

Title	Uniform boundedness of the radially symmetric solutions of the Navier-Stokes equations for isentropic compressible fluids
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Citation	Osaka Journal of Mathematics. 2009, 46(3), p. 863-876
Version Type	VoR
URL	<a href="https://doi.org/10.18910/8561">https://doi.org/10.18910/8561</a>
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# UNIFORM BOUNDEDNESS OF THE RADIALLY SYMMETRIC SOLUTIONS OF THE NAVIER-STOKES EQUATIONS FOR ISENTROPIC COMPRESSIBLE FLUIDS

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(Received February 27, 2008, revised June 30, 2008)

## Abstract

We study the isentropic compressible Navier-Stokes equations with radially symmetric data and non-negative initial density in an annular domain. We prove the global existence of strong solutions for any  $\gamma \geq 1$ . Moreover, we obtain the uniform in time  $L^\infty$ -boundedness of the density and  $H^1$ -boundedness of the velocity, improving therefore the corresponding result in [2], where the condition  $\gamma \geq 2$  is required to guarantee the existence.

## 1. Introduction

The Navier-Stokes equations with external forces for the isentropic motion of a compressible viscous gas in Eulerian coordinates read:

$$(1.1) \quad \rho_t + \operatorname{div}(\rho u) = 0,$$

$$(1.2) \quad (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u = \rho f \quad \text{in } \Omega \times (0, \infty).$$

Here  $\Omega$  is a domain in  $\mathbb{R}^n$  ( $n \geq 2$ ),  $\rho$  and  $u$  denote the unknown density and velocity, respectively,  $p \equiv p(\rho) = A\rho^\gamma$  ( $A > 0$ ,  $\gamma > 1$ ) is the pressure,  $\mu$  and  $\lambda$  are the constant viscosity coefficients satisfying the usual physical requirements  $\mu > 0$ ,  $2\mu + n\lambda \geq 0$ .

In this paper, we are interested in spherically symmetric strong solutions to spherically symmetric initial boundary value problems. Thus, we consider that the domain  $\Omega$  and the external force  $f$  are given by

$$\Omega := \{x \in \mathbb{R}^n \mid a < |x| < b\}, \quad f(x, t) := f(|x|, t) \frac{x}{|x|}$$

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2000 Mathematics Subject Classification. 76N15, 35Q30, 35Q35, 35M20.

Supported by the NSFC (Grant No. 10301014, 10225105) and the National Basic Research Program (Grant No. 2005CB321700).

for some constants  $a, b$  with  $0 < a < b$ , and the initial and boundary conditions are imposed as follows:

$$(1.3) \quad (\rho, u)|_{t=0} = (\rho_0, u_0) \quad \text{in } \Omega,$$

$$(1.4) \quad u|_{\partial\Omega} = 0, \quad t > 0,$$

where

$$(1.5) \quad \rho_0(x) = \rho_0(|x|) \geq 0 \quad \text{and} \quad u_0(x) = u_0(|x|) \frac{x}{|x|} \quad \text{for } x \in \Omega.$$

A spherically symmetric solution to (1.1)–(1.4) is of the form

$$(1.6) \quad \rho(x, t) = \rho(r, t), \quad u(x, t) = u(r, t) \frac{x}{r}.$$

Hence the spherically symmetric solution  $(\rho, u)(r, t)$  should satisfy the following system:

$$(1.7) \quad \rho_t + (\rho u)_r + (n-1) \frac{\rho u}{r} = 0,$$

$$(1.8) \quad (\rho u)_t + (\rho u^2)_r + (n-1) \frac{\rho u^2}{r} - (\lambda + 2\mu) \left( u_r + (n-1) \frac{u}{r} \right)_r + p_r = \rho f, \quad r \in (a, b)$$

with initial and boundary conditions:

$$(1.9) \quad u(a, t) = u(b, t) = 0,$$

$$(1.10) \quad \rho(r, 0) = \rho_0(r), \quad u(r, 0) = u_0(r),$$

where  $f \equiv f(r)$  is assumed to be independent of time for simplicity.

The spherically symmetric Cauchy and initial boundary value problems for the system (1.1), (1.2) have been studied by a number of mathematicians in the last decades. Global spherically symmetric solutions of the compressible isentropic Navier-Stokes equations for the Cauchy problem was obtained by Jiang and Zhang [7] for any  $\gamma > 1$  provided that initial data are spherically symmetric. For isothermal flows, Hoff [5] proved the global existence of spherically symmetric weak solutions for initial density in the  $BV$  space, while Jiang and Zhang [8] obtained the same result to the Cauchy problem when  $\rho_0$  is in the Orlicz space  $L_M(\mathbb{R}^2)$ .

For the initial boundary value problem (1.7)–(1.10), Weigant [12] constructed a radially symmetric strong solution  $(\rho, u)$  in  $(0, 1) \times B_R$  in the case  $a = 0$  and  $1 < \gamma < 1 + 1/(n-1)$ , such that  $\|\rho(\cdot, t)\|_{L^\infty(B_R)} \rightarrow \infty$  as  $t \rightarrow 1$ , where  $B_R := \{x \in \mathbb{R}^n, |x| < R\}$ . Higuchi [4] and Matsumura [9] proved that a global solution to the problem (1.7)–(1.10) exists for  $\rho_0(r) > 0$  in  $[a, b]$  and converges exponentially to the corresponding stationary solution as time tends to infinity. Very recently, Choe and Kim [2] showed the global existence of strong solutions to the problem (1.7)–(1.10)

under the technical restriction  $\gamma \geq 2$  and the compatibility condition

$$(1.11) \quad -\mu \Delta u_0 - (\lambda + \mu) \nabla \operatorname{div} u_0 + \nabla(A\rho_0^\gamma) = \rho_0^{1/2} g$$

for some radially symmetric  $g \in L^2(\Omega)$ , and moreover, their a priori estimates of the solutions depend on the time.

The aim of this paper is to drop the technical condition  $\gamma \geq 2$  in [2], and moreover, to give the uniform in time  $L^\infty$ -upper bounds of the density and  $H^1$ -bounds of the velocity.

Our main result in this paper reads:

**Theorem 1.1.** *Let  $\rho_0 \in H^2$ ,  $\nabla \rho_0^{\gamma/2} \in L^4$ ,  $u_0 \in D_0^1 \cap D^2$ . Assume that  $(\rho_0, u_0)$  satisfies the natural compatibility condition (1.11) for some  $\gamma \geq 1$ . Then, there exists a global unique strong solution  $(\rho, u)$  satisfying*

$$(1.12) \quad \sup_{0 \leq t \leq T} (\|\rho(\cdot, t)\|_{H^2} + \|\nabla \rho^{\gamma/2}(\cdot, t)\|_{L^4} + \|\rho_t(\cdot, t)\|_{H^1}) \leq C(T),$$

$$(1.13) \quad \sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} u_t(\cdot, t)\|_{L^2} + \|u(\cdot, t)\|_{D_0^1 \cap D^2} + \int_0^T (\|u_t(\cdot, t)\|_{D_0^1}^2 + \|u(\cdot, t)\|_{D^3}^2) dt \right) \leq C(T).$$

Moreover, there holds

$$(1.14) \quad \|\rho(\cdot, t)\|_{L^\infty} + \|\nabla u(\cdot, t)\|_{L^2} \leq C, \quad \forall t \geq 0,$$

where  $C$  is a positive constant independent of  $t$ .

REMARK 1.1. Recently, Cho and Kim [1] proved the existence of local strong solutions when  $1 < \gamma < 2$ , and thus our paper also improves the result in [1].

By the arguments in [2], we easily see that to prove the global existence and uniqueness in Theorem 1.1, it suffices to derive the a priori estimates (1.12) and (1.13). Therefore, in Section 2 we only prove (1.12) and (1.13) to complete the proof of the existence and uniqueness. In Section 3, we prove the uniform in time estimate (1.14).

As the end of this section, we introduce the notation used throughout the paper.  $L^p(\Omega)$  and  $H^m(\Omega)$  denote the standard Lebesgue and Sobolev spaces in  $\Omega$  with the norms  $\|\cdot\|_{L^p}$  and  $\|\cdot\|_{H^m}$ , respectively. We will frequently use the following abbreviations:

$$L^p := L^p(\Omega), \quad H^m := H^m(\Omega), \quad \|\cdot\| := \|\cdot\|_{L^2}.$$

For a detailed study of homogeneous Sobolev spaces  $D_0^1 = D_0^{1,2}$  and  $D^k$ , we refer to the reference [3].

**2. Global existence of strong solutions for  $\gamma \geq 1$**

To begin with, we first recall the following standard (energy) estimates which can be obtained by integrating (1.1) and multiplying (1.2) by  $u$  in  $L^2((0, t) \times \Omega)$ :

$$(2.1) \quad \int_{\Omega} \rho(x, t) dx = \int_{\Omega} \rho_0(x) dx,$$

and

**Lemma 2.1.**

$$(2.2) \quad \begin{aligned} & \int_{\Omega} \left[ \frac{1}{2} \rho u^2 + \tilde{p}(\rho) - \rho \int_0^r f(\xi) d\xi \right] dx + \int_0^t \int_{\Omega} [\mu(\nabla u)^2 + (\lambda + \mu)(\operatorname{div} u)^2] dx \\ &= \int_{\Omega} \left[ \frac{1}{2} \rho_0 u_0^2 + \tilde{p}(\rho_0) - \rho_0 \int_0^r f(\xi) d\xi \right] dx, \quad \forall t \geq 0. \end{aligned}$$

Here  $\tilde{p}(\rho) = \begin{cases} (A/(\gamma - 1))\rho^\gamma, & \gamma > 1 \\ \rho \ln \rho, & \gamma = 1 \end{cases}$ .

The following Lemmas 2.2–2.4 are shown in [2], the proof of which is therefore omitted here.

**Lemma 2.2** ([2]).

$$(2.3) \quad \sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^\infty} \leq C_1(T),$$

where  $C_1$  is a positive constant depending possibly on  $T$ .

**Lemma 2.3** ([2]).

$$(2.4) \quad \sup_{0 \leq t \leq T} (\|u(\cdot, t)\|_{L^\infty} + \|u(\cdot, t)\|_{D_0^1}) + \int_0^T (\|\sqrt{\rho}u_t(\cdot, t)\|^2 + \|G(\cdot, t)\|_{H^1}^2) dt \leq C_2(T)$$

where  $G := (\lambda + 2\mu) \operatorname{div} u - p(\rho)$  is called the effective viscous flux.

**Lemma 2.4** ([2]).

$$(2.5) \quad \sup_{0 \leq t \leq T} \|\nabla \rho(\cdot, t)\| + \int_0^T (\|\nabla u(\cdot, t)\|_{L^\infty}^2 + \|u(\cdot, t)\|_{D^2}^2) dt \leq C_2(T),$$

$$(2.6) \quad \sup_{0 \leq t \leq T} (\|\sqrt{\rho}u_t(\cdot, t)\| + \|u(\cdot, t)\|_{D^2}) + \int_0^T (\|u_t(\cdot, t)\|_{D_0^1}^2 + \|G(\cdot, t)\|_{H^2}^2) dt \leq C_2(T).$$

In what follows, we shall use the following Sobolev inequalities for radially symmetric functions defined in  $\Omega$ :

$$(2.7) \quad \|\rho\|_{L^\infty} \leq C\|\rho\|_{H^1}, \quad \|f\|_{L^\infty} \leq C\|f\|_{H^1}, \quad \|u\|_{L^\infty} \leq C\|\nabla u\|.$$

In the following lemma we give an upper-bound of the density gradient.

**Lemma 2.5.**

$$(2.8) \quad \int_{\Omega} |\nabla \rho^{\gamma/2}(x, t)|^4 dx \leq C_2(T), \quad t \in [0, T].$$

Proof. Multiplying (1.1) by  $\rho^{\gamma/2-1}$ , we obtain

$$(2.9) \quad (\rho^{\gamma/2})_t + u \cdot \nabla \rho^{\gamma/2} + \frac{2}{\gamma} \rho^{\gamma/2} \operatorname{div} u = 0.$$

Applying  $\partial/\partial x_j$  to (2.9), we arrive at

$$(2.10) \quad (\rho^{\gamma/2})_{x_j t} + u_{x_j} \cdot \nabla \rho^{\gamma/2} + u \cdot \nabla (\rho^{\gamma/2})_{x_j} + \frac{2}{\gamma} (\rho^{\gamma/2})_{x_j} \operatorname{div} u + \frac{2}{\gamma} \rho^{\gamma/2} \operatorname{div} u_{x_j} = 0.$$

Now, multiplying the above equation by  $[(\rho^{\gamma/2})_{x_j}]^2 (\rho^{\gamma/2})_{x_j}$ , integrating then over  $\Omega$  and summing over  $j$ , we deduce that

$$(2.11) \quad \begin{aligned} & \frac{1}{4} \frac{d}{dt} \int_{\Omega} |\nabla \rho^{\gamma/2}|^4 dx \\ & \leq \|\nabla u\|_{L^\infty} \int_{\Omega} |\nabla \rho^{\gamma/2}|^4 dx + \left(\frac{1}{4} + \frac{2}{\gamma}\right) \|\operatorname{div} u\|_{L^\infty} \int_{\Omega} |\nabla \rho^{\gamma/2}|^4 dx \\ & \quad - \frac{2}{\gamma} \int_{\Omega} \rho^{\gamma/2} \frac{1}{\lambda + 2\mu} \sum_j (G_{x_j} + p(\rho)_{x_j}) [(\rho^{\gamma/2})_{x_j}]^3 dx \\ & \leq \left( \|\nabla u\|_{L^\infty} + \left(\frac{1}{4} + \frac{2}{\gamma}\right) \|\operatorname{div} u\|_{L^\infty} \right) \int_{\Omega} |\nabla \rho^{\gamma/2}|^4 dx \\ & \quad + \frac{2}{\gamma} \frac{1}{\lambda + 2\mu} \|\rho^{\gamma/2}\|_{L^\infty} \|\nabla G\|_{L^4} \|\nabla \rho^{\gamma/2}\|_{L^4}^3, \end{aligned}$$

where we have used the following fact:

$$\frac{2}{\gamma} \int_{\Omega} \rho^{\gamma/2} \frac{1}{\lambda + 2\mu} \sum_j p(\rho)_{x_j} [(\rho^{\gamma/2})_{x_j}]^3 dx \geq 0.$$

Recalling that by virtue of the Sobolev inequality,

$$(2.12) \quad \|\nabla G\|_{L^4} \leq C \|G\|_{H^2},$$

we apply the Gronwall inequality to (2.11) and use (2.6) to obtain (2.8). This completes the proof.  $\square$

Next, we derive bounds of higher derivatives of  $\rho$  and  $u$ .

**Lemma 2.6.**

$$(2.13) \quad \sup_{0 \leq t \leq T} (\|\rho(\cdot, t)\|_{H^2} + \|\rho_t(\cdot, t)\|_{H^1}) + \int_0^T \|u(\cdot, t)\|_{D^3}^2 dt \leq C_2(T).$$

REMARK 2.1. In [2], the estimate (2.13) is proved under the technical restriction  $\gamma \geq 2$ . Here we will use (2.8) to show that it still holds for any  $\gamma \geq 1$ .

Proof. Applying  $\nabla^2$  to (1.1) and multiplying by  $\nabla^2 \rho$  in  $L^2(\Omega)$ , and then integrating by parts, we get

$$(2.14) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla^2 \rho|^2 dx \\ & \leq C \int_{\Omega} |\nabla u| |\nabla^2 \rho|^2 + |\nabla^2 u| |\nabla \rho| |\nabla^2 \rho| + \rho |\nabla^2 \operatorname{div} u| |\nabla^2 \rho| dx \\ & \leq C \left( \|\nabla u\|_{H^1} \|\nabla \rho\|_{H^1}^2 + \int_{\Omega} \rho (|\nabla^2 G| + |\nabla^2 p(\rho)|) |\nabla^2 \rho| dx \right) \\ & \leq C (\|\nabla u\|_{H^1} \|\nabla \rho\|_{H^1}^2 + \|\nabla^2 G\| \|\nabla^2 \rho\| + \|\nabla^2 \rho\|^2 + \|\nabla \rho\|_{L^\infty} \|\nabla \rho\| \|\nabla^2 \rho\|) \\ & \leq C (\|\nabla \rho\|_{H^1}^2 + \|G\|_{H^2}^2), \end{aligned}$$

whence, an application of the Gronwall inequality and the use of (2.6) give

$$(2.15) \quad \sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{H^2} \leq C(T).$$

On the other hand, the estimate of  $\rho_t$  follows immediately from the continuity equation (1.1) and (2.15).

Finally, by virtue of

$$\nabla^2 p(\rho) = p'(\rho) \nabla^2 \rho + p''(\rho) (\nabla \rho)^2 = p'(\rho) \nabla^2 \rho + \frac{4A(\gamma - 1)}{\gamma} (\nabla \rho^{\gamma/2})^2$$

and

$$(\lambda + 2\mu) \nabla^2 \operatorname{div} u = \nabla^2 G + \nabla^2 p(\rho),$$

the estimates (2.6), (2.8) and (2.15) imply

$$(2.16) \quad \int_0^T \|u(\cdot, t)\|_{D^3}^2 dt \leq C(T),$$

which completes the proof. □

Now, having had the a priori estimates Lemma 2.1–2.6, we can follow the same procedure as in [2] to obtain the existence and uniqueness of global radially symmetric strong solutions. Thus, we have proved Theorem 1.1.

### 3. Uniform in time boundedness

In this section we will prove the uniform in time boundedness of the density and velocity. First, we use and adapt the techniques developed by Straskraba and Zlotnik [10] to obtain uniform upper bounds of the density.

We define the mean value

$$\langle w \rangle := \frac{1}{b-a} \int_a^b w(r) dr$$

and the operator

$$Iw(r) := \int_a^r w(\xi) d\xi, \quad I^{(1)}w := Iw - \langle Iw \rangle \quad \text{for } w \in L^1(a, b)$$

and denote  $D_t w := w_t + u w_r$ . It is easy to see that

$$(3.1) \quad \|I^{(1)}w\|_{C(\bar{\Omega})} \leq C\|w\|_{L^1} \quad \text{for any } w \in L^1(\Omega).$$

We will use the following lemma on uniform boundedness of solutions to an ordinary differential equation, the proof of which can be found in [10].

**Lemma 3.1** ([10]). *Let the function  $y$  be a solution to the Cauchy problem*

$$\frac{dy}{dt} = g(y) + \frac{db}{dt} \quad \text{on } \mathbb{R}^+, \quad y(0) = y_0$$

with  $g \in C(\mathbb{R})$ , and  $y, b \in W^{1,1}(0, T)$  for all  $T > 0$ . If  $g(\infty) = -\infty$  and  $b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1)$  for all  $0 \leq t_1 < t_2$  with some  $N_0 \geq 0$  and  $N_1 \geq 0$ , then

$$y(t) \leq \bar{y} := \bar{y}(y_0, g, N_0, N_1) < +\infty \quad \text{on } [0, +\infty].$$

(More precisely,  $\bar{y} := \max\{y_0, \bar{\zeta}\} + N_0$ , where  $\bar{\zeta}$  is such that  $g(\zeta) \leq -N_1$ , for  $\zeta \geq \bar{\zeta}$ .)



Now, we are able to prove the *main result* of this section:

**Theorem 3.2.** *Let  $f \in L^\infty(\Omega)$ , then there exists a positive constant  $C$  independent of  $t$ , such that*

$$\|\rho(\cdot, t)\|_{L^\infty} \leq C, \quad \forall t \geq 0.$$

*Proof.* Following calculations similar to those in [10], we easily get (see Section 4):

$$(3.2) \quad D_t(M(\rho) + B_0) = -p(\rho) + B_1 + B_2,$$

where

$$M(\rho) := (\lambda + 2\mu) \ln \rho, \quad B_0 := I^{(1)}(\rho u), \quad B_1 := I^{(1)}\left(\rho f - \frac{n-1}{r} \rho u^2\right),$$

$$B_2 := \left\langle \rho u^2 + p(\rho) - (\lambda + 2\mu) \frac{n-1}{r} u \right\rangle.$$

By Lemma 2.1 and (3.1), we have

$$\|B_0\|_{C(\bar{\Omega})} \leq \|\rho u\|_{L^1} \leq \|\rho\|_{L^1}^{1/2} \|\sqrt{\rho} u\| = \|\rho_0\|_{L^1}^{1/2} \|\sqrt{\rho} u\| \leq C,$$

$$\|B_1\|_{C(\bar{\Omega})} \leq \|\rho f\|_{L^1} + \frac{n-1}{a} \|\rho u^2\|_{L^1} \leq C,$$

$$|B_2| \leq C + C \|\operatorname{div} u\|,$$

which imply

$$\int_{t_1}^{t_2} (\|B_1\|_{C(\bar{\Omega})} + |B_2|) dt \leq C + C(t_2 - t_1).$$

Now, if we transfer the equation (3.2) to the form in Lagrangian coordinates, and take

$$y := M(\rho), \quad g := -p\left(\frac{y}{\lambda + 2\mu}\right), \quad b(t) := \int_0^t (B_1 + B_2)(\tau) d\tau - B_0(t)$$

in Lemma 3.1, then we obtain the uniform upper boundedness of  $M(\rho)$  by applying Lemma 3.1. Consequently, the uniform boundedness of  $\rho$  follows immediately.  $\square$

**REMARK 3.1.** If  $b = \infty$  (the exterior domain), we can decompose  $[a, \infty) = \bigcup_{i=0}^\infty [a+i, a+i+1)$ , and repeat the above process in each unit interval  $(a+i, a+i+1)$  to obtain

$$\|\rho(\cdot, t)\|_{L^\infty(a+i, a+i+1)} \leq C, \quad \forall t \geq 0,$$

where  $C$  is independent of  $i$  and  $t$ . Hence, Theorem 3.2 also holds for the case  $b = \infty$ .

Next, we derive uniform in time bounds of the velocity. In the derivation of these bounds, we will make use of the following uniform Gronwall lemma [11]:

**Lemma 3.3** (Uniform Gronwall lemma). *Let  $g, h, y$  be three positive locally integrable functions on  $(t_0, +\infty)$  such that  $y'$  is locally integrable on  $(t_0, +\infty)$ , and satisfy*

$$\frac{dy}{dt} \leq gy + h \quad \text{for } t \geq t_0,$$

$$\int_t^{t+1} g(s) ds \leq a_1, \quad \int_t^{t+1} h(s) ds \leq a_2, \quad \int_t^{t+1} y(s) ds \leq a_3 \quad \text{for } t \geq t_0,$$

where  $a_1, a_2, a_3$ , are positive constants. Then

$$y(t + 1) \leq (a_2 + a_3) \exp(a_1), \quad \forall t \geq t_0.$$

By virtue of (2.1), (2.2) and the assumption on  $f$ , we easily find that

$$\frac{d}{dt} \int_{\Omega} \left[ \frac{1}{2} \rho u^2 + \tilde{p}(\rho) - \rho \int_0^\gamma f(\xi) d\xi \right] dx + \int_{\Omega} [\mu(\nabla u)^2 + (\lambda + \mu)(\operatorname{div} u)^2] dx \leq 0,$$

Integrating the above estimates over  $(t, t + 1)$  yields

$$(3.3) \quad \int_t^{t+1} \int_{\Omega} [\mu(\nabla u)^2 + (\lambda + \mu)(\operatorname{div} u)^2] dx dt \leq C, \quad \forall t \geq 0.$$

The uniform boundedness of the velocity reads:

**Theorem 3.4.** *Let  $f \in L^\infty(\Omega)$ , then there exists a positive constant  $C$  independent of  $t$ , such that*

$$(3.4) \quad \|u(\cdot, t)\|_{H^1} \leq C, \quad \forall t \geq 0.$$

Proof. A direct calculation shows that

$$\Delta u = \nabla \operatorname{div} u = \left( u_r + \frac{n-1}{r} u \right) \frac{x}{r},$$

and thus (1.2) can be rewritten as

$$\rho u_t + \rho u \cdot \nabla u - (\lambda + 2\mu) \nabla \operatorname{div} u + \nabla p = \rho f.$$

Multiplying this equation by  $u_t$ , integrating by parts over  $\Omega$  and using Young's inequality, we infer that

$$(3.5) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} \frac{\lambda + 2\mu}{2} (\operatorname{div} u)^2 dx + \frac{1}{2} \int_{\Omega} \rho u_t^2 dx \\ & \leq \int_{\Omega} \rho f^2 dx + \int_{\Omega} \rho |u|^2 |\nabla u|^2 dx + \int_{\Omega} p(\rho) \operatorname{div} u_t dx. \end{aligned}$$

Using the continuity equation (1.1), we obtain in the same manner as in [2] that

$$(3.6) \quad \begin{aligned} & \int_{\Omega} p(\rho) \operatorname{div} u_t dx \\ & = \frac{d}{dt} \int_{\Omega} p \operatorname{div} u dx + \int_{\Omega} (\operatorname{div}(pu) + (\gamma - 1)p \operatorname{div} u) \operatorname{div} u dx \\ & = \frac{d}{dt} \int_{\Omega} p \operatorname{div} u dx - \int_{\Omega} pu \cdot \nabla \operatorname{div} u dx + (\gamma - 1) \int_{\Omega} p(\operatorname{div} u)^2 dx \\ & = \frac{d}{dt} \int_{\Omega} p \operatorname{div} u dx + \frac{4\gamma - 3}{2(\lambda + 2\mu)} \int_{\Omega} p^2 \operatorname{div} u dx \\ & \quad + \frac{\gamma - 1}{(\lambda + 2\mu)^2} \int_{\Omega} p(G^2 - p^2) dx - \frac{1}{\lambda + 2\mu} \int_{\Omega} pu \cdot \nabla G dx \\ & = \frac{d}{dt} \int_{\Omega} \left( p(\rho) \operatorname{div} u - \frac{4\gamma - 3}{2(\lambda + 2\mu)(2\gamma - 1)} p^2(\rho) \right) dx \\ & \quad + \frac{\gamma - 1}{(\lambda + 2\mu)^2} \int_{\Omega} p(\rho)(G^2 - p^2(\rho)) dx - \frac{1}{\lambda + 2\mu} \int_{\Omega} p(\rho)u \cdot \nabla G dx. \end{aligned}$$

Substituting (3.6) into (3.5), we have

$$(3.7) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} \left[ \frac{\lambda + 2\mu}{2} (\operatorname{div} u)^2 - p(\rho) \operatorname{div} u + \frac{4\gamma - 3}{2(\lambda + 2\mu)(2\gamma - 1)} p^2(\rho) \right] dx + \frac{1}{2} \int_{\Omega} \rho u_t^2 dx \\ & \leq \int_{\Omega} \rho f^2 dx - \frac{1}{\lambda + 2\mu} \int_{\Omega} p(\rho)u \cdot \nabla G dx + \frac{\gamma - 1}{(\lambda + 2\mu)^2} \int_{\Omega} p(\rho)(G^2 - p^2(\rho)) dx \\ & \quad + \int_{\Omega} \rho |u|^2 |\nabla u|^2 dx. \end{aligned}$$

For simplicity, we denote

$$(3.8) \quad y(t) := \int_{\Omega} \left[ (\lambda + 2\mu)(\operatorname{div} u)^2 - 2p(\rho) \operatorname{div} u + \frac{4\gamma - 3}{(\lambda + 2\mu)(2\gamma - 1)} p^2(\rho) \right] dx.$$

Now, we estimate each term on the right-hand side of (3.7). Recalling  $\|\operatorname{div} u\| = \|\nabla u\|$ , and using the identity

$$\nabla G = \rho u_t + \rho u \cdot \nabla u - \rho f$$

and the obvious inequality

$$\begin{aligned}
 (\lambda + 2\mu)\frac{2\gamma - 2}{4\gamma - 3}(\operatorname{div} u)^2 &\leq (\lambda + 2\mu)(\operatorname{div} u)^2 - 2p(\rho) \operatorname{div} u + \frac{4\gamma - 3}{(\lambda + 2\mu)(2\gamma - 1)}p^2(\rho) \\
 &\leq (\lambda + 2\mu + 1)(\operatorname{div} u)^2 + \left(1 + \frac{4\gamma - 3}{(\lambda + 2\mu)(2\gamma - 1)}\right)p^2(\rho),
 \end{aligned}$$

we derive that

$$\begin{aligned}
 (3.9) \quad \int_{\Omega} \rho|u|^2|\nabla u|^2 dx &\leq \|\rho\|_{L^\infty}\|u\|_{L^\infty}^2\|\nabla u\|^2 \\
 &\leq Cy^2,
 \end{aligned}$$

$$\begin{aligned}
 (3.10) \quad \int_{\Omega} p(\rho)G^2 dx &\leq \|p(\rho)\|_{L^\infty} \int_{\Omega} ((\lambda + 2\mu) \operatorname{div} u - p(\rho))^2 dx \\
 &\leq C + Cy,
 \end{aligned}$$

$$\begin{aligned}
 (3.11) \quad \frac{1}{\lambda + 2\mu} \int_{\Omega} p(\rho)u \cdot \nabla G dx &\leq C\|\rho\|_{L^\infty}^{\gamma-1/2}\|\sqrt{\rho}u\| \cdot \|\nabla G\| \\
 &\leq C(\|\rho u_t\| + \|\rho u \cdot \nabla u\| + \|\rho f\|) \\
 &\leq C + \frac{1}{4} \int_{\Omega} \rho u_t^2 dx + C\|\nabla u\|^2 \\
 &\leq C + Cy + \frac{1}{4} \int_{\Omega} \rho u_t^2 dx.
 \end{aligned}$$

Substitution of (3.9)–(3.11) into (3.7) results in

$$(3.12) \quad \frac{dy}{dt} + \int_{\Omega} \rho u_t^2 dx \leq Cy^2 + Cy + C.$$

Recalling the definition of  $y(t)$ , we use Theorem 3.2 and (3.3) to conclude

$$(3.13) \quad \int_t^{t+1} y(s) ds \leq C,$$

which, by applying the uniform Gronwall lemma to (3.12), gives (3.4). □

REMARK 3.2. From (3.4), (3.8), (3.12), it is easy to see that

$$(3.14) \quad \int_t^{t+1} \int_{\Omega} (\rho u_t^2 + |\nabla G|^2) dx dt \leq C, \quad \forall t \geq 0.$$

**4. Proof of (3.2)**

This section is devoted to the proof of (3.2).

We apply the operator  $I^{(1)}$  to equation (1.8) and obtain

$$I^{(1)}(\rho D_t u) - (\lambda + 2\mu)I^{(1)}\left(\left(u_r + \frac{n-1}{r}u\right)_r\right) + I^{(1)}(p_r) = I^{(1)}(\rho f).$$

It is easy to compute that

$$\begin{aligned} I^{(1)}\left(\left(u_r + \frac{n-1}{r}u\right)_r\right) &= u_r + \frac{n-1}{r}u - \left\langle \frac{n-1}{r}u \right\rangle, \\ I^{(1)}(p_r) &= p - \langle p \rangle, \\ \frac{1}{\rho}(\rho_t + u\rho_r) + u_r + \frac{n-1}{r}u &= 0, \end{aligned}$$

which yield

$$(4.1) \quad I^{(1)}(\rho D_t u) + D_t M(\rho) + (\lambda + 2\mu)\left\langle \frac{n-1}{r}u \right\rangle + p - \langle p \rangle = I^{(1)}(\rho f).$$

On the other hand, it is easy to compute that

$$\begin{aligned} D_t I(\rho u) &= I(\rho u)_t + \rho u^2, \\ I(\rho D_t u) &= I(\rho u_t + \rho u u_r) = I((\rho u)_t - \rho_t u + \rho u u_r) \\ &= I(\rho u)_t + I\left((\rho u^2)_r + \frac{n-1}{r}\rho u^2\right) = I(\rho u)_t + \rho u^2 + I\left(\frac{n-1}{r}\rho u^2\right), \end{aligned}$$

which imply

$$(4.2) \quad D_t I(\rho u) = I(\rho D_t u) - I\left(\frac{n-1}{r}\rho u^2\right).$$

Using (4.2) and noting that

$$D_t \langle I(\rho u) \rangle = \langle I(\rho u) \rangle_t,$$

we deduce that

$$\begin{aligned}
 D_t I^{(1)}(\rho u) &= D_t I(\rho u) - D_t \langle I(\rho u) \rangle \\
 &= I(\rho D_t u) - I\left(\frac{n-1}{r} \rho u^2\right) - \langle I(\rho u) \rangle_t \\
 &= I^{(1)}(\rho D_t u) - I^{(1)}\left(\frac{n-1}{r} \rho u^2\right) \\
 (4.3) \quad &+ \langle I(\rho D_t u) \rangle - \left\langle I\left(\frac{n-1}{r} \rho u^2\right) \right\rangle - \langle I(\rho u) \rangle_t \\
 &= I^{(1)}(\rho D_t u) - I^{(1)}\left(\frac{n-1}{r} \rho u^2\right) + \left\langle I\left[\rho D_t u - \frac{n-1}{r} \rho u^2 - (\rho u)_t\right] \right\rangle \\
 &= I^{(1)}(\rho D_t u) - I^{(1)}\left(\frac{n-1}{r} \rho u^2\right) + \langle \rho u^2 \rangle.
 \end{aligned}$$

Summing up (4.1) and (4.3) gives 3.2.

This completes the proof. □

ACKNOWLEDGEMENTS. The authors are indebted to the referee for giving nice suggestions which improve the presentation of the paper.

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