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ON R-ALGEBRAS WHICH ARE R FINITELY GENERATED

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Let K be a field and R a ring with 1. We know several conditions under which an R-algebra is a finitely generated R-module. In [6] Rosenberg and Zelinsky obtained, for a K-algebra A, those conditions in a case where $A\underset{K}{\otimes}A^*/N(A\underset{K}{\otimes}A^*)$ is Artinian, where A^* is an anti-isomorphic algebra of A and $N(*)$ is the radical of $*$.

In §1 we shall study a similar problem in a case where $A\underset{K}{\otimes}A^*$ is Noetherian and obtain, for an algebraic algebra A over K such that $A/N(A)$ is a semi-simple ring with minimum condition, that $[A:K]<\infty$ if and only if $A\underset{K}{\otimes}A^*$ is right Noetherian.

In §2 we consider a primitive K-algebra with minimal one sided ideals. We give a condition that the associated division ring is of a finite K-dimension.

Finally we consider a separable R-algebra A which is a submodule in a free R-module. If R is Noetherian, then we show that A is R-finitely generated as R-module.

1. Algebras of finite type

In this paper we always assume that K means a field and R a commutative ring with 1.

Let $A_2\supseteq A_1$ be R-algebras. Then we have a natural homomorphism Φ: $A_1\underset{R}{\otimes}A_1^*\to A_2\underset{R}{\otimes}A_2^*$. We denote also the image of Φ by $A_1\underset{R}{\otimes}A_1^*$ if there are no confusions. Furthermore, we have a natural right $A_i\underset{R}{\otimes}A_i^*$-homomorphism φ_i: $A_i\underset{R}{\otimes}A_i^*\to A_i$ by setting $(a\otimes b^*)=ba$. We denote its kernel by J_i.

The following lemma is based on a suggestion of M. Auslander.

Lemma 1. *Let A_3 be an R-algebra and $A_2\supsetneqq A_1$ proper R-subalgebras contained in the center of A_3. We assume that A_{i+1} is A_i-projective for $i=1,2$. Then $J_3\supsetneqq J_2A_3^e\supsetneqq J_1A_3^e$, where $A_3^e=A_3\underset{R}{\otimes}A_3^*$.*

Proof. We consider a natural A_3^e-homomorphism α_2: $A_3 \underset{R}{\otimes} A_3^* \to A_3 \underset{A_2}{\otimes} A_3^*$. If $\alpha_2(J_3)=(0)$, then we obtain easily $A_3 \underset{A_2}{\otimes} A_3^* = A_3$. Let $\mathfrak{p}$ be a prime ideal of A_2. Then $A_{3\mathfrak{p}} \underset{A_{1\mathfrak{p}}}{\otimes} A_{3\mathfrak{p}}^*$. Since $A_{3\mathfrak{p}}$ is $A_{2\mathfrak{p}}$-projective, $A_{3\mathfrak{p}}$ is a free $A_{2\mathfrak{p}}$-module by [5], Theorem 2. Hence $A_{3\mathfrak{p}} = A_{2\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$, which is a contradiction. On the other hand $\alpha_2(J_2 A_3^e)=(0)$. Therefore, $J_2 A_3^e \subsetneqq J_3$. Next we consider a commutative diagram:

$$\begin{tikzcd}
A_2 \underset{R}{\otimes} A_2 \arrow[r, "\beta"] \arrow[dr, "\Phi"'] & A_2 \underset{A_1}{\otimes} A_2 \arrow[r, "\beta'"] & A_3 \underset{A_1}{\otimes} A_3^* \\
& A_3 \underset{R}{\otimes} A_3^* \arrow[ur, "\alpha_1"'] &
\end{tikzcd}$$

From the above argument we know that $\beta(J_2)=(0)$. Since A_2, A_3 are A_1-projective, β' is monomorphic. Therefore, $\alpha_1(J_2)=\alpha_1\Phi(J_2)=\beta'\beta(J_2)\neq(0)$. On the other hand $\alpha_1(J_1)=(0)$. Hence we have $J_2 A \supsetneqq J_1 A_3^e$.

Corollary 1. *Let A be an R-projective R-algebra. We assume that $A \underset{R}{\otimes} A^*$ is right Noetherian* (*resp. Artinian*). *Then a length of ascending* (*resp. descending*) *chain of R-projective, R-separable algebras in the center of A is finite,* (cf. [7], Theorem 2).

Proof. From a fact for a separable R-algebra C that R-projective C-module is C-projective, we have the corollary.

Corollary 2. *Let A be an extension field of K. Then A is a finite type, i.e. A is generated by a finite number of elements if and only if $A \underset{K}{\otimes} A$ is Noetherian,* (cf. [1], p. 99).

Proof. If A is a finite type, then A is an algebraic extension of a rational function field $K(x_1, x_2, \cdots, x_t)$. It is clear that $K(x) \underset{K}{\otimes} K(x)$ is Noetherian. Since $A \underset{K}{\otimes} A$ is a finitely generated $K(x)^e$-module, A^e is Noetherian. The converse is clear from Lemma 1.

REMARK 1. Lemma 1 is valid in a case where A's are division rings. Because, we may take $A_2 \underset{A_1^*}{\otimes} A_2^*$ in a place of $A_2 \underset{A_1}{\otimes} A_2$ and so on.

Lemma 2. *Let A be a right Noetherian, algebraic algebra over a field K. Then the radical of A is nilpotent.*

Proof. By the assumption and [4], p. 212, Proposition 3, the radical

N is nil. Furthermore, since A is Noetherian, N is nilpotent by [4], p. 199, Theorem 1.

Proposition 1. *Let A be a commutative algebraic algebra over a field K. Then the following conditions are equivalent.*

a) $[A:K]<\infty$,

b) $A\underset{K}{\otimes}A$ *is Noetherian,*

c) $A\underset{K}{\otimes}F$ *is Noetherian for any algebraic extension field F of K.*

Proof. First, we assume A^e is Noetherian. Since A^e is algebraic over K, its radical $N(A^e)$ is nilpotent by Lemma 2. Similarly we know that $N=N(A)$ is nilpotent. Hence, if we show $[A/N:K]<\infty$, then by the standard argument we obtain $[A:K]<\infty$ (cf. the proof of [3], Theorem 1). Therefore, we may assume that A is a semi-simple ring in a sense of Jacobson. From [4], p. 210 we know that A is an I-ring, namely every non-nilpotent ideal contains an idempotent element. Hence, since A is a commutative Noetherian semi-simple ring, every ideal is generated by an idempotent element. Therefore, A is a semi-simple ring with minimum conditions. Hence, we may assume that A is a field. Then $[A:K]<\infty$ by Corollary 2. By the similar argument as above, we obtain $[A:K]<\infty$ if A satisfies c).

Theorem 1. *Let A be an algebraic algebra over a field K. We assume A/N is a semi-simple ring with minimum conditions, where N is the radical of A. Then we have the following equivalent conditions*:

a) $[A:K]<\infty$,

b) $A\underset{K}{\otimes}A^*$ *is right Noetherian,*

c) $A\underset{K}{\otimes}F$ *is right Noetherian for every algebraic extension field F of K.*

Proof. In both cases b) and c) we know that N is nilpotent by Lemma 2. Hence, we may assume that A is a division algebra over K. Let L be a maximal subfield of A and Z the center of A. Let $A=\sum\oplus Lu_i$ and $A^*=\sum\oplus L^*v_i$. Since $A\underset{K}{\otimes}A^*=\sum\oplus L\underset{K}{\otimes}L^*(u_i\otimes v_j)$ is right Noetherian, so is $L\underset{K}{\otimes}L^*$. Hence $[L:K]<\infty$ by Proposition 1. If we consider A as a left A- and right L-module, A is a right $A^*\underset{K}{\otimes}L$-module. Since $A^*\underset{K}{\otimes}L$ is a simple ring with minimum conditions and A is a simple faithful $A^*\underset{K}{\otimes}L$-module, A has a finite right base over $A^*\underset{K}{\otimes}L$-endomorphism division ring of A, which is equal to $V_A(L)=\{a\in A\,|\,al=la$ for all $l\in L\}$. Since L is a maximal subfield of A, $V_A(L)=L$. Therefore, $[A:K]<\infty$.

Corollary 3. *Let A be an algebra over a field K. L_1 is an algebraic closure of K and $L_2=K(x)$ a rational function field over K. Then* $[A:K]<\infty$ *if and only if* $A\underset{K}{\otimes}L_i$ $(i=1, 2)$ *is right Artinian,* ([3], Theorem 1).

Proof. By the same reason as in the proof of Proposition 1, we may assume that A is a division ring if $A\underset{K}{\otimes}L_i$ is right Artinian. Furthermore, it is clear that A is algebraic over K. Hence $[A:K]<\infty$ by Theorem 1.

Proposition 2. *Let A be a division algebra over a field K. If* $A\underset{K}{\otimes}A^*/N(A\underset{K}{\otimes}A^*)$ *is right Noetherian, then the center Z of A is of a finite transcendental degree over K and A is a finite type over Z,* (cf. [7], Theorem 2).

Proof. By the proof of [2], Lemma 4, we have $N(A^e)=\mathfrak{a}A^e$, where $\mathfrak{a}$ is an ideal contained in the radical $N(Z^e)$ of Z^e. Since there is a lattice isomorphism between two-sided ideals of A^e and Z^e by [4], p. 114, Theorem 1, $Z^e/N(Z^e)$ is Noetherian. We shall show that the transcendental degree of A over K is finite. We consider again an exact sequence as in Lemma 1. $0\to J_i\to L_i\underset{K}{\otimes}L_i\to L_i\to 0$, where $L_i=K(x_1, \cdots, x_i)$ and the x's are indeterminants in Z over K. Then we shall show that $J_iZ^e+N(Z^e)\neq J_{i+1}Z^e+N(Z^e)$. Otherwise, for any element j in $J_{i+1}(Z^e)$, we have $j=y+r$, $y\in J_iZ^e$, $r\in N(Z^e)$. Since $N(Z^e)$ is nil ([1], p. 85, Proposition 4), $j^n\in J_iZ^e$ for some integer n. Therefore, $(x_{i+1}\otimes 1-1\otimes x_{i+1})^{n'}=x_{i+1}^{n'}\otimes 1-n'(x_{i+1}^{n'-1}\otimes x_{i+1})+\cdots+(-1)^{n'}(1\otimes x_{i+1}^{n'})$ is contained in J_iZ^e. On the other hand, $J_iZ^e=\sum\oplus u_\alpha J_i(L_{i+1}\otimes L_{i+1})$, where $\{u_\alpha\}$ *is a basis of* Z^e over $L_{i+1}\underset{K}{\otimes}L_{i+1}$ and we assume $u_1=1\otimes 1$. Extending $x_{i+1}^k\otimes x_{i+1}^l$, $k, l=0, 1, \cdots$ to a basis $\{x, v\}$ of $L_{i+1}\underset{K}{\otimes}L_{i+1}$ over $L_i\underset{K}{\otimes}L_i$, $J_iZ^e=\sum\oplus(x_{i+1}^k\otimes x_{i+1}^l)J_i\oplus\sum\oplus vJ_i\oplus\sum_{\alpha\neq 1}\oplus u_\alpha J_i(L_{i+1}\underset{K}{\otimes}L_{i+1})$. Hence J_i must contain 1, which is a contradiction. Therefore, the transcendental degree of Z over K is finite. From the assumption, it is clear that $A\underset{Z}{\otimes}A^*$ is right Noetherian. Hence by Lemma 1, A is a finite type over Z.

REMARK 2. The following example shows that A is not a finite type even if A is algebraic commutative field over K and $A^e/N(A^e)$ is Artinian.

Let $A=\bigcup_n K(x^{1/p^n})$, where K is a field of characteristic $p\neq 0$ and x is an indeterminant over K. Then it is clear the $N(A^e)=J_A$ and $A^e/N(A^e)=A$.

2. Primitive algebras

Let A be a simple algebra over K with minimum conditions. Then it is clear that $[A:K]<\infty$ if and only if $N(A^e)$ is nilpotent and $A^e/N(A^e)$ is Artinian. We shall generalize this property as follows:

Theorem 2. *Let A be a primitive K-algebra with minimal one-sided ideals and Δ its associated division ring,* (*see* [4]). *Then $[\Delta:K]<\infty$ if and only if the radical $N(A^e)$ of A^e is nilpotent and $N(A^e)$ is the intersection of a finite number of primitive rings with one-sided ideals.*

Proof. We assume $[\Delta:K]<\infty$. Let $\mathfrak{l}$ and $\mathfrak{r}$ be minimal left and right ideals in A, respectively. Then $\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*=\sum_\alpha\oplus(x_\alpha\otimes y_\alpha)\Delta^e$ is a faithful $A\underset{K}{\otimes}A^*$-module, and $A\underset{K}{\otimes}A^*$ is a dense ring in the Δ^e-endomorphism ring $M_I(\Delta^e)$ of $\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*$ by [4], p. 113, Theorem 1. By the assumption, the radical $N(\Delta^e)$ of Δ^e is nilpotent. We consider a factor module of $\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*$ by its radical: $\overline{\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*}=\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*/N(\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*)=\sum\oplus(x_\alpha\otimes y_\alpha)\Delta^e/N(\Delta^e)$. By a well known theorem, the radical $N(M_I(\Delta^e))$ of $M_I(\Delta^e)$ is contained in $M_I(N(\Delta^e))$, and since $N(\Delta^e)$ is nilpotent, $M_I(N(\Delta^e))$ is equal to $N(M_I(\Delta^e))$. We can easily show that $M_I(\Delta^e)/N(M_I(\Delta^e))=M_I((A_1)_{n_1})\oplus\cdots\oplus M_I((A_r)_{n_r})$, where the A's are division algebras over K. Furthermore, it is clear that $\overline{\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*}$ is a faithful $M_I(\Delta^e)/N(M_I(\Delta^e))$-module. On the other hand, we have $\overline{\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*}=\sum_i\sum_\alpha\sum_{j=1}^{n_i}\oplus(x_\alpha\otimes y_\alpha)\mathfrak{b}_{i,j}$, where the $\mathfrak{b}$'s are irreducible left ideals in $(A^*)_{n_i}$. Put $L_i=\sum_\alpha(x_\alpha\otimes y_\alpha)\mathfrak{b}_{i,1}$, then $\sum_i\oplus L_i$ is a faithful $M_I(\Delta^e)/N(M_I(\Delta^e))$-module. By the above argument, the L's are also A^e-irreducible modules. Hence $N(M_I(\Delta^e))$ contains $N(A^e)$. Since $N(M_I(\Delta^e))$ is nilpotent, we have $N(A^e)=N(M_I(\Delta^e))\cap A^e$. Therefore, $\sum_i\oplus L_i$ is also a faithful $\bar{A}^e=A^e/N(A^e)$-module. Furthermore, $N(\mathfrak{r}\otimes\mathfrak{l}^*)=(\mathfrak{r}\otimes\mathfrak{l}^*)N(M_I(\Delta^e))\subseteq A^e\cap N(M_I(\Delta^e))=N(A^e)$, and since we can represent $\mathfrak{r}$ and $\mathfrak{l}$ by eA and Ae', where e, e' are primitive idempotents in A, then $(\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*)\cap N(A^e)=(e\otimes e'^*)A^e\cap N(A^e)=(e\otimes e'^*)N(A^e)$ $=(\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*)N(A^e)\subseteq N(\mathfrak{r}\underset{K}{\otimes}\mathfrak{l})$. Hence, we have a monomorphism of $\overline{\mathfrak{r}\underset{K}{\otimes}\mathfrak{l}^*}$ into $\bar{A}^e$. Therefore, $\bar{A}^e$ has a faithful complete reducible module $\sum\oplus L_i$. Let $\mathfrak{a}_i$ be the annihilator ideal of L_i in $\bar{A}^e$. Then $\bar{A}^e/\mathfrak{a}_i$ contains $L_i+\mathfrak{a}_i/\mathfrak{a}_i$. Since L_i is irreducible and $\bar{A}^e$ is semi-simple, $L_i+\mathfrak{a}_i/\mathfrak{a}_i\approx L_i$. Hence $\bar{A}^e/\mathfrak{a}_i$ is a primitive ring with minimal one-sided ideals. Furthermore, we have $\bigcap\mathfrak{a}_i=(0)$, which proves the first half of the theorem. Let e be an idempotent. They by [4], p. 48, Proposition 1, $N((e\otimes e^*)(A\underset{K}{\otimes}A^*)(e\otimes e^*))$

$=(e\otimes e^*)N(A\underset{K}{\otimes}A^*)(e\otimes e^*)$, and hence, $N(\Delta^e)$ is nilpotent, where $eAe=\Delta$. Let $\mathfrak{p}_i$'s be a primitive ideals with the property as in the theorem. Then by [2], Lemma 1, $(e\otimes e^*)\mathfrak{p}_i(e\otimes e^*)$ are primitive ideals in $(e\otimes e^*)A\underset{K}{\otimes}A^*(e\otimes e^*)$ with the same property as above. Furthermore, if $\bigcap_i \mathfrak{p}_i=N(A^e)$, then $\bigcap_i (e\otimes e^*)\mathfrak{p}_i(e\otimes e^*)=N(\Delta^e)$. Let Z be the center of Δ. Then by [4], p. 114, Theorem 1, there is a lattice isomorphism between two-sided ideals of Δ^e and those of Z^e. Put $\mathfrak{q}_i=(e\otimes e^*)\mathfrak{p}_i(e\otimes e^*)$. Then there exist ideals $\mathfrak{b}$ and $\mathfrak{c}$ in Z^e which correspont to $\mathfrak{q}_i$ and an ideal $\mathfrak{s}$ in Δ^e such that $\mathfrak{s}\supseteq\mathfrak{q}_i$ and $\mathfrak{s}/\mathfrak{q}_i$ is the socle of $\Delta^e/\mathfrak{q}_i$. We shall show that $\bar{Z}^e=Z^e/\mathfrak{b}$ is a field. Since $\bar{\mathfrak{c}}$ is a unique minimal ideal in $\bar{Z}^e$, $\bar{\mathfrak{c}}$ is contained in $N(\bar{Z}^e)$ if $\bar{\mathfrak{c}}\neq\bar{Z}^e$. $\bar{\mathfrak{s}}=\bar{\mathfrak{s}}^2$, $\bar{\mathfrak{c}}=\bar{\mathfrak{c}}^2$. Hence $\bar{\mathfrak{c}}$ is generated by idempotent element, which is a contradiction. Therefore, $\bar{Z}^e$ is a field. Hence, $\Delta^e/\mathfrak{q}_i$ is a simple ring. Since $\Delta^e/\mathfrak{q}_i$ has the socle, $\Delta^e/\mathfrak{q}_i$ satisfies the minimum conditions. $\bigcap \mathfrak{q}_i=N(\Delta^e)$ implies that $\Delta^e/N(\Delta^e)$ is a semi-simple ring with minimum condition. Therefore, $[\Delta : K]<\infty$ by [7], Theorem 7.

3. Separable algebras

Let R be a Noetherian ring and $\mathfrak{a}$ an ideal in R. For any finitely generated R-module E and its submodule F, there exists an integer r such that $\mathfrak{a}^n E\cap F=\mathfrak{a}^{n-r}(\mathfrak{a}^r E\cap F)$ for all $n>r$ by the Artin-Rees theorem. Thus we shall call an R-module E "*an Artin-Rees module with respect to* $\mathfrak{a}$ (*briefly A-R module*)", if for any finitely generated R-submodule F in E, there exists an integer r such that $F\mathfrak{a}^n\cap F\subseteq F\mathfrak{a}^{n-r}$ for $n>r$.

By definition we have the following lemmas:

Lemma 3. *If E is an A-R module, then any submodule of E and any quotient module of E with respect to a finitely generated submodule of E are A-R modules.*

Lemma 4. *Every submodule of a free R-module is an A-R module.*

Lemma 5. *If $\mathfrak{a}$ is contained in the radical of R and E is an A-R module, then $\bigcap_n E\mathfrak{a}^n=(0)$.*

Proof. Let x be in $\bigcap_n E\mathfrak{a}^n$. Then $xR=xR\cap E\mathfrak{a}^n\subseteq xR\mathfrak{a}^{n-r}$, and hence $xR=(0)$.

Proposition 3. *Let $\mathfrak{a}$ be an ideal contained in the radical of R. For any A-R module E, if $E/E\mathfrak{a}$ is finitely generated then so is E.*

Proof. If $E/E\mathfrak{a}$ is finitely generated, then we have a finitely generated R-submodule F such that $E=E\mathfrak{a}+F$. Let $\bar{E}=E/F$, then $\bar{E}$ is an A-R module by Lemma 3. Hence $(0)=\bigcap \bar{E}\mathfrak{a}^n=\bar{E}$ by Lemma 5.

Corollary 4. *Let R and $\mathfrak{a}$ be as above. If R is not complete with respect to $\{\mathfrak{a}^n\}$, then the completion $\hat{R}$ of R is not contained in a free R-module,* (cf. [1], p. 95).

Proof. If $\hat{R}$ is contained in a free R-module then $\hat{R}$ is an A-R module by Lemma 4. Furthermore, $\hat{R}/\hat{R}\mathfrak{a}\approx R/\mathfrak{a}$, and hence $\hat{R}$ is a finitely generated R-module by the proposition. Then since $\hat{R}=R+\hat{R}\mathfrak{a}$, $\hat{R}=R$ by Nakayama's Lemma, which is a contradiction.

Lemma 6. *Let S be a multiplicative system consisting of non-zero-divisors in R (not necessarily Noetherian) and E a submodule of a free R-module. If $E_s=E\otimes_R R_s$ is finitely generated R_s-module, then E is contained in an finitely generated R-module.*

It is clear.

Theorem 3. *Let R be a Noetherian ring and let A be a separable R-algebra such that A is contained in a free R-module. Then A is a finitely generated R-module.*

Proof. Let S be the set of all non zero-divisors in R. Then A_s is separable R_s-algebra and is contained in a free R_s-module. Hence, we may assume by Lemma 6 that R is semi-local. Let $\mathfrak{p}$ be a maximal ideal in R. Then $A/A\mathfrak{p}$ is a separable algebra over $R/\mathfrak{p}$, (it may be zero). Hence, $A/A\mathfrak{p}$ is a finitely generated $R/\mathfrak{p}$-module by [6], Theorem 1. On the other hand, $A\otimes R_{\mathfrak{p}}/(A\otimes R_{\mathfrak{p}}\mathfrak{p})=A/A\mathfrak{p}$ and $A\otimes R_{\mathfrak{p}}$ is an A-R module by Lemma 4. Therefore, $A\otimes R_{\mathfrak{p}}$ is finitely generated $R_{\mathfrak{p}}$-module, and hence A is a finitely generated R-module by the simple argument.

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