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ON R -ALGEBRAS WHICH ARE R FINITELY GENERATED

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Let K be a field and R a ring with 1. We know several conditions under which an R -algebra is a finitely generated R -module. In [6] Rosenberg and Zelinsky obtained, for a K -algebra A , those conditions in a case where $A \otimes_K A^*/N(A \otimes_K A^*)$ is Artinian, where A^* is an anti-isomorphic algebra of A and $N(^*)$ is the radical of * .

In §1 we shall study a similar problem in a case where $A \otimes_K A^*$ is Noetherian and obtain, for an algebraic algebra A over K such that $A/N(A)$ is a semi-simple ring with minimum condition, that $[A:K] < \infty$ if and only if $A \otimes_K A^*$ is right Noetherian.

In §2 we consider a primitive K -algebra with minimal one sided ideals. We give a condition that the associated division ring is of a finite K -dimension.

Finally we consider a separable R -algebra A which is a submodule in a free R -module. If R is Noetherian, then we show that A is R -finitely generated as R -module.

1. Algebras of finite type

In this paper we always assume that K means a field and R a commutative ring with 1.

Let $A_2 \supseteq A_1$ be R -algebras. Then we have a natural homomorphism $\Phi: A_1 \otimes_R A_1^* \rightarrow A_2 \otimes_R A_2^*$. We denote also the image of Φ by $A_1 \otimes_R A_1^*$ if there are no confusions. Furthermore, we have a natural right $A_i \otimes_R A_i^*$ -homomorphism $\varphi_i: A_i \otimes_R A_i^* \rightarrow A_i$ by setting $(a \otimes b^*) = ba$. We denote its kernel by J_i .

The following lemma is based on a suggestion of M. Auslander.

Lemma 1. *Let A_3 be an R -algebra and $A_2 \supseteq A_1$ proper R -subalgebras contained in the center of A_3 . We assume that A_{i+1} is A_i -projective for $i=1, 2$. Then $J_3 \supseteq J_2 A_3^e \supseteq J_1 A_3^e$, where $A_3^e = A_3 \otimes_R A_3^*$.*

Proof. We consider a natural A_3^e -homomorphism $\alpha_2: A_3 \otimes_R A_3^* \rightarrow A_3 \otimes_{A_2} A_3^*$. If $\alpha_2(J_3) = (0)$, then we obtain easily $A_3 \otimes_{A_2} A_3^* = A_3$. Let \mathfrak{p} be a prime ideal of A_2 . Then $A_{3\mathfrak{p}} \otimes_{A_{1\mathfrak{p}}} A_{3\mathfrak{p}}^*$. Since $A_{3\mathfrak{p}}$ is $A_{2\mathfrak{p}}$ -projective, $A_{3\mathfrak{p}}$ is a free $A_{2\mathfrak{p}}$ -module by [5], Theorem 2. Hence $A_{3\mathfrak{p}} = A_{2\mathfrak{p}}$ for every prime ideal \mathfrak{p} , which is a contradiction. On the other hand $\alpha_2(J_2 A_3^e) = (0)$. Therefore, $J_2 A_3^e \subseteq J_3$. Next we consider a commutative diagram:

$$\begin{array}{ccccc}
 A_2 \otimes_R A_2 & \xrightarrow{\beta} & A_2 \otimes_{A_1} A_2 & \xrightarrow{\beta'} & A_3 \otimes_{A_1} A_3^* \\
 & \searrow \Phi & & \nearrow \alpha_1 & \\
 & & A_3 \otimes_R A_3^* & &
 \end{array}$$

From the above argument we know that $\beta(J_2) = (0)$. Since A_2, A_3 are A_1 -projective, β' is monomorphic. Therefore, $\alpha_1(J_2) = \alpha_1 \Phi(J_2) = \beta' \beta(J_2) = (0)$. On the other hand $\alpha_1(J_1) = (0)$. Hence we have $J_2 A_3^e \subseteq J_1 A_3^e$.

Corollary 1. *Let A be an R -projective R -algebra. We assume that $A \otimes_R A^*$ is right Noetherian (resp. Artinian). Then a length of ascending (resp. descending) chain of R -projective, R -separable algebras in the center of A is finite, (cf. [7], Theorem 2).*

Proof. From a fact for a separable R -algebra C that R -projective C -module is C -projective, we have the corollary.

Corollary 2. *Let A be an extension field of K . Then A is a finite type, i.e. A is generated by a finite number of elements if and only if $A \otimes_K A$ is Noetherian, (cf. [1], p. 99).*

Proof. If A is a finite type, then A is an algebraic extension of a rational function field $K(x_1, x_2, \dots, x_t)$. It is clear that $K(x) \otimes_K K(x)$ is Noetherian. Since $A \otimes_K A$ is a finitely generated $K(x)^e$ -module, A^e is Noetherian. The converse is clear from Lemma 1.

REMARK 1. Lemma 1 is valid in a case where A 's are division rings. Because, we may take $A_2 \otimes_{A_1^*} A_2^*$ in a place of $A_2 \otimes_{A_1} A_2$ and so on.

Lemma 2. *Let A be a right Noetherian, algebraic algebra over a field K . Then the radical of A is nilpotent.*

Proof. By the assumption and [4], p. 212, Proposition 3, the radical

N is nil. Furthermore, since A is Noetherian, N is nilpotent by [4], p. 199, Theorem 1.

Proposition 1. *Let A be a commutative algebraic algebra over a field K . Then the following conditions are equivalent.*

- a) $[A:K] < \infty$,
- b) $A \otimes_K A$ is Noetherian,
- c) $A \otimes_K F$ is Noetherian for any algebraic extension field F of K .

Proof. First, we assume A^e is Noetherian. Since A^e is algebraic over K , its radical $N(A^e)$ is nilpotent by Lemma 2. Similarly we know that $N=N(A)$ is nilpotent. Hence, if we show $[A/N:K] < \infty$, then by the standard argument we obtain $[A:K] < \infty$ (cf. the proof of [3], Theorem 1). Therefore, we may assume that A is a semi-simple ring in a sense of Jacobson. From [4], p. 210 we know that A is an I -ring, namely every non-nilpotent ideal contains an idempotent element. Hence, since A is a commutative Noetherian semi-simple ring, every ideal is generated by an idempotent element. Therefore, A is a semi-simple ring with minimum conditions. Hence, we may assume that A is a field. Then $[A:K] < \infty$ by Corollary 2. By the similar argument as above, we obtain $[A:K] < \infty$ if A satisfies c).

Theorem 1. *Let A be an algebraic algebra over a field K . We assume A/N is a semi-simple ring with minimum conditions, where N is the radical of A . Then we have the following equivalent conditions:*

- a) $[A:K] < \infty$,
- b) $A \otimes_K A^*$ is right Noetherian,
- c) $A \otimes_K F$ is right Noetherian for every algebraic extension field F of K .

Proof. In both cases b) and c) we know that N is nilpotent by Lemma 2. Hence, we may assume that A is a division algebra over K . Let L be a maximal subfield of A and Z the center of A . Let $A = \sum \oplus L u_i$ and $A^* = \sum \oplus L^* v_i$. Since $A \otimes_K A^* = \sum \oplus L \otimes L^* (u_i \otimes v_j)$ is right Noetherian, so is $L \otimes_K L^*$. Hence $[L:K] < \infty$ by Proposition 1. If we consider A as a left A - and right L -module, A is a right $A^* \otimes_K L$ -module. Since $A^* \otimes_K L$ is a simple ring with minimum conditions and A is a simple faithful $A^* \otimes_K L$ -module, A has a finite right base over $A^* \otimes_K L$ -endomorphism division ring of A , which is equal to $V_A(L) = \{a \in A \mid al = la \text{ for all } l \in L\}$. Since L is a maximal subfield of A , $V_A(L) = L$. Therefore, $[A:K] < \infty$.

Corollary 3. *Let A be an algebra over a field K . L_1 is an algebraic closure of K and $L_2=K(x)$ a rational function field over K . Then $[A:K]<\infty$ if and only if $A\otimes_K L_i$ ($i=1, 2$) is right Artinian, ([3], Theorem 1).*

Proof. By the same reason as in the proof of Proposition 1, we may assume that A is a division ring if $A\otimes_K L_i$ is right Artinian. Furthermore, it is clear that A is algebraic over K . Hence $[A:K]<\infty$ by Theorem 1.

Proposition 2. *Let A be a division algebra over a field K . If $A\otimes_K A^*/N(A\otimes_K A^*)$ is right Noetherian, then the center Z of A is of a finite transcendental degree over K and A is a finite type over Z , (cf. [7], Theorem 2).*

Proof. By the proof of [2], Lemma 4, we have $N(A^e)=\alpha A^e$, where α is an ideal contained in the radical $N(Z^e)$ of Z^e . Since there is a lattice isomorphism between two-sided ideals of A^e and Z^e by [4], p. 114, Theorem 1, $Z^e/N(Z^e)$ is Noetherian. We shall show that the transcendental degree of A over K is finite. We consider again an exact sequence as in Lemma 1. $0\rightarrow J_i\rightarrow L_i\otimes_K L_i\rightarrow L_i\rightarrow 0$, where $L_i=K(x_1, \dots, x_i)$ and the x 's are indeterminants in Z over K . Then we shall show that $J_i Z^e + N(Z^e) \neq J_{i+1} Z^e + N(Z^e)$. Otherwise, for any element j in $J_{i+1}(Z^e)$, we have $j=y+r$, $y\in J_i Z^e$, $r\in N(Z^e)$. Since $N(Z^e)$ is nil ([1], p. 85, Proposition 4), $j^n\in J_i Z^e$ for some integer n . Therefore, $(x_{i+1}\otimes 1 - 1\otimes x_{i+1})^{n'} = x_{i+1}^{n'}\otimes 1 - n'(x_{i+1}^{n'-1}\otimes x_{i+1}) + \dots + (-1)^{n'}(1\otimes x_{i+1}^{n'})$ is contained in $J_i Z^e$. On the other hand, $J_i Z^e = \sum \oplus u_\alpha J_i(L_{i+1}\otimes_K L_{i+1})$, where $\{u_\alpha\}$ is a basis of Z^e over $L_{i+1}\otimes_K L_{i+1}$ and we assume $u_1=1\otimes 1$. Extending $x_{i+1}^k\otimes x_{i+1}^l$, $k, l=0, 1, \dots$ to a basis $\{x, v\}$ of $L_{i+1}\otimes_K L_{i+1}$ over $L_i\otimes_K L_i$, $J_i Z^e = \sum \oplus (x_{i+1}^k\otimes x_{i+1}^l) J_i \oplus \sum \oplus v J_i \oplus \sum_{\alpha=1} \oplus u_\alpha J_i(L_{i+1}\otimes_K L_{i+1})$. Hence J_i must contain 1, which is a contradiction. Therefore, the transcendental degree of Z over K is finite. From the assumption, it is clear that $A\otimes_Z A^*$ is right Noetherian. Hence by Lemma 1, A is a finite type over Z .

REMARK 2. The following example shows that A is not a finite type even if A is algebraic commutative field over K and $A^e/N(A^e)$ is Artinian.

Let $A = \bigcup_n K(x^{1/p^n})$, where K is a field of characteristic $p \neq 0$ and x is an indeterminate over K . Then it is clear the $N(A^e) = J_A$ and $A^e/N(A^e) = A$.

2. Primitive algebras

Let A be a simple algebra over K with minimum conditions. Then it is clear that $[A:K] < \infty$ if and only if $N(A^e)$ is nilpotent and $A^e/N(A^e)$ is Artinian. We shall generalize this property as follows:

Theorem 2. *Let A be a primitive K -algebra with minimal one-sided ideals and Δ its associated division ring, (see [4]). Then $[\Delta:K] < \infty$ if and only if the radical $N(A^e)$ of A^e is nilpotent and $N(A^e)$ is the intersection of a finite number of primitive rings with one-sided ideals.*

Proof. We assume $[\Delta:K] < \infty$. Let I and r be minimal left and right ideals in A , respectively. Then $r \otimes_K I^* = \sum_{\alpha} \oplus (x_{\alpha} \otimes y_{\alpha}) \Delta^e$ is a faithful $A \otimes_K A^*$ -module, and $A \otimes_K A^*$ is a dense ring in the Δ^e -endomorphism ring $M_I(\Delta^e)$ of $r \otimes_K I^*$ by [4], p. 113, Theorem 1. By the assumption, the radical $N(\Delta^e)$ of Δ^e is nilpotent. We consider a factor module of $r \otimes_K I^*$ by its radical: $\overline{r \otimes_K I^*} = r \otimes_K I^* / N(r \otimes_K I^*) = \sum \oplus (x_{\alpha} \otimes y_{\alpha}) \Delta^e / N(\Delta^e)$. By a well known theorem, the radical $N(M_I(\Delta^e))$ of $M_I(\Delta^e)$ is contained in $M_I(N(\Delta^e))$, and since $N(\Delta^e)$ is nilpotent, $M_I(N(\Delta^e))$ is equal to $N(M_I(\Delta^e))$. We can easily show that $M_I(\Delta^e) / N(M_I(\Delta^e)) = M_I((A_1)_{n_1}) \oplus \cdots \oplus M_I((A_r)_{n_r})$, where the A 's are division algebras over K . Furthermore, it is clear that $\overline{r \otimes_K I^*}$ is a faithful $M_I(\Delta^e) / N(M_I(\Delta^e))$ -module. On the other hand, we have $\overline{r \otimes_K I^*} = \sum_i \sum_{\alpha} \sum_{j=1}^{n_i} \oplus (x_{\alpha} \otimes y_{\alpha}) b_{i,j}$, where the b 's are irreducible left ideals in $(A^*)_{n_i}$. Put $L_i = \sum_{\alpha} (x_{\alpha} \otimes y_{\alpha}) b_{i,1}$, then $\sum_i \oplus L_i$ is a faithful $M_I(\Delta^e) / N(M_I(\Delta^e))$ -module. By the above argument, the L 's are also A^e -irreducible modules. Hence $N(M_I(\Delta^e))$ contains $N(A^e)$. Since $N(M_I(\Delta^e))$ is nilpotent, we have $N(A^e) = N(M_I(\Delta^e)) \cap A^e$. Therefore, $\sum_i \oplus L_i$ is also a faithful $\bar{A}^e = A^e / N(A^e)$ -module. Furthermore, $N(r \otimes_K I^*) = (r \otimes_K I^*) N(M_I(\Delta^e)) \subseteq A^e \cap N(M_I(\Delta^e)) = N(A^e)$, and since we can represent r and I by eA and Ae' , where e, e' are primitive idempotents in A , then $(r \otimes_K I^*) \cap N(A^e) = (e \otimes e'^*) A^e \cap N(A^e) = (e \otimes e'^*) N(A^e) = (r \otimes_K I^*) N(A^e) \subseteq N(r \otimes_K I)$. Hence, we have a monomorphism of $\overline{r \otimes_K I^*}$ into \bar{A}^e . Therefore, \bar{A}^e has a faithful complete reducible module $\sum \oplus L_i$. Let α_i be the annihilator ideal of L_i in \bar{A}^e . Then \bar{A}^e / α_i contains $L_i + \alpha_i / \alpha_i$. Since L_i is irreducible and \bar{A}^e is semi-simple, $L_i + \alpha_i / \alpha_i \approx L_i$. Hence \bar{A}^e / α_i is a primitive ring with minimal one-sided ideals. Furthermore, we have $\bigcap \alpha_i = (0)$, which proves the first half of the theorem. Let e be an idempotent. Then by [4], p. 48, Proposition 1, $N((e \otimes e^*)(A \otimes_K A^*)(e \otimes e^*))$

$= (e \otimes e^*) N(A \otimes_{\kappa} A^*) (e \otimes e^*)$, and hence, $N(\Delta^e)$ is nilpotent, where $eAe = \Delta$. Let \mathfrak{p}_i 's be a primitive ideals with the property as in the theorem. Then by [2], Lemma 1, $(e \otimes e^*) \mathfrak{p}_i (e \otimes e^*)$ are primitive ideals in $(e \otimes e^*) A \otimes_{\kappa} A^* (e \otimes e^*)$ with the same property as above. Furthermore, if $\bigcap_i \mathfrak{p}_i = N(A^e)$, then $\bigcap_i (e \otimes e^*) \mathfrak{p}_i (e \otimes e^*) = N(\Delta^e)$. Let Z be the center of Δ . Then by [4], p. 114, Theorem 1, there is a lattice isomorphism between two-sided ideals of Δ^e and those of Z^e . Put $q_i = (e \otimes e^*) \mathfrak{p}_i (e \otimes e^*)$. Then there exist ideals \mathfrak{b} and \mathfrak{c} in Z^e which correspond to q_i and an ideal \mathfrak{s} in Δ^e such that $\mathfrak{s} \supseteq q_i$ and \mathfrak{s}/q_i is the socle of Δ^e/q_i . We shall show that $\bar{Z}^e = Z^e/\mathfrak{b}$ is a field. Since $\bar{\mathfrak{c}}$ is a unique minimal ideal in \bar{Z}^e , $\bar{\mathfrak{c}}$ is contained in $N(\bar{Z}^e)$ if $\bar{\mathfrak{c}} \neq \bar{Z}^e$. $\bar{\mathfrak{s}} = \bar{\mathfrak{s}}^2$, $\bar{\mathfrak{c}} = \bar{\mathfrak{c}}^2$. Hence $\bar{\mathfrak{c}}$ is generated by idempotent element, which is a contradiction. Therefore, \bar{Z}^e is a field. Hence, Δ^e/q_i is a simple ring. Since Δ^e/q_i has the socle, Δ^e/q_i satisfies the minimum conditions. $\bigcap_i q_i = N(\Delta^e)$ implies that $\Delta^e/N(\Delta^e)$ is a semi-simple ring with minimum condition. Therefore, $[\Delta : K] < \infty$ by [7], Theorem 7.

3. Separable algebras

Let R be a Noetherian ring and α an ideal in R . For any finitely generated R -module E and its submodule F , there exists an integer r such that $\alpha^n E \cap F = \alpha^{n-r} (\alpha^r E \cap F)$ for all $n > r$ by the Artin-Rees theorem. Thus we shall call an R -module E "*an Artin-Rees module with respect to α (briefly A - R module)*", if for any finitely generated R -submodule F in E , there exists an integer r such that $F \alpha^n \cap F \subseteq F \alpha^{n-r}$ for $n > r$.

By definition we have the following lemmas:

Lemma 3. *If E is an A - R module, then any submodule of E and any quotient module of E with respect to a finitely generated submodule of E are A - R modules.*

Lemma 4. *Every submodule of a free R -module is an A - R module.*

Lemma 5. *If α is contained in the radical of R and E is an A - R module, then $\bigcap_n E \alpha^n = (0)$.*

Proof. Let x be in $\bigcap_n E \alpha^n$. Then $xR = xR \cap E \alpha^n \subseteq xR \alpha^{n-r}$, and hence $xR = (0)$.

Proposition 3. *Let α be an ideal contained in the radical of R . For any A - R module E , if $E/E\alpha$ is finitely generated then so is E .*

Proof. If $E/E\alpha$ is finitely generated, then we have a finitely generated R -submodule F such that $E = E\alpha + F$. Let $\bar{E} = E/F$, then \bar{E} is an A - R module by Lemma 3. Hence $(0) = \bigcap \bar{E}\alpha^n = \bar{E}$ by Lemma 5.

Corollary 4. *Let R and α be as above. If R is not complete with respect to $\{\alpha^n\}$, then the completion \hat{R} of R is not contained in a free R -module, (cf. [1], p. 95).*

Proof. If \hat{R} is contained in a free R -module then \hat{R} is an A - R module by Lemma 4. Furthermore, $\hat{R}/\hat{R}\alpha \approx R/\alpha$, and hence \hat{R} is a finitely generated R -module by the proposition. Then since $\hat{R} = R + \hat{R}\alpha$, $\hat{R} = R$ by Nakayama's Lemma, which is a contradiction.

Lemma 6. *Let S be a multiplicative system consisting of non-zero-divisors in R (not necessarily Noetherian) and E a submodule of a free R -module. If $E_s = E \otimes_R R_s$ is finitely generated R_s -module, then E is contained in an finitely generated R -module.*

It is clear.

Theorem 3. *Let R be a Noetherian ring and let A be a separable R -algebra such that A is contained in a free R -module. Then A is a finitely generated R -module.*

Proof. Let S be the set of all non zero-divisors in R . Then A_s is separable R_s -algebra and is contained in a free R_s -module. Hence, we may assume by Lemma 6 that R is semi-local. Let \mathfrak{p} be a maximal ideal in R . Then $A/A\mathfrak{p}$ is a separable algebra over R/\mathfrak{p} , (it may be zero). Hence, $A/A\mathfrak{p}$ is a finitely generated R/\mathfrak{p} -module by [6], Theorem 1. On the other hand, $A \otimes R_{\mathfrak{p}} / (A \otimes R_{\mathfrak{p}} \mathfrak{p}) = A/A\mathfrak{p}$ and $A \otimes R_{\mathfrak{p}}$ is an A - R module by Lemma 4. Therefore, $A \otimes R_{\mathfrak{p}}$ is finitely generated $R_{\mathfrak{p}}$ -module, and hence A is a finitely generated R -module by the simple argument.

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