Generalizations of Nakayama ring. VI. (Right US-n rings; n=3,4)

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Osaka University
GENERALIZATIONS OF NAKAYAMA RING VI

(RIHGT US-n RINGS; n=3, 4)

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

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We have studied artinian right US-3 rings in [5] and right US-4 algebras over an algebraically closed field in [7]. We shall continue, in this paper, to study a right US-3 (resp. US-4) ring \( R \) when \( R \) is either hereditary or left serial.

In the first two sections, we shall give the characterization of a right US-3 (resp. US-4) ring \( R \), when \( R \) satisfies a weaker condition \((\ast, 1')\) (see § 1) than \( R \) being either hereditary or left serial. In the next two sections, we shall specify the characterizations given in the previous sections to hereditary rings and left serial rings. We shall exhibit several examples in the final section to illustrate the above characterizations.

1. US-3 rings

Throughout this paper we deal with an artinian ring \( R \) and every \( R \)-module is a unitary right \( R \)-module. We shall use the same terminologies and definitions given in [2]~[8].

As a generalization of right serial rings, we considered

\[(\ast, n)\quad \text{Every maximal submodule in a direct sum } D \text{ of } n \text{ hollow modules contains a non-zero direct summand of } D \quad \text{[5].}\]

It is clear that if \( D/f(D) \) is not homogeneous, \( D \) satisfies \((\ast, n)\). Hence we may restrict ourselves to hollow modules of a form \( eR/E \), where \( e \) is a primitive idempotent and \( E \) is a submodule of \( eR \). If \((\ast, n)\) holds for any direct sum of \( n \) hollow modules, we call \( R \) a right US-\( n \) ring [5]. Since the concept of US-\( n \) rings is Morita equivalent, we study always a basic ring.

We studied right US-\( n \) algebras over an algebraically closed field for \( n=3 \) and 4 in [5] and [7], respectively. In this and next sections we shall give a complete list of the structure of right US-3 (resp. US-4) rings with \((\ast, 1')\) below. We can give theoretically the complete structure, however as we know a few properties of division rings, we can not give the complete examples for each structure.
We quote here a particular property of a semisimple module (cf. [8] and [9]).

Let e be a primitive idempotent in R and D a semisimple R-module and (\#, m) a left eRe-module. For any two R-submodules \(V_1\) and \(V_2\) with \(|V_1|=|V_2|=m\), there exists a unit x in eRe such that \(xV_1=V_2\).

Further we consider one more property:

\[(1')\] \(eJ^i\) is a direct sum of hollow modules for each primitive idempotent e and each i.

If R satisfies \((1')\), then \((1')\) holds. Moreover, if R is hereditary or left serial, \((1')\) holds by [11], Corollary 4.2. Under the assumption \((1')\), we obtain the following diagram (cf. [8]):

\[
\begin{array}{cccc}
A_1 & A_2 & \cdots & A_m \\
A_{11} & A_{12} & \cdots & \cdots & eJ^1 \\
\end{array}
\]

where the A are hollow.

Let \(A_1, A_2\) be submodules in \(eR\). If there exists a unit x in eRe such that \(xA_1\supset A_2\) or \(xA_1\subset A_2\), we indicate this situation by \(A_1\sim A_2\) [4]. We put \(\Delta = eRe/eJ^\infty (=e\overline{Re})\) and \(\Delta(A_1) = \{x\in\Delta, xA_1\subset A_2\}\) [2].

Let \(D=A_1\oplus A_2\); the \(A_i\) are uniserial. A submodule \(B=B_1\oplus B_2\ (A_1\supset B_i)\) is called a standard submodule in D [3].

**Lemma 1.** Let \(A_1\) and \(A_2\) be as in \((1)\). If \(A_1\sim A_2, A_1=xA_2\) for some unit element x in eRe, and hence \(A_1\sim A_2\).

**Proof.** Since \(A_1\sim A_2\), there exists a unit x in eRe such that \(xA_1\supset A_2\) or \(xA_1\subset A_2\). We may assume that \(xA_1\subset A_2\). If \(xA_1\supset A_2, xA_1\subset J(A_2)\subset eJ^{i+1}\), since \(A_2\) is hollow. Hence \(A_1\subset x^{-1}eJ^{i+1}=eJ^{i+1}\), a contradiction. Therefore \(xA_1= A_2\).

**Lemma 2.** Let \(A_1\) and \(A_2\) be as in \((1)\). Let B be a hollow submodule in \(A_2\), which appears on the level \(eJ^{k+i} (k\geq 0)\) in \((1)\). If \(\Delta(A_1)=\Delta, A_1\sim B\).

**Proof.** First assume \(k\geq 1\) and \(A_1\sim B\), i.e., there exists a unit x in eRe such that \(xA_1\supset B\) or \(xA_1\subset B\). In the latter case \(A_1\subset eJ^{i+1}\). Hence \(xA_1\supset B\). Since \(\Delta(A_1)=\Delta\), there exists an element j in eRe with \((x+j)A_1=A_1\). Let b be a generator of B. Then we obtain a in \(A_1\) with \(xa=b, b=(x+j)a=(x+j)a-ja\). Let \(p_1\) be the projection of \(eJ^i\) to \(A_1\). \(0=p_1(b)=(x+j)a-p_1(ja)\). Assume \(a\in J^{k+i-1}\), and \(p(ja)\in J^{k+i}\), which is a contradiction, since \(x+j\) is a unit in eRe. Finally assume \(B=A_2\). Then \(A_2=x'A_1\) for some unit \(x'\) in eR. Hence we obtain the same situation as above, which is a contradiction.
From [2], Theorem 2 we have

**Lemma 3.** If $R$ is a right $US$-$n$ ring, then $[\Delta : \Delta(A)] \leq n-1$ for any submodule $A$ in $eR$.

Put $\mathcal{R} = R/J^{t+k}$. Then $\widetilde{eRe}_e J = eRe|J^{t+k} = \Delta$. Let $A_1$ be as in (1). Then we can define $\Delta(A_1) = \Delta((A_1 + J^{t+k})/J^{t+k}) = \{x \in \Delta, x(A_1 + J^{t+k}) \subseteq (A_1 + J^{t+k})\}$. It is clear that $\Delta(A_1)$ is a division subring of $\Delta(A)$. 

**Lemma 4.** Let $A_1$ and $A_2$ be as in (1). If $\Delta(A_1) = \Delta$, $\Delta(A_2) = \Delta$.

Next assume that $A_2 = xA_1$ for some unit $x$ in $eRe$. If $[\Delta : \Delta(A_1)] = 2$ (resp. 3), $[\Delta : \Delta(A_2)] = 2$ (resp. 3), where $A_1 = (A_1 + J^{t+k})/J^{t+k} \subseteq R/R^{J^{t+k}}$.

Proof. The first part is clear from the remark above. Assume $(x+j)A_1 \subseteq A_1$ for some $j$ in $eRe$. Since $(x+j)A_1 \subseteq A_2 + jA_1 \subseteq A_2 + eJ^{t+1}$, $(x+j)A_1 \subseteq (A_1 + eJ^{t+k}) \cap (A_2 + eJ^{t+1}) = eJ^{t+1}$, a contradiction. Hence $x \not\in \Delta(A_1)$. Further $[\Delta : \Delta(A_1)]$ is prime, and so $[\Delta : \Delta(A_1)] = [\Delta : \Delta(A)]$.

**Remark 5.** We shall study a right $US$-$n$ ring and observe $[\Delta : \Delta(A_1)]$. Since $[\Delta : \Delta(A_1)] \leq 3$, we may assume $J^{t+1} = 0$ by Lemma 4, [3], Lemma 1 and its proof, when we observe $[\Delta : \Delta(A_1)]$ (the $x$ in Lemma 4 exists, provided $[\Delta : \Delta(A_1)] \geq 2$).

**Theorem 1.** $R$ is a right (basic) $US$-$3$ ring with $(*, 1')$ if and only if $eR$ has one of the following structures for each primitive idempotent $e$.

1) $eRe|J^t$ is uniserial for some $t$ and 
2) $eJ^t = 0$ or $eJ^t = A \oplus B$, where $A$ is simple and $B$ is uniserial, such that 
   a) $[\Delta : \Delta(A)] = 2$ or b) $A \cong B$.

In case a) $B$ is simple and $A \oplus B$ satisfies $(\#1, 1)$.

In case b) 
   i) $B$ is simple and $A \cong B$ or 
   ii) $B$ is not simple, and if $A$ is isomorphic to a simple subfactor module $B_i/B_{i+1}$ of $B$, $B_{i+1} = 0$ (i.e., $B_i$ is the socle of $B$) and this isomorphism is given by $j_i$: the left-sided multiplication of $j$ in $eRe$.

Proof. We assume that $R$ is a right $US$-$3$ ring. From $(*, 1')$ and [5], Proposition 1,3) $eJ = A \oplus B$, where $A$ and $B$ are hollow. We may assume $|A| \leq |B|$. $[\Delta : \Delta(C)] \leq 2$ for any submodule $C$ in $eR$ by Lemma 3. Hence we divide ourselves into two cases: I) $[\Delta : \Delta(A)] = 2$ and II) $\Delta = \Delta(A)$.

Case I). Since $[\Delta : \Delta(A)] = 2$, by [5], Proposition 1,2) there exists a unit element $x$ in $eRe$ such that $xA \subseteq J(A) \oplus J(B)$ or $xA \supseteq J(A) \oplus J(B)$. However $A \subseteq eJ^{t+1}$ and so $xA \supseteq J(A) \oplus J(B)$. On the other hand, $|A| = |J(A)+1|$ and $xA = J(A) \oplus J(B)$. Hence $J(B) = 0$. Further $A \cong B$ by Lemma 1 and [5], Proposition 1,2). Therefore $A$ and $B$ are simple and $eJ^{t+1} = 0$. Which means that every
(simple) submodule $C$ in $e^j$ is characteristic if and only if $\Delta(C)=\Delta$. Hence $[\Delta: \Delta(C)]=2$ and $e^j$ satisfies $(\#$, 1) by [5], Proposition 1,2).

Case II). We know from the above argument that $\Delta=\Delta(A)=\Delta(B)$ (note that we did not use the assumption $|A|\leq |B|$). Let $y$ be any unit element in $eRe$.

Since $\Delta=\Delta(A)$, there exists an element $j$ in $eFe$ such that $(y+j)A=A$. Then $(y+j)(A\oplus J(B)) \subseteq A\oplus (y+j)J(B) \subseteq A\oplus e^{j+1} = A\oplus J(B)$. Hence $\Delta(A\oplus J(B)) = \Delta$. Assume that $B$ is not simple. $A\oplus J(B)$ or $J(A)\oplus B$ is hollow by [5], Proposition 1,4)-iv). Hence $J(A) = 0$, i.e., $A$ is simple.

We shall show that $B$ is uniserial. Assume $e^{j+i}=BJ = C_1 \oplus C_2 \oplus \cdots$; the $C_i$ are hollow. If $\Delta(C_1)\neq \Delta$, $C_1 \sim A_i$ by [5], Proposition 1,2), which is a contradiction from Lemma 2. Hence $\Delta=\Delta(C_1)=\Delta(C_2)$. However $\{A, C_1, C_2\}$ derives a contradiction by Lemma 2 and [4], Corollary 2 of Theorem 2, provided $C_2 \neq 0$. Therefore $B$ is uniserial.

Next assume $g: A\approx B_{i+1} \supseteq B_i \supseteq B_{i+1}$. Take $\{A, B_i, B_i(g^{-1})\}$; the graph of $B_i$ with respect to $g^{-1}$). Since $A$ is simple (and hence $eFeB \subseteq B$) and $\Delta(B)=\Delta$, $B$ is characteristic. Hence $A\sim B_i(g^{-1})$, and so there exists a unit $x_i$ in $eRe$ such that $x_iA \subseteq B_i(g^{-1})$. If $B_{i+1} \neq 0$, $x_iA \subseteq B_{i+1} \subseteq e^{j+1}$, a contradiction. Hence $B_{i+1} = 0$ and $g: A\approx B_n$, the socle of $B$. Let $j$ be an element in $eFe$ such that $(x_1+j)A=A$, and put $x_1=x_1+j$. Then $A(g)=x_1A=(x_1-j)A$. Put $A=aR$.

Then $a+g(a)=(x_2-j)a$ for some $r$ in $R$. $eFeA \subseteq e^{j+1}$ and $e^{j+1}=BJ$ imply $eFeA \subseteq B_n$.

Hence $a=x_2ar$ and $g(a)=-jar$,

and so $g(a)=-(x_2^{-1})a$. Therefore $g=(-x_2^{-1})_i$ and $-x_2^{-1}\in eFe$ (b-ii)). Finally assume that $B$ is simple. If $f: A\approx B$, $\{A, B, A(f)\}$ derives a contradiction from [5], Lemma 1, (note $e^{j+i}=0$ and use Lemma 8 below). Hence $A\approx B$ (b-i)). Conversely, assume that $eR$ has one of the structures given in the theorem. Clearly $(\ast, 1')$ holds. Let $\{E_i\}_{i=1}^n$ be any set of submodules in $eR$.

Case a): If $E_i \supseteq e^j$ and $E_i \supseteq e^j$, $\Delta(E_i)=\Delta$ for $i=1, 2$ and $E_1 \supseteq E_2$ or $E_1 \subseteq E_2$. Hence $D=\bigoplus_{i=1}^n E_i$ contains a non-zero direct summand of $D$ by [4], Corollary 1 of Theorem 2. If $E_i \subseteq e^j$ and $E_2 \supseteq e^j$, $E_2=xE_1 (=A)$ for some $x$ in $eRe$ by $(\#$, 1). Hence $D$ satisfies $(\ast\ast, 3)$ again by [4], Corollary 1 of Theorem 2. Case b-i): If $E_i \subseteq e^j$, $x_iE_i$ is a standard submodule in $e^j$ for a unit $x_i=(e+j)$ in $eRe$ by assumption. Hence $E_i \sim E_j$ for some pair $i, j$. Further $\Delta=\Delta(E)$ by assumption. Therefore $D$ satisfies $(\ast\ast, 3)$ by [4], Corollary 1 of Theorem 2. Case b-ii): This is much simpler than the above. Thus $R$ is right US-3.
In the last paragraph of the proof of "only if part" in Theorem 1, we have shown

**Lemma 6.** Assume that $eJ' = A \oplus A' \oplus B$ and 1) $A$ and $A'$ are simple modules with $\Delta(A) = \Delta$, and 2) $B$ is non-simple and uniserial. If $g: A \approx B_i \approx B_{i+1}$ and $A \sim B_i(g^{-1})$, $B_{i+1} = 0$ and $g$ is given by $j; j \in eJ$, and hence $i > 1$ (cf. [7], Lemma 16).

We shall illustrate the structure in Theorem 1 as the following diagram:

1) $\begin{array}{ccc} eR & eJ & eJ' \\ \cdot & \cdot & \cdot \\ 0 \end{array}$

2) $\begin{array}{ccc} eR & eJ & eJ' & eJ' \\ \cdot & \cdot & A & 0 \\ A & 0 \end{array}$

where the straight line means uniserial.

It is clear that if $R$ has the structure above, $(*)$, 1) (and hence $(*, 1')$) holds. We note that if $(*, 1')$ does not hold, Theorem 1 is not true (see [6]). We shall give examples of a) and b) in § 5.

2. US-4 rings

Next we shall characterize a right US-4 ring with $(*, 1')$.

**Lemma 7.** Let $R$ be a right US-4 ring and \{A_i\}$_{i=1}^t$ a set of submodules in $eJ$. Then 1) if $\Delta(A_i) = \Delta$ or all $i \leq 3$ and $A_k \sim A_k'$, for $k \neq k' \leq 3$, then $A_k \sim$ (some $A_i$). 2) $A_i \sim A_j$ for some pair $i, j$. 3) If $[\Delta: \Delta(A_i)] = 2$ for $i = 1, 2, A_i \sim A_2$. 4) If $[\Delta: \Delta(A_i)] = 3, A_i \sim A_j$ for all $j$. 5) If $[\Delta: \Delta(A_i)] = 2, A_i \sim A_j$ for some $i, j \leq 3$.

Proof. This is clear from [4], Corollary 2 of Theorem 2.

**Lemma 8.** Let $A_1$ and $A_2$ be as in (1). Assume $J^{i+1} = 0$. If $\Delta(A_i) = \Delta$, $A_i$ is characteristic.

Proof. This is clear.

**Lemma 9.** Let $R$ be a right US-4 (basic) ring, and \{A_i\}$_{i=1}^t$ a set of hollow submodules on the level $eJ'$ in (1). If $\Delta(A_i) = \Delta$ for all $i, t \leq 3$.

Proof. This is clear from Lemmas 7, 8 and Remark 5.

From now on we assume that $R$ is a right US-4 (basic) ring satisfying $(*, 1')$. Let $D = (eJ' =)A_1 \oplus A_2 \oplus \cdots \oplus A_t$, where the $A_i$ are hollow. In the
following lemmas, we mainly assume that $D$ is characteristic. We note $[\Delta: \Delta(A_1)] < 3$ for all $i$ by Lemma 3.

**Lemma 10.** Assume $[\Delta: \Delta(A_1)] = 2$ for all $i$. Then i) $t = 2$. ii) There exists a unit $x$ in $eRe$ such that $xA_1 = A_2$. iii) $A_1$ is a uniserial module with $|A_1| \leq 2$. iv) If there are characteristic submodules in $A_1 \oplus A_2$, they are linear with respect to the inclusion. v) If $B$ is not a characteristic submodule in $A_1 \oplus A_2$, $[\Delta: \Delta(B)] = 2$ and those submodules are related by $\sim$.

**Proof.** We may assume $|A_1| \leq |A_2| \leq \cdots \leq |A_t|$ (note $t \geq 2$). By Lemmas 1 and 7, $A_k = x_k A_1$ for all $k$. Hence

$(\alpha)$ if $[\Delta: \Delta(A_1)] > 2$ for all $i$, there exists a unit $x_i$ in $eRe$ such that $x_i A_1 = A_i$ for all $i$. On the other hand, since $[\Delta: \Delta(A_1)] = 2$, $\Delta = \Delta(A_1) + x_2(\Delta(A_1))$. Assume $e^{t+i+1} = 0$ from Remark 5. Since $D = \Delta A_1 = \Delta(A_1)A_1 + x_2(\Delta(A_1))A_1 = A_1 \oplus A_2$, $t = 2$. We note that from the above argument and Lemma 3 we obtain

$(\beta)$ If $[\Delta: \Delta(A_1)] > 2$ for all $i$, $t \leq 3$. Assume that $A_1/A_1J^*$ is uniserial and $A_1J^* = B_1 \oplus B_2 \oplus \cdots \oplus B_s$, where the $B_j$ are hollow and $s \geq 2$. In order to show $s \leq 1$, we may assume $e^{t+i+1} = 0$ by Remark 5. First we note that there exists a unit $x$ in $eRe$ such that $xA_1 = A_2$. Hence $\Delta(B) = \Delta$ for all $p$. On the other hand, $D J^* = A_1J^* \oplus A_2J^* = \sum \oplus B_1 \oplus \sum \oplus xB_p$, which is a contradiction to $(\beta)$. Therefore $A_1$ and $A_2$ are uniserial. Next assume $A_1J^* \neq 0$. $\Delta(A_1J^* \oplus A_2J^*) = \Delta$ by existence of $x_2$. Hence $\{A_1, A_1J^* \oplus A_2J^*\}$ derives a contradiction by Lemma 7. Therefore $|A_1| \leq 2$. Since $\Delta(A_1J^* \oplus A_2J^*) \subset \Delta(A_1)$, $\Delta(A_1J^* \oplus A_2J^*) = \Delta(A_1)$ for $\Delta(J(A_1)) = 2$. Similarly $[\Delta: \Delta(J(A_1))] = 2$. Let $E$ be a submodule with $[\Delta: \Delta(E)] = 3$. Then there exists a unit element $x$ in $eRe$ such that $xE \subset A_1$ or $xE \supset A_1$ by Lemma 7. In the former case $[\Delta: \Delta(E)] = [\Delta: \Delta(xE)] = 2$. If $xE \supset A_1$, $xE = A_1 \oplus E'$; $E' \subset A_2$. Hence $[\Delta: \Delta(xE)] = 2$ from the above. Therefore there are no submodules $E$ with $[\Delta: \Delta(E)] = 3$. Finally assume that $A_1 \oplus A_2$ contains two characteristic submodules $C_1, C_2$ such that $C_1 \sim C_2$. Consider $\{A_1, A_2, C_1, C_2\}$, and $A_1 \sim A_2$. If $A_1 \supset C_1, C_1 = 0$ and if $A_1 \subset C_1, C_1 = A_1 \oplus F$; $F \subset A_1$, and so $C_1 \supset C_2$ or $C_1 \subset C_2$. Let $\Delta(E) = \Delta$. If $|A_1| = 1$, $E$ is characteristic. Assume $|A_1| = 2$. Put $C_1 = A_1 \oplus B_2$. Then $E \sim C_1$ from the above. Hence $E \subset C_1$ or $E \supset C_1$, and so $E$ is characteristic.

**Lemma 11.** Assume $[\Delta: \Delta(A_1)] = 3$ for all $i$. Then $t \leq 3$, and the $A_i$ are simple and there exists a unit $x_i$ in $eRe$ such that $x_i A_1 = A_i$ for each $i$. If $t = 3$, $D$ satisfies $(\# 1)$ and $(\# 2)$ and $[\Delta: \Delta(C)] \leq 3$ for every submodule $C$ in $D$. If $t = 2$, $D$ satisfies $(\# 1)$.

**Proof.** Since $[\Delta: \Delta(A_1)] = 3$, there exists a unit $x_i$ in $eRe$ such that $x_i A_1 = A_i$.
from (α) and \( t \leq 3 \) by (β). Assume \( t=3 \). Taking \( \{A_i, J(D)\} \), we know from Lemma 7 that \( A_i \) is simple and hence \( e f^{i+1} = 0 \). It is clear from Lemmas 7 and 8 that there are no simple submodules \( B \) in \( D \) with \( \Delta(B) = \Delta \). Hence \( D \) satisfies (\#, 1). Let \( C \) be a submodule of \( D \) with \( |C| = 2 \). Then \( D = C \oplus A_i \) for some \( i \). Hence \( \Delta(C) \neq \Delta \) by Lemma 7, and so \( D \) satisfies (\#, 2). We obtain the similar result for \( t=2 \).

**Lemma 12.** Assume \( [\Delta: \Delta(A_i)] = 1 \) and \( \Delta(A_i) \neq \Delta \) for \( i \geq 2 \). Then \( A_i \) is uniserial and \( t \leq 3 \).

i) \( t=3 \):
Then all \( A_i \) are simple, \( [\Delta: \Delta(A_i)] = 2 \) for \( i = 2, 3 \), \( A_1 \cong A_2 \) and \( A_2 \oplus A_3 \) satisfies (\#, 1).

ii) \( t=2 \):
\( A_1 \) is not simple.
Then \( [\Delta: \Delta(A_2)] = 2 \), and \( A_2 \) is a simple submodule isomorphic to \( B \), the socle of \( A_1 \). If \( A_2 \cong E_i \oplus E_{i+1} \), \( E_i = B \) and \( E_{i+1} = 0 \). Further \( B \oplus A_2 \) satisfies (\#, 1) except \( B \).

b) \( A_1 \) is simple.
Then
1) \( [\Delta: \Delta(A_2)] = 2 \), \( A_1 \oplus A_2 \| J(A_2) \) satisfies (\#, 1) except \( A_1 \).
2) \( A_2 \downarrow A_2 f^t \) is uniserial for some \( t \) and
   a) \( A_2 f^t = 0 \) or
   b) \( A_2 f^t \) is simple and \( B_2 \) is uniserial.
2-ii-1) \( \Delta(B_1) = \Delta(B_2) = \Delta \).
2-ii-1-1) \( B_1 \cong B_2 \| J(B_2) \).
2-ii-1-2) \( A_1 \cong F_i \oplus F_{i+1} \) (\( A_2 \cong F_i \oplus F_{i+1} \oplus B_1 \oplus B_2 \)).
2-ii-1-3) If \( f \cdot A_i \cong G_j \| G_{j+1} \) (\( f' \cdot B_i \cong G_j \| G_{j+1} \) (\( B_2 \cong G_j \| G_{j+1} \)), then \( G_{j+1} = 0 \) and \( f(f') \) is given by \( j_i \); \( j \in eJ \).
2-ii-1-4) If \( f \cdot A_i \cong B_1 \), we have the same result as 2-ii-1-3).
2-ii-2) \( [\Delta: \Delta(B_2)] = 2 \).
2-ii-2-1) \( B_1 \) and \( B_2 \) are simple and \( B_1 \oplus B_2 \) satisfies (\#, 1).
2-ii-2-2) \( A_2 \cong F_i \oplus F_{i+1} \) (\( A_2 \cong F_i \oplus F_{i+1} \oplus B_2 \)).
2-ii-2-3) If \( A_1 \cong B_1 \), then \( f \) is given by \( j_i \); \( j \in eJ \).

Proof. It is clear, from the assumption and Lemmas 1 and 7, that \( [\Delta: \Delta(A_i)] = 2 \) for all \( i \geq 2 \). Assume that \( A_i \) contains two independent submodules \( B_1, B_2 \). If \( \Delta(B_1) = \Delta(B_2) = \Delta \), \( \{B_1, B_2, A_2, A_3\} \) derives a contradiction by Lemmas 7, 8 and Remark 5. On the other hand, if \( \Delta(B_1) \neq \Delta \), \( \{B_1, B_2, A_2, A_3\} \) derives again a contradiction. Hence \( A_1 \) is uniserial by (\#, 1).

a) \( J(A_i) = 0 \): Consider \( \{A_i, A_3, J(D)\} \). Then \( A_i \cong A_2 \) by Lemma 2. Hence \( J(D) \sim A_i \) or \( J(D) \sim A_i \) by Lemma 7. However \( J(D) \sim A_2 \), since \( J(D) \) is
characteristic and \( J(A_i) \neq 0 \). Hence
\[
\text{the } A_i \text{ are simple for all } i \geq 2.
\]
Since \([\Delta: \Delta(A_i)] = 2\), there exists \( x_i \) in \( eRe \) such that \( x_i A_2 = A_i \) for \( i \geq 2 \) by Lemmas 1 and 7. Hence in order to show \( t \leq 3 \), we may assume \( J^{i+1} = 0 \) by Remark 5. Noting \( x \notin \Delta(A_2), \Delta = \Delta(A_2) \oplus x \Delta(A_2) \), which implies that \( A_2 \oplus A_3 = \Delta A_2 \supset \bigoplus_{i=1}^t A_i \). Hence \( t \leq 3 \). Assume \( t = 3 \). Now we resume to the original situation. We note \( eRe \subseteq J(A_i) \), and hence \( A_i \) is characteristic. Since \( \Delta(J(A_i)) = \Delta, \Delta(J(A_i) \oplus A_2) \neq \Delta \). Consider \( \{A_1, J(A_i) \oplus A_2, A_2 \oplus A_3\} \). \( \Delta(A_1) = \Delta \) and \( \Delta(J(A_i) \oplus A_2) \neq \Delta \) imply \( (J(A_i) \oplus A_2) \supset (A_2 \oplus A_3) \). Hence there exists a unit \( y \) in \( eRe \) such that \( xJ(A_i) \subset (A_2 \oplus A_3) \) or \( xJ(A_i) \supset (A_2 \oplus A_3) \). Hence \( xJ(A_i) \subset (A_2 \oplus A_3) \). Taking \( R = R[J^{i+1}] \), we know that it is impossible. Therefore \( t = 2 \) provided \( J(A_i) \neq 0 \), i.e.,
\[
D = A_1 \oplus A_2 \quad (J(A_i) \neq 0).
\]

Now we take the similar manner to Lemma 6. Assume \( f: A_2 \approx E_i/E_{i+1}; A_i \supset E_i \supset E_{i+1} \). We note that \( A_i \) is characteristic. \( \{A_1, A_2, E_i(f^{-1})\} \) implies \( A_2 \sim E_i(f^{-1}) \) from the above remark and Lemma 7. Hence \( E_{i+1} = 0 \) as the proof of Lemma 6. Further since \( \Delta(A_2) \neq \Delta, A_2 \approx E_n \); the socle of \( A_1 \). Let \( C(\neq E_n) \) be a simple submodule in \( E_n \oplus A_2 \). Consider \( \{A_1, C, A_2, A_3\} \). It is clear that if \( C \sim A_1, C \subset A_1 \). Hence \( C \sim A_2 \) by Lemmas 2 and 7, and so \( E_n \oplus A_2 \) satisfies \((\#), 1)\) except \( E_n \).

b) \( J(A_i) = 0, t \geq 3 \). Assume \( J(A_i) = 0 \). Since \( t \geq 3 \), there exists a unit \( x \) in \( eRe \) with \( xA_2 = A_3 \) by Lemmas 1 and 7, and so \( \Delta(A_1 \oplus J(A_2)) \neq \Delta \). Then \( A_2 \sim A_1 \oplus J(A_2) \) by Lemma 7. Assume \( A_2 \supset y(A_1 \oplus J(A_2)) \) for some unit \( y \). Since \( A_1 \) is simple and \( \Delta(A_1) = \Delta, p_i(yA_1) = A_1, \) where \( p_i: eJ^i \to A_i \) the projection, which is a contradiction. Similarly, since \( A_1 \) is simple and \( A_2 \) is not, \( p_2(yA_1) \subset J(A_2) \) for any unit \( y' \) in \( eRe \). Hence \( A_2 \not\subset y'(A_1 \oplus J(A_2)) \). Therefore
\[
A_2 \quad \text{(and so } A_i \text{ (} i \geq 2 \text{)) is simple.}
\]

Accordingly \( t = 3 \) from the initial paragraph of a). If \( f: A_1 \approx A_2, \{A_1, A_2(f), A_2, A_3\} \) derives a contradiction, since \( \Delta A_2 = A_2 \oplus A_3 \) as before (note \( eJ^{i+1} = 0 \)). Hence \( A_1 \sim A_2 \). Further if \( A_2 \oplus A_3 \) contains a characteristic submodule \( B \neq 0 \), \( \{A_1, B, A_2, A_3\} \) derives a contradiction. Therefore \( A_2 \oplus A_3 \) satisfies \((\#), 1)\).

Case \( t = 2 \) and \( J(A_i) = 0 \). First we shall show that \( A_i \oplus A_2 \middle| J(A_2) \) satisfies \((\#), 1)\) except \( A_1 \). Since \( \Delta(A_2) \neq \Delta \), there exists a unit \( x \) in \( eRe \) such that \( p_i(xA_2) = A_1, \) where \( p_i: eJ^i \to A_i \) the projection. Further \( eReA_2 \subset A_2 \), since \( A_i \) is simple. Hence \( (x + j)(A_2 + J^{i+1}) = A_2 + J^{i+1} \) for any \( j \) in \( eRe \), and so \( \Delta(A_2) = \Delta((A_2 + J^{i+1})/J^{i+1}) \). Therefore we may assume \( J^{i+1} = 0 \) (cf. Remark 5). Then
$A_1 \oplus A_2$ satisfies $(\#_1, 1)$ except $A_1$ from Lemma 7. Now we resume the original situation. Since $A_1$ is simple, $eJ^{i+1} = A_2J$. Assume that $A_2/A_2J'$ is uniserial and $eJ^{i+1} = B_1 \oplus B_2 \oplus \cdots \oplus B_n$, where the $B_i$ are hollow. Then from Lemmas 10–16 below, $s \leq 3$. Further $[\Delta : \Delta(B_i)] \leq 2$ by Lemmas 2 and 7. Assume $s = 3$. Then $\Delta(B_j) = \Delta$ (resp. $[\Delta : \Delta(B_j)] = 2$) for some $i$ (resp. $j$) by Lemmas 7 and 10. Hence we remain two cases $\Delta(B_i) = \Delta$, $[\Delta : \Delta(B_i)] = 2$ for $i = 1, 2$, and $\Delta(B_j) = \Delta$ for $j = 2, 3$ and $[\Delta : \Delta(B_j)] = 2$. On the other hand, since $\Delta(A_1) = \Delta$, we do not have such cases by Lemmas 2 and 7. Therefore $s \leq 2$. Similarly we do not have a case $\Delta(B_i) = \Delta$ and $[\Delta : \Delta(B_j)] = 2$. Thus we obtain two cases; 2-ii-1): $\Delta(B_i) = \Delta$ for $i = 1, 2$ and 2-ii-2): $[\Delta : \Delta(B_i)] = 2$ for $i = 1, 2$.

2-ii-1) We assume $|B_1| < |B_2|$. {A, B, J(B)_1} gives $J(B_1) = 0$ from Lemmas 2 and 7. Assume $B_2J^s = C_1C_2 \oplus \cdots \oplus C_r$; $s \geq 2$ and the $C_i$ are hollow. If $[\Delta : \Delta(C_i)] \geq 2$, {A, B, C, C} derives a contradiction from Lemmas 2 and 7. Hence $\Delta(C_i) = \Delta(C_j) = \Delta$. Taking $R/J^{i+2}$, we obtain again a contradiction from \{A, B, C, C\} and Lemmas 2, 7 and 8. Accordingly $s \leq 2$. In the same manner given in the proof of 2-ii-1-3), we have 2-ii-1-4).

2-ii-2) Since $[\Delta : \Delta(B_i)] = [\Delta : \Delta(B_j)] = 2$, $B_1 \cong B_2$ by Lemmas 1 and 7. {A, B, J(B)_1} gives $J(B_1) = J(B_2) = 0$. Accordingly $B_1$ and $B_2$ are simple. Let $C$ be any simple submodule in $B_1 \oplus B_2$. Then \{A, B, C\} shows $C = xB$ for some unit $x$ in $eRe$ by Lemmas 2 and 7. Hence $B_1 \oplus B_2$ satisfies $(\#_1, 1)$ (2-ii-2-1).

2-ii-2-2) is same to 2-ii-1-2). If $f$: $A \cong B$, {A, f, B, B} gives $A \cong A(f)$. Hence $f$ is given by $j; j \in eRe$ (2-ii-2-3)).

REMARK 13. We shall consider the situation of ii-b) of Lemma 12. Taking $\tilde{R} = R/J^{i+1}$, we may assume that $eJ^{i+1} = V = A_1 \oplus A_2$: the $A_i$ are simple, $\Delta(A_i) = \Delta$, and $[\Delta : \Delta(A_2)] = 2$. Then $A_1 \cong A_2 \cong g_2Rg_1 = \Delta'$. We shall express $\text{End}_{\Delta'}(V)$ as elements of matrices $(\Delta')^2$. Since $A_1$ is characteristic, for any element $x$ in $\Delta$, $x = \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}$ if $x_1 \in \Delta'$. Hence we
obtain two monomorphisms as rings $f_1, f_2$ of $\Delta$ to $\Delta'$ such that $f_i(x) = x_i$ and a homomorphism $g$ as additive groups of $\Delta$ to $\Delta'$ such that

$$g(x^i) = f_i(x)g(x') + g(x)f_2(x').$$

Then $\Delta(A_2) = g^{-1}(0)$ (note, from i), that $g^{-1}(0)$ is a division subring of $\Delta$.

Hence $[\Delta: \Delta(A_2)] = 2$ is equivalent to

ii) $[\Delta: g^{-1}(0)] = 2$.

Further $(\#)_1$ holds if and only if, for any $\alpha$ in $\Delta$, there exists $x \neq 0$ in $\Delta$ such that

$$\alpha = -f_1(x^{-1})g(x) (= g(x^{-1})f_2(x)), \text{ i.e., } F: \Delta \rightarrow \Delta' (F(x) = f_1(x^{-1})g(x)) \text{ is surjective.}$$

If $\alpha \neq 0, x \neq g^{-1}(0)$. Hence if either $|\Delta|$, cardinal of $\Delta$, ($|\Delta| \leq |\Delta'|$) is finite or $|\Delta| < |\Delta'|$, i), iii) does not hold. Hence we assume that $|\Delta|$ is infinite. Further, since $f_i$ is a monomorphism, we may assume that $\Delta \subset \Delta'$ and $f_i$ is the inclusion.

Now assume that $\Delta'$ is commutative. Then $g$ is a $K$-linear mapping from i), where $K = g^{-1}(0)$. Using those facts and $|\Delta| \geq \infty$, for any $g$ we can show by computation that there exists $\alpha$ in $\Delta'$ not satisfying iii) for any $x \in \Delta$. Therefore if $\Delta'$ is commutative, we do not have the case of ii) of Lemma 12.

REMARK 14. Next we consider the case $t = 2$ in Lemma 11. Let $K$ be a field and $R$ a $K$-algebra. If $[\Delta': K]$ is not divided by 3, this case does not occur. Because, since $V = A_1 \oplus A_2$ and $A_i \cong A_2$, $\text{End}_A(V) = (\Delta')_2$ and $\Delta \subset (\Delta')_2$.

$[\Delta: \Delta(A_1)] = 3$ implies that $4[\Delta': K]$ is divided by 3.

Finally we take division rings given by [10]. Let $D \supset D_1$ be division rings such that $[D: D_1] = 3$ and $[D: D_2] = 2$. Put $D = D_1 + D_2u$, and $D^* = \text{Hom}_{D_1}(D, D_1D)$. Then $[D^*: D_1] = 2$ and $D^*$ is a left $D$-vector space. Define $1^* \in D^*$ by setting $1^*(1) = 1, 1^*(u) = 0$, and put $A_1 = 1^*D_1$. Then $D(A_1) = \{d \in D, dA_1 \subset A_1\} = u^{-1}D_1u$, and so $[D: D(A_1)] = 3$. For any $h$ in $D^*$ and $h^{-1}(0) = D_2u$, we have $D = D_1u \oplus D_1v$. Put $d = h(v_1)$. Then $(u_1^{-1}u)1^*(u_1) = 0$ and $(u_1^{-1}u)1^*(v_1) = d' \neq 0$. Hence $h = (u_1^{-1}u)^*d'^{-1}$, and so $hD_1 = (u_1^{-1}u)A_1$. Therefore $D^*$ satisfies $(\#1)$, $[D: D(A_1)] = 3$ and $[D^*: D_1] = 2$. We shall use $D^*$ in §5, Example 3'.

Now we resume to study the structure of right US-4 rings.

Lemma 15. If $R$ is a US-4 ring with $(\ast, 1')$. $D$ has one of the structures in Lemmas 10, 11, 12 and 16 below.

Proof. Assume 1) $\Delta(A_1) = \Delta(A_2) = \Delta(A_3) = \Delta$. Then $t = 3$ by Lemmas 7 and 8 (the case of Lemma 11 below). 2) $\Delta(A_1) = \Delta(A_2) = \Delta$ and $\Delta(A_i) \neq \Delta$ for $i \geq 3$. Then $\{A_1, A_2, A_i, A_3\}$ derives a contradiction from Lemmas 2 and 7. 3) $\Delta(A_1) = \Delta$ and $\Delta(A_i) \neq \Delta$ for $i \geq 2$. This is a case of Lemma 11. 4) $[\Delta: \Delta(A_i)] = 2$ for $i \leq \text{some } l, [\Delta: \Delta(A_j)] = 3$ for $j > l$. Since $[\Delta: \Delta(A_k)] \geq 2$ for all $k$, from $(\alpha)$ there exists a unit $x_i$ in $eRe$ such that $x_iA_i = A_i$ for all $i$. Hence $\Delta(A_j) = x_i\Delta(A)x_j^{-1}$, and so we obtain the cases of Lemmas 10 and 11.
Lemma 16. Assume $\Delta(A_i) = \Delta$ for all $i$. Then $t \leq 3$, and

1) $t = 3:$

$A_3$ is uniserial and $A_1$, $A_2$ are simple, $A_3 \cong A_2$. If $A_3$ is simple, $A_3 \cong A_1$ and $A_2 \cong A_3$. If $A_3$ is not simple and $f: A_3 \cong F_i/F_{i+1}$, then $F_{i+1} = 0$ and $f$ is given by $j_1$; $j \in eF_0$, and hence $i \geq 1$.

2) $t = 2:$  

i) $A_1 \cong A_2$ (\text{\textcolor{red}{\textit{\textcolor{red}{\textit{\textcolor{red}{\textit{\frac{g_1}{R_1}f_1}}}}}}) if $A_3$ is simple, $A_3 \cong A_2 \bigcirc A_2$. If $A_3$ is not simple and $f: A_3 \cong F_i/F_{i+1}$, then $F_{i+1} = 0$ and $f$ is given by $j_1$; $j \in eF_0$, and hence $i \geq 1$.

Then $A_1$ and $A_2$ are simple and $\Delta = g_1Rg_1 \cong \mathbb{Z}/2$.

ii) $A_1 \cong A_2$ (\text{\textcolor{red}{\textit{\textcolor{red}{\textit{\textcolor{red}{\textit{\frac{g_1}{R_1}f_1}}}}}}) if $A_3$ is uniserial; $A_2 = F_1 \bigcirc F_2 \bigcirc \cdots \bigcirc F_p \bigcirc F_{i+1} = 0$. Then $A_1$ is a uniserial module with $|A_1| < 2$; $A_1 = E_1 \bigcirc E_2 \bigcirc E_3 = 0$.

a-1) $|A_1| = 2$.

a-1-1) If $f: A_1/E_2 \cong A_2/F_2$ (\text{\textcolor{red}{\textit{\textcolor{red}{\textit{\textcolor{red}{\textit{\frac{g_1}{R_1}f_1}}}}}}), $\Delta \cong g_2Rg_2 \cong \mathbb{Z}$. $f$ is a unique isomorphism. In this case put $B_1' = \{x + y | A_1 \bigoplus A_2, j(x) = y\}$.

a-1-2) If $A_1/E_2 \cong A_2/F_2$, $i > 1$, then $i \geq p - 1$.

a-1-3) If $f: E_2 \cong F_i/F_{i+1}$ (\text{\textcolor{red}{\textit{\textcolor{red}{\textit{\textcolor{red}{\textit{\frac{g_1}{R_1}f_1}}}}}} $p > i \geq 2$), $\Delta \cong g_2Rg_2 \cong \mathbb{Z}$. We have the same result as a-2-1) below, replacing $A_1$ with $E_2$. In this case put $B_i' = \{x + y | E_2 \bigoplus E_2, j(x) = y\}$.

a-1-4) $f: E_2 \cong F_p$. If $p = 2$, $\Delta \cong g_2Rg_2 \cong \mathbb{Z}$, where $E_2 \cong F_2 \cong g_4Rg_4$. Further $f' : A_1/E_2 \cong F_2 (A_2/F_2 \cong E_2), A_1(f) = xA_1$ for some unit $x$ in $eR_E$. If $p > 2$, we have the same result as a-2-1) below, replacing $A_1$ with $E_2$. If $f$ is not given by $j_1$, put $B' = E_p(f)$.

a-1-5) Further every submodule in $eF_1$ except $B_1$, $B_1'$ and $B_1''$ is isomorphic to a standard submodule in $eF_1$ via $x_1$; $x$ is a unit in $eR_E$.

a-2). $|A_1| = 1$.

a-2-1) If $A_1 \cong F_i/F_{i+1}$ (\text{\textcolor{red}{\textit{\textcolor{red}{\textit{\textcolor{red}{\textit{\frac{g_1}{R_1}f_1}}}}}}), $\Delta \cong g_2Rg_2 \cong \mathbb{Z}$. Further $A_1 \cong F_i/F_{i+1}$ for any $(i \neq j) f < p$.

a-2-2) Assume $f_1$, $f_2$: $A_1 \cong F_p$ (\text{\textcolor{red}{\textit{\textcolor{red}{\textit{\textcolor{red}{\textit{\frac{g_1}{R_1}f_1}}}}}}).$ If the $f_i$ are not given by $f_1$ in $eF_0$, there exists a unit $x$ in $eR_E$ such that $xA_1 = A_i$ and $xf_1 - f_2 x_1 = j_1 (j \in eF_0)$. In this case $A_1 \cong F_i/F_{i+1}$ (i < p). In particular if $eF_0A_1 = 0$, $\Delta \cong g_2Rg_2 \cong \mathbb{Z}$.

b) $A_2/F_1$ is uniserial and $A_2/F_1$ is not uniserial, i.e., $A_2/F_1 = B_1 \bigoplus B_2 \bigoplus \cdots \bigoplus B_s$, where the $B_i$ are hollow. Then $A_1$ is simple and $s = 2$. Further

$\Delta(B_1) = \Delta(B_2) = \Delta.$

Then $B_1 \cong B_2$ (\text{\textcolor{red}{\textit{\textcolor{red}{\textit{\textcolor{red}{\textit{\frac{g_1}{R_1}f_1}}}}}}) and $B_1$ is simple, $B_2$ is uniserial.

b-1) If $f: A_1 \cong F_i/F_{i+1}$ (\text{\textcolor{red}{\textit{\textcolor{red}{\textit{\textcolor{red}{\textit{f_1}}}}}} $A_2/F_i \cong F_{i+1} \bigoplus B_1 \bigoplus B_2$, then we obtain the same result modulo $B_1 \bigoplus B_2$ as given in a-2-1).

b-1-1) If $f: A_1 \cong B_1$, $f$ is given by $j_1$; $j \in eF_0$.

b-1-3) If $f: A_1 \cong H_i/H_{i+1}$ (\text{\textcolor{red}{\textit{\textcolor{red}{\textit{\textcolor{red}{\textit{\frac{g_1}{R_1}f_1}}}}}} $H_{i+1} = 0$ and $f$ is given by $j_1$; $j \in eF_0$.

b-1-4) If $f: B_1 \cong H_i/H_{i+1}$, then $H_{i+1} = 0$ and $f$ is given by $j_1$; $j \in eF_0$.

b-2) $[\Delta: \Delta(B_i)] = 2$ for $i = 1, 2.$
Then $B_1 \cong B_2$ and $B_1, B_2$ are simple and $V = B_1 \oplus B_2$ satisfies ($\#$, 1).

b_2-1) $A_i \cong F_i/F_{i+1} (A_1 \supset F_i \supset F_{i+1} \supset V)$.

b_2-2) If $f$: $A_i \cong B_i$, $f$ is given by $j_i$; $j \in$ e(7) (cf. [7], Theorem 17.)

Proof. We know $t \geq 3$ by Lemma 9. Assume that $|A_1| \leq |A_2| \leq |A_3|$.

i) $t = 3$. Consider $\{A_1, A_2, A_3, J(D) = J(A_1) \oplus J(A_2) \oplus J(A_3)\}$. Since $\Delta(A_i) = \Delta$, $J(A_1) \oplus J(A_2) \oplus J(A_3)$ is contained in some $A_i$ by Lemmas 2 and 7. Hence $J(A_i) = J(A_2) = 0$ (note $|A_3| \geq |A_1|$). Assume that $A_3$ contains two independent submodules $B_1$ and $B_2$. Assume $f$: $A_i \cong B_i$. Then both $[\Delta: \Delta(A_i)]$ and $[\Delta: \Delta(J(A_i))]$ are not equal to 1 and $[\Delta: \Delta(J(A_i))] = 2$ for $i = 1$ or 2 by Lemma 3, (say $i = 1$). Then $\{B_1, B_2, A_i\}$ contradicts Lemmas 2 and 7, since $B_2 \cong B_1$. Hence $A_3$ is uniserial. Assume $f$: $A_i \cong A_2$. Then $\{A_1, A_2, A_3, A_4(f)\}$ implies $A_4(f)$ is contained in some $A_i$. Since $A_i$ is characteristic (we may assume $J(A_i) = 0$ by Remark 5), $A_i(f) \subset A_i$, which is a contradiction. Finally assume $g$: $A_i \cong F_i/F_{i+1}$. Since $\Delta(A_i) = \Delta$ and $A_3$, $A_2$ are simple, $A_i$ is characteristic. Hence $\{A_1, A_2, F_i(g^{-1}), A_3\}$.

2) $t = 2$.

ii) $A_i \cong A_2$. Assume $\Delta \neq \Delta(A_i(f))$. Then $\{A_1, A_2, A_3(f), A_i(f)\}$ implies $A_i(f) \cong A_i$ for some $i$, say 1 from Lemma 7. Since $A_2 \cong A_1 \cong A_i(f), A_i(f) = xA_i$ for some unit $x$ in $e_\mathfrak{Re}$. Hence $\Delta(A_i(f)) = \Delta(A_i(f)) = \Delta$, a contradiction. Accordingly $\Delta(A_i(f)) = \Delta$. Consider $\{A_1, A_2, A_3(f), J(A_1) \oplus J(A_2)\}$, and $J(A_i) = 0$ by Lemma 7 (note $\Delta(A_i) = \Delta(A_i(f)) = \Delta$). Hence $A_1$ and $A_2$ are simple, and so $e_j^{i+1} = 0$. Let $f$ and $f'$ be two isomorphisms of $A_1$ to $A_2$ and consider $\{A_1, A_2, A_4(f), A_4(f')\}$. Since $e_j^{i+1} = 0$, they are characteristic, and so $A_i(f) = A_i(f')$ by Lemmas 7 and 8. Hence $f = f'$. Considering an isomorphism $\delta f$ for $\delta \in \Delta$, $\Delta = \{0, 1\}$. Since $\text{Hom}_\mathfrak{R}(A_1, A_2) = \{0, 1\}, \Delta = \delta \mathfrak{R} \mathfrak{G} = \{0, 1\}$, where $A \cong \mathfrak{R} \mathfrak{G}$. Assume $A_i \cong A_i$. Assume $A_iJ \neq 0$ and $A_iJ = C_1 \oplus C_2 \oplus \cdots \oplus C_s (s \geq 1)$, where the $C_i$ are hollow. Consider $\{A_1, A_2, A_4(J + C_1), A_4(J + C_s)\} (s \geq 2)$. Then $A_1J + C_1 \cong A_1J + C_2$ by Lemmas 2 and 7, provided $A_1J \neq 0$, Assume $\Delta(A_1J + C_i) = \Delta$ for $i = 1, 2$ and $x(A_1J + C_i) \subset A_1J + C_2$ for some unit $x$. We may assume $J^{i+1} = 0$. There exists $j$ in $e_\mathfrak{Re}$ such that $(x + j)(A_1J + C_1) = A_1J + C_2$. Then $xC_1 \subset (A_1J + C_1) \cap (A_1J + C_2) = A_1J$, and so $C_1 \cong A_1J$, a contradiction by Lemma 2. Hence $\Delta(A_1J + C_i) = \Delta$ for some $i$, say 1. Consequently $A_1J + C_1 \cap (A_1J + C_2) = A_1J$, and so $C_1 \cong A_1J$, a contradiction by Lemma 2. Hence $s = 1$, and so $A_2$ is uniserial, provided $A_1J \neq 0$.

Similarly $A_3$ is also uniserial, provided $A_2J \neq 0$. Now assume that $A_2$ is uniserial ($|A_1| \geq 2$ and hence so is $A_3$). We shall show $|A_2| \leq 2$. Assume $A_2J^s = 0$ and...
hence $A_2J^2+0$. Consider $\{A_1,A_1f+A_2f^2, A_1f^2+A_2J, A_2\}$. Since $A_1\sim A_2$ by Lemma 2, 1) $A_1\sim A_1f+A_2f^2$ or 2) $A_1\sim A_1f^2+A_2J$ ($A_1$ and $A_2$ are symmetry) or 3) $A_1f+A_2J^2\sim A_1J^2+A_2J$.

1) It is clear that $xA_i\supset A_1f+A_2f$ for a unit $x$. However $A_i$ is uniserial, and so $A_2J^2=0$ (note $|A_1|<|A_2|$). 2) This is similar. 3) Assume $x(A_1J^2+A_2J)\supset A_1J^2+A_2J$. Since $\Delta(A_1)=\Delta$, there exists $j$ in $eF$ such that $(x+j)A_1=A_1$. Let $a_3j$ be an element in $A_3$ ($a_3\in A_3, j_3\in J$). Then $x(a_3j+a_3j)=a_3j$ for some $a_3\in A_3, a_3\in A_3, j_3\in J$ and $j_3\in J$. Hence $(x+j)a_3j-a_3j_3+xa_3j_3=a_3j_3$. On the other hand, $(x+j)a_3j-j_3a_3j+xa_3j_3=a_3j_3$. Similarly if $x(A_1J^2+A_2J)\supset A_1J^2+A_2J, A_1J=0$. Therefore $|A_1|<2$.

We observe isomorphisms between sub-factor modules of $A_1$ and $A_2$, and then investigate submodules $X$ in $eF$. It is well known that there exist sub-modules $A_1\supset C\supset C'$ and $A_2\supset D\supset D'$ such that $h: C/C'\sim D/D'$ and $X=\{c+d | \in C\oplus D, h(c+C')=d+D\}$ (cf. [3]). We denote $X$ by $C(h)D$.

a-1) Let $|A_1|=2$.

a-1-1) $f_1, A_1E_2=A_1F_2, (\approx gR/gJ)$.

Then $\Delta=gRg=Z$ from 2-i) and $f$ is a unique isomorphism.

a-1-2) Assume $f_1, A_1E_2=A_1F_2, (\approx gR/gJ)$.

Consider $\{A_1, A_2, E_2 \oplus F_2, A_1(f)F_1\}$. Since $\Delta(A_1)=\Delta(A_2)=\Delta, A_2 \sim A_1(f)F_1$. Further $E_2 \oplus F_2$ being characteristic, from Lemma 7 there exists a unit $x$ in $eR$ such that $x^2A_1 \subset A_1(f)F_1$. Let $p_1, eF^1 \rightarrow A_1$ be the projection and $x=i+2; xA_1=A_1, j \in eF$ as usual. Then for a generator $a$ in $A_1$

$$(x+j)a=ar+f(ar)+z_1+z_2, r \in R, z_1 \in E_2 and z_2 \in F_{i+1}.$$ 

Hence $\quad xa+p_1(ja)=ar+z_1 and p_2(ja)=f(ar)+z_2$.

Since $p_1(ja) \in E_2, xa \equiv ar (\mod E_2)$. Assume $i<\rho-1$. Since $ja \in F_{i-1} \subset F_{i+1}, f(ar)\equiv f(xa) \equiv 0 (\mod F_{i+1})$. However $xa$ is a generator of $A_1$, and hence $f=0$. Therefore $i \equiv \rho-1$.

a-1-3) See a-2-1) below.

a-1-4) $E_2=\approx F_2 (\rho=2)$. We have the situation of 2-i).

Assume further $f, A_1E_2=\approx F_2 (A_1F_2=\approx F_2)$, and consider $\{A_1, A_2, A_1(f), E_2 \oplus F_2\}$. Then $A_1 \sim A_1(f)$ by Lemma 7 and so $A_1(f)=xA_1$ for some unit $x$ in $eRe, since A_1 \sim A_1(f)$. If $\rho>2$, see a-2-2) below.

a-1-5) Let $X$ be a submodule in $eF^1$.

i) $X=A_1(f_1)F_1=F_1(f_1^{-1}) (f_1: A_1 \approx F_1/F_{i+2})$. If $i=1$, consider $R/F_{i+3}$. Then this contradicts 2-i). Hence $i \neq 1, F_i=F_{i-1}$ and $F_{i+2}=0$ from a-1-2). $\{A_1, A_2, E_2 \oplus F_2, A_1(f_1)\}$ shows $A_1(f_1)=xA_1$ for some unit $x$ in $eRe$.

ii) $X=A_1(f_2)A_2 (f_2: A_1(E_2 \approx A_1/F_0)$. Then $X=B_1$ from a-1-1).

iii) $X=A_1(f_3)F_1 (f_3: A_1/E_2 \approx F_1/F_{i+1}, i>1)$ and hence $i=\rho-1 or \rho$ by a-1-2). Then $\{A_2 \oplus F_{i+1}, A_2, E_2 \oplus F_2, A_1(f_3)F_1\}$ shows $A_1(f_3)F_1=x(A_1 \oplus F_{i+1})$.
iv) $X = A_2(f_{i}^\circ)$ ($f_i$: $E_2 \cong A_2/F_2$). \{A_1$, $A_2$, $E_2 \oplus F_2$, $A_4(f_{i}^\circ)$\} shows $A_2 = xA_2(f_{i}^\circ)$.

v) $X = F_i(f_{i}^\circ)$ ($f_i$: $E_2 \cong F_i/F_{i+1}$, $i \geq 2$). In this case $eJ^{i+1} = E_2 \oplus F_2$. Hence this is the case of a-2). Accordingly $X = B_1$, $B'_1$ or $B''$, provided $X$ is not isomorphic to a standard submodule in $eJ^{i+1}$ via $x_i$.

Thus we have shown that $X$ is isomorphic to a standard submodule in $eJ^i$ via $x_i$ except $B_j$, $B'_j$ and $B''$. 

a-2) $|A_1| = 1$.

a-2-1) Let $f$: $A_1 \approx F_i/F_{i+1}$ ($i < p$). If $F_i(f_{i}^\circ) \supset xA_1$ for some unit $x$ in $eRe$, $xA_1 \subset F_{i+1} \subset A_2$, since $F_i(f_{i}^\circ) = F_{i+1}$, which is a contradiction from Lemma 2. We note further that $A_2$ is characteristic, since $\Delta(A_2) = \Delta$ and $A_1$ is simple. Assume $\Delta(F_i(f_{i}^\circ)) \neq \Delta$. Then $\{A_2$, $A_1$, $F_i(f_{i}^\circ)\}$ derives a contradiction from the above remarks and Lemma 7. It is clear that $eRe(F_i(f_{i}^\circ)) \subset eRe(F_i \otimes A_1) \subset F_{i+1}$. Hence $F_i(f_{i}^\circ)$ is also characteristic. Let $f': A_1 \approx F_i/F_{i+1}$ be another isomorphism. $\{A_2$, $A_1$, $F_i(f_{i}^\circ)$, $F_i(f'_i(f_{i}^\circ))\}$ gives $F_i(f_{i}^\circ) = F_{i}F_i(f'_{i}(f_{i}^\circ))$ since they are characteristic. Therefore $f = f'$. Accordingly, $\Delta \approx \bar{g}_z \bar{R}_z g \approx \bar{Z}$ as given in the proof of 2-i). Further assume $g$: $A_1 \approx F_i/F_{i+1}$ ($j < p$). Again consider $\{A_2$, $A_1$, $F_i(f_{i}^\circ)$, $F_j(g_{j}^\circ)\}$. Then $F_i(f_{i}^\circ) \supset F_j(g_{j}^\circ)$ if $i < j$, and so $F_j(g_{j}^\circ) \subset F_{i+1}$, a contradiction.

a-2-2) Assume that $f_1$, $f_2$: $A_1 \approx F_p$ and they are not given by $j_1'$ in $eRe$. Then $\{A_2$, $A_1$, $A_1(f_1)$, $A_1(f_2)\}$ gives, from Lemmas 6 and 7, that $A_1(f_1) = x'A_1(f_2)$ for some unit $x'$ in $eRe$. Since $\Delta(A_1) = \Delta$, there exists $j$ in $eRe$ such that $(x' + j)A_1 = A_1$. Put $x = x' + j$. Then for a generator $a$ in $A_1$

$$(x-j)(a_2 + f_2(a)) = ar + f_1(ar); r \in R.$$ 

Hence

$$xa = ar, x f_2(a) - ja = f_1(ar).$$

Next assume further that $g$: $A_1 \approx F_i/F_{i+1}$ ($i < p$). Consider $\{A_2$, $A_1$, $F_i(q_{-1})$, $A_1(f_{i})\}$. and $F_{i}(q_{-1}) \sim A_1(f_i)$ since $F_{i}(q_{-1})$ is characteristic. Which is a contradiction. In particular, if $eReA_1 = 0$, $A_1(f_{i})$ is characteristic, since $\Delta(A_1(f_{i})) = \Delta$ (if $\Delta \neq \Delta(A_1(f_{i}))$). $\{A_1$, $A_2$, $A_1(f_{i})\}$ gives $A_1 \sim A_1(f_{i})$. Hence $f_1 = f_2$ from the first paragraph, and so $\Delta \approx \bar{g}_z \bar{R}_z g \approx \bar{Z}$ as in the proof of 2-i).

b) $A_2/A_2J^k$ ($k \geq 1$) is uniserial and $A_2J^k = \sum_{i=1}^{s} B_i$ ($s \geq 2$). Then $A_1$ is simple from the initial paragraph of ii). Then $Df^k = eJ^{i+k} = A_2J^k$. Since $eJ^{i+k} = B_1 \oplus \cdots \oplus B_s$, $s \leq 3$ from Lemma 15, and $[\Delta: \Delta(B_i)] \leq 2$ for all $i$ by Lemmas 2 and 7. If $[\Delta: \Delta(B_1)] = 1$ and $[\Delta: \Delta(B_2)] = 2$, $\{A_1$, $B_1$, $B_2$, $B_3\}$ derives a contradiction. Hence either $[\Delta: \Delta(B_1)] = 1$ for all $i$ ($b_i$) or $[\Delta: \Delta(B_2)] = 2$ for all $i$ ($b_2$). In the former case $s = 2$ by Lemma 7 and in the latter case also $s = 2$ and $B_2 = xB_1$ for some unit $x$ in $eRe$ by Lemma 10.
Then \( \{A_1, B_1, B_2, J(B_1) \oplus J(B_2) \} \) implies \( J(B_1) = 0 \) (\(|B_1| \leq |B_2|\)). If \( f: B_1 \approx B_2, \{A_1, B_1, B_2, B_1(f)\} \) derives a contradiction. Hence \( B_1 \approx B_2 \). We can show as before that \( B_2 \) is uniserial.

b) \( \Delta(B_1) = \Delta(B_2) = \Delta \).

Then \( \{A_1, B_1, B_2, J(B_1) \oplus J(B_2) \} \) implies \( J(B_1) = 0 \) (\(|B_1| \leq |B_2|\)). If \( f: B_1 \approx B_2, \{A_1, B_1, B_2, B_1(f)\} \) derives a contradiction. Hence \( B_1 \approx B_2 \). We can show as before that \( B_2 \) is uniserial.

b1) This is the case of a-2-1).

b1-2) Assume \( f: A_1 \approx B_1 \). \( \{A_1, A_1(f), B_1, B_1(f)\} \) derives \( A_1 \approx A_1(f) \), i.e., \((x + j)A_1 = A_1(f)\): \( xA_1 = A_1 \) and \( j \in e \). \( (x + j)a = ar + f(ar) \), \( A_1 = aR, r \in R \).

Hence \( xa = ar \) and \( ja = f(ar) \). Put \( xa = b, A_1 = bR, f(b) = jx^{-1}b \).

b1-3) Assume \( f: A_1 \approx H_i/A_{i+1} \). \( \{A_1, B_1, B_2, H_i(f^{-1})\} \) shows \( A_1 \approx H_i(f^{-1}) \).

Hence \( H_{i+1} = 0 \) and \( f \) is given by \( j_i \) as above (cf. Lemma 6).

b1-4) Assume \( f: B_i \approx H_i/A_{i+1} \). \( \{A_1, B_1, B_2, H_i(f^{-1})\} \) derives \( B_1 \approx H_i(f^{-1}) \), since \( \Delta(B_1) = \Delta \). Hence \( H_{i+1} = 0 \) and \( f \) is given by \( j_i \) from Lemma 6.

b2) \( [\Delta: \Delta(B_i)] = 2 \) for \( i = 1, 2, (B_2 = xB_1) \).

\( \{A_1, B_1, B_2, J(B_1) \oplus J(B_2) \} \) shows, from Lemma 2, that \( J(B_2) = 0 \), i.e., \( B_2 \) is simple. Further since \( \Delta(A_1) = \Delta \) and \( [\Delta: \Delta(B_i)] = 2 \), \([\Delta: \Delta(E)] = 2 \) for all simple submodules \( E \) in \( V = B_1 \oplus B_2 \) by Lemmas 2 and 8. Hence \( V \) satisfies (\# 1) by Lemma 7.

b2-1) If \( A_1 \approx F_i/F_{i+1}, \Delta = \tilde{Z} \) by a-1-3). Hence \( \Delta(B_1) = \Delta \).

b2-2) Assume \( f: A_1 \approx B_1 \). \( \{A_1, B_1, B_1, A_1(f)\} \) derives \( A_1 \approx A_1(f) \). Hence \( f \) is given by \( j_i \) as b1-2).

**Remark 17.** If \( R \) is an algebra over an algebraically closed field \( K, \Delta = \bar{Z} \) and the first part of a-2-2) does not occur (take \( f_2 = k f_1, k \neq 1; k \in K \)). We can express \( f \) in a-1-2) as an element in \( efe \), however it is little complicated (cf. [7], Theorem 17).

In order to make the converse version clear, we illustrate the structure of Lemmas 10~16 as follows:

1) (Lemma 10)

\[
\begin{array}{c|c|c}
\text{eR} & \text{eJ}^i & \text{eJ}^{i+1} \\
\hline
& A_1 & B_1 \\
\hline
A_2 = xA_1 - B_2 = xB_1 - 0
\end{array}
\]

\( [\Delta: \Delta(A_1)] = [\Delta: \Delta(A_2)] = 2 \), every characteristic submodule in \( eJ^i \) is linear with respect to the inclusion and \( [\Delta: \Delta(C)] = 2 \) for any non-characteristic submodule \( C \) in \( eJ^i \). Further those \( C \) are related to one another with respect to \( \approx \).

2) (Lemma 11)

\[
\begin{array}{c|c|c}
\text{eR} & \text{eJ}^i & \text{eJ}^{i+1} \\
\hline
& A_1 & 0 \\
\hline
A_1 = A_1 & 0 & 0 \\
A_2 = x_2 A_1 - 0 & 0 & 0 \\
A_3 = x_3 A_1 - 0 & 0 & 0 \\
\end{array}
\]

(3)
\[ \Delta : \Delta(A_1) = [\Delta : \Delta(A_2)] = [\Delta : \Delta(A_3)] = 3 \text{ and } A_1 \oplus A_2 \oplus A_3 \text{ satisfies (1), and } \text{and (2). Further } [\Delta : \Delta(C)] \leq 3 \text{ for every submodule } C \text{ in } A_1 \oplus A_2 \oplus A_3. \text{ (}A_s \text{ may be zero.)} \]

3) (Lemma 12, i))

\[
\begin{array}{c|c}
\text{eR} & eJ^i \\
\hline
A_1 & -0 \\
A_2 & -0 \\
A_s = x_s A_2 & -0 \\
\end{array}
\]

\[ [\Delta : \Delta(A_1)] = 1 \text{ and } [\Delta : \Delta(A_2)] = [\Delta : \Delta(A_3)] = 2. \text{ Further } A_2 \oplus A_3 \text{ satisfies (1).} \]

4) (Lemma 12, ii-a))

\[
\begin{array}{c|c}
\text{eR} & eJ^i \\
\hline
A_1 & E_1 -0 \\
A_2 & 0 \\
\end{array}
\]

\[ [\Delta : \Delta(A_1)] = 1, [\Delta : \Delta(A_2)] = 2 \text{ and } A_2 \oplus E_1 \text{ satisfies (1) except } E_1. \]

5) (Lemma 12, ii-b ii-b-2-ii-1))

\[
\begin{array}{c|c|c|c}
\text{eR} & eJ^i & eJ^{i+k} & eJ^p \\
\hline
A_1 & 0 & B_1 & 0 \\
A_2 & B_2 & 0 \\
\end{array}
\]

\[ [\Delta : \Delta(A_1)] = 1, [\Delta : \Delta(A_2)] = 2 \text{ and } [\Delta : \Delta(B_1)] = [\Delta : \Delta(B_2)] = 1. A_1 \oplus A_2 \oplus ](A_2) \text{ satisfies (1) except } A_1. \text{ (}B_2 \text{ may be zero.)} \]

5')

\[
\begin{array}{c|c|c}
\text{eR} & eJ^i & eJ^{i+k} \\
\hline
A_1 & 0 & B_1 & B_2 \\
A_2 & 0 & 0 \\
\end{array}
\]
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\[ [\Delta : \Delta(B_1)] = [\Delta : \Delta(B_2)] = \text{2 and } B_1 \approx B_2. \quad B_1 \oplus B_2 \text{ satisfies } (\#, 1). \]

6) (Lemma 16, 1))

\[
\begin{array}{c|c|c}
\Delta : \Delta & eJ^i & eJ^{i+n-1} \\
\hline
\Delta : \Delta(B_1) & eJ^i & eJ^{i+n-1} \\
\hline
A_1 & E_n = 0 & 0 \\
\hline
A_2 & 0 & \\
\hline
A_3 & \\
\end{array}
\]

\[ [\Delta : \Delta(A_1)] = [\Delta : \Delta(A_2)] = 1. \quad \text{If } f : A_2 \approx E_n, \quad f \text{ is given by } j; j \in efe. \quad (A_3, A_4 \text{ and } E_4 \text{ may be zero).} \]

7) (Lemma 16, 2-i))

\[
\begin{array}{c|c|c}
\Delta : \Delta & eJ^i \\
\hline
\Delta : \Delta(A_1) & eJ^i \\
\hline
A_1 & 0 \\
\hline
A_2 & 0 \\
\end{array}
\]

\[ [\Delta : \Delta(A_1)] = [\Delta : \Delta(A_2)] = 1 \text{ and } \Delta \approx \Delta' \approx \mathbb{Z}. \]

8) (Lemma 16, 2-ii-a-1))

\[
\begin{array}{c|c|c|c}
\Delta : \Delta & eJ^i & eJ^{i+1} & eJ^{i+n-1} \\
\hline
\Delta : \Delta(A_1) & eJ^i & eJ^{i+1} & eJ^{i+n-1} \\
\hline
A_1 & E_2 = 0 & 0 & \\
\hline
A_2 & F_2 & F_n = 0 & \\
\end{array}
\]

\[ [\Delta : \Delta(A_1)] = [\Delta : \Delta(A_2)] = 1. \quad \text{If } f : A_2/[E_2 \approx A_3]/F_2, \Delta \approx \Delta' \approx \mathbb{Z}. \quad \text{Every submodule except } B_1, B_1' \text{ and } B'' \text{ is isomorphic to a standard submodule via } x_t. \quad (\text{If } n=2 \text{ and } E_2 \approx F_2, \Delta \approx \Delta' \approx \mathbb{Z}.) \quad \text{If } E_2=0, \text{ the conditions in a-2) of Lemma 16 are fulfilled.} \]

9) (Lemma 16, 2-ii-b_1))

\[
\begin{array}{c|c|c|c}
\Delta : \Delta & eJ^i & eJ^{i+n} & eJ^{i+n-1} \\
\hline
\Delta : \Delta(A_1) & eJ^i & eJ^{i+n} & eJ^{i+n-1} \\
\hline
A_1 & B_1 = 0 & 0 & \\
\hline
A_2 & B_2 & F_n = 0 & \\
\end{array}
\]

\[ [\Delta : \Delta(A_1)] = [\Delta : \Delta(A_2)] = 1, [\Delta : \Delta(B_1)] = [\Delta : \Delta(B_2)] = 1. \quad \text{If } f : A_1 \approx B_1, f \text{ is given by } j; j \in efe. \quad \text{Similar facts hold for other cases.} \]

10) (Lemma 16, 2-ii-b_2))
We shall show that if \( eR \) has one of the structures of the above diagrams (1)~(10), then \( R \) is a right US-4 ring with \( (*, 1') \). It is clear from the diagrams that \( (*, 1') \) holds. Let \( \{U_j\}_{i=1}^4 \) be a set of submodules in \( eR \).

Diagram 1). If \( U_1 \) and \( U_2 \) are characteristic, \( U_1 \cup U_2 \) or \( U_1 \cup U_2 \). Hence \( U_1 \cup U_2 \) satisfies \((**, 2)\) by [4], Corollary 1 of Theorem 2. Hence \( D = \sum_{i=1}^4 U_i \) satisfies \((**, 4)\) by [2], Lemma 1. Assume that \( U_1 \cup U_2 \supset eJ_i \). Then \( U_i \) for \( i=1, 2 \) is characteristic, and hence \( D \) satisfies \((**, 4)\) from the above. Next assume that \( U_1 \supset eJ_i \) and \( eJ_i \supset U_j \) for \( j > 1 \). Since \( \Delta(U_i) = \Delta, U_1 \oplus U_2 \) satisfies \((**, 2)\) by [4], Corollary 1 of Theorem 2. Finally assume \( eJ_i \supset U_j \) for all \( i \). If \( \{U_j\}_{j=1}^4 \) is a set of non-characteristic submodules, then we may assume \( U_1 \supset x_2 U_2 \supset x_3 U_3 \) for some units \( x_j \) in \( eRe \) by assumption. Since \( [\Delta: \Delta(U_j)] = 2 \), \( U_1 \oplus U_2 \oplus U_3 \) satisfies \((**, 3)\) by [4], Corollary 3 of Theorem 2. Therefore \( D \) satisfies \((**, 4)\).

2) As is shown in 1), we may assume that \( eJ_i \supset U_j \) for all \( i \). Then \( U_1 \supset x_2 U_2 \supset x_3 U_3 \supset x_4 U_4 \) by assumption, where the \( x_j \) are units in \( eRe \). Then from the assumption \([\Delta: \Delta(C)] \leq 3\) and the argument of the proof of [4], Corollary 3 of Theorem 2, \( D \) satisfies \((**, 4)\).

3) Let \( eJ_i \supset U_j \) for all \( i \). Then \( U_i = A_i \oplus B_i \) or \( U_i \subset A_i \oplus A_3 \) by assumption, where \( B_i \subset A_i \oplus A_3 \). First assume \( U_j \subset A_2 \oplus A_3 \) or \( U_j = A_i \oplus B_j \) \((B_j \neq 0)\) for all \( j < 3 \). Then \( D \) satisfies \((**, 4)\) by [4], Corollary 3 of Theorem 2 (note \( A_i \) and \( A_2 \oplus A_3 \) are characteristic and see the remark above). If \( U_1 = A_1 \) and \( U_2 = A_1 \oplus B_2 \), \( U_1 \oplus U_2 \) satisfies \((**, 2)\) by [4], Corollary 1 of Theorem 2. Thus \( D \) satisfies \((**, 4)\).

4) Every submodule in \( eJ_i \) is isomorphic to a standard submodule in \( eJ_i \) via \( x_j \). Hence we may assume that all \( U_j \) are standard. Then \( D \) satisfies \((*, 4)\) by [4], Corollaries 1~3 of Theorem 2.

5) and 5') Let \( eJ_i \supset U_i \supset A_2 J \) and \( U_i = A_1 \oplus A_4 \). Then \( U_i/A_2 J = x(A_2/A_2 J) \), and so \( xA_2 = U_i \). Further \( A_1 \oplus A_2 J \) is characteristic. If \( U_i = A_1 \oplus A_2 J \) and \( U_i \subset A_2 J \), \( U_i \oplus U_2 \) satisfies \((*, 2)\). Accordingly we may assume that \( U_i \) is \( A_4 \) or a submodule of \( A_2 \). Therefore \( D \) satisfies \((**, 4)\).

6) and 7) These are clear.
8) First we note $B_1 \supset E_2 \oplus F_2 \supset B'_1$ ($E_2 \oplus F_2 \supset B''$) and $B_1, B''$ do not appear simultaneously. If the $U_i$ are standard for all $i$, $U_i \sim U_j$ for some pair $i, j$. Hence $D$ satisfies $(\ast \ast, 4)$ by [4], Corollary 2 of Theorem 2. The conditions given in Lemma 16 show that $A_i \sim A_i(f), F_p(f^{-1}) \sim F_p(g^{-1}), \ldots$ etc. Hence we obtain the desired result.

9) and 10) These are simpler than 8), (if $A_1 \approx F_i/F_{i+1} (F_{i+1} \supset B_1 \oplus B_2)$, $\Delta \approx \hat{Z}$. Hence $\Delta(C) = \Delta$ for any submodule $C$ in $eR$).

Thus we obtain

**Theorem 2.** $R$ is a right US-4 (basic) ring with $(\ast, 1')$ if and only if $eR$ has one of the structures given in Lemmas 10~16 (cf. Diagrams 1)~10)) for each primitive idempotent $e$.

3. Hereditary rings

In this section, we shall study a hereditary and right US-3 (resp. US-4) ring $R$. If $R$ is hereditary, $(\ast, 1')$ holds, and hence we can make use of the results in the previous sections.

**Lemma 18.** Assume that $R$ is basic and hereditary. Then a submodule $A$ in $eR$ is characteristic if and only if $\Delta(A) = \Delta$. Every non-zero element in $\text{Hom}_R(eR, fR)$ is a monomorphism, where $e$ and $f$ are primitive idempotents.

Proof. The second half is clear (see [9], Lemma 2). Hence, since $ejfe = 0$, the first one is clear

From now on we assume that $R$ is a hereditary and basic ring. First we assume further that $R$ is right US-3.

**Theorem 3.** Let $R$ be a hereditary (and basic) ring. Then $R$ is a right US-3 ring if and onyl if $eR$ has the following structure for each primitive idempotent $e$:

i) $eR/eJ'$ is uniserial for some $t$ and

ii) $eJ' = 0$ or $eJ' = A \oplus B$ such that either

a) $A$ and $B$ are simple and $A \oplus B$ satisfies $(\#, 1)$, and $[\Delta : \Delta(A)] = 2$, or

b) $A$ is simple, $B$ is uniserial and $A$ is not isomorphic to any sub-factor modules of $B$ (and hence $\Delta(A) = \Delta(B) = \Delta$).

Proof. If $R$ is right US-3, $eR$ has the structure in Theorem 1. We consider the case b) of Theorem 1. Assume that $f: A \cong (the socle of B)$. Then $\{A, A(f), B\}$ derives a contradiction, since $A$ and $B$ are characteristic by Lemma 18. Thus we obtain the theorem from Theorem 1.

Let $R$ be a basic hereditary ring. Then
where the $\Delta_i$ are division rings and the $M_{ij}$ are left $\Delta_i$- and right $\Delta_j$-modules [1].

We shall express explicitly the content of Theorem 3 for $M_{ij}$ in a row of the above ring.

1) 
\[ (0 \cdots \Delta_i0\Delta_{t_1}0 \cdots 0\Delta_{t_p}0) \]

2) 
\[ (0 \cdots \Delta_i0 \cdots \Delta_{t_1}0 \cdots \Delta_{t_p}0 \cdots 0) \]

(4) 
\[ \begin{pmatrix} u_{\rho_1} \Delta_{t_1} \\ v_{\rho_1} \Delta_{t_1} \end{pmatrix} = u_{\rho_1} \Delta_{t_1} \oplus v_{\rho_1} \Delta_{t_1} \text{ satisfies } (\#, 1). \]

3) 
\[ \begin{pmatrix} u_{\rho_{t+1}} \Delta_{t+1} \\ v_{\rho_{t+2}} \Delta_{t+2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \cdots \Delta_{t+2}0 \cdots \Delta_{t_{t_p}}0 \cdots 0 \end{pmatrix} \]

As is given in the proof of [9], Theorem 1, we can show a ring monomorphism $\rho_{rs}: \Delta_r \to \Delta_i$ for $r<s<k$ such that $xu_r = u_{\rho_{rs}}(x)$ for $x \in \Delta_r$ and $\rho_{rs} \rho_{rs} = \rho_{rs}$.

Next we shall characterize a hereditary (basic) and right US-4 ring. If $R$ is hereditary, some results in the previous sections may not occur as shown in Theorem 3. We shall observe them.

In the case b) of Lemma 12, $A_2$ is simple. Because, since $A_1$ is simple and $[\Delta: \Delta(A_2)] = 2$, $A_1 \cong A_2/J(A_2)$. Hence $A_1 \cong A_2$ by Lemma 18.

We shall observe the conditions in Lemma 16 for a hereditary ring. a-1-1), a-1-2), a-1-3), any of b-1-1)~4) and b-2-2) do not occur from Lemma 18. For instance, if $f': A_1/E_2 \approx F_{\rho_{-1}}/F_\rho$ (a-1-2), $f_1 \approx F_{\rho_{-1}}$ by Lemma 18. Then $A_1 \approx A_1(f)$ by a-1-5). However, $A_i$ is characteristic, and so $A_1 = A_1(f)$. Therefore $\rho_{rs} = 0$.

We shall use the notations after Theorem 3.

**Lemma 19.** In case 2-i) in Lemma 16, $e_i R$ is of the form $(0, \cdots, \tilde{Z}, \cdots$
0 \cdots \bar{Z} \cdots 0). In case of 2-a-1-4) in Lemma 16, \( A_1 \) (resp. \( A_2 \)) is of the form (0, \cdots, \bar{Z}, 0, \bar{Z}, 0 \cdots) (resp. (0, \cdots, \bar{Z}, 0, \bar{Z}, 0 \cdots)).

Proof. Let \( E_2 \approx F_2 \approx e_{ik}R \). Then \( e_{ik}R \approx \bar{Z} \) by Lemma 16. Let \( A_2 \approx e_{ik}R \). Then \( e_{ik}R \) is uniserial and \( M_{ik} = u_{ik} \bar{Z} \approx F_2 \). Since \( M_{ik} \) is a left \( \Delta_s \)-module, \( \Delta_s \subset \bar{Z} \). Hence \( \Delta_s = \bar{Z} \). We have the same for 2-i).

Thus we have

**Theorem 4.** Let \( R \) be a hereditary (basic) ring. Then \( R \) is right US-4 if and only if for each \( e = e_{ii} \), \( eR \) has one of the following structures: 1~11

1) 
\[
(0 \cdots 0 \Delta_i 0 \Delta_i 0 \cdots 0)
\]

2) (Lemma 10)
\[
(0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0) \quad \cdots A_1
\]
\[
(0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0) \quad \cdots A_2
\]

\([\Delta : \Delta(A_i)] = 2 \) (i=1, 2) and \( u_{i+2}, v_{i+2} \) may be zero. The conditions in Lemma 10 are satisfied.

3) (Lemma 11)
\[
(0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0) \quad \cdots A_1
\]
\[
(0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0) \quad \cdots A_2
\]
\[
(0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0) \quad \cdots A_3
\]

\([\Delta : \Delta(A_i)] = 3 \) for each \( i \) and \( A_1 \oplus A_2 \oplus A_3 \) satisfies \((\#, 1)\) and \((\#, 2)\). \( v_{i+1} \) may be zero.

4) (Lemma 12-i)
\[
(0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0) \quad \cdots A_1
\]
\[
(0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0) \quad \cdots A_2
\]
\[
(0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0) \quad \cdots A_3
\]

\( \Delta(A_4) = \Delta, [\Delta : \Delta(A_i)] = 2 \) (i=1, 2) and \( A_1 \oplus A_2 \) satisfies \((\#, 1)\).

5) (Lemma 12-ii-a) and b)
\[
(0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0) \quad \cdots A_1
\]
\[
(0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0 \Delta_i 0 \cdots 0) \quad \cdots A_2
\]

\( \Delta(A_5) = \Delta, [\Delta : \Delta(A_i)] = 2, \) and \( u_p \Delta_i \oplus v_p \Delta_i \) satisfies \((\#, 1)\), except \( u_p \Delta_i \)
\((u_{i+1}, \ldots, u_{p-1}) \) may be zero.

6) (Lemma 16, 1)}
\[
\begin{pmatrix}
\Delta_i \Delta_i \Delta_i \Delta_i \\
v_t + 1 \Delta_{i+1} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_{i+2} \Delta_{i+2} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
v_{i+3} \Delta_{i+3} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_{\beta} \Delta_{\beta} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[\Delta(A_i) = \Delta (i = 1, 2, 3) \text{ and } u_{i+1} \text{ may be zero.}\]

\[\Delta_i \subseteq \Delta_i \subseteq \cdots \subseteq \Delta_i \subseteq \Delta_i \subseteq \Delta_i \subseteq \cdots \subseteq \Delta_i \]

7) (Lemma 16, 2-i))

\[\begin{pmatrix}
0 \\
v_{i+1} \Delta_{i+1} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_{i+2} \Delta_{i+2} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[\begin{pmatrix}
0 \\
v_{i+3} \Delta_{i+3} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_{\beta} \Delta_{\beta} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[\Delta(A_i) = \Delta (i = 1, 2), u_{i+3} \text{ or } \{v_{i+4}, \ldots, v_{\beta}\} \text{ may be zero.}\]

\[\Delta_i \subseteq \Delta_i \subseteq \cdots \subseteq \Delta_i \subseteq \Delta_i \subseteq \cdots \subseteq \Delta_i \]

8) (Lemma 16, 2-ii-a))

\[\begin{pmatrix}
0 \\
v_{i+1} \Delta_{i+1} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_{i+2} \Delta_{i+2} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[\begin{pmatrix}
0 \\
v_{i+3} \Delta_{i+3} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_{\beta} \Delta_{\beta} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[u_{i+2} \text{ may be zero.}\]

9) (Lemma 16, 2-ii-a')

\[\begin{pmatrix}
0 \\
v_{i+1} \Delta_{i+1} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_{i+2} \Delta_{i+2} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[\begin{pmatrix}
0 \\
v_{i+3} \Delta_{i+3} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_{\beta} \Delta_{\beta} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[u_{i+2} \text{ may be zero.}\]

10) (Lemma 16, 2-ii-b))

\[\begin{pmatrix}
0 \\
v_{i+1} \Delta_{i+1} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_{i+2} \Delta_{i+2} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[\begin{pmatrix}
0 \\
v_{i+3} \Delta_{i+3} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_{\beta} \Delta_{\beta} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[u_{i+2} \text{ may be zero.}\]
\[ \Delta(A_i) = \Delta(B_i) = \Delta \quad (i = 1, 2). \]

\[
\Delta_i \subset \Delta_i \subset \cdots \subset \Delta_i \subset \Delta_{i+1} \subset \cdots \subset \Delta_{i+t} \subset \Delta_{i+t+1} \subset \cdots \subset \Delta_{i+t+1+1}. 
\]

11) (Lemma 16, 2-ii-b))

\[
(0 \cdots \Delta, \cdots \Delta, 0 \cdots) 
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
w_{i+1} \Delta_{i+1} & 0 \\
0 & w_{i+1} \Delta_{i+1}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\Delta_i \Delta_i \Delta_i \Delta_i \Delta_{i+1} \Delta_{i+1} \\
\Delta_i \Delta_i \Delta_i \Delta_i \Delta_{i+1} \Delta_{i+1}
\end{pmatrix}
\]

\[ \Delta(A_i) = \Delta \quad \text{and} \quad \left[ \Delta : \Delta : (B_i) \right] = 2 \quad (i = 1, 2). \]

\[ \Delta_i \subset \Delta_i \subset \cdots \subset \Delta_i \subset \Delta_{i+t} \subset \Delta_{i+t+1} \subset \cdots \subset \Delta_{i+t+1} \subset \Delta_{i+t+1+1}. \]

4. Left serial rings

We shall investigate the same problem for a left serial ring \( R \). In this case \((*, 1')\) holds, too by [11], Corollarly 4, 2. Therefore we can make use of the results in §§ 1 and 2.

From now on we always assume that \( R \) is a left serial ring.

**Lemma 20.** If \( eJ^i = A_i \oplus A_2 \) and the \( A_i \) are uniserial, every submodule \( E \) in \( eJ^i \) is isomorphic to a standard submodule \( B_1 \oplus B_2 \) via \( x_1 : x \) is a unit in \( eRe \), where \( B_i \subset A_i \).

See the proof of [3], Theorem 1.

**Lemma 21.** Let \( eJ^i = A_i \oplus A_2 \) and the \( A_i \) hollow. If \( \Delta(A_i) \neq \Delta \), there exists a unit \( x \) in \( eRe \) such that \( xA_1 = A_2 \).

Proof. Since \( \Delta(A_i) \neq \Delta \), there exists a unit \( y \) in \( eRe \) such that \((y + j')A_i \subset A_i\) for all \( j' \) in \( eJ^e \). Let \( p \) be the projection of \( eJ^i \) onto \( A_2 \). Then \( f = py_j | A_1 \) is an element in \( \text{Hom}_e(A_1, A_2) \). If \( f \) is not an epimorphism, \( f = j \) for some \( j \) in \( eJ^e \), since \( A_2 \) is a hollow module \((\subset eJ^{i+2}) \) and \( R \) is left serial. Then \((y - j)A_1 \subset A_1\),
a contradiction. Hence there exists a unit $x$ in $eRe$ such that $x_1=f$, and so $x_2A=A_1$.

**Lemma 22.** Let $eJ=A_1\oplus A_2$ be as in Lemma 21. If $\Delta(A_1)=\Delta, \Delta(A_1J^k \oplus A_2J^\nu)=\Delta$.

Proof. From Lemma 21, $\Delta(A_2)=\Delta$. Hence we may assume $k\leq k'$. Let $x$ be any unit element in $eRe$. Since $\Delta(A_2)=\Delta$, there exists $j$ in $J(e)$ such that $(x+j)A_1=A_1$. Hence $(x+j)(A_1J^k \oplus A_2J^\nu) \subset A_1J^k + (x+j)A_2J^\nu \subset A_1J^k \oplus A_2J^\nu$, and so $x=x+j \in \Delta(A_1J^k \oplus A_2J^\nu)$.

From Theorem 1, Lemmas 21, 22 and [8], Proposition 2, we obtain

**Theorem 5.** Let $R$ be a left serial ring. Then $R$ is a right US-3 ring if and only if $eR$ has the following structure for each primitive idempotent $e$:

There exists an integer $t$ such that

i) $eR/eJ^i$ is uniserial and

ii) $eJ'=0$ or $eJ^i$ is a direct sum of a simple module and a uniserial module.

Finally we shall give a characterization of a left serial and right US-4 ring. As was shown in the previous section, we shall refine the results in § 2.

In Lemma 10, every submodule in $eJ^i$ is standard up to $x_i$ ($x$ is a unit in $eRe$) by Lemma 20. Further since $\Delta(A_1 \oplus A_2J) \neq \Delta$,

$$A_1 \oplus A_2 \supset J(A_1) \oplus J(A_2) \supset 0$$

is the set of all characteristic submodules in $eJ^i$.

From the above proof we have

**Remark 23.** Let $R$ be left serial and assume $eJ^i=A_1 \oplus A_2$; the $A_i$ are uniserial. If $[\Delta: \Delta(A_1)]=2, [\Delta: \Delta(C)]\leq 2$ for every submodule $C$ in $eJ^i$ and $\{eJ^{i+1}\}$ is the set of characteristic submodules in $eJ^i$. Hence, if $R$ is left serial, i), ii) and iii) in Lemma 10 imply iv) and v). However hereditariness does not as is shown from the following example:

Let $K \subset L$ be fields such that $[L: K]=2$. Put

$$R = \begin{pmatrix} L & L \otimes L & L \otimes L \\ 0 & K & L \\ 0 & 0 & L \\ 0 & 0 & 0 & L \end{pmatrix}$$

Then $R$ is hereditary. Put $L=1K+uK, e_1=e$, and $eJ=A_1 \oplus A_2; A_1=1e_1R, A_2=ue_1R$ satisfy i), ii) and iii) in Lemma 10. Further $[\Delta: \Delta(B)]=2$ for any submodule $B$ in $eJ^i$ if $\Delta \neq \Delta(B)$, since $[L: K]=2$. $\{eJ, eJ^*, eJ^3, (1 \otimes u \pm u \otimes 1)e_3R,$
The set of characteristic submodules provided $u \in K$, and $(1 \otimes u \otimes 1) e_4 R$ is the set of characteristic submodules provided $u \in K$, and $(1 \otimes 1) e_4 R \sim (1 \otimes 1 + u \otimes x) e_4 R$, provided $x \in K$.

**Lemma 24.** Let $B_1$ and $B_2$ be simple submodules in $eJ^i$ and $V = B_1 \oplus B_2$. If $B_1 \cong B_2$, $V$ always satisfies $(\#, 1)$.

Proof. Since $R$ is left serial, every simple submodule in $V$ is isomorphic to $B_1$ via $x_i$; $x$ is a unit in $eRe$. Hence $V$ satisfies $(\#, 1)$.

In Lemma 12, we do not have the case $t=2$ by Lemma 21.

In Lemma 16, we have always $A_1 \cong A_2$, since $\Delta(A_1) = \Delta(A_2) = \Delta$. Hence $2-i)$, $2-a-1-1)$, $2-a-2-3)$ and $p=2$ in $2-a-2-4)$ do not occur. Similarly $2-a-2-1)$ does not occur.

Thus we obtain

**Theorem 6.** Let $R$ be a left serial ring. Then $R$ is right US-4 if and only if, for each primitive idempotent $e$, $eR$ has one of the following structures:

1) $eR$ is uniserial: $eR$ $eJ$ $eJ^p$

2) $eR$ $eJ^{i-1}$ $eJ^i$ $eJ^{i+1}$

$\Delta: \Delta(A_1)] = 2$. In this case $A_1 \cong A_2$ and $B_1$ may be zero.

3) $eR$ $eJ^{i-1}$ $eJ^i$

$\Delta: \Delta(A_i)] = 3$ and $A_1 \oplus A_2 \oplus A_3$ satisfies $(\#, 2)$. In this case $A_1 \cong A_2 \cong A_3$.

4) $eR$ $eJ^{i-1}$ $eJ^i$

$\Delta(A_i) = \Delta$, $[\Delta: \Delta(A_i)] = 2 (i=2, 3)$. In this case $A_2 \cong A_3$. 

\[ \Delta(A_i) = \Delta, [\Delta: \Delta(A_i)] = 2 (i=2, 3). \] In this case $A_2 \cong A_3$. 

\[ \Delta(A_i) = \Delta, [\Delta: \Delta(A_i)] = 2 (i=2, 3). \] In this case $A_2 \cong A_3$.
5) \[ eR \ eJ^{i-1} \ eJ^i \ eJ^{i+1} \ eJ^p \]

\[
\begin{array}{l}
A_1 - 0 \\
\cdots \cdots \\
A_2 - 0 \\
A_3 \cdots \cdots 0
\end{array}
\]

\[ \Delta(A_i) = \Delta \ (i=1, 2, 3). \] 
In this case \( A_1 \approx A_3 \) and \( A_2 \) may be zero.

6) \[ eR \ eJ^{i-1} \ eJ^i \ eJ^{i+1} \ eJ^p \]

\[
\begin{array}{l}
A_1 - B_1 - 0 \\
\cdots \cdots \\
A_2 - B_2 \cdots \cdots 0
\end{array}
\]

\[ \Delta(A_i) = \Delta \ (i=1, 2). \]

7) \[ eR \ eJ^{i-1} \ eJ^i \ eJ^{i+1} \ eJ^p \]

\[
\begin{array}{l}
A_1 - 0 \\
\cdots \cdots \\
A_2 \\
B_1 - 0 \\
B_2 \cdots \cdots 0
\end{array}
\]

\[ \Delta(A_i) = \Delta \ (i=1, 2, 3) \] \text{and} \[ \Delta(B_j) = \Delta \ (j=1, 2). \]

8) \[ eR \ eJ^{i-1} \ eJ^i \ eJ^{i+1} \ eJ^p \]

\[
\begin{array}{l}
A_1 - 0 \\
\cdots \cdots \\
A_2 \\
B_1 - 0 \\
B_2 - 0
\end{array}
\]

\[ \Delta(A_1) = \Delta(A_2) = \Delta \] \text{and} \[ [\Delta : \Delta(B_j)] = 2. \] 
In this case \( B_1 \approx B_2, \)
where each straight line means "uniserial".

5. Examples

We shall give examples of hereditary (resp. left serial) and right US-3 (resp. US-4) rings. Let \( K \) be a field. By \( L \) and \( L' \) we denote extension fields of \( K \) with \([L : K]=2\) and \([L' : K]=3\), respectively, and \( \mathbb{Z} = \mathbb{Z}/2 \), where \( \mathbb{Z} \) is the ring of integers.

The following two rings are hereditary, left serial and right US-3 rings.
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\[
\begin{pmatrix}
K & K & K & K \\
K & K & K & K \\
K & 0 & K \\
0 & K
\end{pmatrix}
\]
is the second type b) of Theorem 1 and
\[
\begin{pmatrix}
L & L & L \\
L & L & L \\
0 & K
\end{pmatrix}
\]
is the first type a) of Theorem 1.

On the other hand
\[
\begin{pmatrix}
K & L & L \\
0 & L & L \\
0 & 0 & K
\end{pmatrix}
\]
is a hereditary, non-left serial and right US-3 ring, and
\[
\begin{pmatrix}
L & L & 0 \\
0 & K & K \\
0 & 0 & K
\end{pmatrix}
\]
with \(e_{12}e_{23} = 0\) is a left serial, non-hereditary and right US-3 ring.

Next we shall give hereditary and right US-4 rings for each structure in Theorem 4. However, we cannot construct an example of the case 5) from the reason given in Remark 13.

\[
\begin{pmatrix}
1 & K & K & K & K \\
K & K \\
0 & K \\
0 & K
\end{pmatrix}
\]
\[
\begin{pmatrix}
2 & L & L & L & L \\
L & L & L \\
K & K \\
0 & K
\end{pmatrix}
\]
\[
\begin{pmatrix}
3 & L' & L' & L' \\
L' & L' \\
0 & K
\end{pmatrix}
\]

3' \[
\begin{pmatrix}
D & D^* \\
D_1
\end{pmatrix}
\]
where \(D, D_1\) and \(D^*\) are given in Remark 14.

\[
\begin{pmatrix}
4 & L & L & L \\
L & 0 \\
0 & K
\end{pmatrix}
\]
\[
\begin{pmatrix}
6 & K & K & K & K & K \\
K & 0 & 0 \\
K & 0 \\
0 & K
\end{pmatrix}
\]

7 \[
\begin{pmatrix}
\bar{Z} & \bar{Z} & \bar{Z} & \bar{Z} \\
0 & \bar{Z} & \bar{Z} \\
0 & \bar{Z}
\end{pmatrix}
\]
8 \[
\begin{pmatrix}
K & K & K & K & K \\
K & 0 & 0 \\
K & 0 \\
0 & K
\end{pmatrix}
\]

where $L$ is an extension of $\mathbb{Z}$ with $[L: \mathbb{Z}]=2$. $e_{11}R$ is of the form 2–1) in Lemma 16 and $e_{22}R$ is of the form in Lemma 10.

The rings of 1)~6), 8), 10) and 11) are left serial.

If $R$ is either hereditary or left serial, $A_1/E_2 \cong A_2/F_2$ implies $A_1 \cong A_2$ in Lemma 16. In general this is not true for US-4 rings.

We shall give rings of the type a) in Lemma 16. Let $R=\sum \oplus e_i R$ and $e_i e_j = \delta_{ij} e_i$ (the $e_i$ are primitive idempotents).

1) $A_1/E_2 \cong A_2/F_2$ and $E_2 \cong F_2$

\begin{align*}
A_1 &= (1, 2)\mathbb{Z}+(1, 2)(2, 3)\mathbb{Z} \\
E_2 &= (1, 2)(2, 3)\mathbb{Z} \\
F_2 &= (1, 2)'(2, 3)'\mathbb{Z}
\end{align*}

\[e_1 R = e_1 \mathbb{Z} + e_1 J\]

\[e_2 R = e_2 \mathbb{Z} + e_2 J\]

and $(1, 2) (2, 3)' = (1, 2)'(2, 3) = 0$. This is a type of a-1-1) and a-1-4). ($R$ is a finite ring.)

2) $A_1/E_2 \cong A_2/F_2$, $E_2 \cong F_2$
\[ e_1 R = e_1 \bar{Z} + e_1 J \]

\[ A_1 = (1, 2)\bar{Z} + (1, 2)(2, 3)K \quad A_2 = (1, 2)'\bar{Z} + (1, 2)'(2, 4)K \]

\[ E_2 = (1, 2)(2, 3)K \quad F_2 = (1, 2)'(2, 4)K \]

\[ e_2 R = e_2 K + e_2 J \quad e_3 R = e_3 K \quad e_4 R = e_4 K \]

\[ (2, 3)K \quad (2, 4)K \]

\[ 0 \quad 0 \]

and \((1, 2)(2, 4) = (1, 2)'(2, 3) = 0\), where \(K\) is a finite field of characteristic 2. This is a type of a-1-1).

3)

\[ e_1 R = e_1 \bar{Z} + e_1 J \]

\[ A_1 = (1, 2)\bar{Z} + E_2 \quad A_2 = (1, 1)\bar{Z} + F_2 \]

\[ E_2 = (1, 2)(2, 3)K \quad F_2 = (1, 1)(1, 2)\bar{Z} + F_3 \]

\[ 0 \quad F_3 = (1, 1)(1, 2)(2, 3)K \]

\[ e_2 R = e_2 \bar{Z} + e_2 J \quad e_3 R = e_3 K \]

\[ (2, 3)K \quad 0 \]

This is a type of a-1-2). If \(K = \bar{Z}\), \(R\) is a left serial and finite ring.

4)

\[ e_1 R = e_1 \bar{Z} + e_1 J \]

\[ A_1 = (1, 2)\bar{Z} + E_2 \quad A_2 = (1, 3)\bar{Z} + F_2 \]

\[ E_2 = (1, 2)(2, 4)\bar{Z} \quad F_2 = (1, 3)(3, 5)\bar{Z} + F_3 \]

\[ 0 \quad F_3 = (1, 3)(3, 5)(5, 4)\bar{Z} + F_4 \]

\[ F_4 = (1, 3)(3, 5)(5, 4)(4, 6)K \]

\[ 0 \]
This is a type of a-1-3).

Other products among \((i, j)\) are zero (e.g. \((1, 1)(1, 1)=0\)). In the above \(e_i(k, l)e_j=(k, l)\delta_{ik}\), \(\delta_{ij}\) is Kronecker delta.

Similarly we can construct a US-4 ring of a-2-1) in Lemma 16. Finally we shall give an example concerning ii) of Lemma 12.

Let \(K\) be a field of characteristic 2 and \(L\) an extension of \(K\); \(L=K(a)\) and \(a^2\in K\). Put \(g(a)=b\neq 0\) in \(L\) and \(g(1)=0\). Then \(g\) is a derivation of \(L\) over \(K\). Put

\[
R = \begin{pmatrix}
L \\
L
\end{pmatrix},
\]

where \(l = (l_1, l_2)\) and \(g(l) = (g_1, g_2)\) as in Remark 13. Then \(e_{11}J=A_1 \oplus A_2\) and \(\Delta(A_1)=\Delta, [\Delta: \Delta(A_2)]=2\). However, \(e_{11}R\) does not satisfy \((\#, 1)\) as an \(L-L\)-module. Hence \(e_{11}R\) has the similar form to ii) of Lemma 12, but \(R\) is not right US-4.

References
