3J & \qquad & e_4R=e_4\bar{Z} \\ | & & | & & | \\ (2, 4)\bar{Z} & & (3, 5)\bar{Z} & & 0 \\ | & & | & & \\ 0 & & 0 & & \end{array}$$

$$\begin{array}{ccc} e_5R = e_5\bar{Z}+e_5J & \qquad & e_6R = e_6K+e_6J \\ | & & | \\ (5, 4)\bar{Z} & & (6, 4)K \\ | & & | \\ 0 & & 0 \end{array}$$

This is a type of a-1-3).

Other products among (i, j) are zero (e.g. $(1, 1)(1, 1)=0$). In the above $e_i(k, l)e_j=(k, l)\delta_{ik}\delta_{lj}$, ($\delta_{ij}$ is Kronecker delta).

Similarly we can construct a US-4 ring of a-2-1) in Lemma 16. Finally we shall give an example concerning ii) of Lemma 12.

Let K be a field of characteristic 2 and L an extension of K; $L=K(a)$ and $a^2\in K$. Put $g(a)=b\neq 0$ in L and $g(1)=0$. Then g is a derivation of L over K. Put

$$R=\begin{pmatrix} L & \begin{pmatrix} L \\ L \end{pmatrix} \\ 0 & L \end{pmatrix},$$

where $l\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}=\begin{pmatrix} l & g(l) \\ 0 & l \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \quad (l_1, l_2\in L)$

as in Remark 13. Then $e_{11}J=A_1\oplus A_2$ and $\Delta(A_1)=\Delta$, $[\Delta\colon \Delta(A_2)]=2$. However, $e_{11}J$ does not satisfy (#, 1) as an $L-L$-module. Hence $e_{11}R$ has the similar form to ii) of Lemma 12, but R is not right US-4.

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