<table>
<thead>
<tr>
<th>Title</th>
<th>Generalizations of Nakayama ring. VI. (Right US-n rings; n=3,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Harada, Manabu</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 1987, 24(4), p. 785-815</td>
</tr>
<tr>
<td>Version Type</td>
<td>VoR</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/8552">https://doi.org/10.18910/8552</a></td>
</tr>
<tr>
<td>Note</td>
<td></td>
</tr>
</tbody>
</table>
We have studied artinian right US-3 rings in [5] and right US-4 algebras over an algebraically closed field in [7]. We shall continue, in this paper, to study a right US-3 (resp. US-4) ring \( R \) when \( R \) is either hereditary or left serial.

In the first two sections, we shall give the characterization of a right US-3 (resp. US-4) ring \( R \), when \( R \) satisfies a weaker condition \((*, 1')\) (see § 1) than \( R \) being either hereditary or left serial. In the next two sections, we shall specify the characterizations given in the previous sections to hereditary rings and left serial rings. We shall exhibit several examples in the final section to illustrate the above characterizations.

1. US-3 rings

Throughout this paper we deal with an artinian ring \( R \) and every \( \mathfrak{M} \)-module is a unitary right \( \mathfrak{M} \)-module. We shall use the same terminologies and definitions given in [2]~[8].

As a generalization of right serial rings, we considered

\[(**, n) \quad \text{Every maximal submodule in a direct sum } D \text{ of } n \text{ hollow modules contains a non-zero direct summand of } D \text{ [5].}\]

It is clear that if \( D/J(D) \) is not homogeneous, \( D \) satisfies \((**, n)\). Hence we may restrict ourselves to hollow modules of a form \( e\mathfrak{R}/E \), where \( e \) is a primitive idempotent and \( E \) is a submodule of \( e\mathfrak{R} \). If \((**, n)\) holds for any direct sum of \( n \) hollow modules, we call \( R \) a right \( US-n \) ring [5]. Since the concept of \( US-n \) rings is Morita equivalent, we study always a basic ring.

We studied right \( US-n \) algebras over an algebraically closed field for \( n=3 \) and 4 in [5] and [7], respectively. In this and next sections we shall give a complete list of the structure of right US-3 (resp. US-4) rings with \((*, 1')\) below. We can give theoretically the complete structure, however as we know a few properties of division rings, we can not give the complete examples for each structure.
We quote here a particular property of a semisimple module (cf. [8] and [9]).

Let \( e \) be a primitive idempotent in \( R \) and \( D \) a semisimple \( R \)-module and \( (\#, m) \) a left \( eRe \)-module. For any two \( R \)-submodules \( V_1 \) and \( V_2 \) with \( \mid V_1 \mid = \mid V_2 \mid = m \), there exists a unit \( x \) in \( eRe \) such that \( xV_1 = V_2 \).

Further we consider one more property:

\( (\ast, 1') \) \( eJ^i \) is a direct sum of hollow modules for each primitive idempotent \( e \) and each \( i \).

If \( R \) satisfies \( (\ast, 1) \), then \( (\ast, 1') \) holds. Moreover, if \( R \) is hereditary or left serial, \( (\ast, 1') \) holds by [11], Corollary 4.2. Under the assumption \( (\ast, 1') \), we obtain the following diagram (cf. [8]):

\[
\begin{array}{cccc}
A_1 & | & A_2 & \cdots & A_m & eJ^i \\
\downarrow & | & \downarrow & & & \\
A_{11} & \cdots & A_{1s_1} & A_{21} & \cdots & A_{2s_2} & \cdots & eJ^{i+1}
\end{array}
\]

where the \( A \) are hollow.

Let \( A_1, A_2 \) be submodules in \( eR \). If there exists a unit \( x \) in \( eRe \) such that \( xA_1 \subseteq A_2 \) or \( xA_1 \supset A_2 \), we indicate this situation by \( A_1 \sim A_2 \) [4]. We put \( \Delta = eRe(eRe) \) and \( \Delta(A_1) = \{ x \in \Delta \mid xA_1 \subseteq A_2 \} \) [2].

Let \( D = A_1 \oplus A_2 \); the \( A \) are uniserial. A submodule \( B = A_1 \oplus A_2 \) is called a standard submodule in \( D \) [3].

**Lemma 1.** Let \( A_1 \) and \( A_2 \) be as in (1). If \( A_1 \sim A_2 \), \( A_1 = xA_2 \) for some unit element \( x \) in \( eRe \), and hence \( A_1 \sim A_2 \).

**Proof.** Since \( A_1 \sim A_2 \), there exists a unit \( x \) in \( eRe \) such that \( xA_1 \supset A_2 \) or \( xA_1 \subseteq A_2 \). We may assume that \( xA_1 \subseteq A_2 \). If \( xA_1 \supset A_2 \), \( xA_1 \subseteq f(A_2) \subseteq eJ^{i+1} \), since \( A_2 \) is hollow. Hence \( A_1 \subseteq x^{-1}eJ^{i+1} = eJ^{i+1} \), a contradiction. Therefore \( xA_1 = A_2 \).

**Lemma 2.** Let \( A_1 \) and \( A_2 \) be as in (1). Let \( B \) be a hollow submodule in \( A_2 \), which appears on the level \( eJ^{k+i} \) \( (k \geq 0) \) in (1). If \( \Delta(A_1) = \Delta \), \( A_1 \sim B \).

**Proof.** First assume \( k \geq 1 \) and \( A_1 \sim B \), i.e., there exists a unit \( x \) in \( eRe \) such that \( xA_1 \supset B \) or \( xA_1 \subseteq B \). In the latter case \( A_1 \subseteq eJ^{i+1} \). Hence \( xA_1 \supset B \). Since \( \Delta(A_1) = \Delta \), there exists an element \( j \) in \( eRe \) with \( (x+j)A_1 = A_1 \). Let \( b \) be a generator of \( B \). Then we obtain \( a \) in \( A_1 \) with \( xa = b \). \( b = (x+j)a = (x+j)a - ja \). Let \( p_i \) be the projection of \( eJ^i \) to \( A_i \). \( 0 = p_i(b) = (x+j)a - p_i(ja) \). Assume \( a \in eJ^i - eJ^{i+1} \), and \( p(ja) \in eJ^{i+1} \), which is a contradiction, since \( x+j \) is a unit in \( eRe \). Finally assume \( B = A_2 \). Then \( A_2 = x'A_1 \) for some unit \( x' \) in \( eR \). Hence we obtain the same situation as above, which is a contradiction.
From [2], Theorem 2 we have

**Lemma 3.** If \( R \) is a right US-\( n \) ring, then \( [\Delta : \Delta(A)] \leq n - 1 \) for any submodule \( A \) in \( eR \).

Put \( R = R/J^{t+k} \). Then \( eRe/eJ = eRe/eJ = \Delta \). Let \( A_1 \) be as in (1). Then we can define \( \Delta(A_1) = \Delta((A_1 + J^{t+k})(J^{t+k})) = \{ x \in \Delta, x(A_1 + J^{t+k}) \subseteq (A_1 + J^{t+k}) \} \). It is clear that \( \Delta(A_1) \) is a division subring of \( \Delta(A) \).

**Lemma 4.** Let \( A_1 \) and \( A_2 \) be as in (1). If \( \Delta(A_1) = \Delta, \Delta(A_2) = \Delta \).

Next assume that \( A_2 = xA_1 \) for some unit \( x \) in \( eRe \). If \( [\Delta : \Delta(A_1)] = 2 \) (resp. \( 3 \)), \( [\Delta : \Delta(A_1)] = 2 \) (resp. \( 3 \)), where \( A_1 = (A_1 + J^{t+k})(J^{t+k}) \subseteq R - R/J^{t+k} \).

**Proof.** The first part is clear from the remark above. Assume \((x + j)A_1 \subseteq A_1 \) for some \( j \) in \( eJ \). Since \((x + j)A_1 \subseteq A_2 + jA_1 \subseteq A_2 + eJ^{t+k}, (x + j)A_1 \subseteq (A_1 + eJ^{t+k}) \cap (A_2 + eJ^{t+k}) = eJ^{t+k} \), a contradiction. Hence \( x \notin \Delta(A_2) \). Further \([\Delta : \Delta(A_1)] \) is prime, and so \([\Delta : \Delta(A_1)] = [\Delta : \Delta(A_2)] \).

**Remark 5.** We shall study a right US-\( n \) ring and observe \([\Delta : \Delta(A_1)] \).

Since \([\Delta : \Delta(A_1)] \leq 3 \), we may assume \( J^{t+1} = 0 \) by Lemma 4, [3], Lemma 1 and its proof, when we observe \([\Delta : \Delta(A_1)] \) (the \( x \) in Lemma 4 exists, provided \([\Delta : \Delta(A_1)] \geq 2 \).

**Theorem 1.** \( R \) is a right (basic) US-3 ring with \((*, 1')\) if and only if \( eR \) has one of the following structures for each primitive idempotent \( e \).

1. \( eRe/[J^t] \) is uniserial for some \( t \) and
2. \( eJ = eJ = A \oplus B \), where \( A \) is simple and \( B \) is uniserial, such that
   a) \([\Delta : \Delta(A)] = 2 \) or b) \( \Delta = \Delta(A) = \Delta(B) \).
   In case a) \( B \) is simple and \( A \oplus B \) satisfies \((#, 1) \).
   In case b)
   i) \( B \) is simple and \( A \cong B \) or
   ii) \( B \) is not simple, and if \( A \) is isomorphic to a simple subfactor module \( B_i/B_{i+1} \) of \( B \), \( B_{i+1} = 0 \) (i.e., \( B_i \) is the socle of \( B \)) and this isomorphism is given by \( j_i \): the left-sided multiplication of \( j \) in \( eJ \).

**Proof.** We assume that \( R \) is a right US-3 ring. From \((*, 1')\) and [5], Proposition 1,3) \( eJ = A \oplus B \), where \( A \) and \( B \) are hollow. We may assume \(|A| \leq |B| \).

[\([\Delta : \Delta(C)] \leq 2 \) for any submodule \( C \) in \( eR \) by Lemma 3. Hence we divide ourselves into two cases: I) \([\Delta : \Delta(A)] = 2 \) and II) \( \Delta = \Delta(A) \).

**Case I.** Since \([\Delta : \Delta(A)] = 2 \), by [5], Proposition 1,2) there exists a unit element \( x \) in \( eRe \) such that \( xA \subseteq J(A) \oplus J(B) \) or \( xA \supset J(A) \oplus J(B) \). However \( A \subseteq eJ^{t+1} \) and so \( xA \supset J(A) \oplus J(B) \). On the other hand, \( |A| = |J(A) + 1| \) and \( xA \supset J(A) \oplus J(B) \). Hence \( J(B) = 0 \). Further \( A \cong B \) by Lemma 1 and [5], Proposition 1,2). Therefore \( A \) and \( B \) are simple and \( eJ^{t+1} = 0 \). Which means that every
(simple) submodule $C$ in $ej^j$ is characteristic if and only if $\Delta(C)=\Delta$. Hence $[\Delta: \Delta(C)]=2$ and $ej^j$ satisfies (2, 1) by [5], Proposition 1,2).

Case II). We know from the above argument that $\Delta=\Delta(A)=\Delta(B)$ (note that we did not use the assumption $|A| \leq |B|$). Let $y$ be any unit element in $eRe$. Since $\Delta=\Delta(A)$, there exists an element $j$ in $eJ$ such that $(y+j)A=A$. Then $(y+j)(A\oplus J(B)) \subseteq A\oplus (y+j)J(B) \subseteq A\oplus ej^{i+1}=A\oplus J(B)$. Hence $\Delta(A\oplus J(B))=\Delta$. Assume that $B$ is not simple. $A\oplus J(B)$ or $J(A)\oplus B$ is hollow by [5], Proposition 1,4)-iv).

Hence

$$J(A)=0,$$

i.e., $A$ is simple.

We shall show that $B$ is uniserial. Assume $ej^{i+k}=BJ^*=C_1\oplus C_2\oplus \cdots$; the $C_i$ are hollow. If $\Delta(C_i)=\Delta, C_1 \sim A_1$ by [5], Proposition 1,2), which is a contradiction from Lemma 2. Hence $\Delta=\Delta(C_1)=\Delta(C_2)$. However $\{A, C_1, C_2\}$ derives a contradiction by Lemma 2 and [4], Corollary 2 of Theorem 2, provided $C_2 \neq 0$. Therefore

$$B$$
is uniserial.

Next assume $g: A\sim B_i/B_{i+1}$; $B\supset B_i \supset B_{i+1}$. Take $\{A, B_i, B_i(g^{-1})\}$; the graph of $B_i$ with respect to $g^{-1})$. Since $A$ is simple (and hence $eJeB \subseteq B$) and $\Delta(B)=\Delta$, $B$ is characteristic. Hence $A\sim B_i(g^{-1})$, and so there exists a unit $x_i$ in $eRe$ such that $x_iA \subseteq B_i(g^{-1})$. If $B_{i+1}=0, x_iA \subseteq B_{i+1} \subseteq ej^{i+1}$, a contradiction. Hence $B_{i+1}=0$ and $g: A\sim B_i$, the socle of $B$. Let $j$ be an element in $eJe$ such that $(x_i+j)A=A$, and put $x_i=x_i+j$. Then $A(g)=x_iA=(x_i-j)A$. Put $A=aR$. Then $a+g(a)=(x_i-j)ar$ for some $r$ in $R$. $eJeA \subseteq ej^{i+1}$ and $ej^{i+1}=BJ$ imply $eJeA \subseteq B_i$. Hence

$$a=x_iar$$

and $g(a)=−jar$, Therefore $g=−jx^{-1}a$. Therefore $g=(-jx^{-1})_i$ and $−x^{-1}_i \subseteq eJe$ (b-ii)). Finally assume that $B$ is simple. If $f: A\approx B, \{A, B, A(f)\}$ derives a contradiction from [5], Lemma 1, (note $ej^{i+1}=0$ and use Lemma 8 below). Hence $A \not\sim B$ (b-i)). Conversely, assume that $eR$ has one of the structures given in the theorem. Clearly (2, 1') holds. Let $\{E_i\}_{i=1}^{\infty}$ be any set of submodules in $eR$. Case a): If $E_i \supset ej^j$ and $E_2 \supset ej^j, \Delta(E_i)=\Delta$ for $i=1, 2$ and $E_1 \supset E_2$ or $E_1 \subseteq E_2$. Hence

$$D=\sum_{i=1}^{\infty} E_i$$
contains a non-zero direct summand of $D$ by [4], Corollary 1 of Theorem 2. If $E_i \supset ej^j$ and $E_2 \supset ej^j, E_2=xE_1 (\approx A)$ for some $x$ in $eRe$ by (2, 1). Hence $D$ satisfies (2, 3) again by [4], Corollary 1 of Theorem 2. Case b-ii): If $E_i \supset ej^j, x_iE_i$ is a standard submodule in $ej^j$ for a unit $x_i=(e+j)A$ in $eJe$ by assumption. Hence $E_i \sim E_j$ for some pair $i, j$. Further $\Delta=\Delta(E)$ by assumption. Therefore $D$ satisfies (2, 3) by [4], Corollary 1 of Theorem 2. Case b-i): This is much simpler than the above. Thus $R$ is right US-3.
In the last paragraph of the proof of "only if part" in Theorem 1, we have shown

**Lemma 6.** Assume that \( eJ' = A \oplus A' \oplus B \) and 1) \( A \) and \( A' \) are simple modules with \( \Delta(A) = \Delta \), and 2) \( B \) is non-simple and uniserial. If \( g: A \sim B_i | B_{i+1} \) and \( A \sim B_i (g^{-1}) \), \( B_{i+1} = 0 \) and \( g \) is given by \( j; j \in eJ, \) and hence \( i > 1 \) (cf. [7], Lemma 16).

We shall illustrate the structure in Theorem 1 as the following diagram:

1) \[ eR \quad eJ \quad eJ' \quad eJ^b \]
2) \[ eR \quad eJ \quad eJ' \quad eJ^b \]

where the straight line means uniserial.

It is clear that if \( R \) has the structure above, \((*, 1)\) (and hence \((*, 1')\)) holds. We note that if \((*, 1')\) does not hold, Theorem 1 is not true (see [6]). We shall give examples of a) and b) in § 5.

2. **US-4 rings**

Next we shall characterize a right US-4 ring with \((*, 1')\).

**Lemma 7.** Let \( R \) be a right US-4 ring and \( \{A_i\}_{i=1}^3 \) a set of submodules in \( eJ. \) Then
1) if \( \Delta(A_i) = \Delta \) or all \( i \leq 3 \) and \( A_k \sim A_k \), for \( k \neq k' \leq 3 \), then \( A_k \sim \) (some \( A_i \)).
2) \( A_i \sim A_j \) for some pair \( i, j \).
3) If \( [\Delta: \Delta(A_i)] = 2 \) for \( i = 1, 2, A_i \sim A_j \).
4) If \( [\Delta: \Delta(A_i)] = 3 \), \( A_i \sim A_j \) for all \( i, j \).
5) If \( [\Delta: \Delta(A_i)] = 2, A_i \sim A_j \) for some \( i, j \leq 3 \).

Proof. This is clear from [4], Corollary 2 of Theorem 2.

**Lemma 8.** Let \( A_1 \) and \( A_2 \) be as in (1). Assume \( J^{i+1} = 0 \). If \( \Delta(A_i) = \Delta \), \( A_i \) is characteristic.

Proof. This is clear.

**Lemma 9.** Let \( R \) be a right US-4 (basic) ring, and \( \{A_i\}_{i=1}^3 \) a set of hollow submodules on the level \( eJ' \) in (1). If \( \Delta(A_i) = \Delta \) for all \( i, t \leq 3 \).

Proof. This is clear from Lemmas 7, 8 and Remark 5.

From now on we assume that \( R \) is a right US-4 (basic) ring satisfying \((*, 1')\). Let \( D = (eJ' =) A_1 \oplus A_2 \oplus \cdots \oplus A_n \), where the \( A_i \) are hollow. In the
following lemmas, we mainly assume that $D$ is characteristic. We note $[\Delta: \Delta(A_i)] \leq 3$ for all $i$ by Lemma 3.

**Lemma 10.** Assume $[\Delta: \Delta(A_i)] = 2$ for all $i$. Then i) $t = 2$. ii) There exists a unit $x$ in $eRe$ such that $xA_1 = A_2$. iii) $A_1$ is a uniserial module with $|A_1| \leq 2$. iv) If there are characteristic submodules in $A_1 \oplus A_2$, they are linear with respect to the inclusion. v) If $B$ is not a characteristic submodule in $A_1 \oplus A_2$, $[\Delta: \Delta(B)] = 2$ and those submodules are related by $\sim$.

Proof. We may assume $|A_1| \leq |A_2| \leq \cdots \leq |A_t|$ (note $t \geq 2$). By Lemmas 1 and 7, $A_k = x_kA_1$ for all $k$. Hence

$(a)$ if $[\Delta: \Delta(A_i)] \geq 2$ for all $i$, there exists a unit $x_i$ in $eRe$ such that $x_iA_1 = A_i$ for all $i$.

On the other hand, since $[\Delta: \Delta(A_1)] = 2$, $\Delta = \Delta(A_1) + x_2\Delta(A_1)$. Assume $e_j^{t+1} = 0$ from Remark 5. Since $D = \Delta(A_1) + x_2\Delta(A_1)A_1 = A_1 \oplus A_2$, $t = 2$. We note that from the above argument and Lemma 3 we obtain

$(\beta)$ If $[\Delta: \Delta(A_i)] \geq 2$ for all $i$, $t \leq 3$.

Assume that $A_1J^s$ is uniserial and $A_1J^s = B_1 \oplus B_2 \oplus \cdots \oplus B_s$, where the $B_j$ are hollow and $s \geq 2$. In order to show $s \leq 1$, we may assume $e_j^{t+1} = 0$ by Remark 5. First we note that there exists a unit $x$ in $eRe$ such that $xA_1 = A_2$. Hence $\Delta(B_p) = \Delta$ for all $p$. On the other hand, $DJ^s = A_1J^s \oplus A_2J^s = \sum_{p=1}^s B_p \oplus \sum_{p=1}^s xB_p$, which is a contradiction to $(\beta)$. Therefore $A_1$ and $A_2$ are uniserial. Next assume $A_1J^s = 0$. $\Delta(A_1J \oplus A_2J^s) = \Delta$ by existence of $x_2$. Hence $\{A_1, A_1J \oplus A_2J^s\}$ derives a contradiction by Lemma 7. Therefore $|A_1| \leq 2$. Since $\Delta(A_1J \oplus (A_2J^s)) \subseteq \Delta(A_1)$, $\Delta(A_1J \oplus (A_2J^s)) = \Delta(A_1)$ for $\Delta(A_1J \oplus (A_2J^s)) = \Delta$. Similarly $[\Delta: \Delta(J(A_1))] = 2$. Let $E$ be a submodule with $[\Delta: \Delta(E)] = 3$. Then there exists a unit element $x$ in $eRe$ such that $xA_1 = A_2$. Hence $\Delta(E) = \Delta$ for all $p$. On the other hand, $DJ^s = A_1J^s \oplus A_2J^s = \sum_{p=1}^s B_p \oplus \sum_{p=1}^s xB_p$, which is a contradiction to $(\beta)$. Therefore $A_1$ and $A_2$ are uniserial. Next assume $A_1J^s = 0$. $\Delta(A_1J \oplus A_2J^s) = \Delta$ by existence of $x_2$. Hence $\{A_1, A_1J \oplus A_2J^s\}$ derives a contradiction by Lemma 7. Therefore $|A_1| \leq 2$. Since $\Delta(A_1J \oplus (A_2J^s)) \subseteq \Delta(A_1)$, $\Delta(A_1J \oplus (A_2J^s)) = \Delta(A_1)$ for $\Delta(A_1J \oplus (A_2J^s)) = \Delta$. Similarly $[\Delta: \Delta(J(A_1))] = 2$. Let $E$ be a submodule with $[\Delta: \Delta(E)] = 3$. Then there exists a unit element $x$ in $eRe$ such that $xE \subseteq A_1$ or $xE \supseteq A_1$ by Lemma 7. In the former case $[\Delta: \Delta(E)] = [\Delta: \Delta(xE)] = 2$. If $xE \supseteq A_1$, $xE = A_1 \oplus E'$; $E' \subseteq A_2$. Hence $[\Delta: \Delta(xE)] = 2$ from the above. Therefore there are no submodules $E$ with $[\Delta: \Delta(E)] = 3$. Finally assume that $A_1 \oplus A_2$ contains two characteristic submodules $C_1$, $C_2$ such that $C_1 \sim C_2$. Consider $\{A_1, A_1 \oplus C_2, A_2 \oplus C_1\}$ and $A_1 \sim A_2$, or $A_2 \sim C_1$ by Lemma 7. If $A_1 \supseteq C_1, C_1 = 0$ and if $A_1 \subset C_1, C_1 = A_1 \oplus F$; $F \subseteq A_2$, and so $C_1 = A_1 \oplus A_2$. Hence $C_1 \supseteq C_2$ or $C_1 \subseteq C_2$. Let $\Delta(E) = \Delta$. If $|A_1| = 1$, $E$ is characteristic. Assume $|A_1| = 2$. Put $C_1 = A_2 \oplus B_2$. Then $E \sim C_1$ from the above. Hence $E \supseteq C_1$ or $E \supseteq C_1$, and so $E$ is characteristic.

**Lemma 11.** Assume $[\Delta: \Delta(A_i)] = 3$ for all $i$. Then $t \leq 3$, and the $A_i$ are simple and there exists a unit $x_i$ in $eRe$ such that $x_iA_1 = A_i$ for each $i$. If $t = 3$, $D$ satisfies $(\# 1)$ and $(\# 2)$ and $[\Delta: \Delta(C)] \leq 3$ for every submodule $C$ in $D$. If $t = 2$, $D$ satisfies $(\# 1)$.

Proof. Since $[\Delta: \Delta(A_i)] = 3$, there exists a unit $x_i$ in $eRe$ such that $x_iA_1 = A_i$.
from \((a)\) and \(t \leq 3\) by \((b)\). Assume \(t = 3\). Taking \(\{A_i, J(D)\}\), we know from Lemma 7 that \(A_i\) is simple and hence \(ef^{i+1} = 0\). It is clear from Lemmas 7 and 8 that there are no simple submodules \(B\) in \(D\) with \(\Delta(B) = \Delta\). Hence \(D\) satisfies \((\# , 1)\). Let \(C\) be a submodule of \(D\) with \(|C| = 2\). Then \(D = C \oplus A_i\) for some \(i\). Hence \(\Delta(C) = \Delta\) by Lemma 7, and so \(D\) satisfies \((\#, 2)\). We obtain the similar result for \(t = 2\).

Lemma 12. Assume \([\Delta: \Delta(A_i)]=1\) and \(\Delta(A_i) \neq \Delta\) for \(i \geq 2\). Then \(A_i\) is uniserial and \(t \leq 3\).

i) \(t = 3\):
Then all \(A_i\) are simple, \([\Delta: \Delta(A_i)] = 2\) for \(i = 2, 3, A_i \approx A_2\) and \(A_2 \oplus A_3\) satisfies \((\#, 1)\).

ii) \(t = 2\):
\(a)\) \(A_1\) is not simple.
Then \([\Delta: \Delta(A_2)] = 2\), and \(A_2\) is a simple submodule isomorphic to \(B\), the socle of \(A_i\).
If \(A_2 \approx E_i / E_{i+1} (A_i \supset E_i \supset E_{i+1}), E_i = B\) and \(E_{i+1} = 0\). Further \(B \oplus A_2\) satisfies \((\#, 1)\) except \(B\).

\(b)\) \(A_1\) is simple.
Then
1) \([\Delta: \Delta(A_2)] = 2, A_1 \oplus A_2 | J(A_2)\) satisfies \((\#, 1)\) except \(A_1\).
2) \(A_2 | A_2 f^*\) is uniserial for some \(t\) and
2-i) \(A_2 f^* = 0\) or
2-ii) \(A_2 f^* = B_1 \oplus B_2; B_1\) is simple and \(B_2\) is uniserial.
2-ii-1) \(\Delta(B_1) = \Delta(B_2) = \Delta\).
2-ii-1-1) \(B_1 \approx B_2 | J(B_2)\).
2-ii-1-2) \(A_2 \approx F_i / F_{i+1} (A_2 \supset F_i \supset F_{i+1} \supset B_1 \oplus B_2\).
2-ii-1-3) If \(f: A_1 \approx G_j / G_{j+1} (f^*: B_1 \approx G_j / G_{j+1} (G_j \supset G_{j+1}), then G_{j+1} = 0\)
and \(f (f^*)\) is given by \(j; j' \in e J e\).
2-ii-1-4) If \(f: A_1 \approx B_1, we have the same result as 2-ii-1-3)\).
2-ii-2) \([\Delta: \Delta(B_2)] = [\Delta: \Delta(B_2)] = 2\).
2-ii-2-1) \(B_1\) and \(B_2\) are simple and \(B_1 \oplus B_2\) satisfies \((\#, 1)\).
2-ii-2-2) \(A_2 \approx F_i / F_{i+1} (A_2 \supset F_i \supset F_{i+1} \supset B_1 \oplus B_2\).
2-ii-2-3) If \(A_1 \approx B_1, then f is given by j; j' \in e J e\).

Proof. It is clear, from the assumption and Lemmas 1 and 7, that \([\Delta: \Delta(A_i)] = 2\) for all \(i \geq 2\). Assume that \(A_i\) contains two independent submodules \(B_1, B_2\). If \(\Delta(B_1) = \Delta(B_2) = \Delta, \{B_1, B_2, A_2, A_3\}\) derives a contradiction by Lemmas 7, 8 and Remark 5. On the other hand, if \(\Delta(B_1) = \Delta, \{B_1, B_2, A_2, A_3\}\) derives again a contradiction. Hence

\(A_1\) is uniserial

by \((*, 1')\).

\(a)\) \(J(A_i) = 0:\) Consider \(\{A_1, A_3, J(D)\}\). Then \(A_1 \not\sim A_3\) by Lemma 2. Hence \(J(D) \sim A_1\) or \(J(D) \sim A_1\) by Lemma 7. However \(J(D) \not\sim A_2, since J(D)\) is
characteristic and $J(A_i) \neq 0$. Hence

the $A_i$ are simple for all $i \geq 2$.

Since $[\Delta : \Delta(A_i)] = 2$, there exists $x_i$ in $eRe$ such that $x_iA_2 = A_i$ for $i > 2$ by Lemmas 1 and 7. Hence in order to show $t \leq 3$, we may assume $J^{t+1} = 0$ by Remark 5. Noting $A_i \in \Delta(A_2)$, $\Delta = \Delta(A_2) \oplus A_3\Delta(A_2)$, which implies that $A_2 \oplus A_3 = \Delta A_2 \supset \sum_{i=1}^{t} \oplus A_i$. Hence $t \leq 3$. Assume $t = 3$. Now we resume to the original situation. We note $eRe \subset J(A_i)$, and hence $A_i$ is characteristic. Since $\Delta(J(A_i)) = \Delta, \Delta(J(A_i) \oplus A_2) \neq \Delta$. Consider $\{A_i, J(A_i) \oplus A_2, A_2 \oplus A_3\}$. $\Delta(A_i) = \Delta$ and $\Delta(J(A_i) \oplus A_2) \neq \Delta$ imply $(J(A_i) \oplus A_2) \sim (A_2 \oplus A_3)$ by Lemma 7. Hence there exists a unit $x$ in $eRe$ such that $x(J(A_i) \oplus A_2) \subset (A_2 \oplus A_3)$ or $x(J(A_i) \oplus A_2) \supset (A_2 \oplus A_3)$. However, $\Delta(J(A_i)) = \Delta$ implies $xJ(A_i) \subset A_2 \oplus A_3$. Hence $x(J(A_i) \oplus A_2) \supset A_2 \oplus A_3$. Taking $R = R(J^{t+1})$, we know that it is impossible. Therefore $t = 2$ provided $J(A_i) \neq 0$, i.e.,

$$D = A_1 \oplus A_2 \quad (J(A_i) \neq 0).$$

Now we take the similar manner to Lemma 6. Assume $f: A_2 \approx E_i \oplus E_{i+1}$; $A_i \supset E_{i+1}$ from the above remark and Lemma 7. $E_i = E_i(f^{-1})$ and $E_i = E_i(f^{-1})$ imply $(J(A_i) \oplus A_2) \sim (A_2 \oplus A_3)$ by Lemma 7. Hence $E_i = E_i$ as the proof of Lemma 6. Further since $\Delta(A_2) \neq \Delta, A_2 \approx E_i$; the socle of $A_i$. Let $C(\neq E_i)$ be a simple submodule in $E_i \oplus A_2$. Consider $\{A_i, C, A_2, A_3\}$. It is clear that if $A_i \sim A_2, C \subset A_3$. Hence $C \sim A_2$ by Lemmas 2 and 7, and so $E_i \oplus A_2$ satisfies $(\#1)$ except $E_i$.

b) $J(A_i) = 0, t \geq 3$. Assume $J(A_i) = 0$. Since $t \geq 3$, there exists a unit $x$ in $eRe$ with $xA_2 = A_3$ by Lemmas 1 and 7, and so $\Delta(A_1 \oplus J(A_2)) \neq \Delta$. Then $A_2 \sim A_1 \oplus J(A_2)$ by Lemma 7. Assume $A_2 \supset y(A_1 \oplus J(A_2))$ for some unit $y$. Since $A_1$ is simple and $\Delta(A_1) = \Delta$, $p_i(yA_1) = A_1$, where $p_i: eJ^{-1} \rightarrow A_i$ the projection, which is a contradiction. Similarly, since $A_1$ is simple and $A_2$ is not, $p_i(yA_1) \subset J(A_2)$ for any unit $y'$ in $eRe$. Hence $A_2 \subset y'(A_1 \oplus J(A_2))$. Therefore $A_2$ (and so $A_1 (i \geq 2)$) is simple.

Accordingly $t = 3$ from the initial paragraph of a). If $f: A_1 \approx A_2, \{A_i, A_i(f), A_2, A_3\}$ derives a contradiction, since $\Delta(A_2) = A_2 \oplus A_3$ as before (note $J^{t+1} = 0$). Hence $A_1 \approx A_2$. Further if $A_2 \oplus A_3$ contains a characteristic submodule $B \neq 0, \{A_i, B, A_2, A_3\}$ derives a contradiction. Therefore $A_2 \oplus A_3$ satisfies $(\#1)$.

Case $t = 2$ and $J(A_i) = 0 (D = A_1 \oplus A_2)$. First we shall show that $A_1 \oplus A_2 \subset J(A_2)$ satisfies $(\#1)$ except $A_1$. Since $\Delta(A_2) \neq \Delta$, there exists a unit $x$ in $eRe$ such that $p_i(xA_2) = A_1$, where $p_i: eJ^{-1} \rightarrow A_i$ is the projection. Further $eFeA_2 \subset A_2$, since $A_1$ is simple. Hence $(x+j)(A_2 + J^{t+1}) = A_2 + J^{t+1}$ for any $j$ in $eFe$, and so $\Delta(A_2) = \Delta((A_2 + J^{t+1})/J^{t+1})$. Therefore we may assume $J^{t+1} = 0$ (cf. Remark 5). Then
\(A_1 \oplus A_2\) satisfies \((\#, 1)\) except \(A_1\) from Lemma 7. Now we resume the original situation. Since \(A_1\) is simple, \(eJ^{i+1} = A_2 J\). Assume that \(A_2 / A_2 J^i\) is uniserial and \(eJ^{i+1} = B_1 \oplus B_2 \oplus \cdots \oplus B_n\), where the \(B_i\) are hollow. Then from Lemmas 10~16 below, \(s \leq 3\). Further \([\Delta : \Delta(B_j)] \leq 2\) by Lemmas 2 and 7. Assume \(s = 3\). Then \(\Delta(B_j) = \Delta\) (resp. \([\Delta : \Delta(B_j)] = 2\)) for some \(i\) (resp. \(j\)) by Lemmas 7 and 10. Hence we remain two cases \(\Delta(B_j) = \Delta\), \([\Delta : \Delta(B_j)] = 2\) for \(j = 2, 3\) and \(\Delta(B_i) = \Delta\) for \(i = 1, 2\). On the other hand, since \(\Delta(A_1) = \Delta\), we do not have such cases by Lemmas 2 and 7. Therefore \(s \leq 2\). Similarly we do not have a case \(\Delta(B_j) = \Delta\) and \([\Delta : \Delta(B_j)] = 2\). Thus we obtain two cases; 2-ii-1): \(\Delta(B_j) = \Delta\) for \(j = 2, 3\) and \([\Delta : \Delta(B_j)] = 2\) for \(j = 1, 2\).

2-ii-1) We assume \(\|B_1\| = \|B_2\|\). \(\{A_1, B_1, B_2, J(B_1) \oplus J(B_2)\}\) gives \(J(B_1) = 0\) from Lemmas 2 and 7. Assume \(B_2 J^* = C_1 \oplus C_2 \oplus \cdots \oplus C_s\); \(s \geq 2\) and the \(C_i\) are hollow. Then from Lemmas 10~16 below, \(\Delta(C_1) = \Delta(C_2) = \Delta\). Further \([\Delta : \Delta(C_i)] = 2\) for \(i = 1, 2\). Hence \(B_1 \otimes B_2\) satisfies \((\#, 1)\) (2-ii-1-2). If \(h : A_1 \approx B_1 \oplus B_2\), \(\{A_1, B_1, B_2, B_3(f\pm 1)\}\) derives a contradiction. Accordingly \(A_1 = A_2\) (2-ii-2). If \(j : A_1 \approx B_1 \oplus B_2\), \(\{A_1, B_1, B_2, B_3(f\pm 1)\}\) gives \(A_1 \approx G_i(f\pm 1)\) since \(\Delta(A_1) = \Delta(G_i)\). Then \(xA_1 \subseteq F_i(h\pm 1)\) for some unit \(x\) in \(eRe\), since \(A_1 \neq F_i\) and \(\Delta(A_1) = \Delta\). Hence \(xA_1 \subseteq F_i h \subseteq J(A_2)\), a contradiction. Accordingly \(A_1 \approx F_i h\) (2-ii-2-2). In the same manner given in the proof of 2-ii-1-3), we have 2-ii-1-4).

2-ii-2) Since \([\Delta : \Delta(B_1)] = [\Delta : \Delta(B_2)] = 2\), \(B_1 \approx B_2\) by Lemmas 1 and 7. \(\{A_1, B_1, B_2, J(B_1) \oplus J(B_2)\}\) gives \(J(B_1) = J(B_2) = 0\). Accordingly \(B_1\) and \(B_2\) are simple. Let \(C\) be any simple submodule in \(B_1 \oplus B_2\). Then \(\{A_1, B_1, B_2, C\}\) shows \(C = xB_1\) for some unit \(x\) in \(eRe\) by Lemmas 2 and 7. Hence \(B_1 \oplus B_2\) satisfies \((\#, 1)\) (2-ii-2-1). 2-ii-2-2) is same to 2-ii-1-2). If \(f : A_1 \approx B_1\), \(\{A_1, A_1(f), B_1, B_1, B_1\}\) gives \(A_1 \approx A_i(f)\). Hence \(f\) is given by \(j_i; j \in eRe\) (2-ii-2-3)).

Remark 13. We shall consider the situation of ii-b) of Lemma 12. Taking \(R = R / J^{i+1}\), we may assume that \(eJ^i = V\); the \(A_i\) are simple, \(\Delta(A_1) = \Delta\), and \([\Delta : \Delta(A_2)] = 2\). Then \(A_1 \approx A_2 (\approx g_1 Rg_1 = \Delta')\). We shall express \(\text{End}_{\Delta'}(V)\) as elements of matrices \((\Delta')_2\). Since \(A_1\) is characteristic, for any element \(x\) in \(\Delta\),

\[
x = \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix} : x_1 \in \Delta'.
\]
obtain two monomorphisms as rings $f_1, f_2$ of $\Delta$ to $\Delta'$ such that $f_i(x) = x_i$ and a homomorphism $g$ as additive groups of $\Delta$ to $\Delta'$ such that

i) $g(x^i) = f_i(x)g(x') + g(x)f_2(x').$

Then $\Delta(A_2) = g^{-1}(0)$ (note, from i), that $g^{-1}(0)$ is a division subring of $\Delta$.

Hence $[\Delta: \Delta(A_2)] = 2$ is equivalent to

ii) $[\Delta: g^{-1}(0)] = 2$.

Further (i, 1) holds if and only if, for any $\alpha$ in $\Delta$, there exists $x \neq 0$ in $\Delta$ such that

iii) $\alpha = -f_i(x^-1)g(x) (= g(x^-1)f_2(x))$, i.e., $F: \Delta \to \Delta'$ ($F(x) = f_i(x)^{-1}g(x)$) is surjective.

If $\alpha \neq 0, x \not\in g^{-1}(0)$. Hence if either $|\Delta|$, cardinal of $\Delta$, ($|\Delta| \leq |\Delta'|$) is finite or $|\Delta| < |\Delta'|$, iii) does not hold. Hence we assume that $|\Delta|$ is infinite. Further, since $f_i$ is a monomorphism, we may assume that $\Delta \subset \Delta'$ and $f_i$ is the inclusion. Now assume that $\Delta'$ is commutative. Then $g$ is a $K$-linear mapping from i), where $K = g^{-1}(0)$. Using those facts and $|\Delta| \geq \infty$, for any $g$ we can show by computation that there exists $\alpha$ in $\Delta'$ not satisfying iii) for any $x \in \Delta$. Therefore if $\Delta'$ is commutative, we do not have the case of ii) of Lemma 12.

Remark 14. Next we consider the case $t = 2$ in Lemma 11. Let $K$ be a field and $R$ a $K$-algebra. If $[\Delta': K]$ is not divided by 3, this case does not occur. Because, since $V = A_1 \oplus A_2$ and $A_i \cong A_2$, $\text{End}_A(V) = (\Delta')_2$ and $\Delta \subset (\Delta')_2$.

$[\Delta: (\Delta(A_1))] = 3$ implies that $4[\Delta': K]$ is divided by 3.

Finally we take division rings given by [10]. Let $D \supset D_1$ be division rings such that $[D: D_1] = 3$ and $[D: D_1^*] = 2$. Put $D = D_1 + D_1u$ and $D^* = \text{Hom}_{D_1}(D_1D, D_1D)$. Then $[D^*: D_1] = 2$ and $D^*$ is a left $D$-vector space. Define $1^* \in D^*$ by setting $1^*(1) = 1, 1^*(u) = 0$, and put $A_i = 1^*D_1$. Then $D(A_i) = \{d \in D, dA_i \subset A_i\} = u^{-1}D_1u$, and so $[D: D(A_i)] = 3$. For any $h \in D^*$ and $h^{-1}(0) = D_1u_0$, we have $D = D_1u_0 \oplus D_1v_1$. Put $d = h(v_1)$. Then $(u_i^{-1}u_0)1^*(u_0) = 0$ and $(u_i^{-1}u)1^*(v_1) = d - u_0$. Hence $h = (u_i^{-1}u_0)^*(d^{-1})d$ and so $hD_1 = (u_i^{-1}u_0)A_1$. Therefore $D^*$ satisfies (4, 1), $[D: D(A_i)] = 3$ and $[D^*: D_1] = 2$. We shall use $D^*$ in §5, Example 3'.

Now we resume to study the structure of right US-4 rings.

Lemma 15. If $R$ is a US-4 ring with $(\ast, 1')$. $D$ has one of the structures in Lemmas 10, 11, 12 and 16 below.

Proof. Assume 1) $\Delta(A_i) = \Delta(A_2) = \Delta(A_3) = \Delta$. Then $t = 3$ by Lemmas 7 and 8 (the case of Lemma 16 below). 2) $\Delta(A_i) = \Delta(A_2) = \Delta$ and $\Delta(A_3) = \Delta$ for $i \geq 3$. Then $\{A_1, A_2, A_3, A_4\}$ derives a contradiction from Lemmas 2 and 7.

3) $\Delta(A_i) = \Delta$ and $\Delta(A_j) = \Delta$ for $i \geq 2$. This is a case of Lemma 12. 4) $[\Delta: \Delta(A_i)] = 2$ for $i \leq \text{some } l$, $[\Delta: \Delta(A_j)] = 3$ for $j > l$. Since $[\Delta: \Delta(A_k)] \geq 2$ for all $k$, from (4) there exists a unit $x_i$ in $eRe$ such that $x_iA_i = A_i$ for all $i$. Hence $\Delta(A_i) = x_i\Delta(A)A_i^{-1}$, and so we obtain the cases of Lemmas 10 and 11.
Lemma 16. Assume $\Delta(A_i) = \Delta$ for all $i$. Then $t \leq 3$, and

1) $t = 3$:

$A_3$ is uniserial and $A_1$, $A_2$ are simple, $A_3 \cong A_2$. If $A_3$ is simple, $A_3 \cong A_1$ and $A_3 \cong A_2$. If $A_3$ is not simple and $f$: $A_3 \cong F_{i+1} (A_3 \supseteq F_{i+1} \supseteq F_{i+1})$, then $F_{i+1} = 0$ and $f$ is given by $j_i$; $j \in eJe$, and hence $i > 1$.

2) $t = 2$:

i) $A_1 \cong A_2 (\approx g_1 R[g_1])$.

Then $A_1$ and $A_2$ are simple and $\Delta = g_1 R[g_1] \cong \mathbb{Z} = \mathbb{Z}/2$.

ii) $A_3 \cong A_2 (\approx |A_1| = |A_2|)$

a) $A_3$ is uniserial; $A_2 = F_{i} \supseteq F_{i} \supseteq F_{i} \supseteq F_{i+1} = 0$.

Then $A_1$ is a uniserial module with $|A_1| = 2$; $A_1 = E_1 \supseteq E_2 \supseteq E_3 = 0$.

a-1) $|A_1| = 2$.

i) $f$: $A_1/E_3 \cong A_2 (\approx g_1 R[g_1])$, $\Delta = g_1 R[g_1] \cong \mathbb{Z}$. $f$ is a unique isomorphism. In this case put $B_1 = \{x+y | x \in A_1 \cong A_2, f(x) = y\}$.

ii) $A_1/E_2 \cong F_{i+1}$, $i > 1$, then $i > p - 1$.

a-1-3) $f$: $E_2 \cong F_{i+1}$ (assuming $E_2 \cong g_1 R[g_1]$). $\Delta = g_1 R[g_1] \cong \mathbb{Z}$. We have the same result as a-2-1 below, replacing $A_1$ with $E_2$. In this case put $B'_1 = \{x+y | x \in E_2 \cong F_{i+1}, f(x) = y\}$.

a-1-4) $f$: $E_2 \cong F_{i+1}$. If $p = 2$, $\Delta = g_1 R[g_1] \cong \mathbb{Z}$, where $E_2 \cong F_{i+1}$ (assuming $E_2 \cong g_1 R[g_1]$). Further if $f'$: $A_1/E_2 \cong F_{i+1} (A_2/F_{i+1} \cong E_2)$, $A_1(f) = xA_1$ for some unit $x$ in $eRe$.

If $p > 2$, we have the same result as a-2-2 below, replacing $A_1$ with $E_2$. If $f$ is not given by $j_1$, put $B'' = E_2(j)$.

a-1-5) Further every submodule in $eF^i$ except $B_1$, $B'_1$ and $B''$ is isomorphic to a standard submodule in $eF^i$ via $x_1$; $x$ is a unit in $eRe$.

a-2) $|A_1| = 1$:

i) $A_1 \cong F_{i+1}$ (assuming $A_1 \cong F_{i+1}$). $\Delta = g_1 R[g_1] \cong \mathbb{Z}$. Further $A_i \cong F_{i+1}$ for any ($i \neq j$) $j > p$.

ii) $A_1 \cong F_{i+1}$ (assuming $A_1 \cong F_{i+1}$). $\Delta = g_1 R[g_1] \cong \mathbb{Z}$. Further $A_i \cong F_{i+1}$ for any ($i \neq j$) $j > p$.

a-2-2) Assume $f_1$, $f_2$: $A_1 \cong F_{i+1}$ (assuming $A_1 \cong F_{i+1}$). $\Delta = g_1 R[g_1] \cong \mathbb{Z}$. Further $A_i \cong F_{i+1}$ for any ($i \neq j$) $j > p$.

b) $A_1/A_2J^k$ is uniserial and $A_2J^k$ is not uniserial, i.e., $A_2J^k = B_1 \oplus B_2 \oplus \cdots \oplus B_s$, where the $B_i$ are hollow. Then $A_1$ is simple and $s = 2$. Further $B_1 = A_1$ and $B_2 = A_2$.

Then $B_1 \cong B_2 (|B_1| = |B_2|)$, and $B_1$ is simple, $B_2$ is uniserial.

b-1) $f$: $A_1 \cong F_{i+1} (A_2 \supseteq F_{i+1} \supseteq F_{i+1} \supseteq B_1 \oplus B_2)$, then we obtain the same result modulo $B_1 \oplus B_2$ as given in a-2-1).

b-2) $f$: $A_1 \cong B_1$, $f$ is given by $j_1$; $j \in eJe$.

b-3) $f$: $A_1 \cong H_{i+1} (B_1 \supseteq H_{i+1} \supseteq H_{i+1})$, then $H_{i+1} = 0$ and $f$ is given by $j_1$; $j \in eJe$.

b-4) $f$: $B_1 \cong H_{i+1}$, then $H_{i+1} = 0$ and $f$ is given by $j_1$; $j \in eJe$.

b) $[\Delta: \Delta(B_i)] = 2$ for $i = 1, 2$. 

Then \( B_1 \cong B_2 \) and \( B_1, B_2 \) are simple and \( V = B_1 \oplus B_2 \) satisfies (1).

\[ b_2 \]
\[ A_1 \cong F_i/F_{i+1} (A_3 \supset F_i \supset F_{i+1} \supset V) . \]

\[ b_2 \]
\[ a) \text{If: } A_i \cong B_i, \text{ f is given by } j; j \in e \text{. (cf. [7], Theorem 17.)} \]

Proof. We know \( t \leq 3 \) by Lemma 9. Assume that \( |A_1| \leq |A_2| \leq |A_3| \).

i) \( t = 3 \). Consider \( \{A_1, A_2, A_3, J(D) = J(A_3) \oplus J(A_2) \oplus J(A_1)\} \). Since \( \Delta(A_i) = \Delta, J(A_1) \oplus J(A_3) \oplus J(A_2) \) is contained in some \( A_i \) by Lemmas 2 and 7. Hence \( J(A_1) = J(A_2) = 0 \) (note \( |A_3| \geq |A_1| \)). Assume that \( A_3 \) contains two independent submodules \( B_1 \) and \( B_2 \) in \( e F_i \) on the same level in (1). Take \( R = R/J_{i+1}^+ \) . Then both \( [\Delta: \Delta] \) and \( [\Delta: \Delta(J_2)] \) are not equal to 1 and \( [\Delta: \Delta(J_3)] \neq 2 \) for any \( i \) by Lemmas 2 and 7, and hence \( [\Delta: \Delta(B_1)] = 2 \) for \( i = 1 \) or 2 by Lemma 3, (say \( i = 1 \)). Then \( \{B_1, B_2, B_3\} \) contradicts Lemmas 2 and 7, since \( B_1 \neq B_2 \).

Hence \( A_3 \) is uniserial. Assume \( f: A_1 \cong A_2 \). Then \( \{A_1, A_2, A_3, A_i(f)\} \) implies \( A_i(f) \cong \text{some } A_i \). Since \( A_1 \) is characteristic (we may assume \( f_i^{+1} = 0 \) by Remark 5), \( A_i(f) \) is a contradiction. Finally assume \( g: A_1 \cong F_i/F_{i+1}^+ \). Since \( \Delta(A_3) = \Delta \) and \( A_1, A_2 \) are simple, \( A_3 \) is characteristic. Hence \( \{A_1, A_2, F_i(g^{-1})\} \) derives from Lemmas 2 and 7 that \( A_i \cong F_i(g^{-1}) \). Therefore \( g \) is given by \( j \) from Lemma 6.

2) \( t = 2 \).

i) \( f: A_1 \cong A_2 \). Assume \( \Delta \neq \Delta(A_1(f)) \). Then \( \{A_1, A_2, A_1(f), A_2(f)\} \) implies \( A_i(f) \cong A_i \) for some \( i \), say 1 from Lemma 7. Since \( A_2 \cong A_2(f) \cong A_1(f) \cong x A_i \), for some unit \( x \) in \( e R \) . Hence \( \Delta(A_i(f)) = x \Delta(A_i(f)) = \Delta, \) a contradiction. Accordingly \( \Delta(A_1(f)) = \Delta \). Consider \( \{A_1, A_2, A_1(f), J(A_1) \oplus J(A_2)\} \), and \( J(A_1) = 0 \) by Lemma 7 (note \( \Delta(A_i) = \Delta(A_i(f)) = \Delta \)). Hence \( A_1 \) and \( A_2 \) are simple, and so \( e F_i = 0 \). Let \( f \) and \( f' \) be two isomorphisms of \( A_1 \) to \( A_2 \) and consider \( \{A_1, A_2, A_i(f), A_i(f')\} \). Since \( e F_i = 0 \), they are characteristic, and so \( A_i(f) = A_i(f') \) by Lemmas 7 and 8. Hence \( f = f' \). Considering an isomorphism \( f \) for \( \Xi \in \Delta, \Delta = \{0, 1\} \). Since \( \text{Hom}_R(A_1, A_1) = \{0, 1\} \), \( \Delta = \text{gRg} = \{0, 1\} \), where \( A \cong g R(g) \).

ii) \( A_i \cong A_i \) (\( |A_i| \leq |A_2|\)). Assume \( A_1 \neq 0 \) and \( A_1 = C_i + C_1 + \cdots + C_i \) \( (s \geq 1) \), where the \( C_i \) are hollow. Consider \( \{A_1, A_2, A_1 \oplus C_1, A_1 \oplus C_1 \} \) \( (s \geq 2) \). Then \( A_1 \oplus C_1 = A_1 \oplus C_2 \) by Lemmas 2 and 7, provided \( A_1 \neq 0 \). Assume \( \Delta(A_1 \oplus C_i) = 0 \) for \( i = 1, 2 \) and \( x(A_1 \oplus C_i) \subset A_1 \oplus C_2 \) for some unit \( x \). We may assume \( J_i^{+1} = 0 \). There exists \( j \) in \( e F_i \) such that \( (x+j)(A_1 \oplus C_i) = A_1 \oplus C_i \). Then \( xC_1 \cap (A_1 \oplus C_2) = A_1 \oplus C_i \), and so \( C_1 \cong C_i \), a contradiction by Lemma 2. Hence \( \Delta(A_1 \oplus C_i) = 0 \). Assume \( A_2 \cong A_2 \) and \( A_2 \neq 0 \). Now assume that \( A_2 \) is uniserial \( (|A_2| \geq 2 \) and hence so is \( A_2 \)) . We shall show \( |A_i| < 2 \). Assume \( A_i \neq 0 \) and
hence $A_i J^i \neq 0$. Consider $\{A_1, A_1 J \oplus A_2 J^i, A_1 J^i \oplus A_2 J, A_2 J\}$. Since $A_i \sim A_2$ by Lemma 2, 1) $A_i \sim A_1 J \oplus A_2 J^i$ or 2) $A_i \sim A_1 J^i \oplus A_2 J$ ($A_1$ and $A_2$ are symmetry) or 3) $A_1 J \oplus A_2 J^i \sim A_1 J^i \oplus A_2 J$.

1) It is clear that $x A_i \supset A_i J \oplus A_2 J$ for a unit $x$. However $A_i$ is uniserial, and so $A_2 J^i = 0$ (note $|A_i| \leq |A_2|$). 2) This is similar. 3) Assume $x (A_1 J \oplus A_2 J^i) \supset A_1 J^i \oplus A_2 J$. Since $\Delta(A_i) = \Delta$, there exists $j$ in $e J e$ such that $(x + j) A_i = A_i$. Let $a_{j_2}$ be an element in $A_2 (a_{j_2} e A_2, j_2 e J)$. Then $x(a_{j_2} + a_{j_2} i) = a_{j_2}$ for some $a_{j_2} e A_1, a_{j_2} e A_2, j_3 e J$ and $j_4 e J^i$. Hence $(x + j) a_{j_3} - (a_{j_3} + x a_{j_4} = a_{j_2}$. On the other hand, $j a_{j_3}, x a_{j_4}$ are contained in $J^{i+2}$. Take the projection of $e f^i$ onto $A_2$, and $a_{j_2} j \in A_2 \cap J^{i+2} = A_2 J$. Hence $A_2 J = 0$. Similarly if $x (A_1 J \oplus A_2 J^i) \supset A_1 J^i \oplus A_2 J, A_1 J = 0$. Therefore $|A_i| \neq 2$.

We observe isomorphisms between sub-factor modules of $A_1$ and $A_2$, and then investigate submodules $X$ in $e J^i$. It is well known that there exist submodules $A_i \supset C \supset C'$ and $A_2 \supset D \supset D'$ such that $h: C / C' \sim D / D'$ and $X = \{c + d | \in C \oplus D, h(c + C') = d + D'\}$ (cf. [3]). We denote $X$ by $C(h) D$.

a-1) Let $|A_i| = 2$.

a-1-1) $f: A_i E_2 \cong A_i F_2 \approx g R / g f$. Then $\Delta' = g R / g$ is from 2-i) and $f$ is a unique isomorphism.

a-1-2) Assume $f: A_i E_2 \approx E_2 / E_{i+1} (i > 1)$. Consider $\{A_1, A_2, E_2 \oplus F_2, A_i J, A_i f\}$. Since $\Delta(A_i) = \Delta(A_2) = \Delta$, $A_2 \approx A_2 (f) f_i$. Further $E_2 \oplus F_2$ being characteristic, from Lemma 7 there exists a unit $x'$ in $e R$ such that $x' A_1 \subset A_1 (f) f_i$. Let $p_i: e f^i \rightarrow A_i$ be the projection and $x' = x + j; x A_i = A_i, j e F_i$ as usual. Then for a generator $a$ in $A_i$

$$\begin{align*}
(x + j) a &= ar + f (ar) + z_1 + z_2; r e R, z_1 e E_2 \text{ and } z_2 e F_{i+1}.
\end{align*}$$

Hence $x a + p_i (j a) = ar + z_1$ and $p_i (j a) = f (ar) + z_2$.

Since $p_i (j a) e E_2, x a \equiv ar \pmod{E_2}$. Assume $i < p - 1$. Then $j e F_{i+1} \subset F_{i+1}, f (ar) \equiv f (xa) \equiv 0 \pmod{F_{i+1}}$. However $x a$ is a generator of $A_i$, and hence $f = 0$. Therefore $i > p - 1$.

a-1-3) See a-2-1) below.

a-1-4) $E_2 \approx F_2 (p = 2)$. We have the situation of 2-i).

Assume further $f: A_i E_2 \approx F_2 (A_i F_2 \approx E_2)$, and consider $\{A_1, A_2, A_i (f), E_2 \oplus F_2\}$. Then $A_i \sim A_i (f)$ by Lemma 7 and so $A_i (f) = x A_i$ for some unit $x$ in $e R$, since $A_i \approx A_i (f)$. If $p > 2$, see a-2-2) below.

a-1-5) Let $X$ be a submodule in $e F^i$.

i) $X = A_i (f_i) F_i \approx F_i (f_i)^{-1} (f_i: A_i \approx F_i / F_{i+2})$. If $i = 1$, consider $R / f^{i+3}$. Then this contradicts 2-i). Hence $i \neq 1, F_i = F_{i-1}$ and $F_{i+2} = 0$ from a-1-2). $\{A_1, A_2, E_2 \oplus F_2, A_i (f_i)\}$ shows $A_i (f_i) = x A_i$ for some unit $x$ in $e R$.

ii) $X = A_i (f_i) A_i (f_a: A_i E_2 \approx A_i F_2)$. Then $X = B_i$ from a-1-1).

iii) $X = A_i (f_i) F_i (f_i A_i E_2 \approx F_i / F_{i+1}, i > 1)$ and hence $i = p - 1$ or $p$ by a-1-2). Then $\{A_i \oplus F_{i+1}, A_2, E_2 \oplus F_2, A_i (f_i) F_i\}$ shows $A_i (f_i) F_i = x (A_i \oplus F_{i+1})$. 


iv) $X = A_2(k^{-1})$ $(f_4: E_2 \cong A_2/F_2)$. \{A_1, A_2, E_2 \oplus F_2, A_4(k^{-1})\} shows $A_2 = xA_2(k^{-1})$.

v) $X = F_i(f^{-1})$ $(f_5: E_2 \cong F_i/F_{i+1}, i \geq 2)$. In this case $eJ^{i+1} = E_2 \oplus F_2$. Hence this is the case of a-2). Accordingly $X = B_1$, $B_1'$ or $B''$, provided $X$ is not isomorphic to a standard submodule in $eJ^{i+1}$ via $x_i$.

Thus we have shown that $X$ is isomorphic to a standard submodule in $eJ^i$ via $x_i$ except $B_1$, $B_1'$ and $B''$.

a-2) $|A_1| = 1$.

a-2-1) Let $f: A_1 = F_i/F_{i+1}$ $(i < p)$. If $F_i(f^{-1}) \supset xA_1$ for some unit $x$ in $eRe$, $xA_1 \subset F_{i+1} \subset A_2$, since $(F_i(f^{-1}) = F_{i+1}$, which is a contradiction from Lemma 2. We note further that $A_2$ is characteristic, since $\Delta(A_2) = \Delta$ and $A_1$ is simple. Assume $\Delta(F_1(f^{-1})) = \Delta$. Then $A_2, A_1, F_1(f^{-1})$ derives a contradiction from the above remarks and Lemma 7. It is clear that $eJe(F_i(f^{-1}) \subset eJe(F_i \oplus A_1) \subset F_{i+1}$. Hence $F_i(f^{-1})$ is also characteristic. Let $f': A_2 = F_i/F_{i+1}$ be another isomorphism. $\{A_2, A_1, F_i(f^{-1}), F_i(f'^{-1})\}$ gives $F_i(f^{-1}) = F_i(f'^{-1})$ since they are characteristic. Therefore $f = f'$. Accordingly, $\Delta \cong \bar{g}_r \Lambda g \cong \bar{Z}$ as given in the proof of 2-i). Further assume $g: A_1 = F_i/F_{i+1}$ $(j < p)$. Again consider $\{A_2, A_1, F_i(f^{-1}), F_j(g^{-1})\}$. Then $F_i(f^{-1}) \supset F_j(g^{-1})$ if $i < j$, and so $F_i(g^{-1}) \subset F_{i+1}$, a contradiction.

a-2-2) Assume that $f_1, f_2: A_1 = F_1$, and they are not given by $j_1$ in $eRe$. Then $\{A_2, A_1, A_1(f_1), A_1(f_2)\}$ gives, from Lemmas 6 and 7, that $A_1(f_1) = x'A_1(f_2)$ for some unit $x'$ in $eRe$. Since $\Delta(A_1) = \Delta$, there exists $j$ in $eRe$ such that $(x' + j)A_1 = A_1$. Put $x = x' + j$. Then for a generator $a$ in $A_1$

$$(x-j)(a + f_2(a)) = ar + f_1(ar); r \in R.$$ 

Hence

$$xa = ar, xf_2(a) - ja = f_1(ar).$$

Next assume further that $g: A_1 = F_i/F_{i+1}$ $(i < p)$. Consider $\{A_2, A_1, F_i(q^{-1}), A_1(f)\}$, and $F_i(q^{-1}) \sim A_1(f)$ since $F_i(q^{-1})$ is characteristic. Which is a contradiction. In particular, if $eJeA_1 = 0$, $A_1(f)$ is characteristic, since $\Delta(A_1(f)) = \Delta$ (if $\Delta \neq \Delta(A_1(f))$, $\{A_1, A_2, A_1(f)\}$ gives $A_1 \sim A_1(f)$). Then $f$ is given by $j_1$ from Lemma 6). Hence $f_1 = f_2$ from the first paragraph, and so $\Delta \cong \bar{g}_r \Lambda g \cong \bar{Z}$ as in the proof of 2-i).

b) $A_2/A_2J^k$ $(k \geq 1)$ is uniserial and $A_2J^k = \sum_{i=1}^s \oplus B_i$ $(s \geq 2)$. Then $A_1$ is simple from the initial paragraph of ii). Then $Df^k = eJ^{i+k} = A_2J^k$. Since $eJ^{i+k} = B_1 \oplus \cdots \oplus B_s$, $s \leq 3$ from Lemma 15, and $[\Delta: \Delta(B_i)] \leq 2$ for all $i$ by Lemmas 2 and 7. If $[\Delta: \Delta(B_1)] = 1$ and $[\Delta: \Delta(B_2)] = 2$, $\{A_1, B_1, B_2, B_3\}$ derives a contradiction. Hence either $[\Delta: \Delta(B_i)] = 1$ for all $i$ (b1) or $[\Delta: \Delta(B_i)] = 2$ for all $i$ (b2). In the former case $s = 2$ by Lemma 7 and in the latter case also $s = 2$ and $B_2 = xB_1$ for some unit $x$ in $eRe$ by Lemma 10.
Then \{A_1, B_1, B_2, J(B_1) \oplus J(B_2)\} implies J(B_1) = 0 (\mid B_1 \mid \leq \mid B_2 \mid). If f: B_1 \rightarrow B_2, \{A_1, B_1, B_2, B_1(f)\} derives a contradiction. Hence B_1 \approx B_2. We can show as before that B_2 is uniserial.

b_1) \Delta(B_i) = \Delta(B_2) = \Delta.

Then \{A_1, B_1, B_2, J(B_1) \oplus J(B_2)\} implies J(B_1) = 0 (\mid B_1 \mid \leq \mid B_2 \mid). If f: B_1 \rightarrow B_2, \{A_1, B_1, B_2, B_1(f)\} derives a contradiction. Hence B_1 \approx B_2. We can show as before that B_2 is uniserial.

b_1-1) This is the case of a-2-1).

b_1-2) Assume f: A_1 \approx B_1. \{A_1, A_1(f), B_1, B_2\} derives A_1 \approx A_1(f), i.e.,

\((x+j)A_1 = A_1(f): xA_1 = A_1 and j \in eJf. (x+j)a = ar + f(ar); A_1 = aR, r \in R.

Hence xa = ar and ja = f(ar). Put xa = b, and A_1 = bR. f(b) = jx^{-1}b.

b_1-3) Assume f: A_1 \approx H_i / H_{i+1}. \{A_1, B_1, B_2, H_1(f^{-1})\} shows A_1 \approx H_1(f^{-1}).

Hence H_{i+1} = 0 and f is given by j_i as above (cf. Lemma 6).

b_1-4) Assume f: B_1 \approx H_i / H_{i+1}. \{A_1, B_1, B_2, H_1(f^{-1})\} derives B_1 \approx H_1(f^{-1}), since \Delta(B_1) = \Delta. Hence H_{i+1} = 0 and f is given by j_i from Lemma 6.

b_2) [\Delta: \Delta(\xi)] = 2 for i = 1, 2, (B_2 = xB_1).

{A_1, B_1, B_2, J(B_1) \oplus J(B_2)} shows, from Lemma 2, that J(B_2) = 0, i.e., B_2 is simple. Further since \Delta(A_1) = \Delta and [\Delta: \Delta(A_1)] = 2, [\Delta: \Delta(E)] = 2 for all simple submodules E in V = B_1 \oplus B_2 by Lemmas 2 and 8. Hence V satisfies (\#1) by Lemma 7.

b_2-1) If A_1 \approx F_1 / F_{i+1}, \Delta = \bar{Z} by a-1-3). Hence \Delta(B_1) = \Delta.

b_2-2) Assume f: A_1 \approx B_1. \{A_1, B_1, B_1, A_1(f)\} derives A_1 \approx A_1(f). Hence f is given by j_i as b_1-2).

REMARK 17. If R is an algebra over an algebraically closed field K, \Delta \neq \bar{Z} and the first part of a-2-2) does not occur (take f_2 = kf_1, k \neq 1; k \in K). We can express f in a-1-2) as an element in eJf, however it is little complicated (cf. [7], Theorem 17).

In order to make the converse version clear, we illustrate the structure of Lemmas 10-16 as follows:

1) (Lemma 10)

\[
\begin{array}{c|c|c|c}
  eR & eJ & eJ^{i+1}  \\
  \hline
  A_1 & B_1 & 0  \\
  \hline
  A_2 = xA_1 - B_2 = xB_1 - 0
\end{array}
\]

[\Delta: \Delta(A_1)] = [\Delta: \Delta(A_2)] = 2, every characteristic submodule in eJ is linear with respect to the inclusion and [\Delta: \Delta(C)] = 2 for any non-characteristic submodule C in eJ. Further those C are related to one another with respect to ~.

2) (Lemma 11)

\[
\begin{array}{c|c|c|c}
  eR & eJ  \\
  \hline
  A_1 & 0  \\
  \hline
  A_2 = x_2A_1 - 0  \\
  A_3 = x_3A_1 - 0
\end{array}
\]

(3)
M. HARADA

\[ [\Delta: \Delta(A_1)] = [\Delta: \Delta(A_2)] = [\Delta: \Delta(A_3)] = 3 \] and \( A_1 \oplus A_2 \oplus A_3 \) satisfies (#, 1) and (#, 2). Further \( [\Delta: \Delta(C)] \leq 3 \) for every submodule \( C \) in \( A_1 \oplus A_2 \oplus A_3 \). (\( A_3 \) may be zero.)

3) (Lemma 12, i))

\[ eR \quad eJ^i \]
\[ \begin{array}{c|c}
A_1 & 0 \\
\hline
\hat{\theta} & 0 \\
\hline
A_3 & x_3 A_2 - 0
\end{array} \]

\( [\Delta: \Delta(A_1)] = 1 \) and \( [\Delta: \Delta(A_2)] = [\Delta: \Delta(A_3)] = 2 \). Further \( A_2 \oplus A_3 \) satisfies (#, 1).

4) (Lemma 12, ii-a))

\[ eR \quad eJ^i \quad eJ^{i+1} \]
\[ \begin{array}{c|c}
A_1 & E_2 - 0 \\
\hline
A_2 & 0
\end{array} \]

\( [\Delta: \Delta(A_1)] = 1 \), \( [\Delta: \Delta(A_2)] = 2 \) and \( A_2 \oplus E_2 \) satisfies (#, 1) except \( E_2 \).

5) (Lemma 12, ii-b ii-b-2-ii-1))

\[ eR \quad eJ^i \quad eJ^{i+k} \quad eJ^p \]
\[ \begin{array}{c|c}
A_1 & 0 \\
\hline
B_1 & 0 \\
\hline
A_2 & \\
\hline
B_2 & 0
\end{array} \]

\( [\Delta: \Delta(A_1)] = 1 \), \( [\Delta: \Delta(A_2)] = 2 \) and \( A_1 \oplus A_2 / J(A_2) \) satisfies (#, 1) except \( A_1 \). (\( B_2 \) may be zero.)

5')

\[ eR \quad eJ^i \quad eJ^{i+k} \]
\[ \begin{array}{c|c}
A_1 & 0 \\
\hline
B_1 & 0 \\
\hline
A_2 & \\
\hline
B_2 & 0
\end{array} \]
\[\Delta: \Delta(B_1) = \Delta(\Delta(B_2)) = 2 \text{ and } B_1 \approx B_2. \quad B_1 \oplus B_2 \text{ satisfies } (\#, 1)\)

6) (Lemma 16, 1))

\[
\begin{array}{ccc}
eR & e^{\mathcal{J}^i} & e^{\mathcal{J}^i+\mathcal{J}^{i+1}} \\
\hline
A_1 & - & E_n - 0 \\
A_2 & - & 0 \\
A_3 & - & 0
\end{array}
\]

\[\Delta: \Delta(A_i) = \Delta(\Delta(A_{i+1})) = 1. \text{ If } f: A_2 \approx E_n, \text{ } f \text{ is given by } j; j \in eFe. \quad (A_2, A_3 \text{ and } E_2 \text{ may be zero.)}
\]

7) (Lemma 16, 2-i))

\[
\begin{array}{ccc}
eR & e^{\mathcal{J}^i} \\
\hline
A_1 & - & 0 \\
A_2 & - & 0
\end{array}
\]

\[\Delta: \Delta(A_i) = \Delta(\Delta(A_{i+1})) = 1 \text{ and } \Delta \approx \Delta' \approx \mathbb{Z}.
\]

8) (Lemma 16, 2-ii-a-1))

\[
\begin{array}{ccc}
eR & e^{\mathcal{J}^i} & e^{\mathcal{J}^i+1} & e^{\mathcal{J}^i+\mathcal{J}^{i+1}} \\
\hline
A_1 - E_2 & - & 0 \\
A_2 - F_2 & - & F_n - 0
\end{array}
\]

\[\Delta: \Delta(A_i) = \Delta(\Delta(A_{i+1})) = 1. \text{ If } f: A_1[E_2 \approx A_3]/F_2, \Delta \approx \Delta' \approx \mathbb{Z}. \text{ Every submodule except } B_1, B'_1 \text{ and } B'' \text{ is isomorphic to a standard submodule via } x_i. \text{ (If } n=2 \text{ and } E_2 \approx F_2, \Delta \approx \Delta' \approx \mathbb{Z}.) \text{ If } E_2 = 0, \text{ the conditions in a-2) of Lemma 16 are fulfilled.}
\]

9) (Lemma 16, 2-ii-b_1))

\[
\begin{array}{ccc}
eR & e^{\mathcal{J}^i} & e^{\mathcal{J}^i+\mathcal{J}^{i+1}} & e^{\mathcal{J}^i+\mathcal{J}^{i+1}} \\
\hline
A_1 & - & 0 \\
A_2 & - & B_1 - 0 \\
B_2 & - & F_2 - 0
\end{array}
\]

\[\Delta: \Delta(A_i) = \Delta(\Delta(A_{i+1})) = 1, \Delta: \Delta(B_1) = \Delta(\Delta(B_2)) = 1. \text{ If } f: A_1 \approx B_1, \text{ } f \text{ is given by } j; j \in eFe. \quad \text{Similar facts hold for other cases.}
\]

10) (Lemma 16, 2-ii-b_2))
We shall show that if \( eR \) has one of the structures of the above diagrams 1) \~ 10), then \( R \) is a right US-4 ring with (\(*, 1') \). It is clear from the diagrams that (\(*, 1') \) holds. Let \( \{U_i\}_{i=1}^k \) be a set of submodules in \( eR \).

Diagram 1). If \( U_1 \) and \( U_2 \) are characteristic, \( U_1 \supseteq U_2 \) or \( U_1 \subseteq U_2 \). Hence \( U_1 \oplus U_2 \) satisfies (**, 2) by [4], Corollary 1 of Theorem 2. Hence \( D = \sum_{i=1}^k U_i \)

satisfies (**, 4) by [2], Lemma 1. Assume that \( U_1 \cap U_2 \ni eJ^i \). Then \( U_i \) for \( i=1, 2 \) is characteristic, and hence \( D \) satisfies (**, 4) from the above. Next assume that \( U_1 \supseteq \varepsilon J^i \) and \( eJ^i \supseteq U_j \) for \( j > 1 \). Since \( \Delta(U_1) = \Delta, U_1 \oplus U_2 \) satisfies (**, 2) by [4], Corollary 1 of Theorem 2. Finally assume \( eJ^i \supseteq U_j \) for all \( j \). If \( \{U_j\}_{j=1}^n \) is a set of non-characteristic submodules, then we may assume \( U_1 \supseteq x_2 U_2 \supseteq x_3 U_3 \) for some units \( x_i \) in \( eR \) by assumption. Since \( \Delta(\Delta(U_j)) = 2 \), \( U_1 \oplus U_2 \oplus U_3 \) satisfies (**, 3) by [4], Corollary 3 of Theorem 2. Therefore \( D \) satisfies (**, 4).

2) As is shown in 1), we may assume that \( eJ^i \supseteq U_j \) for all \( j \). Then \( U_1 \supseteq x_2 U_2 \supseteq x_3 U_3 \supseteq x_4 U_4 \) by assumption, where the \( x_i \) are units in \( eR \). Then from the assumption \( [\Delta: \Delta(C)] \leq 3 \) and the argument of the proof of [4], Corollary 3 of Theorem 2, \( D \) satisfies (**, 4).

3) Let \( eJ^i \supseteq U_j \) for all \( j \). Then \( U_1 = A_1 \oplus B_i \) or \( U_1 \subseteq A_2 \oplus A_3 \) by assumption, where \( B_i \subseteq A_2 \oplus A_3 \). First assume \( U_j \subseteq A_2 \oplus A_3 \) or \( U_j = A_1 \oplus B_j \) for \( j > 0 \) for all \( j \leq 3 \). Then \( D \) satisfies (**, 4) by [4], Corollary 3 of Theorem 2 (note \( A_1 \) and \( A_2 \oplus A_3 \) are characteristic and see the remark above). If \( U_1 = A_1 \) and \( U_2 = A_1 \oplus B_2 \), \( U_1 \oplus U_2 \) satisfies (**, 2) by [4], Corollary 1 of Theorem 2. Thus \( D \) satisfies (**, 4).

4) Every submodule in \( eJ^i \) is isomorphic to a standard submodule in \( eJ^i \) via \( x_1 \). Hence we may assume that all \( U_j \) are standard. Then \( D \) satisfies (*) by [4], Corollaries 1 \~ 3 of Theorem 2.

5) and 5') Let \( eJ^i \supseteq U_1 \supseteq A_2 J \) and \( U_1 \neq A_1 \oplus A_2 \). Then \( U_1 / A_2 J = x(A_2 / A_2 J) \), and so \( xA_2 = U_1 \). Further \( A_1 \oplus A_2 J \) is characteristic. If \( U_i = A_1 \oplus A_2 J \) and \( U_i \subseteq A_2 J, U_1 \oplus U_2 \) satisfies (*) by [4], Corollary 1 of Theorem 2. Accordingly we may assume that \( U_i \) is \( A_1 \) or a submodule of \( A_2 \). Therefore \( D \) satisfies (**, 4).

6) and 7) These are clear.
8) First we note \( B'_i \rightarrow E_2 \oplus F_2 \rightarrow B' \) (\( E_2 \oplus F_2 \rightarrow B'' \)) and \( B', B'' \) do not appear simultaneously. If the \( U_i \) are standard for all \( i \), \( U_i \sim U_j \) for some pair \( i, j \). Hence \( D \) satisfies (**, 4) by [4], Corollary 2 of Theorem 2. The conditions given in Lemma 16 show that \( A_i \sim A_1(f), F_p(f^{-1}) \sim F_p(g^{-1}), \ldots \) etc. Hence we obtain the desired result.

9) and 10) These are simpler than 8), (if \( A_1 \approx F_i \oplus B_{i+1} \oplus B_2 \)), \( \Delta \approx \hat{Z} \). Hence \( \Delta(C) = \Delta \) for any submodule \( C \) in \( eR \).

Thus we obtain

**Theorem 2.** \( R \) is a right US-4 (basic) ring with \((*, 1') \) if and only if \( eR \) has one of the structures given in Lemmas 10~16 (cf. Diagrams 1~10) \) for each primitive idempotent \( e \).

3. Hereditary rings

In this section, we shall study a hereditary and right US-3 (resp. US-4) ring \( R \). If \( R \) is hereditary, \((*, 1') \) holds, and hence we can make use of the results in the previous sections.

**Lemma 18.** Assume that \( R \) is basic and hereditary. Then a submodule \( A \) in \( eR \) is characteristic if and only if \( \Delta(A) = \Delta \). Every non-zero element in \( \text{Hom}_R(eR, fR) \) is a monomorphism, where \( e \) and \( f \) are primitive idempotents.

Proof. The second half is clear (see [9], Lemma 2). Hence, since \( eJe=0 \), the first one is clear.

From now on we assume that \( R \) is a hereditary and basic ring. First we assume further that \( R \) is right US-3.

**Theorem 3.** Let \( R \) be a hereditary (and basic) ring. Then \( R \) is a right US-3 ring if and only if \( eR \) has the following structure for each primitive idempotent \( e \):

i) \( eR/eJ' \) is uniserial for some \( t \) and

ii) \( eJ'=0 \) or \( eJ'=A \oplus B \) such that either

a) \( A \) and \( B \) are simple and \( A \oplus B \) satisfies (\( \delta, 1 \)), and \( [\Delta: \Delta(A)] = 2, \) or

b) \( A \) is simple, \( B \) is uniserial and \( A \) is not isomorphic to any sub-factor modules of \( B \) (and hence \( \Delta(A) = \Delta(B) = \Delta \)).

Proof. If \( R \) is right US-3, \( eR \) has the structure in Theorem 1. We consider the case b) of Theorem 1. Assume that \( f: A \approx (\text{the socle of } B) \). Then \( \{A, A(f), B\} \) derives a contradiction, since \( A \) and \( B \) are characteristic by Lemma 18. Thus we obtain the theorem from Theorem 1.

Let \( R \) be a basic hereditary ring. Then
where the $\Delta_i$ are division rings and the $M_{ij}$ are left $\Delta_i$- and right $\Delta_j$-modules [1].

We shall express explicitly the content of Theorem 3 for $M_{ij}$ in a row of the above ring.

1) 

\[ (0 \cdots \Delta, 0 \Delta, 0 \cdots \Delta, 0) \]

2) 

\[ (0 \cdots \Delta, 0 \cdots \Delta, 0 \cdots \Delta, 0 \cdots \Delta, 0) \]

\[ \begin{pmatrix} u_p \Delta_i, & \cdots \\ v_p \Delta_i, & \cdots \end{pmatrix} \]

......A

......B

(4)

As is given in the proof of [9], Theorem 1, we can show a ring monomorphism $\rho_{rs}: \Delta_r \rightarrow \Delta_s$ for $r<s<k$ such that $xu_r = u_s \rho_{rs}(x)$ for $x \in \Delta_r$ and $\rho_{rs} \rho_{st} = \rho_{st}$.

Next we shall characterize a hereditary (basic) and right US-4 ring. If $R$ is hereditary, some results in the previous sections may not occur as shown in Theorem 3. We shall observe them.

In the case b) of Lemma 12, $A_2$ is simple. Because, since $A_1$ is simple and $[\Delta: \Delta(A_2)] = 2$, $A_1 \approx A_2/J(A_2)$. Hence $A_1 \approx A_2$ by Lemma 18.

We shall observe the conditions in Lemma 16 for a hereditary ring. a-1-1), a-1-2), a-1-3), any of b-1-1) to 4) and b2-2) do not occur from Lemma 18. For instance, if $f': A_1/E_2 \approx F_{p-1}/F_p$ (a-1-2), $f': A_1 \approx F_{p-1}$ by Lemma 18. Then $A_1 \approx A_1(f)$ by a-1-5). However, $A_1$ is characteristic, and so $A_1 = A_1(f)$. Therefore $f = 0$.

We shall use the notations after Theorem 3.

**Lemma 19.** In case 2-i) in Lemma 16, $e_iR$ is of the form $(0, \cdots, \tilde{Z}, \cdots$
In case of 2-a-1-4) in Lemma 16, \( A_i \) (resp. \( A_k \)) is of the form 
\( (0, \ldots, \bar{Z}, 0, \bar{Z}, 0 \ldots) \) (resp. \( (0, \ldots, \bar{Z}, 0, \bar{Z}, 0 \ldots) \)).

Proof. Let \( E \approx F \approx e_{kk} R \). Then \( e_{kk} R \approx \bar{Z} \) by Lemma 16. Let \( A_2 \approx e_{kk} R \). Then \( e_{kk} R \) is uniserial and \( M_{kk} = u_{kk} \bar{Z} \approx F \). Since \( M_{kk} \) is a left \( \Delta_s \)-module, \( \Delta_s \subset \bar{Z} \). Hence \( \Delta_s = \bar{Z} \). We have the same for 2-i).

Thus we have

**Theorem 4.** Let \( R \) be a hereditary (basic) ring. Then \( R \) is right US-4 if and only if for each \( e = e_{ii} \), \( eR \) has one of the following structures: 1~11

1) 
\( (0 \ldots 0\Delta_1 0\Delta_1 0 \ldots \Delta_\ell \ldots 0) \)
2) (Lemma 10)
\( (0 \ldots 0\Delta_1 0 \ldots \Delta_1 0 \ldots (u_{i+1}\Delta_{i+1}) 0 (u_{i+2}\Delta_{i+2}) \ldots 0 \ldots 0) \)
\[ \Delta: \Delta(A_i) = 2 \ (i=1,2) \] and \( u_{i+2}, v_{i+2} \) may be zero. The conditions in Lemma 10 are satisfied.
3) (Lemma 11)
\( (0 \ldots \Delta_1 0 \ldots \Delta_1 0 \ldots (u_{i+1}\Delta_{i+1}) \ldots 0) \)
\[ \Delta: \Delta(A_i) = 3 \] for each \( i \) and \( A_1 \oplus A_2 \) satisfies (\( \# \), 1) and (\( \# \), 2). \( v_{i+1} \) may be zero.
4) (Lemma 12-i)
\( (0 \ldots \Delta_1 0 \ldots \Delta_1 0 \ldots \Delta_1 0 \ldots 0 \ldots (u_{i+1}\Delta_{i+1}) \ldots 0) \)
\( \Delta(A_1) = \Delta, \ [\Delta: \Delta(A_i)] = 2 \) (\( i=1,2 \)) and \( A_1 \oplus A_3 \) satisfies (\( \# \), 1).
5) (Lemma 12-ii-a) and b))
\( (0 \ldots \Delta_1 0 \ldots \Delta_1 0 \ldots \Delta_1 0 \ldots 0 \ldots 0) \)
\( \Delta(A_1) = \Delta, [\Delta: \Delta(A_i)] = 2, \) and \( u_{i+1} \Delta_1 \oplus v_{i+1} \Delta_1 \) satisfies (\( \# \), 1), except \( u_{i+1} \Delta_i \) (\( \{u_{i+1}, \ldots, u_{p-1}\} \) may be zero.)
6) (Lemma 16, 1)
\[ \begin{bmatrix} u_{t+1} \Delta_{t+1} \\ v_{t+2} \Delta_{t+2} \\ v_{t+3} \Delta_{t+3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \Delta(A_i) = \Delta \ (i = 1, 2, 3) \text{ and } u_{t+1} \text{ may be zero.} \]

\[ \subseteq \Delta_{i+1} \subseteq \Delta_i \subseteq \Delta_{i+2} \subseteq \Delta_{i+3} \subseteq \Delta_i \]

7) (Lemma 16, 2-i))

\[ \begin{bmatrix} u_{t+1} \bar{Z} \\ v_{t+1} \bar{Z} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \Delta(A_i) = \Delta \ (i = 1, 2), \ u_{t+3} \text{ or } \{v_{t+4}, \ldots, v_p\} \text{ may be zero.} \]

\[ \subseteq \Delta_{i+1} \subseteq \Delta_{i+3} \subseteq \Delta_{i+2} \subseteq \Delta_i \subseteq \Delta_{i+4} \subseteq \Delta_i \]

8) (Lemma 16, 2-ii-a))

\[ \begin{bmatrix} u_{t+1} \Delta_{t+1} \\ v_{t+2} \Delta_{t+2} \\ v_{t+3} \Delta_{t+3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \Delta(A_i) = \Delta \ (i = 1, 2), \ u_{t+3} \text{ or } \{v_{t+4}, \ldots, v_p\} \text{ may be zero.} \]

\[ \subseteq \Delta_{i+1} \subseteq \Delta_{i+3} \subseteq \Delta_{i+2} \subseteq \Delta_i \subseteq \Delta_{i+4} \subseteq \Delta_i \]

9) (Lemma 16, 2-ii-a')

\[ \begin{bmatrix} 0 \\ v_{t+1} \bar{Z} \\ 0 \end{bmatrix} \begin{bmatrix} u_{t+2} \bar{Z} \\ v_{t+2} \bar{Z} \\ v_{t+3} \bar{Z} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ u_{t+2} \text{ may be zero.} \]

10) (Lemma 16, 2-ii-b1))

\[ \begin{bmatrix} u_{t+1} \Delta_{t+1} \\ v_{t+2} \Delta_{t+2} \\ v_{t+3} \Delta_{t+3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \subseteq \Delta_{i+1} \subseteq \Delta_{i+2} \subseteq \Delta_i \subseteq \Delta_{i+3} \subseteq \Delta_{i+4} \subseteq \Delta_i \]

\[ B_1 \subseteq B_2 \subseteq \Delta_{i+3} \subseteq \Delta_{i+2} \subseteq \Delta_i \subseteq \Delta_{i+3} \subseteq \Delta_{i+2} \subseteq \Delta_i \]

\[ B_1 \subseteq B_2 \subseteq \Delta_i \subseteq \Delta_{i+3} \subseteq \Delta_{i+2} \subseteq \Delta_i \subseteq \Delta_{i+3} \subseteq \Delta_{i+2} \subseteq \Delta_i \]
\[ \Delta(A_i) = \Delta(B_i) = \Delta \quad (i=1,2). \]

\[ \Delta_i \subset \Delta_{i+1} \subset \cdots \subset \Delta_{i+x} \subset \Delta_{i+y} \subset \cdots \subset \Delta_{i+p} \]

11) (Lemma 16, 2-ii-b))

\[ (0 \cdots \Delta_i 0 \cdots \Delta_i 0 \cdots) \]

\[ \begin{pmatrix}
  u_{i+1} & 0 & 0 & 0 \\
  0 & v_{i+2} & 0 & 0 \\
  0 & 0 & w_{i+3} & 0 \\
  0 & 0 & 0 & z_{i+4}
\end{pmatrix} \]

\[ \cdots A_1 \]

\[ \cdots B_1 \]

\[ \cdots A_2 \]

\[ \cdots B_2 \]

\[ \Delta_i \subset \Delta_{i+1} \subset \cdots \subset \Delta_{i+x} \subset \Delta_{i+y} \subset \cdots \subset \Delta_{i+z} \subset \Delta_{i+p} \]

\[ \Delta(A_i) = \Delta \text{ and } [\Delta : \Delta : (B_i)] = 2 \quad (i=1,2). \]

\[ \Delta_{i+1} \Delta_{i+2} \oplus \Delta_{i+3} \Delta_{i+4} \text{ satisfies } (\#1), \]

where \( \Delta \)s are division rings and \( \Delta_i \subset \Delta \) except 6, 8, 10) and 11). The series: \( (0 \cdots \Delta_i 0 \cdots \Delta_i 0 \cdots) \) on the same level means a uniserial module.

4. Left serial rings

We shall investigate the same problem for a left serial ring \( R \). In this case \((\ast,1')\) holds, too by [11], Corollary 4.2. Therefore we can make use of the results in §§ 1 and 2.

From now on we always assume that \( R \) is a left serial ring.

**Lemma 20.** If \( eJ' = A_i \oplus A_2 \) and the \( A_i \) are uniserial, every submodule \( E \) in \( eJ' \) is isomorphic to a standard submodule \( B_1 \oplus B_2 \) via \( x_1 : x \) is a unit in \( eRe \), where \( B_i \subset A_i \).

See the proof of [3], Theorem 1.

**Lemma 21.** Let \( eJ' = A_i \oplus A_2 \) and the \( A_i \) hollow. If \( \Delta(A_i) \neq \Delta \), there exists a unit \( x \) in \( eRe \) such that \( xA_1 = A_2 \).

Proof. Since \( \Delta(A_i) \neq \Delta \), there exists a unit \( y \) in \( eRe \) such that \( (y+j')A_i \subset A_i \) for all \( j' \) in \( eJe \). Let \( p \) be the projection of \( eJ' \) onto \( A_2 \). Then \( f = py_1 | A_1 \) is an element in \( \text{Hom}_R(A_1, A_2) \). If \( f \) is not an epimorphism, \( f = j_i \) for some \( j \) in \( eJe \), since \( A_2 \) is a hollow module \((\subset eJ' \oplus)\) and \( R \) is left serial. Then \((y-j)A_i \subset A_i \),
a contradiction. Hence there exists a unit \( x \) in \( eRe \) such that \( x_1 = f \), and so 
\[ x_2 A = A_1. \]

**Lemma 22.** Let \( eJ' = A_1 \oplus A_2 \) be as in Lemma 21. If \( \Delta(A_1) = \Delta, \Delta(A_1J^* \oplus A_2J') = \Delta. \)

Proof. From Lemma 21, \( \Delta(A_2) = \Delta. \) Hence we may assume \( k \leq k' \). Let \( x \) be any unit element in \( eRe \). Since \( \Delta(A_1) = \Delta \), there exists \( j \) in \( eJ^* \) such that 
\[ (x + j) A_1 = A_1. \]
Hence \( (x + j)(A_1J^* \oplus A_2J') \subseteq A_1J^* + (x + j)A_2J' \subseteq A_1J^* \oplus A_2J' \), and so 
\[ x = x + j \in \Delta(A_1J^* \oplus A_2J'). \]

From Theorem 1, Lemmas 21, 22 and [8], Proposition 2, we obtain

**Theorem 5.** Let \( R \) be a left serial ring. Then \( R \) is a right US-3 ring if and only if \( eR \) has the following structure for each primitive idempotent \( e \):

There exists an integer \( t \) such that

i) \( eR/eJ^* \) is uniserial and

ii) \( eJ^* = 0 \) or \( eJ^* \) is a direct sum of a simple module and a uniserial module.

Finally we shall give a characterization of a left serial and right US-4 ring. As was shown in the previous section, we shall refine the results in § 2.

In Lemma 10, every submodule in \( eJ^* \) is standard up to \( x \) (\( x \) is a unit in \( eRe \)) by Lemma 20. Further since \( \Delta(A_1 \oplus A_2J) = \Delta \),
\[ A_1 \oplus A_2 \supset J(A_1 \oplus J(A_2)) \supset 0 \]
is the set of all characteristic submodules in \( eJ^* \).

From the above proof we have

**Remark 23.** Let \( R \) be left serial and assume \( eJ' = A_1 \oplus A_2 \); the \( A_i \) are uniserial. If \([\Delta: \Delta(A_i)] = 2, [\Delta: \Delta(C)] \leq 2\) for every submodule \( C \) in \( eJ^* \) and \( \{eJ'^* \} \) is the set of characteristic submodules in \( eJ^* \). Hence, if \( R \) is left serial, i), ii) and iii) in Lemma 10 imply iv) and v). However hereditariness does not as is shown from the following example:

Let \( K \subset L \) be fields such that \([L: K] = 2\). Put
\[
R = \begin{pmatrix}
L & L & L & L \\
0 & K & L & L \\
0 & 0 & L & L \\
0 & 0 & 0 & L
\end{pmatrix}
\]

Then \( R \) is hereditary. Put \( L = 1K + uK, e_1 = e, \) and \( eJ = A_1 \oplus A_2; A_1 = 1e_1R, A_2 = ue_1R \) satisfy i), ii) and iii) in Lemma 10. Further \([\Delta: \Delta(B)] = 2\) for any submodule \( B \) in \( eJ^* \) if \( \Delta \neq \Delta(B) \), since \([L: K] = 2\). Let \( eJ, eJ^*, eJ^3, (1 \otimes u \pm u \otimes 1) e_3 R, \)
(1 \otimes u \pm u \otimes 1)e_u R \) is the set of characteristic submodules provided \( u^c \subseteq K \), and 
\((1 \otimes 1)e_u R \sim (1 \otimes 1 + u \otimes x)e_u R \), provided \( x \in K \).

**Lemma 24.** Let \( B_1 \) and \( B_2 \) be simple submodules in \( eJ^i \) and \( V = B_1 \oplus B_2 \). If \( B_1 \approx B_2 \), \( V \) always satisfies (\( \#, \) 1).

Proof. Since \( R \) is left serial, every simple submodule in \( V \) is isomorphic to \( B_1 \) via \( x; x \) is a unit in \( eRe \). Hence \( V \) satisfies (\( \#, \) 1).

In Lemma 12, we do not have the case \( t = 2 \) by Lemma 21.
In Lemma 16, we have always \( A_1 \approx A_2 \), since \( \Delta(A_1) = \Delta(A_2) = \Delta \). Hence 2-i), 2-a-1-1), 2-a-2-3) and \( p = 2 \) in 2-a-2-4) do not occur. Similarly 2-a-2-1) does not occur.
Thus we obtain

**Theorem 6.** Let \( R \) be a left serial ring. Then \( R \) is right US-4 if and only if, for each primitive idempotent \( e \), \( eR \) has one of the following structures:

1) \( eR \) is uniserial: \( eR \) \( eJ \) \( eJ^p \)

\[
\begin{array}{c|ccc}
 & A_1 & A_2 & 0 \\
\hline
A_1 - B_1 & A_2 - B_2 & 0
\end{array}
\]

[\( \Delta: \Delta(A_1) = 2 \). In this case \( A_1 \approx A_2 \) and \( B_1 \) may be zero.

2) \( eR \) \( eJ^{i-1} \) \( eJ^i \) \( eJ^{i+1} \)

\[
\begin{array}{c|ccc}
 & A_1 & A_2 & 0 \\
\hline
A_1 - B_1 & A_2 - B_2 & 0
\end{array}
\]

[\( \Delta: \Delta(A_1) = 2 \). In this case \( A_1 \approx A_2 \) and \( B_1 \) may be zero.

3) \( eR \) \( eJ^{i-1} \) \( eJ^i \)

\[
\begin{array}{c|ccc}
 & A_1 & A_2 & 0 \\
\hline
A_1 & A_2 & 0
\end{array}
\]

[\( \Delta: \Delta(A_1) = 3 \) and \( A_1 \oplus A_2 \oplus A_3 \) satisfies (\( \#, \) 2). In this case \( A_1 \approx A_2 \approx A_3 \).

4) \( eR \) \( eJ^{i-1} \) \( eJ^i \)

\[
\begin{array}{c|ccc}
 & A_1 & A_2 & 0 \\
\hline
A_1 & A_2 & 0
\end{array}
\]

\( \Delta(A_1) = \Delta, [\Delta: \Delta(A_1)] = 2 (i = 2, 3) \). In this case \( A_2 \approx A_3 \).
5) \[ \begin{array}{c|c|c|c|c}
  eR & eJ^{i-1} & eJ^i & eJ^{i+1} & eJ^p \\
  \hline
  A_1 & 0 \\
  \cdots & \cdots \\
  A_2 & 0 \\
  \cdots & \cdots & \cdots & 0 \\
  A_3 & \cdots & \cdots \\
\end{array} \]

\[ \Delta(A_i) = \Delta \text{ (} i = 1, 2, 3 \text{). In this case } A_1 \neq A_2 \text{ and } A_2 \text{ may be zero.} \]

6) \[ \begin{array}{c|c|c|c|c}
  eR & eJ^{i-1} & eJ^i & eJ^{i+1} & eJ^p \\
  \hline
  A_1-B_3 & 0 \\
  \cdots & \cdots \\
  A_2-B_3 & 0 \\
  \cdots & \cdots & \cdots & 0 \\
\end{array} \]

\[ \Delta(A_i) = \Delta \text{ (} i = 1, 2 \text{).} \]

7) \[ \begin{array}{c|c|c|c|c|c}
  eR & eJ^{i-1} & eJ^i & eJ^{i+1} & eJ^p \\
  \hline
  A_1 & 0 \\
  \cdots & \cdots \\
  A_2 & B_1 & 0 \\
  \cdots & \cdots \\
  B_2 & \cdots & \cdots & 0 \\
\end{array} \]

\[ \Delta(A_i) = \Delta \text{ (} i = 1, 2, 3 \text{) and } \Delta(B_j) = \Delta \text{ (} j = 1, 2 \text{).} \]

8) \[ \begin{array}{c|c|c|c|c|c}
  eR & eJ^{i-1} & eJ^i & eJ^{i+1} & eJ_1 & eJ^p \\
  \hline
  A_1 & 0 \\
  \cdots & \cdots \\
  A_2 & \cdots & \cdots \\
  \cdots & \cdots \\
  B_1 & 0 \\
  \cdots & \cdots \\
  B_2 & 0 \\
\end{array} \]

\[ \Delta(A_1) = \Delta(A_2) = \Delta \text{ and } [\Delta: \Delta(B_1)] = 2. \text{ In this case } B_1 \approx B_2, \text{ where each straight line means "uniserial".} \]

5. Examples

We shall give examples of hereditary (resp. left serial) and right US-3 (resp. US-4) rings. Let \( K \) be a field. By \( L \) and \( L' \) we denote extension fields of \( K \) with \([L: K]=2\) and \([L': K]=3\), respectively, and \( \mathbb{Z} = \mathbb{Z}/2 \), where \( \mathbb{Z} \) is the ring of integers.

The following two rings are hereditary, left serial and right US-3 rings.
(K K K K) is the second type b) of Theorem 1 and (L L L) is the first type a) of Theorem 1.

On the other hand

\[
\begin{pmatrix}
K & L & L \\
0 & L & L \\
0 & 0 & K
\end{pmatrix}
\]
is a hereditary, non-left serial and right US-3 ring, and

\[
\begin{pmatrix}
L & L & 0 \\
0 & K & K \\
0 & 0 & K
\end{pmatrix}
\]
with \( e_{12}e_{22} = 0 \) is a left serial, non-hereditary and right US-3 ring.

Next we shall give hereditary and right US-4 rings for each structure in Theorem 4. However, we can not construct an example of the case 5) from the reason given in Remark 13.

1 \[
\begin{pmatrix}
K & K & K \\
K & K & L \\
0 & K & K
\end{pmatrix}
\]
2 \[
\begin{pmatrix}
L & L & L \\
L & L & L \\
K & K & K
\end{pmatrix}
\]
3 \[
\begin{pmatrix}
L' & L' & L' \\
L' & L' & L' \\
0 & K & K
\end{pmatrix}
\]

3' \[
\begin{pmatrix}
D & D^* \\
0 & D_1
\end{pmatrix}
\]
where \( D, D_1 \) and \( D^* \) are given in Remark 14.

4 \[
\begin{pmatrix}
L & L & L \\
L & 0 & L \\
0 & K & K
\end{pmatrix}
\]
6 \[
\begin{pmatrix}
K & K & K & K \\
K & 0 & 0 & K \\
K & 0 & 0 & K \\
0 & K & K & K
\end{pmatrix}
\]

7 \[
\begin{pmatrix}
\bar{Z} & \bar{Z} & \bar{Z} \\
0 & \bar{Z} & \bar{Z} \\
\bar{Z} & \bar{Z} & \bar{Z} \\
0 & \bar{Z} & \bar{Z}
\end{pmatrix}
\]
8 \[
\begin{pmatrix}
K & K & K & K \\
K & K & 0 & 0 \\
K & 0 & 0 & K \\
0 & K & K & K
\end{pmatrix}
\]
where $L$ is an extension of $\mathcal{Z}$ with $[L: \mathcal{Z}]=2$. $e_{11}R$ is of the form 2-1) in Lemma 16 and $e_{12}R$ is of the form in Lemma 10.

The rings of 1)~6), 8), 10) and 11) are left serial.

If $R$ is either hereditary or left serial, $A_1/E_2 \cong A_2/F_2$ implies $A_1 \cong A_2$ in Lemma 16. In general this is not true for US-4 rings.

We shall give rings of the type a) in Lemma 16. Let $R = \sum \oplus e_i R$ and $e_i e_j = \delta_{ij} e_i$ (the $e_i$ are primitive idempotents).

1) $A_1/E_2 \cong A_2/F_2$ and $E_2 \cong F_2$

$$e_1 R = e_1 \mathcal{Z} + e_1 J$$

$$A_1 = (1, 2) \mathcal{Z} + (1, 2)(2, 3) \mathcal{Z}$$
$$E_2 = (1, 2)(2, 3) \mathcal{Z}$$

$$e_2 R = e_2 \mathcal{Z} + e_2 J$$

$$F_2 = (1, 2)'(2, 3)' \mathcal{Z}$$

and $(1, 2) (2, 3)' = (1, 2)'(2, 3) = 0$. This is a type of a-1-1) and a-1-4). ($R$ is a finite ring.)

2) $A_1/E_2 \cong A_2/F_2$, $E_2 \cong F_2$
\[
e_1R = e_1\tilde{Z} + e_1J
\]
\[
A_1 = (1, 2)\tilde{Z} + (1, 2)(2, 3)K \quad A_2 = (1, 2)'\tilde{Z} + (1, 2)'(2, 4)K
\]
\[
E_2 = (1, 2)(2, 3)K \quad F_2 = (1, 2)'(2, 4)K
\]
\[
e_2R = e_2K + e_2J \quad e_3R = e_3K \quad e_4R = e_4K
\]
\[
(2, 3)K \quad (2, 4)K
\]
\[
0 \quad 0
\]

and \((1, 2)(2, 4) = (1, 2)'(2, 3) = 0\), where \(K\) is a finite field of characteristic 2. This is a type of a-1-1).

3)
\[
e_1R = e_1\tilde{Z} + e_1J
\]
\[
A_1 = (1, 2)\tilde{Z} + E_2 \quad A_2 = (1, 1)\tilde{Z} + F_2
\]
\[
E_2 = (1, 2)(2, 3)K \quad F_2 = (1, 1)(1, 2)\tilde{Z} + F_3
\]
\[
0 \quad F_3 = (1, 1)(1, 2)(2, 3)K
\]
\[
e_2R = e_2\tilde{Z} + e_2J \quad e_3R = e_3K
\]
\[
(2, 3)K \quad 0
\]
\[
0
\]

This is a type of a-1-2). If \(K = \tilde{Z}, R\) is a left serial and finite ring.

4)
\[
e_1R = e_1\tilde{Z} + e_1J
\]
\[
A_1 = (1, 2)\tilde{Z} + E_2 \quad A_2 = (1, 3)\tilde{Z} + F_2
\]
\[
E_2 = (1, 2)(2, 4)\tilde{Z} \quad F_2 = (1, 3)(3, 5)\tilde{Z} + F_3
\]
\[
0 \quad F_3 = (1, 3)(3, 5)(5, 4)\tilde{Z} + F_4
\]
\[
F_3 = (1, 3)(3, 5)(5, 4)(4, 6)K
\]
\[
F_4 = (1, 3)(3, 5)(5, 4)(4, 6)K
\]
\[
0
\]
\begin{align*}
e_2R &= e_2\bar{Z} + e_2J \\
(2, 4)\bar{Z} &= 0
\end{align*}
\begin{align*}
e_3R &= e_3\bar{Z} + e_3J \\
(3, 5)\bar{Z} &= 0
\end{align*}
\begin{align*}
e_4R &= e_4\bar{Z} + e_4J \\
(5, 4)\bar{Z} &= 0
\end{align*}
\begin{align*}
e_5R &= e_5\bar{Z} + e_5J \\
(6, 4)K &= 0
\end{align*}

This is a type of \textit{a-1-3}.

Other products among \((i, j)\) are zero (e.g. \((1, 1)(1, 1)=0\)). In the above \(e_i(k, l)e_j=(k, l)\delta_{ik}\delta_{ij}\), \((\delta_{ij}\) is Kronecker delta).

Similarly we can construct a US-4 ring of \textit{a-2-1}) in Lemma 16. Finally we shall give an example concerning ii) of Lemma 12.

Let \(K\) be a field of characteristic 2 and \(L\) an extension of \(K\); \(L=K(\alpha)\) and \(\alpha^2\in K\). Put \(g(\alpha)=b\neq 0\) in \(L\) and \(g(1)=0\). Then \(g\) is a derivation of \(L\) over \(K\). Put

\[ R = \begin{pmatrix} L \mid L \\ 0 \mid L \end{pmatrix}, \]

where \(l \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} l \ g(l) \\ 0 \ l \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \) \((l_1, l_2\in L)\)
as in Remark 13. Then \(e_{11}J=A_1\oplus A_2\) and \(\Delta(A_1)=\Delta, [\Delta: \Delta(A_2)]=2\). However, \(e_{11}R\) does not satisfy \((\# , 1)\) as an \(L-L\)-module. Hence \(e_{11}R\) has the similar form to ii) of Lemma 12, but \(R\) is not right US-4.

References


Department of Mathematics
Osaka City University
Sugimoto-3, Sumiyoshi-ku
Osaka 558, Japan