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<th>Generalizations of Nakayama ring. VI. (Right ( \text{US-n} ) rings; ( n=3,4 ))</th>
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We have studied artinian right US-3 rings in [5] and right US-4 algebras over an algebraically closed field in [7]. We shall continue, in this paper, to study a right US-3 (resp. US-4) ring $R$ when $R$ is either hereditary or left serial.

In the first two sections, we shall give the characterization of a right US-3 (resp. US-4) ring $R$, when $R$ satisfies a weaker condition ($*, 1'$) (see § 1) than $R$ being either hereditary or left serial. In the next two sections, we shall specify the characterizations given in the previous sections to hereditary rings and left serial rings. We shall exhibit several examples in the final section to illustrate the above characterizations.

1. US-3 rings

Throughout this paper we deal with an artinian ring $R$ and every $R$-module is a unitary right $R$-module. We shall use the same terminologies and definitions given in [2]~[8].

As a generalization of right serial rings, we considered

\[ (**, n) \]

Every maximal submodule in a direct sum $D$ of $n$ hollow modules contains a non-zero direct summand of $D$ [5].

It is clear that if $D/J(D)$ is not homogeneous, $D$ satisfies ($**$, $n$). Hence we may restrict ourselves to hollow modules of a form $eR/E$, where $e$ is a primitive idempotent and $E$ is a submodule of $eR$. If ($**$, $n$) holds for any direct sum of $n$ hollow modules, we call $R$ a right $US-n$ ring [5]. Since the concept of US-$n$ rings is Morita equivalent, we study always a basic ring.

We studied right US-$n$ algebras over an algebraically closed field for $n=3$ and 4 in [5] and [7], respectively. In this and next sections we shall give a complete list of the structure of right US-3 (resp. US-4) rings with ($*$, $1'$) below. We can give theoretically the complete structure, however as we know a few properties of division rings, we can not give the complete examples for each structure.
We quote here a particular property of a semisimple module (cf. [8] and [9]).

Let $e$ be a primitive idempotent in $R$ and $D$ a semisimple $R$-module and a left $eR$-module. For any two $R$-submodules $V_1$ and $V_2$ with $|V_1| = |V_2| = m$, there exists a unit $x$ in $eR$ such that $xV_1 = V_2$.

Further we consider one more property:

(*, 1') $ef^i$ is a direct sum of hollow modules for each primitive idempotent $e$ and each $i$.

If $R$ satisfies (*, 1), then (*, 1') holds. Moreover, if $R$ is hereditary or left serial, (*, 1') holds by [11], Corollary 4.2. Under the assumption (*, 1'), we obtain the following diagram (cf. [8]):

\[
\begin{array}{cccc}
A_1 & A_2 & \ldots & A_m \\
\mid & | & & \\
A_{\Pi} & A_{1\Pi} & \ldots & A_{2\Pi} \\
\mid & | & & \\
& & ef^i \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]

where the $A$ are hollow.

Let $A_1$, $A_2$ be submodules in $eR$. If there exists a unit $x$ in $eR$ such that $xA_1 \subset A_2$ or $xA_1 \supset A_2$, we indicate this situation by $A_1 \sim A_2$ [4]. We put $\Delta = eRe/ef^i$ and $\Delta(A_1) = \{x \in \Delta, xA_1 \subset A_2\}$ [2].

Let $D = A_1 \oplus A_2$; the $A_i$ are uniserial. A submodule $B = B_1 \oplus B_2 (A_i \supset B_i)$ is called a standard submodule in $D$ [3].

**Lemma 1.** Let $A_1$ and $A_2$ be as in (1). If $A_1 \sim A_2$, $A_1 = xA_2$ for some unit element $x$ in $eR$, and hence $A_1 \approx A_2$.

**Proof.** Since $A_1 \sim A_2$, there exists a unit $x$ in $eR$ such that $xA_1 \subset A_2$ or $xA_1 \supset A_2$. We may assume that $xA_1 \subset A_2$. If $xA_1 \neq A_2$, $xA_1 \subset J(A_2) \subset ef^{i+1}$, since $A_2$ is hollow. Hence $A_1 \subset x^{-1}ef^{i+1} = ef^{i+1}$, a contradiction. Therefore $xA_1 = A_2$.

**Lemma 2.** Let $A_1$ and $A_2$ be as in (1). Let $B$ be a hollow submodule in $A_2$, which appears on the level $ef^{k-i}$ ($k > 0$) in (1). If $\Delta(A_1) = \Delta, A_1 \sim B$.

**Proof.** First assume $k \geq 1$ and $A_1 \sim B$, i.e., there exists a unit $x$ in $eR$ such that $xA_1 \supset B$ or $xA_1 \subset B$. In the latter case $A_1 \subset ef^{i+1}$. Hence $xA_1 \supset B$. Since $\Delta(A_1) = \Delta$, there exists an element $j$ in $ef^i$ with $(x+j)A_1 = A_1$. Let $b$ be a generator of $B$. Then we obtain $a$ in $A_1$ with $xa = b$. $b = (x+j)a = (x+j)a - ja$. Let $p_1$ be the projection of $ef^i$ to $A_1$, $0 = p_1(b) = (x+j)a - p_1(ja)$. Assume $a \in ef^i - ef^{i+1}$, and $p(ja) \in ef^{i+1}$, which is a contradiction, since $x+j$ is a unit in $eR$. Finally assume $B = A_2$. Then $A_2 = x'A_1$ for some unit $x'$ in $eR$. Hence we obtain the same situation as above, which is a contradiction.
From [2], Theorem 2 we have

**Lemma 3.** If \( R \) is a right US-\( n \) ring, then \( [\Delta; \Delta(A)] \leq n-1 \) for any submodule \( A \) in \( eR \).

Put \( \bar{R} = R/J^{t+k} \). Then \( \bar{R} \cong \bar{e}R/\bar{e}J \bar{e} = \Delta \). Let \( A_1 \) be as in (1). Then we can define \( \Delta(\bar{A}_1) = \Delta((A_1 + J^{t+k})/J^{t+k}) = \{ \{x \mid x \in \Delta, x(A_1 + J^{t+k}) \subseteq (A_1 + J^{t+k}) \} \}. \) It is clear that \( \Delta(A_1) \) is a division subring of \( \Delta(\bar{A}_1) \).

**Lemma 4.** Let \( A_1 \) and \( A_2 \) be as in (1). If \( \Delta(A_1) = \Delta(\bar{A}_1) = \Delta \). Next assume that \( A_2 = xA_1 \) for some unit \( x \) in \( eRe \). If \( [\Delta; \Delta(A)] = 2 \) (resp. 3), \( [\Delta; \Delta(\bar{A}_1)] = 2 \) (resp. 3), there holds \( \bar{A}_1 = (A_1 + J^{t+k})/J^{t+k} \subset \bar{R} = R/J^{t+k} \).

**Proof.** The first part is clear from the remark above. Assume \( (x+j)A_1 \subseteq \bar{A}_1 \) for some \( j \) in \( eJe \). Since \( (x+j)A_1 \subseteq (A_1 + J^{t+k}) \cap (A_2 + J^{t+k}) = J^{t+k+1} \) a contradiction. Hence \( x \notin \Delta(A_1) \). Further \( [\Delta; \Delta(A_1)] \) is prime, and so \( [\Delta; \Delta(A_1)] = [\Delta; \Delta(\bar{A}_1)] \).

**Remark 5.** We shall study a right US-\( n \) ring and observe \( [\Delta; \Delta(A)] \). Since \( [\Delta; \Delta(A)] \leq 3 \) we may assume \( f^{t+k} = 0 \) by Lemma 4, [3], Lemma 1 and its proof, when we observe \( [\Delta; \Delta(A)] \) (the \( x \) in Lemma 4 exists, provided \( [\Delta; \Delta(A)] \geq 2 \)).

**Theorem 1.** \( R \) is a right (basic) US-3 ring with \( (*, 1') \) if and only if \( eRe \) has one of the following structures for each primitive idempotent \( e \).

1) \( eRe/J^t \) is uniserial for some \( t \) and
2) \( eJ^t = 0 \) or \( eJ^t = A \oplus B \), where \( A \) is simple and \( B \) is uniserial, such that
   a) \( [\Delta; \Delta(A)] = 2 \) or b) \( \Delta(\bar{A}) = \Delta(\bar{B}) \).
   In case a) \( B \) is simple and \( A \oplus B \) satisfies \( (#, 1) \).
   In case b)
   i) \( B \) is simple and \( A \cong B \) or
   ii) \( B \) is not simple, and if \( A \) is isomorphic to a simple subfactor module \( B_i/B_{i+1} \) of \( B_i \) \( B_{i+1} = 0 \) (i.e., \( B_i \) is the socle of \( B \)) and this isomorphism is given by \( j_i \) the left-sided multiplication of \( j \) in \( eJe \).

**Proof.** We assume that \( R \) is a right US-3 ring. From \((*, 1') \) and [5], Proposition 1,3) \( eJ = A \oplus B \), where \( A \) and \( B \) are hollow. We may assume \( |A| \leq |B| \). \( [\Delta; \Delta(C)] \leq 2 \) for any submodule \( C \) in \( eR \) by Lemma 3. Hence we divide ourselves into two cases: I) \( [\Delta; \Delta(A)] = 2 \) and II) \( \Delta = \Delta(A) \). Case I). Since \( [\Delta; \Delta(A)] = 2 \), by [5], Proposition 1,2) there exists a unit element \( x \) in \( eRe \) such that \( xA \subseteq J(A) + J(B) \) or \( xA \supset J(A) \oplus J(B) \). However \( A \subseteq eJ^{t+1} \) and so \( xA \supset J(A) \oplus J(B) \). On the other hand, \( |A| = |J(A) + 1| \) and \( xA \neq J(A) \oplus J(B) \). Hence \( J(B) = 0 \). Further \( A \cong B \) by Lemma 1 and [5], Proposition 1.2). Therefore \( A \) and \( B \) are simple and \( eJ^{t+1} = 0 \). Which means that every
(simple) submodule $C$ in $ef^i$ is characteristic if and only if $\Delta(C) = \Delta$. Hence $[\Delta : \Delta(C)] = 2$ and $ef^i$ satisfies $(\#, 1)$ by [5], Proposition 1,2).

Case II). We know from the above argument that $\Delta = \Delta(A) = \Delta(B)$ (note that we did not use the assumption $|A| \leq |B|$). Let $y$ be any unit element in $eRe$. Since $\Delta = \Delta(A)$, there exists an element $j$ in $ef^i$ such that $(y + j)A = A$. Then

$$(y + j)(A \oplus J(B)) \subset A \oplus (y + j)J(B) \subset A \oplus ef^{i+1} = A \oplus J(B).$$

Hence $\Delta(A \oplus J(B)) = \Delta$. Assume that $B$ is not simple. $A \oplus J(B)$ or $J(A) \oplus B$ is hollow by [5], Proposition 1,4)-iv). Hence

$$J(A) = 0,$$

i.e., $A$ is simple.

We shall show that $B$ is uniserial. Assume $ef^{i+k} = BJ^k = C_1 \oplus C_2 \oplus \cdots$; the $C_i$ are hollow. If $\Delta(C_i) \neq \Delta$, $C_1 \sim A_i$ by [5], Proposition 1,2), which is a contradiction from Lemma 2. Hence $\Delta = \Delta(C_1) = \Delta(C_2)$. However $\{A, C_1, C_2\}$ derives a contradiction by Lemma 2 and [4], Corollary 2 of Theorem 2, provided $C_2 \neq 0$. Therefore $B$ is uniserial.

Next assume $g: A \approx B_i/B_{i+1}$; $B \supset B_i \supset B_{i+1}$. Take $\{A, B_i, B_j(g^{-1})\}$; the graph of $B_i$ with respect to $g^{-1}$. Since $A$ is simple (and hence $ef^iB \subset B$) and $\Delta(B) = \Delta$, $B$ is characteristic. Hence $A \sim B_i(g^{-1})$, and so there exists a unit $x_i$ in $eRe$ such that $x_iA \subset B_i(g^{-1})$. If $B_{i+1} \neq 0$, $x_iA \subset B_{i+1} \subset ef^{i+1}$, a contradiction. Hence $B_{i+1} = 0$ and $g: A \approx B_\ast$, the socle of $B$. Let $j$ be an element in $ef^i$ such that $(x_i + j)A = A$, and put $x_i = x_i + j$. Then $A(g) = x_iA = (x_i - j)A$. Put $A = ar$. Then $a + g(a) = (x_i - j)ar$ for some $r$ in $R$. $ef^iA \subset ef^{i+1}$ and $ef^{i+1} = BJ$ imply $ef^iA \subset B_\ast$. Hence

$$a = x_iar \quad \text{and} \quad g(a) = -jar,$$

and so $g(a) = - jx_i^{-1}a$. Therefore $g = (- jx_i^{-1})$, and $- jx_i^{-1} \in ef^i$ (b-ii)). Finally assume that $B$ is simple. If $f: A \approx B_i$, $\{A, B, A(f)\}$ derives a contradiction from [5], Lemma 1, (note $ef^{i+1} = 0$ and use Lemma 8 below). Hence $A \not\approx B$ (b-i)). Conversely, assume that $eR$ has one of the structures given in the theorem. Clearly $(\ast, 1')$ holds. Let $\{E_i\}_{i=1}^3$ be any set of submodules in $eR$. Case a): If $E_1 \supset ef^i$ and $E_2 \supset ef^i$, $\Delta(E_i) = \Delta$ for $i = 1, 2$ and $E_1 \supset E_2$ or $E_1 \subset E_2$. Hence $D = \bigoplus_{i=1}^3 E_i$ contains a non-zero direct summand of $D$ by [4], Corollary 1 of Theorem 2. If $E_i \subseteq ef^i$ and $E_2 \subseteq ef^i$, $E_3 = xE_i (\approx A)$ for some $x$ in $eRe$ by $(\#, 1)$. Hence $D$ satisfies $(**3)$ again by [4], Corollary 1 of Theorem 2. Case b-ii): If $E_i \subseteq ef^i$, $x_iE_i$ is a standard submodule in $ef^i$ for a unit $x_i = (e + j)$ in $eRe$ by assumption. Hence $E_i \sim E_j$ for some pair $i, j$. Further $\Delta = \Delta(E)$ by assumption. Therefore $D$ satisfies $(**3)$ by [4], Corollary 1 of Theorem 2. Case b-i): This is much simpler than the above. Thus $R$ is right US-3.
In the last paragraph of the proof of "only if part" in Theorem 1, we have shown

**Lemma 6. Assume that \( eJ' = A \oplus A' \oplus B \) and 1) \( A \) and \( A' \) are simple modules with \( \Delta(A) = \Delta \), and 2) \( B \) is non-simple and uniserial. If \( g: A \sim B_{i+1} \) and \( A \sim B_{i}(g^{-1}) \), \( B_{i+1} = 0 \) and \( g \) is given by \( j; j \in eJ \), and hence \( i > 1 \) (cf. [7], Lemma 16).

We shall illustrate the structure in Theorem 1 as the following diagram:

1) \[
\begin{array}{ccc}
  eR & eJ & eJ^b \\
  \cdot & \cdot & \cdot \\
  & A & 0 \\
\end{array}
\]

2) \[
\begin{array}{ccc}
  eR & eJ & eJ' \\
  \cdot & \cdot & \cdot \\
  A & 0 \\
  B & B_p & 0 \\
\end{array}
\]

where the straight line means uniserial.

It is clear that if \( R \) has the structure above, \((*, 1)\) (and hence \((*, 1')\)) holds. We note that if \((*, 1')\) does not hold, Theorem 1 is not true (see [6]). We shall give examples of a) and b) in § 5.

2. **US-4 rings**

Next we shall characterize a right US-4 ring with \((*, 1')\).

**Lemma 7.** Let \( R \) be a right US-4 ring and \( \{A_i\}_{i=1}^{4} \) a set of submodules in \( eJ \). Then 1) if \( \Delta(A_i) = \Delta \) or all \( i \leq 3 \) and \( A_k \sim A_k \) for \( k \neq k' \leq 3 \), then \( A_4 \sim (some A_i) \). 2) \( A_i \sim A_j \) for some pair \( i, j \). 3) If \( \Delta(\Delta(A_i)) = 2 \) for \( i = 1, 2, A_1 \sim A_2 \). 4) If \( \Delta(\Delta(A_i)) = 3 \), \( A_i \sim A_j \) for all \( j \). 5) If \( \Delta(\Delta(A_i)) = 2 \), \( A_1 \sim A_j \) for some \( i, j \leq 3 \).

Proof. This is clear from [4], Corollary 2 of Theorem 2.

**Lemma 8.** Let \( A_1 \) and \( A_2 \) be as in (1). Assume \( J^{i+1} = 0 \). If \( \Delta(A_i) = \Delta \), \( A_1 \) is characteristic.

Proof. This is clear.

**Lemma 9.** Let \( R \) be a right US-4 (basic) ring, and \( \{A_i\}_{i=1}^{t} \) a set of hollow submodules on the level \( eJ' \) in (1). If \( \Delta(A_i) = \Delta \) for all \( i, t \leq 3 \).

Proof. This is clear from Lemmas 7, 8 and Remark 5.

From now on we assume that \( R \) is a right US-4 (basic) ring satisfying \((*, 1')\). Let \( D = (eJ' =) A_1 \oplus A_2 \oplus \cdots A_t \), where the \( A_i \) are hollow. In the
following lemmas, we mainly assume that \( D \) is characteristic. We note \([\Delta: \Delta(A_i)]\leq 3\) for all \( i \) by Lemma 3.

**Lemma 10.** Assume \([\Delta: \Delta(A_i)]\leq 2\) for all \( i \). Then i) \( t=2 \). ii) There exists a unit \( x \) in \( eRe \) such that \( xA_1=A_2 \). iii) \( A_1 \) is a uniserial module with \( |A_1|\leq 2 \). iv) If there are characteristic submodules in \( A_1 \oplus A_2 \), they are linear with respect to the inclusion. v) If \( B \) is not a characteristic submodule in \( A_1 \oplus A_2 \), \([\Delta: \Delta(B)]\leq 2\) and those submodules are related by \( \sim \).

Proof. We may assume \(|A_1|\leq |A_2|\leq \cdots \leq |A_t|\) (note \( t\geq 2 \)). By Lemmas 1 and 7, \( A_k=x_kA_1 \) for all \( k \).

On the other hand, since \([\Delta: \Delta(A_1)]\leq 2\), \( \Delta=\Delta(A_1)+x_2\Delta(A_1) \). Assume \( e^{|t+1}|=0 \) from Remark 5. Since \( D=\Delta(A_1)+x_2\Delta(A_1)A_1=A_1 \oplus A_2 \), \( t=2 \). We note that from the above argument and Lemma 3 we obtain

(\( \alpha \)) if \([\Delta: \Delta(A_i)]\geq 2\) for all \( i \), there exists a unit \( x_i \) in \( eRe \) such that \( x_iA_1=A_i \) for all \( i \).

Assume that \( A_1/A_0J^s \) is uniserial and \( A_0J^s=B_1 \oplus B_2 \oplus \cdots \oplus B_\ell \), where the \( B_j \) are hollow and \( s\geq 2 \). In order to show \( s\leq 1 \), we may assume \( e^{|t+i+1}|=0 \) by Remark 5. First we note that there exists a unit \( x \) in \( eRe \) such that \( xA_1=A_2 \). Hence \( \Delta(B)=\Delta \) for all \( p \). On the other hand, \( D^s=A_1J^s \oplus A_2J^s=\sum_{p=1}^s B_p \oplus \sum_{p=1}^s xB_p \), which is a contradiction to (\( \beta \)). Therefore \( A_1 \) and \( A_2 \) are uniserial. Next assume \( A_1J^s\neq 0 \). \( \Delta(A_1 \oplus A_2J^s)\neq \Delta \) by existence of \( x_2 \). Hence \( \{A_1, A_1J \oplus A_2J^s\} \) derives a contradiction by Lemma 7. Therefore \( |A_1|\leq 2 \). Since \( \Delta(A_1 \oplus (A_2J)) \subset \Delta(A_1J) \), \( \Delta(A_1 \oplus (A_2J))=\Delta(A_1J) \) for \( \Delta(A_1 \oplus (A_2J)) \neq \Delta \). Similarly \([\Delta: \Delta(J(A_2J))]=2 \). Let \( E \) be a submodule with \([\Delta: \Delta(E)]=3 \). Then there exists a unit element \( x \) in \( eRe \) such that \( xE \subset A_1 \) or \( xE \supset A_1 \) by Lemma 7. In the former case \([\Delta: \Delta(E)]=\Delta(\Delta(eE))=2 \). If \( xE \subset A_1 \), \( xE=A_1 \oplus E' \); \( E' \subset A_2 \). Hence \([\Delta: \Delta(xE)]=2 \) from the above. Therefore there are no submodules \( E \) with \([\Delta: \Delta(E)]=3 \). Finally assume that \( A_1 \oplus A_2 \) contains two characteristic submodules \( C_1, C_2 \) such that \( C_1 \sim C_2 \). Consider \( \{A_1, A_2, C_1, C_2\} \) and \( A_1 \sim C_1 \) or \( A_1 \sim C_2 \) by Lemma 7. If \( A_1 \supset C_1, C_1=0 \) and if \( A_1 \subset C_1, C_1=A_1 \oplus F; F \subset A_2 \), and so \( C_1=A_1 \oplus A_2 \). Hence \( C_1 \supset C_2 \) or \( C_1 \subset C_2 \). Let \( \Delta(E)=\Delta \). If \( |A_1|=1 \), \( E \) is characteristic. Assume \( |A_2|=2 \). Put \( C_1=A_2 \oplus B_2 \). Then \( E \sim C_1 \) from the above. Hence \( E \subset C_1 \) or \( E \supset C_1 \), and so \( E \) is characteristic.

**Lemma 11.** Assume \([\Delta: \Delta(A_i)]\leq 3\) for all \( i \). Then \( t\leq 3 \), and the \( A_i \) are simple and there exists a unit \( x_i \) in \( eRe \) such that \( x_iA_i=A_i \) for each \( i \). If \( t=3 \), \( D \) satisfies (\# 1) and (\# 2) and \([\Delta: \Delta(C)]\leq 3 \) for every submodule \( C \) in \( D \). If \( t=2 \), \( D \) satisfies (\# 1).

Proof. Since \([\Delta: \Delta(A_i)]\leq 3 \), there exists a unit \( x_i \) in \( eRe \) such that \( x_iA_i=A_i \).
from \((\alpha)\) and \(t \leq 3\) by \((\beta)\). Assume \(t=3\). Taking \(\{A_i, J(D)\}\), we know from Lemma 7 that \(A_i\) is simple and hence \(eJ^{i+1}=0\). It is clear from Lemmas 7 and 8 that there are no simple submodules \(B\) in \(D\) with \(\Delta(B)=\Delta\). Hence \(D\) satisfies \((\#. 1)\). Let \(C\) be a submodule of \(D\) with \(|C|=2\). Then \(D=C \oplus A_i\) for some \(i\). Hence \(\Delta(C)=\Delta\) by Lemma 7, and so \(D\) satisfies \((\#. 2)\). We obtain the similar result for \(t=2\).

Lemma 12. Assume \([\Delta: \Delta(A_i)]=1\) and \(\Delta(A_i) \neq \Delta\) for \(i \geq 2\). Then \(A_i\) is uniserial and \(t \leq 3\).

i) \(t=3\):

Then all \(A_i\) are simple, \([\Delta: \Delta(A_i)]=2\) for \(i=2, 3\), \(A_1 \not\cong A_2\) and \(A_2 \oplus A_3\) satisfies \((\#, 1)\).

ii) \(t=2\):

a) \(A_4\) is not simple.

Then \([\Delta: \Delta(A_2)]=2\), and \(A_2\) is a simple submodule isomorphic to \(B\), the socle of \(A_1\). If \(A_2 \cong E_i/E_{i+1}\), then \(E_i=B\) and \(E_{i+1}=0\). Further \(B \oplus A_2\) satisfies \((\#, 1)\) except \(B\).

b) \(A_1\) is simple.

Then \(\Delta(A_2)\) is uniserial for some \(t\) and

1) \(\Delta(A_2)\) is uniserial for some \(t\)

2) \(A_2A_2J^t\) is uniserial for some \(t\)

2-i) \(A_2J^t=0\) or \(A_2J^t=B_1 \oplus B_2\); \(B_1\) is simple and \(B_2\) is uniserial.

2-ii) \(A_2J^t=B_1 \oplus B_2\); \(B_1\) is simple and \(B_2\) is uniserial.

2-ii-1) \(\Delta(B_1)=\Delta(B_2)=\Delta\).

2-ii-1-1) \(B_1 \cong B_2, J^t(B_1)\).

2-ii-1-2) \(B_1 \cong F_i^tB_{i+1}\), \(A_2 \cong F_i^tB_{i+1} \oplus B_1 \oplus B_2\).

2-ii-1-3) If \(f: A_1 \cong G_j^t, J^t(G_j^t) \oplus G_{j+1}^t \oplus G_{j+1}^t \oplus B_1 \oplus B_2\), then \(G_j^tB_1 \cong G_{j+1}^t\), \(J^t(B_1) \oplus B_2\).

and \(f(J^t)\) is given by \(j; j \in eJ^t\).

2-ii-1-4) If \(f^t: A_1 \cong B_1, \) we have the same result as 2-ii-1-3).

2-ii-2) \([\Delta: \Delta(B_1)]=[\Delta: \Delta(B_2)]=2\).

2-ii-2-1) \(B_1\) and \(B_2\) are simple and \(B_1 \oplus B_2\) satisfies \((\#, 1)\).

2-ii-2-2) \(A_2 \cong F_i^tB_{i+1}\), \(A_2 \cong F_i^tB_{i+1} \oplus B_1 \oplus B_2\).

2-ii-2-3) If \(A_1 \cong B_1, \) then \(f\) is given by \(j; j \in eJ^t\).

Proof. It is clear, from the assumption and Lemmas 1 and 7, that \([\Delta: \Delta(A_i)]=2\) for all \(i \geq 2\). Assume that \(A_i\) contains two independent submodules \(B_1, B_2\). If \(\Delta(B_1)=\Delta(B_2)=\Delta, \{B_1, B_2, A_2, A_3\}\) derives a contradiction by Lemmas 7, 8 and Remark 5. On the other hand, if \(\Delta(B_1) \neq \Delta, \{B_1, B_2, A_2, A_3\}\) derives again a contradiction. Hence

\[A_1\] is uniserial.

by \((*, 1')\).

a) \(J(A_i) \neq 0\): Consider \(\{A_1, A_2, J(D)\}\). Then \(A_1 \not\cong A_2\) by Lemma 2. Hence \(J(D) \cong A_1\) or \(J(D) \cong A_2\) by Lemma 7. However \(J(D) \not\cong A_2\), since \(J(D)\) is
characteristic and \( J(A_i) \neq 0 \). Hence

the \( A_i \) are simple for all \( i \geq 2 \).

Since \( [\Delta: \Delta(A_i)]=2 \), there exists \( x_i \) in \( eRe \) such that \( x_i A_2 = A_i \) for \( i \geq 2 \) by Lemmas 1 and 7. Hence in order to show \( t \leq 3 \), we may assume \( J^{i+1} = 0 \) by Remark 5. Noting \( A_3 \in \Delta(A_2), \Delta = \Delta(A_2) \oplus \mathcal{X}_2 \Delta(A_2) \), which implies that \( A_2 \oplus A_3 = \Delta A_2 \supseteq \sum_{i=1}^{\infty} A_i \). Hence \( t \leq 3 \). Now we resume to the original situation. We note \( eRe \subset J(A_i) \), and hence \( A_1 \) is characteristic. Since \( \Delta(J(A_i)) = \Delta, \Delta(J(A_i) \oplus A_2) \neq \Delta \). Consider \( \{A_1, J(A_i) \oplus A_2, A_3 \oplus A_4\} \). \( \Delta(A_i) = \Delta \) and \( \Delta(J(A_i) \oplus A_2) \neq \Delta \) imply \( (J(A_i) \oplus A_2) \supset (A_2 \oplus A_3) \). However, \( \Delta(J(A_i)) = \Delta \) implies \( xJ(A_i) \subset A_2 \oplus A_3 \). Hence \( xJ(A_i) \oplus A_2 \supset A_2 \oplus A_3 \). Taking \( R = R J^{i+1} \), we know that it is impossible. Therefore \( t = 2 \) provided \( J(A_1) = 0 \), i.e.,

\[ D = A_1 \oplus A_2 \ (J(A_i) = 0) \]

Now we take the similar manner to Lemma 6. Assume \( f: A_2 \cong E_i \oplus E_{i+1} ; A_1 \supseteq E_i \supseteq E_{i+1} \). We note that \( A_1 \) is characteristic. \( \{A_1, A_2, E_i(f^{-1})\} \) implies \( A_2 \sim E_i(f^{-1}) \) from the above remark and Lemma 7. Hence \( E_{i+1} = 0 \) as the proof of Lemma 6. Further since \( \Delta(A_2) \neq \Delta, A_2 \cong E_2 \); the socle of \( A_1 \). Let \( C(\neq E_2) \) be a simple submodule in \( E_2 \oplus A_2 \). Consider \( \{A_1, C, A_2, A_3\} \). It is clear that if \( C \sim A_1, C \subset A_1 \). Hence \( C \sim A_2 \) by Lemmas 2 and 7, and so \( E_2 \oplus A_2 \) satisfies (\# 1) except \( E_2 \).

b) \( J(A_i) = 0, t \geq 3 \). Assume \( J(A_2) = 0 \). Since \( t \geq 3 \), there exists a unit \( x \) in \( eRe \) with \( xA_2 = A_3 \) by Lemmas 1 and 7, and so \( \Delta(A_1 \oplus J(A_2)) \neq \Delta \). Then \( A_2 \sim A_1 \oplus J(A_2) \) by Lemma 7. Assume \( A_2 \supset y(A_1 \oplus J(A_2)) \) for some unit \( y \). Since \( A_1 \) is simple and \( \Delta(A_1) = \Delta, p_1(yA_1) = A_1 \), where \( p_1: E^\perp \rightarrow A_1 \) the projection, which is a contradiction. Similarly, since \( A_1 \) is simple and \( A_2 \) is not, \( p_2(yA_1) \subset J(A_2) \) for any unit \( y \) in \( eRe \). Hence \( A_2 \supset y \ (A_1 \oplus J(A_2)) \). Therefore

\[ A_2 \ (\text{and so } A_i \ (i \geq 2)) \) is simple.

Accordingly \( t = 3 \) from the initial paragraph of a). If \( f: A_i \cong A_2, \{A_i, A_i(f), A_2, A_3\} \) derives a contradiction, since \( \Delta(A_2) = A_2 \oplus A_3 \) as before (note \( eJ^{i+1} = 0 \)). Hence \( A_1 \sim A_2 \). Further if \( A_2 \oplus A_3 \) contains a characteristic submodule \( B \neq 0, \{A_1, B, A_2, A_3\} \) derives a contradiction. Therefore \( A_2 \oplus A_3 \) satisfies (\# 1).

Case \( t = 2 \) and \( J(A_i) = 0 (D = A_i \oplus A_2) \). First we shall show that \( A_1 \oplus A_2 \) satisfies (\# 1) except \( A_i \). Since \( \Delta(A_2) \neq \Delta \), there exists a unit \( x \) in \( eRe \) such that \( p_1(xA_2) = A_1 \), where \( p_1: E^\perp \rightarrow A_1 \) is the projection. Further \( eRe \subset A_2 \), since \( A_1 \) is simple. Hence \( (x+j)(A_2 + J^{i+1}) = A_2 + J^{i+1} \) for any \( j \) in \( eRe \), and so \( \Delta(A_2) = \Delta((A_2 + J^{i+1})/J^{i+1}) \). Therefore we may assume \( J^{i+1} = 0 \) (cf. Remark 5). Then
\( A_1 \oplus A_2 \) satisfies \((\#), 1\) except \( A_1 \) from Lemma 7. Now we resume the original situation. Since \( A_1 \) is simple, \( e^{f+j+1} = A_2 J \). Assume that \( A_2/J = A_2 \) is uniserial and \( e^{f+j+1} = B_1 \oplus B_2 \oplus \cdots \oplus B_n \), where the \( B_i \) are hollow. Then from Lemmas 10 \( \sim \) 16 below, \( s \leq 3 \). Further \([\Delta: \Delta(B_j)] = 2\) by Lemmas 2 and 7. Assume \( s = 3 \). Then \( \Delta(B_j) = \Delta \) (resp. \( [\Delta: \Delta(B_j)] = 2 \)) for some \( i \) (resp. \( j \)) by Lemmas 7 and 10. Hence we remain two cases \( \Delta(B_i) = \Delta \), \([\Delta: \Delta(B_j)] = 2\) for \( i = 1, 2 \), \([\Delta: \Delta(B_j)] = 2\) for \( j = 2 \). On the other hand, since \( \Delta(A_i) = \Delta \), we do not have such cases by Lemmas 2 and 7. Therefore \( s = 2 \). Similarly we do not have a case \( \Delta(B_i) = \Delta \) and \([\Delta: \Delta(B_j)] = 2 \) for \( i = 1, 2 \). Thus we obtain two cases; \( 2-ii-1\): \( \Delta(B_i) = \Delta \) for \( i = 1, 2 \) and \( 2-ii-2\): \([\Delta: \Delta(B_j)] = 2\) for \( j = 1, 2 \).

\( 2-ii-1 \) We assume \( |B_1| = |B_2| \). \( \{A_1, B_1, B_2, J(B_1) \oplus J(B_2)\} \) gives \( J(B_1) = 0 \) from Lemmas 2 and 7. Assume \( B_2J = C_1 \oplus C_2 \oplus \cdots \oplus C_i \); \( s \geq 2 \) and the \( C_i \) are hollow. If \([\Delta: \Delta(C_i)] \geq 2 \), \( \{A_1, B_1, C_1, C_2\} \) derives a contradiction from Lemmas 2 and 7. Hence \( \Delta(C_1) = \Delta(C_2) = \Delta \). Taking \( R/J^{i+j+1} \), we obtain again a contradiction from \( \{A_1, B_1, C_1, C_2\} \) and Lemmas 2, 7 and 8. Accordingly \( A_1 \) is uniserial. \( \{A_1, B_1, C_1, C_2\} \) gives \( A_1 \sim G_i(f_i) \), since \( \Delta(A_i) = \Delta \). Then we can show similarly to Lemma 6 that \( A_1 \sim A_1(f_i) \) and \( C_i \) are simple. Let \( C \) be any simple submodule in \( B_1 \oplus B_2 \). Then \( \{A_1, B_1, C\} \) shows \( C = xB_1 \) for some unit \( x \) in \( eRe \) by Lemmas 2 and 7. Hence \( B_1 \oplus B_2 \) satisfies \( \(\#\), 1\) \( (2-ii-2-1) \). In the same manner given in the proof of \( 2-ii-1-3 \), we have \( 2-ii-1-4 \). In the same manner given in the proof of \( 2-ii-1-3 \), we have \( 2-ii-1-4 \). Similarly \( f_i : A_1 \approx G_i(f_i) \), \( \{A_1, B_1, C_1, C_2\} \) gives \( A_1 \sim A_1(f_i) \). Hence \( f_i \) is given by \( j_{i'} ; j' \in eRe \) (2-ii-2-3)).

Remark 13. We shall consider the situation of \( ii-b\) of Lemma 12. Taking \( R = R/J^{i+j+1} \), we may assume that \( e^{f+i}(=V) = A_1 \oplus A_2 \): the \( A_i \) are simple, \( \Delta(A_i) = \Delta \), and \( \Delta(A_2) = 2 \). Then \( A_i \approx A_2 \approx g_i \approx A_2 \). We shall express \( \text{End}_{A_i}(V) \) as elements of matrices \( (\Delta')_2 \). Since \( A_i \) is characteristic, for any element \( x \) in \( \Delta \),

\[
x = \begin{pmatrix} x_1 & x_2 \\ 0 & x_2 \end{pmatrix} : x_1 \in \Delta'
\]

\( \Delta \) being a division ring, \( x_2 \) and \( x_3 \) are uniquely determined by \( x_1 \). Hence we
obtain two monomorphisms as rings $f_1, f_2$ of $\Delta$ to $\Delta'$ such that $f_i(x) = x_i$ and a homomorphism $g$ as additive groups of $\Delta$ to $\Delta'$ such that

i) \[ g(x^2) = f_1(x)g(x') + g(x)f_2(x'). \]

Then $\Delta(A_2) = g^{-1}(0)$ (note, from i), that $g^{-1}(0)$ is a division subring of $\Delta$). Hence $[\Delta: \Delta(A_2)] = 2$ is equivalent to

ii) \[ [\Delta : g^{-1}(0)] = 2. \]

Further ($\#$, 1) holds if and only if, for any $\alpha$ in $\Delta$, there exists $x \neq 0$ in $\Delta$ such that

iii) $\alpha = -f_1(x^-1)g(x) (= g(x^-1)f_2(x))$, i.e., $F: \Delta \rightarrow \Delta'$ ($F(x) = f_1(x)^{-1}g(x)$) is surjective.

If $\alpha \neq 0, x \notin g^{-1}(0)$. Hence if either $|\Delta|$, cardinal of $\Delta$, ($|\Delta| \leq |\Delta'|$, i) does not hold. Hence we assume that $|\Delta|$ is infinite. Further, since $f_i$ is a monomorphism, we may assume that $\Delta \subset \Delta'$ and $f_i$ is the inclusion. Now assume that $\Delta'$ is commutative. Then $g$ is a $K$-linear mapping from i), where $K = g^{-1}(0)$. Using those facts and $|\Delta| \geq \infty$, for any $g$ we can show by computation that there exists $\alpha$ in $\Delta'$ not satisfying iii) for any $x \in \Delta$. Therefore if $\Delta'$ is commutative, we do not have the case of ii) of Lemma 12.

REMARK 14. Next we consider the case $t = 2$ in Lemma 11. Let $K$ be a field and $R$ a $K$-algebra. If $[\Delta': K]$ is not divided by 3, this case does not occur. Because, since $V = A_1 \oplus A_2$ and $A_1 \cong A_2$, $\text{End}_R(V) = (\Delta')_2$ and $\Delta \subset (\Delta')_2$.

$[\Delta: \Delta(A_1)] = 3$ implies that $4[\Delta': K]$ is divided by 3.

Finally we take division rings given by [10]. Let $D \supseteq D_1$ be division rings such that $[D: D_1] = 3$ and $[D: D_1] = 2$. Put $D = D_1 + D_1u$, and $D^* = \text{Hom}_D(D_1, D_1)$. Then $[D^*: D_1] = 2$ and $D^*$ is a left $D$-vector space. Define $1^* \in D^*$ by setting $1^*(1) = 1, 1^*(u) = 0$, and put $A_1 = 1^*D_1$. Then $D(A_1) = \{d \in D, dA_1 \subset A_1\} = u^{-1}D_1u$, and so $[D: D(A_1)]_r = 3$. For any $h \in D^*$ and $h^{-1}(0) = D_1u$, we have $D = D_1u + D_1v_1$. Put $d = h(v_1)$. Then $(u_i^{-1}u)1^*(u_i) = 0$ and $(u_i^{-1}u)1^*(v_1) = d' \neq 0$. Hence $h = (u_i^{-1}u)^*d'^{-1}d$, and so $hD_1 = (u_i^{-1}u)A_1$. Therefore $D^*$ satisfies ($\#, 1$), $[D: D(A_1)] = 3$ and $[D^*: D_1] = 2$. We shall use $D^*$ in §5, Example 3'.

Now we resume to study the structure of right US-4 rings.

**Lemma 15.** If $R$ is a US-4 ring with $(\ast, 1')$. $D$ has one of the structures in Lemmas 10, 11, 12 and 16 below.

Proof. Assume 1) $\Delta(A_1) = \Delta(A_2) = \Delta(A_3) = \Delta$. Then $t = 3$ by Lemmas 7 and 8 (the case of Lemma 16 below). 2) $\Delta(A_i) = \Delta(A_2) = \Delta$ and $\Delta(A_i) \neq \Delta$ for $i \geq 3$. Then $\{A_1, A_2, A_3, A_4\}$ derives a contradiction from Lemmas 2 and 7. 3) $\Delta(A_1) = \Delta$ and $\Delta(A_i) \neq \Delta$ for $i \geq 2$. This is a case of Lemma 12. 4) $[\Delta: \Delta(A_i)] = 2$ for $i \leq \text{some } l, [\Delta: \Delta(A_j)] = 3$ for $j > l$. Since $[\Delta: \Delta(A_k)] \geq 2$ for all $k$, from (a) there exists a unit $x_i$ in $eRe$ such that $x_iA_i = A_i$ for all $i$. Hence $\Delta(A_i) = x_i \Delta(A_i)x_i^{-1}$, and so we obtain the cases of Lemmas 10 and 11.
Lemma 16. Assume $\Delta(A_i) = \Delta$ for all $i$. Then $t \leq 3$, and

1) $t = 3$:

$A_3$ is uniserial and $A_1, A_2$ are simple, $A_3 \cong A_2$. If $A_3$ is simple, $A_3 \cong A_1$ and $A_3 \cong A_2$. If $A_3$ is not simple and $f$: $A_3 \cong F_i|F_{i+1}$ ($A_3 \supset F_i \supset F_{i+1}$), then $F_{i+1} = 0$ and $f$ is given by $j_i$; $j \in eJ e$, and hence $i > 1$.

2) $t = 2$:

i) $A_1 \cong A_2$ ($\approx g_1 R|g_1 J$).

Then $A_1$ and $A_2$ are simple and $\Delta = g_1 Rg_1 \cong \bar{Z} \cong \bar{Z}/2$.

ii) $A_1 \cong A_2$ ($|A_1| \leq |A_2|$)

a) $A_1$ is uniserial; $A_2 = F_i \supset F_2 \supset \cdots \supset F_{p} \supset F_{p+1} = 0$.

Then $A_1$ is a uniserial module with $|A_1| = 2$; $A_1 = E_1 \supset E_2 \supset E_3 = 0$.

a-1) $|A_1| = 2$.

a-1-1) If $f$: $A_1|E_2 \approx A_2|F_3$ ($\approx g_4 R|g_4 J$), $\Delta \approx g_4 Rg_5 \cong \bar{Z}$. $f$ is a unique isomorphism. In this case put $B_1 = \{ x + y \in A_1 \oplus A_2, f(x) = y \}$.

a-1-2) If $A_1|E_2 \approx F_i|F_{i+1}$, $i > 1$, then $i \geq p - 1$.

a-1-3) If $f$: $E_2 \approx F_i|F_{i+1}$ ($\approx g_4 R|g_4 J$) ($p > i \geq 2$), $\Delta \approx g_4 Rg_5 \cong \bar{Z}$. We have the same result as a-2-1) below, replacing $A_1$ with $E_2$. In this case put $B_1 = \{ x + y \in E_2 \oplus F_i, f(x) = y \}$.

a-1-4) $f$: $E_2 \approx F_p$. If $p = 2$, $\Delta \approx g_4 Rg_4 \cong \bar{Z}$, where $E_2 \approx F_2 \approx g_4 R|g_4 R$. Further if $f'$: $A_1|E_2 \approx F_2$ ($A_2|F_2 \approx E_2$), $A_1(f) = xA_1$ for some unit $x$ in $eRe$.

If $p > 2$, we have the same result as a-2-2) below, replacing $A_1$ with $E_2$. If $f$ is not given by $j_1$, put $B' = E_2(f)$.

a-1-5) Further every submodule in $eJ^i$ except $B_1$, $B_1$ and $B'$ is isomorphic to a standard submodule in $eJ^i$ via $x_1$; $x$ is a unit in $eRe$.

a-2) $|A_1| = 1$:

a-2-1) If $A_1 \approx F_i|F_{i+1}$ ($\approx g_5 R|g_5 J$) for some $i < p$, $\Delta \approx g_5 Rg_6 \cong \bar{Z}$. Further $A_1 \cong F_i|F_{i+1}$ for any $(i = j) j < p$.

a-2-2) Assume $f_1, f_2$: $A_1 \approx F_p$ ($\approx g_6 R|g_6 J$).

If the $f_i$ are not given by $j^1$ in $eJ e$, there exists a unit $x$ in $eRe$ such that $xA_1 = A_1$ and $xf_1 - f_2 x_1 = j_1 (j \in eJ e)$. In this case $A_1 \cong F_i|F_{i+1}$ ($i < p$). In particular if $eJ eA_1 = 0$, $\Delta \approx g_6 Rg_6 \cong \bar{Z}$.

b) $A_2|A_2 f^k$ is uniserial and $A_2 f^k$ is not uniserial, i.e., $A_2 f^k = B_1 \oplus B_2 \oplus \cdots \oplus B_s$, where the $B_i$ are hollow. Then $A_1$ is simple and $s = 2$. Further $B_1$ is a minimal module $B_1 \oplus B_2$, as given in a-2-1).

Then $B_2 \cong B_1$ ($|B_2| \leq |B_1|$), and $B_1$ is simple, $B_2$ is uniserial.

b-1) If $f$: $A_1 \approx F_i|F_{i+1}$ ($A_2 \supset F_i \supset F_{i+1} \supset B_1 \oplus B_2$), then we obtain the same result modulo $B_1 \oplus B_2$ as given in a-2-1).

b-1-2) If $f$: $A_1 \approx B_1, f$ is given by $j_1$; $j \in eJ e$.

b-1-3) If $f$: $A_1 \approx H_i|H_{i+1}$ ($B_2 \supset H_i \supset H_{i+1}$), then $H_{i+1} = 0$ and $f$ is given by $j_1$; $j \in eJ e$.

b-1-4) If $f$: $B_1 \approx H_i|H_{i+1}$, then $H_{i+1} = 0$ and $f$ is given by $j_1$; $j \in eJ e$.

b-2) $[\Delta: \Delta(B_1)] = 2$ for $i = 1, 2$. 
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Then $B_1 \cong B_2$ and $B_1$, $B_2$ are simple and $V = B_1 \oplus B_2$ satisfies ($\ast$, 1).

b-1) $A_1 \cong F_i/F_{i+1}$, $A_2 \cong F_i \oplus F_{i+1} \supseteq V$.

b-2) If $f$: $A_i \cong B_i$, $f$ is given by $j_i$, $j \in eJ_e$. (cf. [7], Theorem 17.)

Proof. We know $t \leq 3$ by Lemma 9. Assume that $|A_1| \leq |A_2| \leq |A_3|$.

i) $t = 3$. Consider $\{A_1, A_2, A_3, J(D) = J(A_1) \oplus J(A_2) \oplus J(A_3)\}$. Since $\Delta(A_1) = A_1$,

$J(A_1) \oplus J(A_2) \oplus J(A_3)$ is contained in some $A_i$ by Lemmas 2 and 7. Hence

$J(A_1) = J(A_2) = 0$ (note $A_1 \not\subseteq A_2$). Assume that $A_3$ contains two independent

submodules $B_1$ and $B_2$ in $eJ_i^{i+k}$ on the same level in (1). Take $R = R/J_{i+k+1}$.

Then both $[\Delta : \Delta(J_a)]$ and $[\Delta : \Delta(J_2)]$ are not equal to 1 and $[\Delta : \Delta(J_3)] \neq 3$ for

any $i$ by Lemmas 2 and 7, and hence $[\Delta : \Delta(J_a)] = 2$ for $i = 1$ or 2 by Lemma 3,

(say $i = 1$). Then $\{B_1, B_2, B_3\}$ contradicts Lemmas 2 and 7, since $B_2 \not\subseteq B_3$.

Hence $A_3$ is uniserial. Assume $f$: $A_1 \cong A_2$. Then $\{A_1, A_2, A_3, A_1(f)\}$ implies

$A_1(f) \sim (\text{some } A_i)$. Since $A_1$ is characteristic (we may assume $J_{i+1} = \mathbb{Q}$

by Remark 5), $A_1(f) \subseteq A_i$ is a contradiction. Finally assume $g$: $A_i \cong F_i/F_{i+1}$. Since

$\Delta(A_3) = \Delta$ and $A_2$ are simple, $A_3$ is characteristic. Hence $\{A_1, A_2, F_i(g^{-1}), A_3\}$

derives from Lemmas 2 and 7 that $A_1 \sim F_i(g^{-1})$. Therefore $g$ is given by $j_i$ from

Lemma 6.

2) $t = 2$.

ii) $A_1 \cong A_2$. Assume $\Delta = \Delta(A_1(f))$. Then $\{A_1, A_2, A_1(f), A_1(f)\}$ implies

$A_1(f) \sim A_i$ for some $i$, say 1 from Lemma 7. Since $A_2 \cong A_1 \cong A_1(f)$, $A_1(f) = xA_i$ for

some unit $x$ in $eR$. Hence $\Delta(A_1(f)) = \Delta(xA_i) = \Delta$, a contradiction. Accordingly

$\Delta(A_1(f)) = \Delta$. Consider $\{A_1, A_2, A_1(f), J(A_1) \oplus J(A_2)\}$, and $J(A_2) = 0$

by Lemma 7 (note $\Delta(A_1) = \Delta(A_1(f)) = \Delta$). Hence $A_1$ and $A_2$ are simple, and so

$eJ_{i+k+1} = 0$. Let $f$ and $f'$ be two isomorphisms of $A_1$ to $A_2$ and consider $\{A_1, A_2,$

$A_1(f), A_2(f')\}$. Since $eJ_{i+k+1} = 0$, they are characteristic, and so $A_1(f) = A_2(f')$ by

Lemmas 7 and 8. Hence $f = f'$. Considering an isomorphism $\delta f$ for $\delta \in \Delta,$

$\Delta = \{0, 1\}$. Since $Hom_R(A_1, A_3) = \{0, 1\}, \Delta' = gRg = \{0, 1\},$ where $A_1 \cong gR [gJ]$

i) $A_1 \cong A_2$ (\{1 \leq |A_1| \leq |A_2|\}). Assume $A_1J \neq 0$ and $A_1J = C_1 \oplus C_2 \oplus \cdots \oplus C_s$ ($s \geq 1$), where the $C_i$ are hollow. Consider $\{A_1, A_2, A_1J \oplus C_1, A_1J \oplus C_s\}$ ($s \geq 2$),

Then $A_1J \oplus C_1 \sim A_2J \oplus C_2$ by Lemmas 2 and 7, provided $A_1J \neq 0$. Assume

$\Delta(A_1J \oplus C_i) = \Delta$ for $i = 1, 2$ and $x(A_1J \oplus C_i) \subseteq A_1J \oplus C_i$ for some unit $x$. We may assume $J_{i+k+1} = 0$. There exists $j$ in $eJ_e$ such that $(x+j)(A_1J \oplus C_1) = A_1J \oplus C_1$. Then $xC_i \subseteq (x+j)C_1 \oplus C_i \subseteq A_1J \oplus C_i$. Hence $xC_i \subseteq (A_1J \oplus C_i) \cap (A_1J \oplus C_2) = A_1J,$ and so $C_i \sim A_1J$, a contradiction by Lemma 2. Hence $\Delta(A_1J \oplus C_i) \neq \Delta$ for some $i$, say 1. $\{A_1, A_2, A_1J \oplus C_1, A_1J \oplus C_s\}$ implies either $A_1 \sim A_1J \oplus C_1$ or $A_2 \sim A_1J \oplus C_2$ by Lemma 7. Which is again a contradiction by Lemma 2. Hence $s = 1$, and so

$A_2$ is uniserial, provided $A_1J \neq 0$.

Similarly $A_1$ is also uniserial, provided $A_2J \neq 0$. Now assume that $A_2$ is uniserial
\(|A_2| \geq 2 \text{ and hence so is } A_1\). We shall show $|A_1| \leq 2$. Assume $A_2J^* \neq 0$ and
hence $A_i J^2 = 0$. Consider \{A_1, A_1 J \oplus A_2 J, A_1 J^2 \oplus A_2 J, A_2 \}. Since $A_1 \sim A_2$ by Lemma 2, 1) $A_1 \sim A_1 J \oplus A_2 J^2$ or 2) $A_1 \sim A_1 J^2 \oplus A_2 J$, $A_1$ and $A_2$ are symmetry) or 3) $A_1 J \oplus A_2 J^2 \sim A_1 J^2 \oplus A_2 J$.

1) It is clear that $x A_1 \supset A_1 J \oplus A_2 J_1$ for a unit $x$. However $A_1$ is uniserial, and so $A_2 J^2 = 0$ (note $|A_1| < |A_2|$). 2) This is similar. 3) Assume $x(A_1 J \oplus A_2 J^2) \supset A_1 J^2 \oplus A_2 J$. Since $\Delta(A_1) = \Delta$, there exists $j$ in $e J e$ such that $(x + j) A_1 = A_1$. Let $a_{2j}$ be an element in $A_2 J (a_{2j} \in A_2, j \in J)$. Then $x(a_{2j} + a_{2j}) = a_{2j}$ for some $a_{1} \in A_1$, $a_{2} \in A_2$, $j_{3} \in J$, and $j \in J^2$. Hence $(x + j)a_{1}j_{3} - ja_{1}j_{3} + xa_{1}j_{4} = a_{2}j_{4}$. On the other hand, $(x + j)a_{1}j_{3} - ja_{1}j_{3} + xa_{1}j_{4} = a_{2}j_{4}$. We observe isomorphisms between sub-factor modules of $A_1$ and $A_2$, and then investigate submodules $X$ in $e J e$. It is well known that there exist submodules $A_1 \supset \phi C \supset C' \supset A_2$ and $A_2 \supset D \supset D'$ such that $h: C/C' \sim D/D'$ and $X = \{c + d \mid c \in C \oplus D, h(c + C') = d + D'\}$ (cf. [3]). We denote $X$ by $C(h) D$.

a-1) Let $|A_1| = 2$.

a-1-1) $f: A_1 E_2 \equiv A_1 F_2 \equiv g R | g J$.

Then $\Delta^e = g R | g J$ from 2-i) and $f$ is a unique isomorphism.

a-1-2) ([7], Theorem 17) Assume $f: A_1 E_2 \approx F_1 / E_{i+1}$ $(i > 1)$. Consider \{A_1, A_2, E_2 \oplus F_2, A_1(f) F_2 \}. Since $\Delta(A_1) = \Delta(A_2) = \Delta$, $A_2 \sim A_1(f) F_2$. Further $E_2 \oplus F_2$ being characteristic, from Lemma 7 there exists a unit $x'$ in $e R e$ such that $x' A_1 \subset A_1(f) F_2$. Let $p_i: e J e$ be the projection and $x' = x + j$; $x A_1 = A_1$, $j \in e J e$ as usual. Then for a generator $a$ in $A_1$

$$(x + j)a = ar + f(ar) + z_1 + z_2; r \in R, z_1 \in E_2 \text{ and } z_2 \in F_{i+1}.$$

Hence

$$x a + p_1( j a) = ar + z_1 \text{ and } p_2( j a) = f( ar) + z_2.$$ 

Since $p_1( j a) \in E_2$, $x a = ar \pmod{E_2}$. Assume $i < p - 1$. Since $j a \in F_{i-1} \subset F_{i+1}$, $f( ar) \equiv f(x a) \equiv 0 \pmod{F_{i+1}}$. However $x a$ is a generator of $A_1$, and hence $f = 0$. Therefore $i > p - 1$.

a-1-3) See a-2-1) below.

a-1-4) $E_2 \approx F_2$ $(p = 2)$. We have the situation of 2-i).

Assume further $f: A_1 / E_2 \approx F_2 (A_2 / F_2 \equiv E_2)$, and consider \{A_1, A_2, A_1(f), E_2 \oplus F_2 \}. Then $A_1 \sim A_1(f)$ by Lemma 7 and so $A_1(f) = x A_1$ for some unit $x$ in $e R e$, since $A_1 \sim A_1(f)$. If $p > 2$, see a-2-2) below.

a-1-5) Let $X$ be a submodule in $e J e$.

i) $X = A_1(f) F_1 \approx F_1 (f^{-1})$ $(f_1: A_1 \approx F_1 / F_{i+2})$. If $i = 1$, consider $R / F_{i+2}$. Then this contradicts 2-i). Hence $i \neq 1$, $F_1 = F_{i-1}$ and $F_{i+2} = 0$ from a-1-2). \{A_1, A_2, E_2 \oplus F_2, A_1(f) \} shows $A_1(f) = x A_1$ for some unit $x$ in $e R e$.

ii) $X = A_1(f) A_2 (f_1: A_1 / E_2 \approx A_2 / F_2)$. Then $X = B_1$ from a-1-1).

iii) $X = A_1(f) F_1 (f_2: A_1 / E_2 \approx F_1 / F_{i+1}, i > 1)$ and hence $i = p - 1$ or $p$ by a-1-2). Then \{A_1 \oplus F_{i+1}, A_2, E_2 \oplus F_2, A_1(f) F_1 \} shows $A_1(f) F_1 = x(A_1 \oplus F_{i+1})$.
iv) \[ X = A_2(f_i^{-1}) \] (\( f_i : E_2 \approx A_2/F_2 \)). \{ A_1, A_2, E_2 \oplus F_2, A_4(f_i^{-1}) \} shows \( A_2 = xA_2(f_i^{-1}) \).

v) \[ X = F_i(f_i^{-1}) \] (\( f_i : E_2 \approx F_i/F_{i+1}, i \geq 2 \)). In this case \( eJ^{i+1} = E_2 \oplus F_2 \). Hence this is the case of a-2). Accordingly \( X = B' \) or \( B'' \), provided \( X \) is not isomorphic to a standard submodule in \( eJ^{i+1} \) via \( x \).

Thus we have shown that \( X \) is isomorphic to a standard submodule in \( eJ^i \) via \( x \) except \( B_1, B' \) and \( B'' \).

a-2) \( |A_1| = 1 \).

a-2-1) Let \( f : A_1 \approx F_i/F_{i+1} (i < p) \). If \( F_i(f^{-1}) \supset xA_1 \) for some unit \( x \) in \( eRe \), \( xA_1 \subset F_{i+1} \subset A_2 \), since \( (F_i(f^{-1})) = F_{i+1} \), which is a contradiction from Lemma 2.

We note further that \( A_2 \) is characteristic, since \( \Delta(A_2) = \Delta \) and \( A_1 \) is simple. Assume \( \Delta(F_i(f^{-1})) \neq \Delta \). Then \( \{ A_2, A_1, F_i(f^{-1}) \} \) derives a contradiction from the above remarks and Lemma 7. It is clear that \( efe(F_i(f^{-1})) \subset \{ f \} \neq A_2 \subset F_{i+1} \).

Hence \( F_i(f^{-1}) \) is also characteristic. Let \( f' : A_1 \approx F_i/F_{i+1} \) be another isomorphism. \( \{ A_2, A_1, F_i(f'^{-1}), F_i(f'^{-1}) \} \) gives \( F_i(f'^{-1}) = F_i(f'^{-1}) \) since they are characteristic. Therefore \( f = f' \). Accordingly, \( \Delta \approx g \approx \bar{g} \approx \bar{Z} \) as given in the proof of 2-i).

Further assume \( g : A_1 \approx F_i/F_{i+1} (j < p) \). Again consider \( \{ A_2, A_1, F_i(f^{-1}), F_i(g^{-1}) \} \). Then \( F_i(f^{-1}) \supset F_i(g^{-1}) \) if \( i < j \), and so \( F_i(f^{-1}) \subset F_{i+1} \), a contradiction.

a-2-2) Assume that \( f_1, f_2 : A_1 \approx F_p \) and they are not given by \( j \) in \( efe \).

Then \( \{ A_2, A_1, A_1(f_1), A_1(f_2) \} \) gives, from Lemmas 6 and 7, that \( A_1(f) = x' A_1(f) \) for some unit \( x' \) in \( eRe \). Since \( \Delta(A_1) = \Delta \), there exists \( j \) in \( efe \) such that \( (x' + j)A_1 = A_1 \).

Put \( x = x' + j \). Then for a generator \( a \) in \( A_1 \)

\[(x - j)(a_2 + f_2(a)) = ar + f_1(ar); r \in R.\]

Hence \( xa = ar, xfa(a) = ja = f_1(ar) \).

Next assume further that \( g : A_1 \approx F_i/F_{i+1} (i < p) \). Consider \( \{ A_2, A_1, F_i(q^{-1}), A_1(f) \} \), and \( F_i(q^{-1}) \sim A_1(f) \) since \( F_i(q^{-1}) \) is characteristic. Which is a contradiction. In particular, if \( efeA_1 = 0, A_1(f) \) is characteristic, since \( \Delta(A_1(f)) = \Delta \) (if \( \Delta \neq \Delta(A_1(f)) \), \( \{ A_1, A_2, A_1(f) \} \) gives \( A_1 \sim A_1(f) \)). Then \( f \) is given by \( j_1 \) from Lemma 6). Hence \( f_1 = f_2 \) from the first paragraph, and so \( \Delta \approx \bar{g} \approx \bar{Z} \) as in the proof of 2-i).

b) \( A_2/A_2J^k (k \geq 1) \) is uniserial and \( A_2J^k = \sum_{i=1}^{s+B_1} (s \geq 2) \). Then \( A_1 \) is simple from the initial paragraph of ii). Then \( Df^k = eJ^{i+k} = A_2J^k \). Since \( eJ^{i+k} = B_1 \oplus \cdots \oplus B_n, s \leq 3 \) from Lemma 15, and \( [\Delta : \Delta(B_1)] \leq 2 \) for all \( i \) by Lemmas 2 and 7. If \( [\Delta : \Delta(B_1)] = 1 \) and \( [\Delta : \Delta(B_2)] = 2 \), \( \{ A_1, B_1, B_2, B_3 \} \) derives a contradiction. Hence either \( [\Delta : \Delta(B_1)] = 1 \) for all \( i \) (b1) or \( [\Delta : \Delta(B_1)] = 2 \) for all \( i \) (b2). In the former case \( s = 2 \) by Lemma 7 and in the latter case \( s = 2 \) and \( B_2 = xB_1 \) for some unit \( x \) in \( eRe \) by Lemma 10.
b_1) \ \Delta(B_1)=\Delta(B_2)=\Delta.

Then \{A_1, B_1, B_2, J(B_1)\oplus J(B_2)\} implies J(B_1)=0 (|B_1| \leq |B_2|). If f: B_1 \approx B_2, \{A_1, B_1, B_2, B_1(f)\} derives a contradiction. Hence B_1 \approx B_2. We can show as before that B_2 is uniserial.

b_1-1) This is the case of a-2-1).

b_1-2) Assume f: A_1 \approx B_1. \{A_1, A_1(f), B_1, B_1\} derives A_1 \sim A_1(f), i.e., 
\((x+j)A_1=A_1(f): xA_1=A_1\) and \(j \in eJf e. (x+j)a=ar+f(ar); A_1=aR, r \in R.\)
Hence \(xa=ar\) and \(ja=f(ar). \) Put \(xa=b, \) and \(A_1=bR. \ f(b)=j^{-1}x.\)

b_1-3) Assume f: A_1 \approx H_i/H_{i+1}. \{A_1, B_1, B_2, H_i(f^{-1})\} shows A_1 \sim H_i(f^{-1}).
Hence \(H_{i+1}=0\) and \(f\) is given by \(j_i\) as above (cf. Lemma 6).

b_1-4) Assume f: B_1 \approx H_i/H_{i+1}. \{A_1, B_1, B_2, H_i(f^{-1})\} derives B_1 \sim H_i(f^{-1}),
since \(\Delta(B_1)=\Delta. \) Hence \(H_{i+1}=0\) and \(f\) is given by \(j_i\) from Lemma 6.

b_2 \ [\Delta: \Delta(B_i)]=2\ for \ i=1, 2, (B_2=xB_1).

\{A_1, B_1, B_2, J(B_1)\oplus J(B_2)\} shows, from Lemma 2, that \(J(B_2)=0, \) i.e., \(B_2\) is simple. Further since \(\Delta(A_1)=\Delta\) and \(\Delta(A_1(B_1))=2, \) \(\Delta(\Delta(E))=2\) for all simple submodules \(E\) in \(V=B_1\oplus B_2\) by Lemmas 2 and 8. Hence \(V\) satisfies (\# 1) by Lemma 7.

b_2-1) If \(A_1 \approx F_i/F_{i+1}, \Delta=\bar{Z} \) by a-1-3). Hence \(\Delta(B_1)=\Delta.\)

b_2-2) Assume f: A_1 \approx B_1. \{A_1, B_1, B_1, A_1(f)\} derives A_1 \sim A_1(f). Hence \(f\) is given by \(j_i\) as b_1-2).

**Remark 17.** If \(R\) is an algebra over an algebraically closed field \(K, \Delta \neq \bar{Z}\) and the first part of a-2-2) does not occur (take \(f_2=KF_1, k \equiv 1; k \in K\)). We can express \(f\) in a-1-2) as an element in \(eJf\), however it is little complicated (cf. [7], Theorem 17).

In order to make the converse version clear, we illustrate the structure of Lemmas 10~16 as follows:

1) (Lemma 10)

\[
\begin{array}{ccc}
  eR & eJ^i & eJ^{i+1} \\
  \hline
  A_1 & B_1 & 0 \\
  A_2=xA_1-B_2=xB_1-0
\end{array}
\]

\([\Delta: \Delta(A_1)]=1, \Delta(\Delta(A_2))=2\) every characteristic submodule in \(eJ^i\) is linear with respect to the inclusion and \([\Delta: \Delta(C)]=2\) for any non-characteristic submodule \(C\) in \(eJ^i\). Further those \(C\) are related to one another with respect to \(\sim\).

2) (Lemma 11)

\[
\begin{array}{ccc}
  eR & eJ^i & eJ^{i+1} \\
  \hline
  A_1 & 0 \\
  A_2=x_2A_1-0 \\
  A_3=x_3A_1-0
\end{array}
\]

(3)
\[ [\Delta : \Delta(A_1)] = [\Delta : \Delta(A_2)] = [\Delta : \Delta(A_3)] = 3 \text{ and } A_1 \oplus A_2 \oplus A_3 \text{ satisfies } (#, 1) \text{ and } (#, 2). \text{ Further } [\Delta : \Delta(C)] \leq 3 \text{ for every submodule } C \text{ in } A_1 \oplus A_2 \oplus A_3. \text{ (}A_s\text{ may be zero.)} \]

3) (Lemma 12, i))

\[
\begin{array}{c|cc}
& eJ^t & eJ^{t-1} \\
\hline
A_1 & -1 & -0 \\
A_2 & -1 & -0 \\
A_3 = x_A A_2 & -1 & -0 \\
\end{array}
\]

\[ [\Delta : \Delta(A_1)] = 1 \text{ and } [\Delta : \Delta(A_2)] = [\Delta : \Delta(A_3)] = 2. \text{ Further } A_2 \oplus A_3 \text{ satisfies } (#, 1). \]

4) (Lemma 12, ii-a))

\[
\begin{array}{c|cc}
& eJ^t & eJ^{t-1} \\
\hline
A_1 & -1 & -0 \\
A_2 & -1 & -0 \\
\end{array}
\]

\[ [\Delta : \Delta(A_1)] = 1, [\Delta : \Delta(A_2)] = 2 \text{ and } A_2 \oplus E_n \text{ satisfies } (#, 1) \text{ except } E_n. \]

5) (Lemma 12, ii-b ii-b-2-ii-l))

\[
\begin{array}{c|cc}
& eJ^t & eJ^{t-1} & eJ^{t+1} & eJ^p \\
\hline
A_1 & -1 & -0 & -1 & -0 \\
A_2 & -1 & -0 & -1 & -0 \\
B_1 & -1 & -0 & -1 & -0 \\
B_2 & -1 & -0 & -1 & -0 \\
\end{array}
\]

\[ [\Delta : \Delta(A_1)] = 1, [\Delta : \Delta(A_2)] = 2 \text{ and } A_1 \oplus A_2 \oplus J(A_2) \text{ satisfies } (#, 1) \text{ except } A_1. \text{ (}B_2\text{ may be zero.)} \]

5')

\[
\begin{array}{c|cc}
& eJ^t & eJ^{t+1} \\
\hline
A_1 & -1 & -0 \\
A_2 & -1 & -0 \\
B_1 & -1 & -0 \\
B_2 & -1 & -0 \\
\end{array}
\]
[Δ: Δ(B_1)]=[Δ: Δ(B_2)]=2 and \( B_1 \cong B_2 \). \( B_1 \oplus B_2 \) satisfies (\#, 1).

6) (Lemma 16, 1))

\[
\begin{array}{ccc}
\cdot & eR & eJ^i \\
A_1 & eJ^{i+n-1} & E_n-0 \\
A_2-0 & \# & A_3-0
\end{array}
\]

\[\Delta: \Delta(A_1)=[\Delta: \Delta(A_2)]=[\Delta: \Delta(A_3)]=1.\] If \( f: A_2 \cong E_n \), \( f \) is given by \( j_i; j \in e\mathcal{F}e \). (\( A_3, A_3 \) and \( E_4 \) may be zero.)

7) (Lemma 16, 2-ii))

\[
\begin{array}{ccc}
\cdot & eR & eJ^i \\
A_1-0 & \# & A_3-0
\end{array}
\]

\[\Delta: \Delta(A_1)=[\Delta: \Delta(A_2)]=1 \text{ and } \Delta \cong \Delta' \cong \tilde{Z}.
\]

8) (Lemma 16, 2-ii-a))

\[
\begin{array}{ccc}
\cdot & eR & eJ^i \ eJ^{i+1} \ eJ^{i+n-1} \\
A_1-E_2-0 & \# & A_2-F_2-0
\end{array}
\]

[\(\Delta: \Delta(A_1)=[\Delta: \Delta(A_2)]=1\). If \( f: A_1/[A_2] \cong \Delta(A_2)/F_2 \), \( \Delta \cong \Delta' \cong \tilde{Z} \). Every submodule except \( B_1, B_1' \) and \( B'' \) is isomorphic to a standard submodule via \( x_i \). (If \( n=2 \) and \( E_2 \cong F_2 \), \( \Delta \cong \Delta' \cong \tilde{Z} \).) If \( E_2=0 \), the conditions in \( a-2 \) of Lemma 16 are fulfilled.

9) (Lemma 16, 2-ii-b))

\[
\begin{array}{ccc}
\cdot & eR & eJ^i \ eJ^{i+k} \ eJ^{i+n-1} \\
A_1-0 & \# & B_1-0
\end{array}
\]

\[\Delta: \Delta(A_1)=[\Delta: \Delta(A_2)]=1, [\Delta: \Delta(B_1)]=[\Delta: \Delta(B_2)]=1.\] If \( f: A_1 \cong B_1, f \) is given by \( j_i; j \in e\mathcal{F}e \). Similar facts hold for other cases.

10) (Lemma 16, 2-ii-b))

\[
\begin{array}{ccc}
\cdot & eR & eJ^i \ eJ^{i+k} \ eJ^{i+n-1} \\
A_1-0 & \# & B_2-F_2-0
\end{array}
\]
We shall show that if \( eR \) has one of the structures of the above diagrams 1)\( \sim \)10), then \( R \) is a right US-4 ring with \((*,1')\). It is clear from the diagrams that \((*,1')\) holds. Let \( \{U_i\}_{i=1}^{10} \) be a set of submodules in \( eR \).

Diagram 1). If \( U_1 \) and \( U_2 \) are characteristic, \( U_1 \supseteq U_2 \) or \( U_1 \subseteq U_2 \). Hence \( U_1 \oplus U_2 \) satisfies \((**,2)\) by [4], Corollary 1 of Theorem 2. Hence \( D=\sum_{i=1}^{4} \oplus U_i \) satisfies \((**,4)\) by [2], Lemma 1. Assume that \( U_1 \cap U_2 \supseteq eJ^i \). Then \( U_i \) for \( i=1,2 \) is characteristic, and hence \( D \) satisfies \((**,4)\) from the above. Next assume that \( U_1 \supseteq eJ^i \) and \( eJ^i \supseteq U_j \) for \( j>1 \). Since \( \Delta(U_1)=\Delta, U_1 \oplus U_2 \) satisfies \((**,2)\) by [4], Corollary 1 of Theorem 2. Finally assume \( eJ^i \supseteq U_j \) for all \( j \). If \( \{U_j\}_{j=1}^{\infty} \) is a set of non-characteristic submodules, then we may assume \( U_1 \supseteq x_1 U_2 \supseteq x_3 U_3 \) for some units \( x_1 \) in \( eR \) by assumption. Since \( \Delta: \Delta(U_1)]=2 \), \( U_1 \oplus U_2 \oplus U_3 \) satisfies \((**,3)\) by [4], Corollary 3 of Theorem 2. Therefore \( D \) satisfies \((**,4)\).

2) As is shown in 1), we may assume that \( eJ^i \supseteq U_j \) for all \( j \). Then \( U_1 \supseteq x_1 U_2 \supseteq x_3 U_3 \supseteq x_4 U_4 \) by assumption, where the \( x_i \) are units in \( eR \). Then from the assumption \( [\Delta: \Delta(C)]=3 \) and the argument of the proof of [4], Corollary 3 of Theorem 2, \( D \) satisfies \((**,4)\).

3) Let \( eJ^i \supseteq U_j \) for all \( j \). Then \( U_i=U_1 \oplus B_i \) or \( U_i \subseteq A_2 \oplus A_3 \) by assumption, where \( B_i \subseteq A_2 \oplus A_3 \). First assume \( U_j \subseteq A_2 \oplus A_3 \) or \( U_j=U_1 \oplus B_j \) \((B_i=0)\) for all \( j \leq 3 \). Then \( D \) satisfies \((**,4)\) by [4], Corollary 3 of Theorem 2 (note \( A_1 \) and \( A_2 \oplus A_3 \) are characteristic and see the remark above). If \( U_i=U_1 \) and \( U_2=U_1 \oplus B, U_1 \oplus U_2 \) satisfies \((**,2)\) by [4], Corollary 1 of Theorem 2. Thus \( D \) satisfies \((**,4)\).

4) Every submodule in \( eJ^i \) is isomorphic to a standard submodule in \( eJ^i \) via \( x_1 \). Hence we may assume that all \( U_j \) are standard. Then \( D \) satisfies \((*,4)\) by [4], Corollaries 1\( \sim \)3 of Theorem 2.

5) and 5') Let \( eJ^i \supseteq U_1 \supseteq A_2 J \) and \( U_1 \supseteq A_1 \oplus A_2 \). Then \( U_1/A_2 J=x(A_2/A_3 J) \), and so \( xA_2 \supseteq U_1 \). Further \( A_1 \oplus A_2 J \) is characteristic. If \( U_1=U_1 \oplus A_2 J \) and \( U_2 \subseteq A_2 J \), \( U_1 \oplus U_2 \) satisfies \((*,2)\). Accordingly we may assume that \( U_i \) is \( A_1 \) or a submodule of \( A_2 \). Therefore \( D \) satisfies \((**,4)\).

6) and 7) These are clear.
8) First we note \( B_1 \oplus E_2 \oplus F_2 \cong B_i' \) and \( B_1', B_i' \) do not appear simultaneously. If the \( U_i \) are standard for all \( i \), \( U_i \sim U_j \) for some pair \( i, j \). Hence \( D \) satisfies (**, 4) by [4], Corollary 2 of Theorem 2. The conditions given in Lemma 16 show that \( A_i \sim A_i(f), F_i(f^{-1}) \sim F_i(g^{-1}), \ldots \) etc.. Hence we obtain the desired result.

9) and 10) These are simpler than 8), (if \( A_1 \cong F_i/F_{i+1} (F_{i+1} \cong B_1 \oplus B_2) \), \( \Delta \cong \mathbb{Z} \). Hence \( \Delta(C) = \Delta \) for any submodule \( C \) in \( eR \).

Thus we obtain

**Theorem 2.** \( R \) is a right US-4 (basic) ring with \((*, 1')\) if and only if \( eR \) has one of the structures given in Lemmas 10~16 (cf. Diagrams 1~10)) for each primitive idempotent \( e \).

3. Hereditary rings

In this section, we shall study a hereditary and right US-3 (resp. US-4) ring \( R \). If \( R \) is hereditary, \((*, 1')\) holds, and hence we can make use of the results in the previous sections.

**Lemma 18.** Assume that \( R \) is basic and hereditary. Then a submodule \( A \) in \( eR \) is characteristic if and only if \( \Delta(A) = \Delta \). Every non-zero element in \( \text{Hom}_R(eR, fR) \) is a monomorphism, where \( e \) and \( f \) are primitive idempotents.

Proof. The second half is clear (see [9], Lemma 2). Hence, since \( eje = 0 \), the first one is clear.

From now on we assume that \( R \) is a hereditary and basic ring. First we assume further that \( R \) is right US-3.

**Theorem 3.** Let \( R \) be a hereditary (and basic) ring. Then \( R \) is a right US-3 ring if and only if \( eR \) has the following structure for each primitive idempotent \( e \):

i) \( eR/eJ_i' \) is uniserial for some \( i \) and

ii) \( eJ' = 0 \) or \( eJ' = A \oplus B \) such that either

a) \( A \) and \( B \) are simple and \( A \oplus B \) satisfies (**, 1), and \([\Delta: \Delta(A)] = 2\), or

b) \( A \) is simple, \( B \) is uniserial and \( A \) is not isomorphic to any sub-factor modules of \( B \) (and hence \( \Delta(A) = \Delta(B) = \Delta \)).

Proof. If \( R \) is right US-3, \( eR \) has the structure in Theorem 1. We consider the case b) of Theorem 1. Assume that \( f: A \cong (\text{the socle of } B) \). Then \( \{A, A(f), B\} \) derives a contradiction, since \( A \) and \( B \) are characteristic by Lemma 18. Thus we obtain the theorem from Theorem 1.

Let \( R \) be a basic hereditary ring. Then
where the $\Delta_i$ are division rings and the $M_{ij}$ are left $\Delta_i$- and right $\Delta_j$-modules [1].

We shall express explicitly the content of Theorem 3 for $M_{ij}$ in a row of the above ring.

1) $(0 \Delta_10 \Delta_0 \cdots 0 \Delta_i0 \cdots \Delta_{i+1}0)$

2) $(0 \Delta_10 \cdots \Delta_{i+1}0 \cdots \Delta_{i+1}0 \cdots (u_{i+1}\Delta_{i+1}) \cdots 0)$

where $(u_{i+1}\Delta_{i+1})=u_{i+1}\Delta_{i+1} \oplus v_{i+1}\Delta_{i+1}$ satisfies $\#1$.

3) $(0 0 \Delta_10 \cdots \Delta_{i+1}0 \cdots 0 \cdots (u_{i+1}\Delta_{i+1}) \cdots 0 \cdots 0)$

As is given in the proof of [9], Theorem 1, we can show a ring monomorphism $\rho_{rs}: \Delta_r \rightarrow \Delta_s$ for $r<s<k$ such that $xu_{rs}=u_{rs}(x)$ for $x \in \Delta_r$ and $\rho_{rs}\rho_{st}=\rho_{st}$.

Next we shall characterize a hereditary (basic) and right US-4 ring. If $R$ is hereditary, some results in the previous sections may not occur as shown in Theorem 3. We shall observe them.

In the case b) of Lemma 12, $A_2$ is simple.

Because, since $A_1$ is simple and $[\Delta: \Delta(A_2)]=2$, $A_1 \approx A_2/J(A_2)$. Hence $A_1 \approx A_2$ by Lemma 18.

We shall observe the conditions in Lemma 16 for a hereditary ring. a-1-1), a-1-2), a-1-3), any of b-1-1)~4) and b-2) do not occur from Lemma 18. For instance, if $f': A_1/E_2 \approx F_{p-1}/F_p$ (a-1-2), $f: A_1 \approx F_{p-1}$ by Lemma 18. Then $A_1 \approx A_1(f)$ by a-1-5). However, $A_1$ is characteristic, and so $A_1=\Delta_1(f)$. Therefore $f=0$.

We shall use the notations after Theorem 3.

**Lemma 19.** In case 2-i) in Lemma 16, $e_iR$ is of the form $(0, \cdots, \bar{Z}, \cdots$
0 \cdots \overline{Z} \cdots 0). In case of 2-a-1-4) in Lemma 16, \( A_1 \) (resp. \( A_2 \)) is of the form 
\((0, \cdots, \overline{Z}, 0, \overline{Z}, 0, \cdots)\) (resp. \((0, \cdots, \overline{Z}, 0, \overline{Z}, 0, \cdots)\)).

Proof. Let \( E_2 \cong F_p \cong e_{sh} R \) by Lemma 16. Let \( A_2 \cong e_{sh} R \). Then \( e_{sh} R \) is uniserial and \( M_{sh} = u_{sh} \overline{Z} \sim F_p \). Since \( M_{sh} \) is a left \( \Delta \)-module, \( \Delta, \subseteq \overline{Z} \). Hence \( \Delta = \overline{Z} \). We have the same for 2-i).

Thus we have

**Theorem 4.** Let \( R \) be a hereditary (basic) ring. Then \( R \) is right US-4 if and only if for each \( e = e_{ii}, eR \) has one of the following structures: 1\( \sim \)11

1) 
\[(0 \cdots 0 \Delta, 0 \Delta, 0 \Delta, 0 \cdots \Delta, 0, \cdots 0)\]

2) (Lemma 10)
\[(0 \cdots 0 \Delta, 0 \Delta, 0 \cdots \Delta, 0 \cdots 0, \cdots 0)\]

\[\Delta: \Delta(A_i) = 2 \ (i=1, 2) \text{ and } u_{i+2}, v_{i+2} \text{ may be zero. The conditions in Lemma 10 are satisfied.}\]

3) (Lemma 11)
\[(0 \cdots 0 \Delta, 0 \cdots \Delta, 0 \cdots 0)\]

\[\Delta: \Delta(A_i) = 3 \text{ for each } i \text{ and } A_1 \oplus A_2 \oplus A_3 \text{ satisfies } (\#, 1) \text{ and } (\#, 2). \ w_{i+1} \text{ may be zero.}\]

4) (Lemma 12-i)
\[(0 \cdots 0 \Delta, 0 \cdots \Delta, 0 \cdots 0)\]

\[\Delta(A_i) = \Delta, \Delta: \Delta(A_i) = 2 \ (i=1, 2) \text{ and } A_1 \oplus A_2 \text{ satisfies } (\#, 1).\]

5) (Lemma 12-ii-a) and b)
\[(0 \cdots 0 \Delta, 0 \cdots \Delta, 0 \cdots 0)\]

\[\Delta(A_i) = \Delta, \Delta: \Delta(A_i) = 2, \text{ and } u_{\Delta}, \Delta_{i+2} \Delta_{i+2} \text{ satisfies } (\#, 1), \text{ except } u_{\Delta}, \Delta_{i+2} \text{ may be zero.}\]

6) (Lemma 16, 1))
$$
\begin{pmatrix}
\Delta_0 & \ldots & \Delta_t & 0 & \ldots & 0
\end{pmatrix}
\begin{pmatrix}
u_{t+1} \\
v_{t+1} \Delta_{i_{t+1}} \\
v_{t+2} \\
v_{t+2} \Delta_{i_{t+2}} \\
v_{t+3} \\
v_{t+3} \Delta_{i_{t+3}}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
x_{t+3} \Delta_{i_{t+3}}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
x_{t+3} \Delta_{i_{t+3}}
\end{pmatrix} = A_1
$$

\Delta(A_i) = \Delta (i=1, 2, 3) and \(u_{t+1}\) may be zero.

\[c \Delta_{i_{t+1}} \subseteq c \Delta_{i_t} \subseteq \cdots \subseteq c \Delta_{i_p}\]

7) (Lemma 16, 2-i))

\[\begin{pmatrix}
\Delta_0 & \ldots & \Delta_t & 0 & \ldots & 0
\end{pmatrix}
\begin{pmatrix}
u_{t+1} \\
u_{t+1} \Delta_{i_{t+1}} \\
v_{t+2} \\
v_{t+2} \Delta_{i_{t+2}} \\
v_{t+3} \\
v_{t+3} \Delta_{i_{t+3}}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
x_{t+3} \Delta_{i_{t+3}}
\end{pmatrix} = A_2
\]

8) (Lemma 16, 2-ii-a))

\[\begin{pmatrix}
\Delta_0 & \ldots & \Delta_t & 0 & \ldots & 0
\end{pmatrix}
\begin{pmatrix}
u_{t+1} \\
u_{t+1} \Delta_{i_{t+1}} \\
v_{t+2} \\
v_{t+2} \Delta_{i_{t+2}} \\
v_{t+3} \\
v_{t+3} \Delta_{i_{t+3}}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
x_{t+3} \Delta_{i_{t+3}}
\end{pmatrix} = A_3
\]

\[\Delta(A_i) = \Delta (i=1, 2), u_{t+3} or \{v_{t+4}, \ldots, v_p\} may be zero.\]

\[c \Delta_{i_{t+1}} \subseteq c \Delta_{i_{t+3}} \subseteq \cdots \subseteq c \Delta_{i_p}\]

9) (Lemma 16, 2-ii-a’)

\[\begin{pmatrix}
\Delta_0 & \ldots & \Delta_t & 0 & \ldots & 0
\end{pmatrix}
\begin{pmatrix}
u_{t+1} \\
u_{t+1} \Delta_{i_{t+1}} \\
v_{t+2} \\
v_{t+2} \Delta_{i_{t+2}} \\
v_{t+3} \\
v_{t+3} \Delta_{i_{t+3}}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
x_{t+3} \Delta_{i_{t+3}}
\end{pmatrix} = A_3
\]

\[u_{t+2} may be zero.\]

10) (Lemma 16, 2-ii-b_1))

\[\begin{pmatrix}
\Delta_0 & \ldots & \Delta_t & 0 & \ldots & 0
\end{pmatrix}
\begin{pmatrix}
u_{t+1} \\
u_{t+1} \Delta_{i_{t+1}} \\
v_{t+2} \\
v_{t+2} \Delta_{i_{t+2}} \\
v_{t+3} \\
v_{t+3} \Delta_{i_{t+3}}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
x_{t+3} \Delta_{i_{t+3}}
\end{pmatrix} = B_2
\]
\[ \Delta(A_i) = \Delta(B_i) = \Delta \quad (i = 1, 2). \]

\[
\begin{align*}
\Delta_i &\subset \Delta_i \subset \cdots \subset \Delta_i \\
\subset \Delta_{i+1} &\subset \cdots \subset \Delta_{i+1} \\
\subset \Delta_{i+2} &\subset \cdots \subset \Delta_{i+2} \\
\subset \Delta_{i+3} &\subset \cdots \subset \Delta_{i+3} \\
\subset \Delta_{i+4} &\subset \cdots \subset \Delta_{i+4} \\
\end{align*}
\]

11) (Lemma 16, 2-ii-b))

\[
\left( \begin{array}{cc}
0 & 0 \\
0 & 0 \\
\xi_{i+2} \Delta_{i+2} & \xi_{i+3} \Delta_{i+3} \\
\end{array} \right)
\]

\[
\begin{array}{cccc}
\cdots & A_1 \\
\cdots & B_1 \\
\cdots & A_2 \\
\cdots & B_2 \\
\end{array}
\]

\[ \Delta(A_i) = \Delta \quad \text{and} \quad [\Delta: \Delta: (B_i)] = 2 \quad (i = 1, 2). \]

4. Left serial rings

We shall investigate the same problem for a left serial ring \( R \). In this case \((*, 1')\) holds, too by [11], Corollary 4,2. Therefore we can make use of the results in §§ 1 and 2.

From now on we always assume that \( R \) is a left serial ring.

**Lemma 20.** If \( ef' = A_1 \oplus A_2 \) and the \( A_i \) are uniserial, every submodule \( E \) in \( ef' \) is isomorphic to a standard submodule \( B_1 \oplus B_2 \) via \( x_1 : x \) is a unit in \( eRe \), where \( B_i \subset A_i \).

See the proof of [3], Theorem 1.

**Lemma 21.** Let \( ef' = A_1 \oplus A_2 \) and the \( A_i \) hollow. If \( \Delta(A_i) \neq \Delta \), there exists a unit \( x \) in \( eRe \) such that \( xA_1 = A_2 \).

Proof. Since \( \Delta(A_i) \neq \Delta \), there exists a unit \( y \) in \( eRe \) such that \( (y + j')A_1 \subset A_i \) for all \( j' \) in \( ef'e \). Let \( p \) be the projection of \( ef' \) onto \( A_2 \). Then \( f = py | A_1 \) is an element in \( \text{Hom}_R(A_1, A_2) \). If \( f \) is not an epimorphism, \( f = j_i \) for some \( j \) in \( ef'e \), since \( A_2 \) is a hollow module \((\subset ef')\) and \( R \) is left serial. Then \((y - j)A_1 \subset A_i \).
a contradiction. Hence there exists a unit $x$ in $eRe$ such that $x_1=f$, and so

$x_2A=\Delta_1$.

**Lemma 22.** Let $eJ'=A_1\oplus A_2$ be as in Lemma 21. If $\Delta(A_1)=\Delta$, $\Delta(A_1J^k\oplus A_2J^k')=\Delta$.

Proof. From Lemma 21, $\Delta(A_2)=\Delta$. Hence we may assume $k\leq k'$. Let $x$ be any unit element in $eRe$. Since $\Delta(A_2)=\Delta$, there exists $j$ in $eRe$ such that $(x+j)A_1=A_1$. Hence $(x+j)(A_1J^k\oplus A_2J^k')\subseteq A_1J^k+(x+j)A_2J^k'\subseteq A_1J^k\oplus A_2J^k'$, and so $x=x+j\in \Delta(A_1J^k\oplus A_2J^k')$.

From Theorem 1, Lemmas 21, 22 and [8], Proposition 2, we obtain

**Theorem 5.** Let $R$ be a left serial ring. Then $R$ is a right US-3 ring if and only if $eR$ has the following structure for each primitive idempotent $e$:

There exists an integer $t$ such that

i) $eR/eJ'$ is uniserial and

ii) $eJ'=0$ or $eJ'$ is a direct sum of a simple module and a uniserial module.

Finally we shall give a characterization of a left serial and right US-4 ring. As was shown in the previous section, we shall refine the results in § 2.

In Lemma 10, every submodule in $eJ'$ is standard up to $x$ ($x$ is a unit in $eRe$) by Lemma 20. Further since $\Delta(A_1\oplus A_2J)\subseteq \Delta$, $A_1\oplus A_2\supseteq \bigcup(A_1\oplus J(A_2)\supseteq 0$

is the set of all characteristic submodules in $eJ'$.

From the above proof we have

**Remark 23.** Let $R$ be left serial and assume $eJ'=A_1\oplus A_2$; the $A_i$ are uniserial. If $[\Delta: \Delta(A_i)]=2$, $[\Delta: \Delta(C)]\leq 2$ for every submodule $C$ in $eJ'$ and \{e\} is the set of characteristic submodules in $eJ'$. Hence, if $R$ is left serial, i), ii) and iii) in Lemma 10 imply iv) and v). However hereditary does not as is shown from the following example:

Let $K\subseteq L$ be fields such that $[L: K]=2$. Put

\[
R = \begin{pmatrix}
L & L \otimes L & L \otimes L \\
0 & K & L \\
0 & 0 & L \\
0 & 0 & 0 & L
\end{pmatrix}
\]

Then $R$ is hereditary. Put $L=1K+uK$, $e_1=e$, and $eJ=\bigcap A_1\oplus A_2$; $A_1=1e_1R$, $A_2=ue_1R$ satisfy i), ii) and iii) in Lemma 10. Further $[\Delta: \Delta(B)]=2$ for any submodule $B$ in $eJ'$ if $\Delta\neq \Delta(B)$, since $[L: K]=2$. \{eJ, eJ^2, eJ^3, (1\otimes u+u\otimes 1)e_1R,$
(1\otimes u \pm u \otimes 1)_{u_4}R\} is the set of characteristic submodules provided \(u^c \in K\), and 
(1 \otimes 1)_{u_4}R \sim (1 \otimes 1 + u \otimes x)_{u_4}R\), provided \(x \in K\).

**Lemma 24.** Let \(B_1\) and \(B_2\) be simple submodules in \(eJ^i\) and \(V = B_1 \oplus B_2\). If \(B_1 \cong B_2\), \(V\) always satisfies \((\#; 1)\).

Proof. Since \(R\) is left serial, every simple submodule in \(V\) is isomorphic to \(B_i\) via \(x_i\); \(x\) is a unit in \(eRe\). Hence \(V\) satisfies \((\#; 1)\).

In Lemma 12, we do not have the case \(t=2\) by Lemma 21.

In Lemma 16, we have always \(A_1 \cong A_2\), since \(\Delta(A_1) = \Delta(A_2) = \Delta\). Hence \(2-i), 2-a-1-1), 2-a-2-3)\) and \(p=2\) in \(2-a-2-4)\) do not occur. Similarly \(2-a-2-1) does not occur. Thus we obtain

**Theorem 6.** Let \(R\) be a left serial ring. Then \(R\) is right US-4 if and only if, for each primitive idempotent \(e\), \(eR\) has one of the following structures:

1) \(eR\) is uniserial: \(eR\) \hspace{1cm} eJ^p \hspace{1cm} eJ^0

2) \hspace{1cm} eR \hspace{0.5cm} eJ^{i-1} \hspace{1cm} eJ^i \hspace{0.5cm} eJ^{i+1} \hspace{1cm} eR

\[\begin{array}{ll}
A_1-B_1-0 & 0 \\
A_2-B_2-0 & 0 \\
\end{array}\]

\([\Delta: \Delta(A_1)]=2. \hspace{1cm} In\hspace{0.5cm}this\hspace{0.5cm}case\hspace{0.5cm}A_1 \cong A_2\hspace{0.5cm}and\hspace{0.5cm}B_1\hspace{0.5cm}may\hspace{0.5cm}be\hspace{0.5cm}zero.\]

3) \hspace{1cm} eR \hspace{0.5cm} eJ^{i-1} \hspace{1cm} eJ^i \\

\[\begin{array}{ll}
A_1-0 & 0 \\
A_2-0 & 0 \\
A_3-0 & 0 \\
\end{array}\]

\([\Delta: \Delta(A_1)]=3\hspace{0.5cm}and\hspace{0.5cm}A_1 \oplus A_2 \oplus A_3\hspace{0.5cm}satisfies\hspace{0.5cm}(\#; 2).\hspace{0.5cm}In\hspace{0.5cm}this\hspace{0.5cm}case\hspace{0.5cm}A_1 \cong A_2 \cong A_3.\]

4) \hspace{1cm} eR \hspace{0.5cm} eJ^{i-1} \hspace{1cm} eJ^i \\

\[\begin{array}{ll}
A_1-0 & 0 \\
A_2-0 & 0 \\
A_3-0 & 0 \\
\end{array}\]

\(\Delta(A_1) = \Delta,\hspace{0.5cm}[\Delta: \Delta(A_1)]=2\hspace{0.5cm}(i=2, \hspace{0.5cm}3).\hspace{0.5cm}In\hspace{0.5cm}this\hspace{0.5cm}case\hspace{0.5cm}A_2 \cong A_3.\)
5) \[ eR \quad eJ^{-1} \quad eJ^i \quad eJ^{i+1} \quad eJ^p \]
\[ \begin{array}{c}
A_1 - 0 \\
\cdots \\
A_2 - 0 \\
\cdots \\
A_3 \cdots \cdots 0
\end{array} \]
\[ \Delta(A_i) = \Delta \quad (i=1, 2, 3). \text{ In this case } A_3 \neq A_2 \text{ and } A_2 \text{ may be zero.} \]

6) \[ eR \quad eJ^{-1} \quad eJ^i \quad eJ^{i+1} \quad eJ^p \]
\[ \begin{array}{c}
A_1 - B_1 - 0 \\
\cdots \\
A_2 - B_2 \cdots \cdots 0
\end{array} \]
\[ \Delta(A_i) = \Delta \quad (i=1, 2). \]

7) \[ eR \quad eJ^{-1} \quad eJ^i \quad eJ^{i+1} \quad eJ^k \quad eJ^{k+1} \quad eJ^p \]
\[ \begin{array}{c}
A_1 - 0 \\
\cdots \\
A_2 \cdots \cdots B_1 - 0 \\
\cdots \\
B_2 \cdots \cdots 0
\end{array} \]
\[ \Delta(A_i) = \Delta \quad (i=1, 2, 3) \text{ and } \Delta(B_j) = \Delta \quad (j=1, 2). \]

8) \[ eR \quad eJ^{-1} \quad eJ^i \quad eJ^{i+1} \quad eJ^{p-1} \quad eJ^p \]
\[ \begin{array}{c}
A_1 - 0 \\
\cdots \\
A_2 \cdots \cdots B_1 - 0 \\
\cdots \\
B_2 - 0
\end{array} \]
\[ \Delta(A_1) = \Delta(A_2) = \Delta \text{ and } [\Delta: \Delta(B_1)] = 2. \text{ In this case } B_1 \approx B_2, \text{ where each straight line means "uniserial".} \]

5. Examples

We shall give examples of hereditary (resp. left serial) and right US-3 (resp. US-4) rings. Let \( K \) be a field. By \( L \) and \( L' \) we denote extension fields of \( K \) with \([L: K] = 2\) and \([L': K] = 3\), respectively, and \( \mathbb{Z} = \mathbb{Z}/2 \), where \( \mathbb{Z} \) is the ring of integers.

The following two rings are hereditary, left serial and right US-3 rings.
is the second type b) of Theorem 1 and is the first type a) of Theorem 1.

On the other hand

\[
\begin{pmatrix}
K & L & L \\
0 & L & L \\
0 & 0 & K \\
\end{pmatrix}
\]
is a hereditary, non-left serial and right US-3 ring, and

\[
\begin{pmatrix}
L & L & 0 \\
0 & K & K \\
0 & 0 & K \\
\end{pmatrix}
\]
with \(e_{12}^2 e_{23} = 0\) is a left serial, non-hereditary and right US-3 ring.

Next we shall give hereditary and right US-4 rings for each structure in Theorem 4. However, we can not construct an example of the case 5) from the reason given in Remark 13.

\[
\begin{array}{cccc}
1 & \begin{pmatrix}
K & K & K & K \\
K & K & K & K \\
0 & K & & \\
\end{pmatrix} & 2 & \begin{pmatrix}
L & L & L & L \\
L & L & L & L \\
K & K & & \\
0 & K & & \\
\end{pmatrix} & 3 & \begin{pmatrix}
L' & L' & L' \\
L' & L' & & \\
0 & K & & \\
\end{pmatrix} \\
3' & \begin{pmatrix}
D & D^* \\
0 & D_1 \\
\end{pmatrix}, & \text{where } D, D_1 & \text{and } D^* & \text{are given in Remark 14.} \\
4 & \begin{pmatrix}
L & L & L \\
L & 0 & \\
0 & K \\
\end{pmatrix} & 6 & \begin{pmatrix}
K & K & K & K & K \\
K & 0 & 0 & & \\
K & 0 & & & \\
K & & & & \\
0 & K & & & \\
\end{pmatrix} \\
7 & \begin{pmatrix}
\bar{Z} & (\bar{Z}) \\
0 & (\bar{Z}) \\
\bar{Z} & (\bar{Z}) \\
0 & \bar{Z} \\
\end{pmatrix} & 8 & \begin{pmatrix}
K & K & K & K & K \\
K & K & 0 & 0 & \\
K & 0 & 0 & & \\
K & & & & \\
0 & K & & & \\
\end{pmatrix}
\end{array}
\]
where $L$ is an extension of $\bar{Z}$ with $[L: \bar{Z}]=2$. $e_{14}R$ is of the form 2–1) in Lemma 16 and $e_{18}R$ is of the form in Lemma 10.

The rings of 1)~6), 8), 10) and 11) are left serial.

If $R$ is either hereditary or left serial, $A_4/E_2 \cong A_2/F_2$ implies $A_1 \cong A_2$ in Lemma 16. In general this is not true for US-4 rings.

We shall give rings of the type a) in Lemma 16. Let $R=\sum \oplus e_iR$ and $e_ie_j=\delta_{ij}e_i$ (the $e_i$ are primitive idempotents).

1) $A_1/E_2 \cong A_2/F_2$ and $E_2 \cong F_2$

\begin{align*}
e_{14} & = e_{14}Z + e_{14}J \\
A_1 & = (1, 2)Z + (1, 2)(2, 3)Z \\
E_2 & = (1, 2)(2, 3)Z \\
F_2 & = (1, 2)(2, 3)Z
\end{align*}

\begin{align*}
e_{18} & = e_{18}Z + e_{18}J \\
(2, 3)Z & \quad (2, 3)Z
\end{align*}

and (1, 2) $(2, 3)^{'}(1, 2)(2, 3)^{'}=0$. This is a type of a-1-1) and a-1-4). ($R$ is a finite ring.)

2) $A_4/E_2 \cong A_2/F_2$, $E_2 \cong F_2$
and $(1, 2)(2, 4) = (1, 2)'(2, 3) = 0$, where $K$ is a finite field of characteristic 2. This is a type of a-1-1).

3)

$$e_1 R = e_1 \mathbb{Z} + e_1 J$$

$$A_1 = (1, 2) \mathbb{Z} + (1, 2)(2, 3)K \quad A_2 = (1, 2)' \mathbb{Z} + (1, 2)'(2, 4)K$$

$$E_2 = (1, 2)(2, 3)K \quad F_2 = (1, 2)'(2, 4)K$$

$$e_2 R = e_2 K + e_2 J$$

$$e_3 R = e_3 K$$

$$e_4 R = e_4 K$$

This is a type of a-1-2). If $K = \mathbb{Z}$, $R$ is a left serial and finite ring.

4)

$$e_1 R = e_1 \mathbb{Z} + e_1 J$$

$$A_1 = (1, 2) \mathbb{Z} + E_2 \quad A_2 = (1, 1) \mathbb{Z} + F_2$$

$$E_2 = (1, 2)(2, 3)K \quad F_2 = (1, 1)(1, 2) \mathbb{Z} + F_3$$

$$F_3 = (1, 1)(1, 2)(2, 3)K$$

$$e_2 R = e_2 \mathbb{Z} + e_2 J$$

$$e_3 R = e_3 K$$

This is a type of a-1-2). If $K = \mathbb{Z}$, $R$ is a left serial and finite ring.
\[ e_2 R = e_2 \mathbb{Z} + e_2 J \quad e_4 R = e_4 \mathbb{Z} + e_4 J \quad e_6 R = e_6 \mathbb{Z} + e_6 J \]
\[
\begin{array}{c|c|c}
(2, 4) & (3, 5) & 0 \\
\hline
(5, 4) & (6, 4) & 0
\end{array}
\]

This is a type of \( a_{1-3} \).

Other products among \((i, j)\) are zero (e.g. \((1, 1)(1, 1) = 0\)). In the above \( e_i(k, l)e_j = (k, l)\delta_{ij} \delta_{ij} \), \((\delta_{ij}\text{ is Kronecker delta})\).

Similarly we can construct a US-4 ring of \( a_{-2-1} \) in Lemma 16. Finally we shall give an example concerning ii) of Lemma 12.

Let \( K \) be a field of characteristic 2 and \( L \) an extension of \( K \); \( L = K(a) \) and \( a^2 \in K \). Put \( g(a) = b \neq 0 \) in \( L \) and \( g(1) = 0 \). Then \( g \) is a derivation of \( L \) over \( K \). Put

\[
R = \begin{pmatrix} L & (L) \\ (L) & L \end{pmatrix},
\]

where \( l \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} l & g(l) \\ 0 & l \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \quad (l_1, l_2 \in L) \)
as in Remark 13. Then \( e_{11} J = A_1 \oplus A_2 \) and \( \Delta(A_1) = \Delta, [\Delta: \Delta(A_2)] = 2 \). However, \( e_{11} R \) does not satisfy \((\#; 1)\) as an \( L-L \)-module. Hence \( e_{11} R \) has the similar form to ii) of Lemma 12, but \( R \) is not right US-4.

References


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