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### ON COHOMOLOGY GROUPS OF NEF LINE BUNDLES TENSORIZED WITH MULTIPLIER IDEAL SHEAVES ON COMPACT KÄHLER MANIFOLDS

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### Introduction

Let X be a compact Kähler manifold of dimension n provided with a Kähler metric  $\omega_X$  and let E be a holomorphic line bundle on X. E is said to be *numerically effective*, "*nef*" for short, if the real first Chern class  $c_{R,1}(E)$  of E is contained in the closure of the Kähler cone of X. If X is projective algebraic, then E is nef if and only if  $C \cdot E = \int_C c_{R,1}(E) \ge 0$  for any irreducible reduced curve C of X (cf.[13], §2 and [1], §6).

If E is nef, then for a fixed smooth metric  $h_E$  of E and a given sequence of positive numbers  $\{\varepsilon_k\}_{k>1}$  decreasing to zero, there exists a sequence of realvalued smooth functions  $\{\varphi_k\}_{k\geq 1}$  such that every form  $\Theta_E + dd^c \varphi_k + \varepsilon_k \omega_X$  yields a Kähler metric. Here  $\Theta_E$  is the curvature form of E relative to  $h_E$  defined by  $\Theta_E = dd^c(-\log h_E)$  with  $d^c = \sqrt{-1}(\bar{\partial} - \partial)/2$ . Normalizing  $\varphi_k$  in such a way that  $\sup_X \varphi_k = 0$ , we can show that  $\varphi_k$  converges to an integrable function  $\varphi_\infty$ on X so that  $\Theta_E + dd^c \varphi_\infty$  is a positive current (cf. §2, Proposition 2.5). Such an integrable function  $\varphi_{\infty}$  is said to be *almost plurisubharmonic*. In general  $\varphi_{\infty}$  has singularities and  $e^{-\varphi_{\infty}}$  is not integrable on X (cf. [11], [18]), which implies that the nefness is strictly weaker than the semi-positivity of line bundle in the sense of Kodaira (cf. [4], Example 1.7). Hence we can define a coherent analytic sheaf of ideal  $\mathcal{I}(\varphi_{\infty})$  associated to  $\varphi_{\infty}$  whose zero variety (possibly empty) is the set of points in a neighborhood of which  $e^{-\varphi_{\infty}}$  is not integrable. The sheaf  $\mathcal{I}(\varphi_{\infty})$ is called the *multiplier ideal sheaf* associated to  $\varphi_{\infty}$  and we obtain the canonical homomorphism  $\iota^q(\varphi_\infty) : H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega^n_X(E)) \longrightarrow H^q(X, \Omega^n_X(E))$  induced by  $\iota(\varphi_{\infty}): \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega^n_X(E) \hookrightarrow \Omega^n_X(E).$ 

Though  $\varphi_{\infty}$  can not be uniquely determined generally, the study of  $H^q(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega_X^n(E))$  is deeply related to several interesting problems in analytic and algebraic geometry (cf. [2], [3], [11], [12], [18]). Nevertheless not much is known about the cohomology group except a vanishing theorem for multiplier ideal sheaves associated to nef and big line bundles by Nadel (cf. [11]). We study the cohomology group by establishing a certain harmonic representation theorem. In particular we

can determine the structure of Image  $\iota^q(\varphi_{\infty})$ . As a consequence we can get the following Lefschetz type theorem (cf. [5], Theorem 0.3).

**Theorem 1.** Let X be a compact Kähler manifold of dimension n provided with a Kähler metric  $\omega_X$  and let E be a nef line bundle on X provided with a smooth hermitian metric  $h_E$ . Let  $\varphi_{\infty}$  be an integrable function determined as above; i.e.,  $\Theta_E + dd^c \varphi_{\infty}$  is a positive current on X, and let  $\mathcal{I}(\varphi_{\infty})$  be the multiplier ideal sheaf associated to  $\varphi_{\infty}$ . Then the homomorphism

$$L^q: \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(E)) \longrightarrow \text{Image } \iota^q(\varphi_\infty) \subset H^q(X, \Omega_X^n(E))$$

is surjective and the Hodge star operator relative to  $\omega_X$  yields a splitting homomorphism

$$\delta^q$$
: Image  $\iota^q(\varphi_\infty) \longrightarrow \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(E))$ 

with  $L^q \circ \delta^q = \text{id } for any q \ge 1$ .

The theorem was formulated and proved by Enoki in the case where E is semipositive, in which case the zero variety defined by  $\mathcal{I}(\varphi_{\infty})$  is empty and  $\iota^q(\varphi_{\infty})$  is isomorphic. Furthermore we can obtain certain injectivity and vanishing theorems for the cohomology groups, which are weaker than the semi-positive line bundle case and are closely linked together to study a Kawamata-Viehweg type vanishing theorem on compact Kähler manifolds (cf. §4, Theorems 4.2 and 4.3). Actually the following vanishing theorem holds (cf. [5], [9], [10], [15], [17], [19]).

**Theorem 2.** Let the situation be the same as in Theorem 1. Then if  $q > n - \kappa_*(E)$ 

$$\iota^{q}(\varphi_{\infty}): H^{q}(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega^{n}_{X}(E)) \longrightarrow H^{q}(X, \Omega^{n}_{X}(E))$$

is the zero homomorphism. Especially if  $\iota^q(\varphi_{\infty})$  is surjective (resp. injective) and  $q > n - \kappa_*(E)$ , then

$$H^{q}(X, \Omega^{n}_{X}(E)) = 0 \ (\textit{resp. } H^{q}(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega^{n}_{X}(E)) = 0),$$

where  $\kappa_*(E)$  is the numerical Kodaira dimension of E defined by

$$\kappa_*(E) := \max\{l : \bigwedge^l c_{R,1}(E) \neq 0 \in H^{2l}(X,R)\}.$$

REMARK. The above vanishing theorem is a variant of Kawamata-Viehweg's vanishing theorem for nef line bundles on projective algebraic manifolds (cf. [9],

[19]). We do not know whether Kawamata-Viehweg's vanishing theorem still holds on any compact Kähler manifold even if E is *nef* and *good* (cf. §3, Comment and §4, Remark 2).

# 1. Harmonic representation theorem for cohomology groups of multiplier ideal sheaves

**1.1.** Let X be a complex manifold of dimension n and let T be a d-closed (1, 1) current on X. Setting  $d^c = \sqrt{-1}(\bar{\partial} - \partial)/2$  we suppose that T is decomposed as follows :

$$T = \Theta + dd^c \varphi_{\infty}$$

for a d-closed smooth real (1,1) form  $\Theta$  and a locally integrable function  $\varphi_{\infty}$  on X. In this article we represent the positivity of T in the sense of current by the notation " $T \gtrsim 0$ " and the semi-positivity (resp. positivity) of  $\Theta$  by the notation " $\Theta \ge 0$ " (resp. " $\Theta > 0$ "). A function  $\varphi$  on X is said to be almost plurisubharmonic if  $\varphi$  is locally equal to the sum of a plurisubharmonic function and of a smooth function (cf. [1], §1). If  $T \gtrsim 0$  and  $d\Theta = 0$ , then locally there exist a plurisubharmonic function  $\psi$  and a smooth function h such that  $T = dd^c\psi$ ,  $\Theta = dd^ch$  and  $h + \varphi_{\infty}$  is equal almost everywhere to  $\psi$ . Hence the function  $\varphi_{\infty}$  is almost plurisubharmonic. The representation  $\varphi_{\infty} = \psi - h$  is not unique. However if  $\varphi_{\infty} = \psi - h = \psi_* - h_*$  with  $\Theta = dd^ch_*$ , then  $\psi - \psi_*$  is pluriharmonic. In particular  $\psi$  is determined uniquely whenever h is fixed. Therefore we can define the following :

DEFINITION. The multiplier ideal sheaf  $\mathcal{I}(\varphi_{\infty}) \subset \mathcal{O}_X$  associated to  $\varphi_{\infty}$  satisfying with  $T = \Theta + dd^c \varphi_{\infty} \gtrsim 0$  is the sheaf of germs of holomorphic functions  $f_x \in \mathcal{O}_{X,x}$  such that  $|f|^2 e^{-\varphi_{\infty}}$  is integrable with respect to the Lebesgue measure in a local coordinates around x for any point x of X.

It is known that  $\mathcal{I}(\varphi_{\infty})$  is a coherent analytic ideal sheaf of  $\mathcal{O}_X$  (cf. [11, 1.2] and [3, Lemma 4.4]). The zero variety  $V(\mathcal{I}(\varphi_{\infty}))$  of  $\mathcal{I}(\varphi_{\infty})$  is the set of points in a neighborhood of which  $e^{-\varphi_{\infty}}$  is not integrable.

### 1.2.

DEFINITION. A holomorphic line bundle E on X is said to be *pseudo effective* (resp. *semi-positive, positive*) if there exists a smooth hermitian metric  $h_E$  and an almost pluri-subharmonic function  $\varphi_{\infty}$  (resp. a smooth hermitian metric  $h_E$ ) such that  $\Theta_E + dd^c \varphi_{\infty} \gtrsim 0$  (resp.  $\Theta_E \geq 0$ ,  $\Theta_E > 0$ ) on X for the curvature form  $\Theta_E$  relative to  $h_E$  defined by  $\Theta_E = dd^c(-\log h_E)$ .

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EXAMPLE. Let  $D = \sum_{j=1}^{k} m_j D_j$  be an effective divisor on X with irreducible components  $D_j$  and non-negative integers  $m_j$ , and let  $[D_j]$  be the line bundle corresponding to each  $D_j$ . Then one can verify that the line bundle  $F := \bigotimes_{j=1}^{k} [D_j]^{\otimes m_j}$ is pseudo effective by the Lelong-Poincaré formula. If D is a divisor with only normal crossings, then one can take a smooth hermitian metric  $h_F$  and an almost plurisubharmonic function  $\varphi_{\infty}$  such that  $\Theta_F + dd^c \varphi_{\infty} \gtrsim 0$  and  $\mathcal{I}(\varphi_{\infty}) = \mathcal{O}_X(F^*)$ , where  $F^*$  is the dual line bundle of F (cf. [3], §5).

**1.3.** To study the cohomology groups of multiplier ideal sheaves of pseudo effective line bundles we need the following Dolbeault's lemma which is formulated for our purpose (cf. [2, Proposition 4.1] and [3, (5.3) Corollary]).

**Theorem.** Let S be a Stein manifold of dimension n provided with a Kähler metric  $\omega_S$  defined by  $\omega_S := dd^c \Phi$  by a smooth strictly plurisubharmonic function  $\Phi \ge 0$  on S. Suppose E (resp. F) be a pseudo effective (resp. positive) line bundle provided with a smooth metric  $h_E$  and an almost plurisubharmonic function  $\varphi_\infty$ (resp. a smooth metric  $h_F$ ) such that  $\Theta_E + dd^c \varphi_\infty \gtrsim 0$  (resp.  $\Theta_F + dd^c \Phi > 0$ ). Set  $(G, h_G) = (E \bigotimes F, h_E \bigotimes h_F)$ . Then for any  $u \in L^{n,q}_{loc}(S,G), q \ge 1$ , with  $\overline{\partial}u = 0$  and

$$\int_{S}|u|_{G}^{2}e^{-\varphi_{\infty}-2\varPhi}dv_{S}<\infty$$

there exists  $v \in L^{n,q-1}_{loc}(S,G)$  with  $\bar{\partial}v = u$  and

$$q \int_{S} |v|_{G}^{2} e^{-\varphi_{\infty} - 2\Phi} dv_{S} \leq \int_{S} |u|_{G}^{2} e^{-\varphi_{\infty} - 2\Phi} dv_{S}.$$

1.4. Let X be an n dimensional complex manifold provided with a hermitian metric  $\omega_X$ . Let E be a pseudo effective line bundle provided with a smooth metric  $h_E$  and an almost plurisubharmonic function  $\varphi_{\infty}$  with  $\Theta + dd^c \varphi_{\infty} \gtrsim 0$  and let  $\mathcal{I}(\varphi_{\infty})$  be the multiplier ideal sheaf associated to  $\varphi_{\infty}$ . Let F be a holomorphic line bundle provided with a smooth metric  $h_F$  and set  $(G, h_G) = (E \bigotimes F, h_E \bigotimes h_F)$ . We denote  $\| \|_{\infty}$  the  $L^2$ -norm of G-valued forms relative to  $\omega_X$  and  $h_G e^{-\varphi_{\infty}}$ , and denote  $\mathcal{F}^q$  the sheaf of germs of G-valued (n, q) forms u with measurable coefficients such that both u and  $\overline{\partial}u$  are locally square integrable relative to  $\| \|_{\infty}$ . By applying 1.3, Theorem to arbitrary small balls one can see that the complex of sheaves  $\{\mathcal{F}^{\bullet}, \overline{\partial}\}$  provides a fine resolution of the sheaf  $\mathcal{I}(\varphi_{\infty}) \bigotimes \Omega_X^n(G)$ . Hence letting  $\Gamma(X, \mathcal{F}^q)$  be the space of global sections with values in  $\mathcal{F}^q$  and seting  $\mathcal{F}^{-1} = 0$ , we obtain the following :

$$H^{q}(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega^{n}_{X}(G)) \cong \frac{\{u \in \Gamma(X, \mathcal{F}^{q}) : \partial u = 0\}}{\{v \in \Gamma(X, \mathcal{F}^{q}) : v = \bar{\partial}w \text{ with } w \in \Gamma(X, \mathcal{F}^{q-1})\}}$$

for any  $q \ge 0$ .

1.5. Let  $C^q(\mathcal{U}, S)$  be the space of q co-chains associated to the locally finite Stein open covering  $\mathcal{U}$  of X with values in the sheaf  $\mathcal{S} := \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega_X^n(G)$ . Combining 1.3, Theorem with the above Dolbeault's theorem in 1.4 the Čech cohomology group  $H^{\bullet}(\mathcal{U}, S)$  defined by the complex  $\{C^{\bullet}(\mathcal{U}, S), \delta\}$  with the co-boundary operator  $\delta$  is isomorphic to the Dolbeault cohomology group  $H^{\bullet}(X, S)$  in view of Leray's theorem ; i.e., the two complexes  $\{\Gamma(X, \mathcal{F}^{\bullet}), \bar{\partial}\}$  and  $\{C^{\bullet}(\mathcal{U}, S), \delta\}$  are quasi-isomorphic. In particular if X is a compact complex manifold, then the Čech cohomology group  $H^{\bullet}(\mathcal{U}, S)$  has finite dimension and so it is a separeted Fréchet topological vector space (cf. [7], Appendix B, 12. Theorem).

**1.6.** From now on we assume that X is a compact complex manifold. Let  $L^{p,q}(X,G)$  (rsep.  $L^{p,q}_{\infty}(X,G)$ ) be the  $L^2$ -space of G-valued square integrable (p,q) forms provided with the inner product (, ) (resp.  $(, )_{\infty}$ ) relative to  $\omega_X$  and  $h_G$  (resp.  $\omega_X$  and  $h_G e^{-\varphi_{\infty}}$ ). We denote  $\vartheta : L^{p,q}(X,G) \to L^{p,q-1}(X,G)$  the adjoint operator of the closed densily defined operator  $\overline{\partial} : L^{p,q}(X,G) \to L^{p,q+1}(X,G)$  relative to (, ). Since  $\varphi_{\infty}$  is bounded from above,  $L^{p,q}_{\infty}(X,G)$  can be regarded as a subspace of  $L^{p,q}(X,G)$ . We denote the restriction of the operator  $\overline{\partial} : L^{n,q}(X,G) \to L^{n,q+1}(X,G) \to L^{n,q+1}(X,G)$  whose domain  $\operatorname{Dom}^{n,q}(\overline{\partial}_{(\infty)})$  coincides with  $\Gamma(X,\mathcal{F}^q) \subseteq L^{n,q}_{\infty}(X,G)$ . We claim the following.

**Lemma.**  $\bar{\partial}_{(\infty)}: L^{n,q}_{\infty}(X,G) \longrightarrow L^{n,q+1}_{\infty}(X,G)$  is a closed densily defined operator.

Proof. By Demailly's regularization result for almost plurisubharmonic functions on compact complex manifolds (cf. [1, Main Theorem 1.1]), there exists a sequence of smooth functions  $\{\varphi_k\}$  on X and an analytic subset A of X such that  $\varphi_k$  decreases to  $\varphi_{\infty}$  on X as k tends to infinity and  $e^{-2\varphi_{\infty}}$  is locally integrable outside A. Set  $(, )_k := (, e^{-\varphi_k})$  and let  $L_k^{n,q}(X,G)$  be the  $L^2$ -space relative to the inner product  $(, )_k$  for any k. Let  $C_0^{n,q}(X \setminus A, G)$  be the space of G-valued smooth (n,q) forms with compact support in  $X \setminus A$ . Take a sequence  $\{w_j\}$  in  $\text{Dom}(\bar{\partial}_{(\infty)})$ such that  $w_j$  and  $\bar{\partial}_{(\infty)}w_j$  converge strongly to w and v respectively. By the decreasing property of  $\varphi_k$ ,  $\bar{\partial}w = v$  in  $L_k^{n,q+1}(X,G)$  for any k. For any  $u \in C_0^{n,q+1}(X \setminus A, G)$ ,  $\langle v, u \rangle_G e^{-\varphi_{\infty}}$  and  $\langle \bar{\partial}w, u \rangle_G e^{-\varphi_{\infty}}$  are integrable on X by Schwarz's inequality. Hence by Lebesgue's dominant convergence theorem we obtain :

$$(v,u)_{\infty} = \lim_{k \to \infty} (v,u)_k = \lim_{k \to \infty} (\bar{\partial}w, u)_k = (\bar{\partial}w, u)_{\infty}.$$

Since  $C_0^{n,q}(X \setminus A, G)$  is dense in  $L_{\infty}^{n,q}(X, G)$ ,  $\bar{\partial}_{(\infty)}$  is densily defined and the above equality implies  $\bar{\partial}_{(\infty)} w = v$  in  $L_{\infty}^{n,q+1}(X,G)$ ; i.e., the closedness of  $\bar{\partial}_{(\infty)}$ .

Hence the adjoint operator  $\vartheta_{(\infty)} := \bar{\partial}_{(\infty)}^*$  of  $\bar{\partial}_{(\infty)}$  can be defined and has the same property as  $\bar{\partial}_{(\infty)}$  with  $\bar{\partial}_{(\infty)} = \bar{\partial}_{(\infty)}^*$ . The domain of  $\vartheta_{(\infty)}$  is defined in the

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following way.

 $v \in \operatorname{Dom}^{n, q}(\vartheta_{(\infty)})$  if and only if there exists a positive constant C such that

$$|(v,\bar{\partial}_{(\infty)}w)_{\infty}| \leq C ||w||_{\infty} \quad for \ any \ w \in \operatorname{Dom}^{n,q-1}(\bar{\partial}_{(\infty)}).$$

For a given linear operator T acting on the Hilbert spaces  $L^{\bullet,\bullet}(X,G)$  and  $L^{\bullet,\bullet}_{\infty}(X,G)$ , we denote  $N^{\bullet,\bullet}(T)$  (resp.  $R^{\bullet,\bullet}(T)$ ) the null space of T (resp. the range of T). Setting  $L^{n,-1}_{\infty}(X,G) = \{0\}$  and  $L^{n,-1}(X,G) = \{0\}$  respectively, we define for any  $q \ge 0$ 

$$H^{n,q}(X,G)\!:=\!N^{n,q}(\bar{\partial})\cap N^{n,q}(\vartheta)\quad\text{and}\quad H^{n,q}_{\infty}(X,G)\!:=\!N^{n,q}(\bar{\partial}_{(\infty)})\cap N^{n,q}(\vartheta_{(\infty)}).$$

 $H^{n,q}(X,G)$  is the *E*-valued (n,q) harmonic space which is isomorphic to  $H^q(X,\Omega^n_X(G))$ . Usually the following weak decomposition of  $L^{n,q}_{\infty}(X,G)$  holds (cf. [8]):

$$L^{n,q}_{\infty}(X,G) = [R^{n,q}(\bar{\partial}_{(\infty)})] \bigoplus H^{n,q}_{\infty}(X,G) \bigoplus [R^{n,q}(\vartheta_{(\infty)})] \text{ for any } q \ge 0,$$

where [] means the closure of space in  $L^{n,q}_{\infty}(X,G)$ . Since X is compact, for any  $q \ge 0$  we note that

$$R^{n,q}(\bar{\partial}_{(\infty)}) = \bar{\partial}\Gamma(X,\mathcal{F}^{q-1}) \text{ and } [R^{n,q}(\bar{\partial}_{(\infty)})] \subset N^{n,q}(\bar{\partial}_{(\infty)}) = \Gamma(X,\mathcal{F}^q) \cap \mathrm{Ker}\bar{\partial}.$$

In view of the compactness of X, it is natural to claim the following strong decomposition.

### **Proposition.**

$$L^{n,q}_{\infty}(X,G) = R^{n,q}(\bar{\partial}_{(\infty)}) \bigoplus H^{n,q}_{\infty}(X,G) \bigoplus R^{n,q}(\vartheta_{(\infty)}) \text{ for any } q \ge 0.$$

Proof. Since the closedness of  $R^{n,q}(\bar{\partial}_{(\infty)})$  is equivalent to the one of  $R^{n,q-1}(\vartheta_{(\infty)})$  (cf. [8, Theorem 1.1.1]), we have only to see that  $[\bar{\partial}\Gamma(X,\mathcal{F}^{q-1})] = \bar{\partial}\Gamma(X,\mathcal{F}^{q-1})$ . Let  $v \in [\bar{\partial}\Gamma(X,\mathcal{F}^{q-1})]$  and let  $\{\bar{\partial}_{(\infty)}w_k\}_{k\geq 1}$  be a sequence in  $\bar{\partial}\Gamma(X,\mathcal{F}^{q-1})$  such that  $\|v - \bar{\partial}_{(\infty)}w_k\|_{\infty} \to 0$  as  $k \to \infty$ . We must find  $w \in \Gamma(X,\mathcal{F}^{q-1})$  with  $v = \bar{\partial}_{(\infty)}w$ . Let  $\mathcal{U}$  be a finite Stein open covering of X taken as in 1.5. Combining the  $L^2$ -estimate in 1.3, Theorem with the quasi-isomorphism theorem in 1.5, there exists a q cocycle  $\sigma(v) \in Z^q(\mathcal{U}, \mathcal{S})$  and a sequence of q-1 cochains  $\{\tau(w_k)\}_{k\geq 1} \subset C^{q-1}(\mathcal{U},\mathcal{S})$  such that  $\sigma(v) - \delta\tau(w_k)$  tends to zero with respect to the uniform convergence topology. From the separability of Fréchet topology induced on  $H^q(\mathcal{U}, \mathcal{S})$ , there is a q-1 cochain  $\tau(w) \in C^{q-1}(\mathcal{U}, \mathcal{S})$  with  $\delta\tau(w) = \sigma(v)$  which implies the conclusion by the compactness of X and the quasi-isomorphism theorem (cf. [17, Proposition 4.6]).

### **1.7.** We obtain the following theorem from the above observations :

**Theorem.** Let X be a compact complex manifold of dimension n provided with a hermitian metric  $\omega_X$  and let E be a pseudo effective line bundle on X provided with a smooth hermitian metric  $h_E$  and an almost plurisubharmonic function  $\varphi_{\infty}$ with  $\Theta_E + dd^c \varphi_{\infty} \gtrsim 0$  on X for  $\Theta_E = dd^c(-\log h_E)$ . Let  $\mathcal{I}(\varphi_{\infty})$  be the multiplier ideal sheaf associated to  $\varphi_{\infty}$ . Then for any holomorphic line bundle F provided with a smooth hermitian metric  $h_F$  on X and  $q \geq 0$ , the space

$$H^{n,q}_{\infty}(X, E\bigotimes F) := \{ u \in \mathrm{Dom}(\bar{\partial}_{(\infty)}) \cap \mathrm{Dom}(\vartheta_{(\infty)}) : \bar{\partial}_{(\infty)}u = 0 \text{ and } \vartheta_{(\infty)}u = 0 \}$$

defined in  $L^{n,q}_{\infty}(X, E \bigotimes F)$  satisfies the following :

$$H^{q}(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega^{n}_{X}(E \bigotimes F)) \cong H^{n,q}_{\infty}(X, E \bigotimes F)$$

and

$$\dim_{\mathbb{C}} H^{n,q}_{\infty}(X, E \bigotimes F) < \infty$$

Furthermore the following diagram is commutative :

$$\begin{array}{ccc} H^{q}(X,\mathcal{I}(\varphi_{\infty})\bigotimes\Omega^{n}_{X}(E\bigotimes F)) & \stackrel{\iota^{q}(\varphi_{\infty})}{\longrightarrow} & H^{q}(X,\Omega^{n}_{X}(E\bigotimes F)) \\ & i^{q}_{\infty} & & i^{q} \\ & & & & \\ H^{n,q}_{\infty}(X,E\bigotimes F) & \stackrel{H^{n,q}}{\longrightarrow} & H^{n,q}(X,E\bigotimes F) \end{array}$$

where  $i_{\infty}^{q}$  and  $i^{q}$  (resp.  $H^{n,q}$ ) are isomorphisms (resp. the orthogonal projection from  $L^{n,q}(X, E \otimes F)$  to  $H^{n,q}(X, E \otimes F)$ ).

# 2. A smoothing of almost plurisubharmonic functions associated to nef line bundles on compact Kähler manifolds

Let X be a compact Kähler manifold of dimension n provided with a Kähler metric  $\omega_X$  and let E be a holomorphic line bundle provided with a smooth hermitian metric  $h_E$  on X.

DEFINITION 2.1.  $(E, h_E)$  is said to be nef if for any  $\varepsilon > 0$  there exists a smooth function  $\psi_{\varepsilon}$  on X such that  $\Theta_E + dd^c \psi_{\varepsilon} + \varepsilon \omega_X$  yields a Kähler metric for  $\Theta_E := dd^c(-\log h_E)$ .

The above definition depends on the choice of neither  $h_E$  nor  $\omega_X$  and is equivalent to that the real first Chern class  $c_{R,1}(E)$  of E is contained in the closure of

the Kähler cone of X (cf. [13], §2). If E has a smooth metric whose curvature is semi-positive, then E is clearly nef. However the converse is not true in general even if X is projective algebraic (cf. [4, Example 1.7]).

We begin with the following lemma suggested by [6], Lemma 2.1 and [18], Proposition 2.1 (compare [2, Lemma 6.6]).

**Lemma 2.2.** Let  $(X, \omega_X)$  be a compact Kähler manifold of dimension n and let  $\Theta$  be a d-closed smooth real (1,1) form on X. Let  $\mathcal{P}(\Theta)$  be the set of real-valued smooth functions  $\psi$  so that  $\Theta + dd^c \psi \ge 0$  and  $\sup_X \psi = 0$ . Then any sequence  $\{\psi_k\}_{k>1}, \psi_k \in \mathcal{P}(\Theta)$ , contains a Cauchy subsequence in  $L^1(X)$ .

**REMARK.** The existence of an  $L^1$  Cauchy subsequence in  $\{\psi_k\}_{k\geq 1}, \psi_k \in \mathcal{P}(\Theta)$ , is not trivial because a local version of such a property is never true (cf. [18, p.238, Remark] and Remark 2 below).

Proof. Let  $\{\psi_k\}_{k\geq 1}$  be a sequence belonging to  $\mathcal{P}(\Theta)$ . Setting  $\tau_X = \omega_X^{n-1}/(n-1)!$  and  $dv_X = \omega_X^n/n!$ , there exists a positive constant  $C(\Theta, \omega_X)$  not depending on k such that

$$\begin{split} 0 &\leq \int_{X} e^{\psi_{k}} d\psi_{k} \wedge d^{c}\psi_{k} \wedge \tau_{X} = -\int_{X} e^{\psi_{k}} dd^{c}\psi_{k} \wedge \tau_{X} \quad \text{by Stokes' theorem} \\ &= -\int_{X} e^{\psi_{k}} \{ dd^{c}\psi_{k} + \Theta \} \wedge \tau_{X} + \int_{X} e^{\psi_{k}} \Theta \wedge \tau_{X} \\ &\leq \int_{X} |\text{Trace}(\Theta, \omega_{X})| dv_{X} \leq C(\Theta, \omega_{X}) < \infty. \end{split}$$

Since  $\{e^{\psi_k/2}\}$  and their first derivatives are bounded in  $L^2(X)$  from the above inequality,  $\{e^{\psi_k/2}\}$  has a Cauchy subsequence in  $L^2(X)$  in view of Rellich's lemma.

On the other hand there are three positive constants  $C_j$  such that  $C_1\omega_X \leq C_2\omega_X + \Theta \leq C_3\omega_X$ . Hence by [18], Proposition 2.1, there exist positive constants  $\alpha$  with  $0 < \alpha \ll 1$  and  $C_*$  not depending on  $\psi \in \mathcal{P}(\Theta)$  such that

(2.3) 
$$\int_X e^{-\alpha\psi} dv_X \le C_* < \infty$$

for any  $\psi \in \mathcal{P}(\Theta)$ . For any  $\beta > 0$  by Schwarz's inequality we obtain

$$\left(\int_X \left| e^{\beta(\psi_j - \psi_k)} - 1 \right| dv_X \right)^2 \le \left(\int_X \left| e^{\beta\psi_j} - e^{\beta\psi_k} \right|^2 dv_X \right) \left(\int_X e^{-2\beta\psi_k} dv_X \right).$$

Taking  $2\beta = \alpha$  the right hand side converges to zero from the above observation and (2.3). In particular we get

(2.4) 
$$\int_X \left| \max\left\{ e^{\beta(\psi_j - \psi_k)}, 1 \right\} - 1 \right| dv_X \to 0 \quad \text{as } j \text{ and } k \to \infty.$$

Here we may assume  $Vol(X, \omega_X) = 1$  and use the following notation :

$$\log^{+} t = \log \max\{t, 1\}$$
 and  $|\log t| = \log^{+} t + \log^{+} \left(\frac{1}{t}\right)$  for  $t > 0$ .

By setting  $\gamma = 1/\beta$  and the concavity of logarithmic functions we obtain :

$$\begin{split} &\int_{X} |\psi_{j} - \psi_{k}| \, dv_{X} \\ &= \gamma \int_{X} \left| \log \left\{ e^{\beta(\psi_{j} - \psi_{k})} \right\} \right| dv_{X} \\ &= \gamma \int_{X} \left\{ \log^{+} e^{\beta(\psi_{j} - \psi_{k})} + \log^{+} e^{\beta(\psi_{k} - \psi_{j})} \right\} dv_{X} \\ &\leq \gamma \log \left\{ \left( \int_{X} \max \left\{ e^{\beta(\psi_{j} - \psi_{k})}, 1 \right\} dv_{X} \right) \left( \int_{X} \max \left\{ e^{\beta(\psi_{k} - \psi_{j})}, 1 \right\} dv_{X} \right) \right\} \end{split}$$

Finally our assertion follows from the above inequality and (2.4).

**Proposition 2.5.** Let  $(E, h_E)$  be a nef line bundle on a compact Kähler manifold  $(X, \omega_X)$ . For a given sequence of positive numbers  $\{\eta_k\}_{k\geq 1}$  decreasing to zero, let  $\{\psi_k\}_{k\geq 1}$  be a sequence of smooth functions on X such that

(2.5) 
$$\Theta_E + dd^c \psi_k + \eta_k \omega_X > 0 \quad \text{on } X \text{ and } \sup_X \psi_k = 0,$$

where  $\Theta_E = dd^c(-\log h_E)$ .

Then there exist an almost plurisubharmonic function  $\varphi_{\infty}$ , a sequence of smooth functions  $\{\varphi_k\}_{k\geq 1}$  on X, and a sequence of positive numbers  $\{\varepsilon_k\}_{k\geq 1}$  decreasing to zero such that

- (i)  $\Theta_E + dd^c \varphi_{\infty} \gtrsim 0$ ; i.e., E is pseudo effective on X
- (ii)  $\Theta_E + dd^c \varphi_k + \varepsilon_k \omega_X > 0$  and  $\varphi_\infty < \varphi_k \le 1$  on X for any  $k \ge 1$
- (iii)  $\varphi_k$  converges to  $\varphi_{\infty}$  in  $L^1(X)$  and almost everywhere on X.

Proof. By applying Lemma 2.2 to  $\Theta_E + \eta_k \omega_X$ , if necessary, taking a subsequence, there exists a limit  $\varphi_{\infty} \in L^1(X)$  such that  $\{\psi_k\}_{k\geq 1}$  converges to  $\varphi_{\infty}$  in  $L^1(X)$ . If necessary, taking a subsequence, we may assume that :

(1) 
$$\|\psi_k - \varphi_\infty\|_{L^1(X)} < \frac{1}{2k}$$

(2) 
$$\Theta_E + dd^c \varphi_\infty \gtrsim 0.$$

(2) follows from the weak continuity of  $\partial \bar{\partial}$  and (2.5) immediately. Locally  $\omega_X$  can be written  $\omega_X = dd^c \Phi$  by a smooth strictly plurisubharmonic function  $\Phi$ . By (2.5) (resp. (2))  $-\log h_E + \eta_k \Phi + \psi_k$  (resp.  $-\log h_E + \varphi_{\infty}$ ) defines locally a smooth

plurisubharmonic function  $\theta_k$  (resp. a plurisubharmonic function  $\theta_{\infty}$ ). For every k we put

$$\lambda_k := \max\{\psi_k, \varphi_\infty\}.$$

Then  $\lambda_k$  satisfies the following properties for any  $k \ge 1$ :

$$\|\lambda_k - \varphi_\infty\|_{L^1(X)} < \frac{1}{2k}$$

(4) 
$$\Theta_E + dd^c \lambda_k + \eta_k \omega_X \gtrsim 0.$$

(3) follows from (1) and (4) follows from the following local equality :

$$\lambda_k = \log h_E - \eta_k \Phi + \max\{\theta_k, \theta_\infty + \eta_k \Phi\}$$

because  $\max\{\theta_k, \theta_\infty + \eta_k \Phi\}$  is plurisubharmonic. Since  $\lambda_k$  is *locally bounded*, the Lelong number of  $\lambda_k$  is zero at any point of X. Therefore by Demailly's regularization result for almost plurisubharmonic functions (cf. [1], §3. the proof of Propositions 3.1 and 3.7), there exist a sequence of smooth functions  $\{\varphi_k\}_{k\geq 1}$  and a sequence of positive numbers  $\{\delta_k\}_{k\geq 1}$  decreasing to zero such that

(5) 
$$\varphi_{\infty} \leq \lambda_k < \varphi_k \leq 1 \quad on \quad X$$

(6) 
$$\Theta_E + dd^c \varphi_k + (\eta_k + \delta_k) \omega_X \ge 0 \quad on \quad X$$

(7) 
$$\|\varphi_k - \lambda_k\|_{L^1(X)} < \frac{1}{2k}$$

for any  $k \ge 1$ . Setting  $\varepsilon_k := \eta_k + 2\delta_k$  and if necessary, taking a subsequence, we obtain the desired sequence  $\{\varphi_k\}_{k\ge 1}$ . This completes the proof of Proposition 2.5.

## 3. On cohomology groups of nef line bundles tensorized with multiplier ideal sheaves on compact Kähler manifolds

Let X be a connected compact Kähler manifold of dimension n provided with a Kähler metric  $\omega_X$ . Let E (resp. F) be a nef (resp. semi-positive) line bundle provided with a smooth metric  $h_E$  (resp.  $h_F$  with  $\Theta_F = dd^c(-\log h_F) \ge 0$ ) on X. Let  $\varphi_{\infty}$  be an almost plurisubharmonic function on X with  $\Theta_E + dd^c \varphi_{\infty} \gtrsim 0$  determined in Proposition 2.5 and let  $\mathcal{I}(\varphi_{\infty})$  be the multiplier ideal sheaf associated to  $\varphi_{\infty}$ . For  $\varphi_{\infty}$  we fix a sequence of smooth almost plurisubharmonic functions  $\{\varphi_k\}_{k\ge 1}$  taken as in Proposition 2.5. We set :

$$G = E \bigotimes F, \quad h_G = h_E \bigotimes h_F, \quad \text{and} \quad h_{G,k} = h_G e^{-\varphi_k}$$

for any k with  $0 \le k \le \infty$ . Here if k = 0, then we set  $\varphi_0 \equiv 0$  and do not specify it in the notations below.

 $L_k^{p,q}(X,G)$  be the  $L^2$ -space of G-valued square integrable (p,q) forms provided with the inner product  $(,)_k$  relative to  $\omega_X$  and  $h_{G,k}$ , and let  $|| ||_k$  denote the norm defined by the inner product.  $L_{\infty}^{p,q}(X,G)$  can be regarded as a subspace of  $L_k^{p,q}(X,G)$  for any k with  $0 \le k < \infty$ . Let  $\vartheta_{(k)}$  denote the adjoint operator of  $\overline{\partial}$ in  $L_k^{p,q}(X,G)$  (cf. 1.6). The space  $N_k^{n,q}(\overline{\partial})$  of null solutions for  $\overline{\partial}$  in  $L_k^{n,q}(X,G)$  is decomposed strongly as follows :

(3.1) 
$$N_k^{n,q}(\bar{\partial}) = R_k^{n,q}(\bar{\partial}) \bigoplus H_k^{n,q}(X,G)$$

where  $H_k^{n,q}(X,G) := \{ u \in L_k^{n,q}(X,G) : \overline{\partial}u = \vartheta_{(k)}u = 0 \}$  for any  $q \ge 1$  and  $0 \le k \le \infty$ . We denote  $H_k^{n,q}$  the orthogonal projection onto  $H_k^{n,q}(X,G)$  for every k with  $0 \le k \le \infty$ .

Setting  $\mathcal{K}^{n,q}_{\infty}(X,G) := \text{Kernel}\{H^{n,q}: H^{n,q}_{\infty}(X,G) \to H^{n,q}(X,G)\}$  (cf. 1.7, Theorem), we define a subspace  $\mathcal{H}^{n,q}_{\infty}(X,G)$  of  $H^{n,q}_{\infty}(X,G)$  by the following orthogonal decomposition relative to  $(, )_{\infty}$ :

$$H^{n,q}_{\infty}(X,G) = \mathcal{H}^{n,q}_{\infty}(X,G) \bigoplus \mathcal{K}^{n,q}_{\infty}(X,G).$$

Since  $\mathcal{K}^{n,q}_{\infty}(X,G) = H^{n,q}_{\infty}(X,G) \cap R^{n,q}(\bar{\partial})$ , the space  $\mathcal{H}^{n,q}_{\infty}(X,G)$  is characterized as follows.

(3.2) 
$$u \in \mathcal{H}^{n,q}_{\infty}(X,G)$$
 if and only if  $u \in N^{n,q}(\bar{\partial}_{\infty})$  and  $(u,\bar{\partial}w)_{\infty} = 0$   
for any  $w \in L^{n,q-1}(X,G)$  with  $\bar{\partial}w \in L^{n,q}_{\infty}(X,G)$ .

We define a homomorphism

$$\mathcal{L}^{q}_{(\infty)}: \Gamma(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega^{n-q}_{X}(G)) \longrightarrow \mathcal{H}^{n,q}_{\infty}(X, G)$$

by the composition of the homomorphism

$$L^q: \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(G)) \longrightarrow N^{n,q}(\bar{\partial}_{(\infty)})$$

induced by the q-times left exterior product by  $\omega_X$  with the orthogonal projection from  $N^{n,q}(\bar{\partial}_{(\infty)})$  to  $\mathcal{H}^{n,q}_{\infty}(X,G)$ .

The following lemma is very useful (cf. [3, (4.10)]).

**Lemma 3.3.** Let W be a holomorphic line bundle on X provided with a smooth hermitian metric  $h_W$ . Let  $\Theta$  be a smooth real (1, 1) differential form on X and let  $\{\lambda_j\}$  be the eigen-values of  $\Theta$  relative to  $\omega_X$  with  $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$  (which are continuous functions on X); i.e.,  $\Theta(x) = \sqrt{-1} \sum_{j=1}^{n} \lambda_j(x) dz^j \wedge d\bar{z}^j$  with  $\omega_X(x) = \sqrt{-1} \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j$ ,  $x \in X$ . Then if  $v(x) = \sum v_{A_n,B_q} dz^{A_n} \wedge d\bar{z}^{B_q} \in C^{n,q}(X,W)$  with  $q \ge 1$ , the following holds

$$\langle \mathbf{e}(\Theta) \Lambda v, v \rangle_W(x) = \sum_{|A_n|=n, |B_q|=q} \left( \sum_{j \in B_q} \lambda_j(x) \right) |v_{A_n, B_q}|_W^2$$

In particular setting  $\delta_q := \sum_{j=1}^q \lambda_j$  with  $q \ge 1$  the following holds

(3.4) 
$$\langle \mathbf{e}(\Theta) \Lambda v, v \rangle_W \ge \delta_q \langle v, v \rangle_W \quad if \ v \in C^{n,q}(X,W).$$

The nefness of E enables us to show the following theorem.

**Theorem 3.5.**  $\mathcal{L}^{q}_{(\infty)}$  is surjective and the Hodge star operator \* relative to  $\omega_X$  yields a splitting homomorphism

$$\delta^q_{(\infty)}: \mathcal{H}^{n,q}_{\infty}(X,G) \longrightarrow \Gamma(X,\mathcal{I}(\varphi_{\infty})\bigotimes \Omega^{n-q}_X(G))$$

with  $\mathcal{L}^q_{(\infty)} \circ \delta^q_{(\infty)} = \mathrm{id}$ . Furthermore  $\mathcal{L}^q_{(\infty)} = L^q$  on  $\mathrm{Image} \delta^q_{(\infty)}$  for any  $q \ge 1$ .

Proof. If  $\mathcal{H}^{n,q}_{\infty}(X,G) = \{0\}$ , then we have nothing to prove. Hence we assume  $\mathcal{H}^{n,q}_{(\infty)}(X,G) \neq \{0\}$  and take  $u \in \mathcal{H}^{n,q}_{\infty}(X,G)$  with  $||u||_{\infty} = 1$ . We claim that  $*u \in \Gamma(X,\mathcal{I}(\varphi_{\infty}) \bigotimes \Omega^{n-q}_X(G))$ , which implies that  $\mathcal{L}^q_{(\infty)} = L^q$  is surjective by  $L^q \circ * = c(n,q)$  id on the space of (n,q) forms for the uniquely determined complex number  $c(n,q) \neq 0$ . We have only to define  $\delta^q_{(\infty)} := c(n,q)^{-1}*$ .

We note that u has the following orthogonal decomposition by (3.1):

(3.6) 
$$u = \bar{\partial}w_k + H_k^{n,q}(u), \|\bar{\partial}w_k\|_k \text{ and } \|H_k^{n,q}(u)\|_k \le 1$$

for any k with  $0 \le k < \infty$ . Setting  $u_k := H_k^{n,q}(u)$ , we may assume  $u_k \ne 0$  for any k. From  $||u_k|| \le e ||u_k||_k \le e$ , taking a subsequence,  $\{u_k\}$  has a weak limit  $u_{\infty} \in L^{n,q}(X,G)$  with  $\bar{\partial}u_{\infty} = 0$ .  $\{\bar{\partial}w_k\}$  also has a weak limit  $v_{\infty}$ . Since  $R^{n,q}(\bar{\partial})$  is closed, there exists  $w_* \in L^{n,q-1}(X,G)$  with  $v_{\infty} = \bar{\partial}w_*$ . Therefore we obtain

(3.7) 
$$u = \bar{\partial}w_* + u_\infty \quad \text{in} \quad L^{n,q}(X,G).$$

We show that  $*u_{\infty} \in \Gamma(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega_X^{n-q}(G))$  and  $u_{\infty} \in \mathcal{H}_{\infty}^{n,q}(X,G)$ , which implies  $\bar{\partial}w_* = 0$  by (3.2); i.e.,  $u_{\infty} = u$ .

By Calabi-Nakano-Vesentini's formula on compact Kähler manifolds (cf. [14, Proposition 1.2]), we obtain the following integral formula :

$$\|\bar{\partial}v\|_k^2 + \|\vartheta_{(k)}v\|_k^2 = \|\bar{\partial}v\|_k^2 + (\mathbf{e}(\Theta_G + dd^c\varphi_k)\Lambda v, v)_k$$

for any G-valued smooth (n,q) form v on X,  $\Theta_G := \Theta_E + \Theta_F$  and  $k \ge 1$ . Since  $q \|v\|_k^2 = (LAv, v)_k$ , by Proposition 2.5, (ii) and the semi-positivity of  $\Theta_F$  (cf. (3.4)), we obtain the following inequality :

$$\varepsilon_k q \|u_k\|_k^2 = \|\bar{\vartheta}u_k\|_k^2 + (\mathbf{e}(\Theta_G + dd^c\varphi_k + \varepsilon_k\omega_X)\Lambda u_k, u_k)_k \\ \ge (\mathbf{e}(\Theta_G + dd^c\varphi_k + \varepsilon_k\omega_X)\Lambda u_k, u_k)_k \ge 0.$$

Therefore when k tends to infinity, we obtain

$$\|\bar{\vartheta}u_k\|_k^2 \le \varepsilon_k q \|u_k\|_k^2 \le \varepsilon_k q \to 0.$$

By  $\bar{\vartheta} = -*\bar{\vartheta}*$  and  $\|\bar{\vartheta}*u_k\|^2 \leq \|\bar{\vartheta}u_k\|_k^2$ ,  $u_\infty$  satisfies  $\bar{\vartheta}*u_\infty = 0$  in the sense of distribution. Therefore  $*u_\infty \in \Gamma(X, \Omega_X^{n-q}(G))$ . Setting  $u^k = u_k e^{-\varphi_k/2}$  and, if necessary taking a subsequence,  $u^k$  converges weakly to  $u^\infty \in L^{n,q}(X,G)$  by  $\|u_k\|_k \leq 1$ . Let V be the analytic subset (might be empty) defined by  $\mathcal{I}(\varphi_\infty)$ . Since  $e^{-\varphi_\infty}$  is locally integrable on  $X \setminus V$ ,  $e^{-\varphi_k}$  converges to  $e^{-\varphi_\infty}$  in  $L^1(K)$  for any compact subset K in  $X \setminus V$  by  $\varphi_\infty < \varphi_k$  and Lebesgue's dominant convergence theorem. For every E-valued smooth (n,q) form v with compact support in  $X \setminus V$ , by setting  $K := \operatorname{Supp}(v)$  and denoting  $|v|_G$  the pointwise length of v relative to  $\omega_X$  and  $h_G$ , we obtain from (3.6) :

$$\begin{split} \lim_{k \to \infty} \left| \left( u_k, \{ e^{-\varphi_{\infty}/2} - e^{-\varphi_k/2} \} v \right) \right| &\leq \lim_{k \to \infty} \sup |v|_G \|u_k\| \|e^{-\varphi_{\infty}/2} - e^{-\varphi_k/2}\|_{L^2(K)} \\ &\leq e \sup_K |v|_G \lim_{k \to \infty} \sqrt{\|e^{-\varphi_{\infty}} - e^{-\varphi_k}\|_{L^1(K)}} = 0. \end{split}$$

Here we have used :  $(a-b)^2 < a^2 - b^2$  if a > b > 0. Hence we get :

$$(u^{\infty}, v) = \lim_{k \to \infty} (u^k, v) = \lim_{k \to \infty} (u_k, v e^{-\varphi_{\infty}/2}) = (u_{\infty} e^{-\varphi_{\infty}/2}, v).$$

This implies  $u^{\infty} = u_{\infty}e^{-\varphi_{\infty}/2}$  on  $X \setminus V$  as current and so  $u_{\infty} \in L^{n,q}_{\infty}(X,G)$  because  $u^{\infty} \in L^{n,q}(X,G)$ . Therefore we get  $*u_{\infty} \in \Gamma(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega^{n-q}_X(G))$ .

Furthermore if  $w \in L^{n,q-1}(X,G)$  with  $\bar{\partial}w \in L^{n,q}_{\infty}(X,G)$ , then  $w \in L^{n,q-1}_k(X,G)$ with  $\bar{\partial}w \in L^{n,q}_k(X,G)$  for any k with  $1 \le k < \infty$  because  $\varphi_k$  is smooth. Therefore by  $\vartheta_k u_k = 0$  and Lebesgue's dominant convergence theorem, we obtain :

$$|(u_{\infty},\bar{\partial}w)_{\infty}| = \lim_{k \to \infty} \left| (u^{k}, \{e^{-\varphi_{\infty}/2} - e^{-\varphi_{k}/2}\}\bar{\partial}w) \right|$$
$$\leq \lim_{k \to \infty} \sqrt{\|\{e^{-\varphi_{\infty}} - e^{-\varphi_{k}}\}|\bar{\partial}w|_{G}^{2}\|_{L^{1}(X)}} = 0.$$

Therefore  $u_{\infty} \in \mathcal{H}_{\infty}^{n,q}(X,G)$  by (3.2). This completes the proof of Theorem 3.5.

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**Proposition 3.8.** Every  $u \in \mathcal{H}^{n,q}_{\infty}(X,G)$  with  $q \ge 1$  satisfies the following :

(3.9) 
$$(\mathbf{e}(\Theta_G + dd^c\varphi)\Lambda u, u)_{\infty} = 0$$

for any smooth real-valued function  $\varphi$  on X.

Proof. By the equations  $\bar{\partial}u = \bar{\vartheta}u = 0$ , we get  $\bar{\partial}\vartheta_G u = \mathbf{e}(\Theta_G)\Lambda u$  and  $\bar{\partial}\mathbf{e}(\bar{\partial}\varphi)^* u = \mathbf{e}(dd^c\varphi)\Lambda u$  by [14], Propositions 1.2 & 1.5. Since  $\Theta_G$  and  $dd^c\varphi$  are smooth on X, we obtain  $\bar{\partial}\vartheta_G u$  and  $\bar{\partial}e(\bar{\partial}\varphi)^* u \in L^{n,q}_{\infty}(X,G)$  by Lemma 3.3. The conclusion follows from (3.2).

In view of the  $L^2$ -estimate (3.9), we can show the following vanishing theorem for  $\mathcal{H}^{n,q}_{\infty}(X,G)$ .

**Theorem 3.10.** If  $q > n - \max\{\kappa_*(E), \kappa_*(F)\}$ , then  $\mathcal{H}^{n,q}_{\infty}(X,G) = 0$ , where  $\kappa_*(E)$  is defined by  $\kappa_*(E) := \max\{l : \bigwedge^l c_{R,1}(E) \neq 0 \in H^{2l}(X,R)\}$  and so on.

Proof. By (3.9), if  $u \in \mathcal{H}^{n,q}_{\infty}(X,G)$ , then for any smooth real-valued function  $\varphi$  on X and  $\varepsilon > 0$  we obtain

(3.11) 
$$0 < (\mathbf{e}(\Theta_G + dd^c \varphi + \varepsilon \omega_X) \Lambda u, u)_{\infty} = q \varepsilon \|u\|_{\infty}$$

and particularly

(3.12) 
$$(\mathbf{e}(\Theta_F)\Lambda u, u)_{\infty} = 0.$$

If  $q > n - \kappa_*(F)$ , then the integrand of (3.12) is non-negative on X and positive at least one point of X by (3.4) (cf. [16], p. 277, Fact 2.7). Therefore u should vanish on X identically because \*u is holomorphic and X is connected.

Assume  $q > n - \kappa_*(E)$  and  $u \neq 0 \in \mathcal{H}^{n,q}_{\infty}(X,G)$ . For any  $\varepsilon > 0$  we set :

$$p(\varepsilon) := \int_X (\Theta_G + \varepsilon \omega_X)^n \left/ \int_X \omega_X^n \right|.$$

Since E is nef, for any  $\varepsilon > 0$  there exists a smooth real-valued function  $\varphi_{\varepsilon}$  on X so that  $\Theta_G + dd^c \varphi_{\varepsilon} + \varepsilon \omega_X$  is a Kähler metric. Furthermore by [21], there exists a smooth real-valued function  $\psi_{\varepsilon}$  on X such that  $\gamma_{\varepsilon} := \Theta_G + dd^c(\varphi_{\varepsilon} + \psi_{\varepsilon}) + \varepsilon \omega_X$  is a Kähler metric on X with

(3.13) 
$$\gamma_{\varepsilon}^{n} = p(\varepsilon)\omega^{n}.$$

Let  $\{\lambda_{\varepsilon,j}\}\$  be the eigenvalues of  $\gamma_{\varepsilon}$  relative to  $\omega_X$  and let  $\delta_{\varepsilon,\mu}$  be a continuous function defined as in Lemma 3.3 relative to  $\{\lambda_{\varepsilon,j}\}\$  for any  $\varepsilon > 0$  and  $1 \le \mu \le n$ .

Set  $U(\varepsilon) := \{\delta_{\varepsilon,q} < 2q\varepsilon\}$  for any  $\varepsilon > 0$ . By applying  $\varphi_{\varepsilon} + \psi_{\varepsilon}$  to (3.11), and Lemma 3.3 we can show

$$0 < \|u\|_{\infty}^2 \le 2 \int_{U(\varepsilon)} |u|_G^2 e^{-\varphi_{\infty}} dv_X.$$

This implies  $U(\varepsilon) \neq \phi$  for any  $\varepsilon > 0$ . We claim that there exists a positive constant  $C_1$  not depending on  $\varepsilon$  such that  $\int_{U(\varepsilon)} dv_X \ge C_1 > 0$  for any  $\varepsilon > 0$ . If  $\int_{U(\varepsilon)} dv_X$  converges to zero, then  $\int_{U(\varepsilon)} |u|^2 e^{-\varphi \infty} dv_X$  also tends to zero because  $|u|_G^2 e^{-\varphi \infty}$  is integrable. However this contradicts to the above inequality.

Furthermore since  $\int_X \mathbf{e}(\gamma_{\varepsilon})\omega_X^{n-1} = \int_X \mathbf{e}(\Theta_G + \varepsilon\omega_X)\omega_X^{n-1}$  is non-negative and bounded from above, there exists positive constant  $C_2$  and  $C_3$  not depending on  $\varepsilon$ such that  $0 < \delta_{\varepsilon,n} \le C_2$  on an open subset  $Q(\varepsilon) \subseteq U(\varepsilon)$  with  $\int_{Q(\varepsilon)} dv_X \ge C_3 > 0$ . Hence we obtain

(3.14) 
$$\prod_{j=1}^{n} \lambda_{\varepsilon,j} \leq (2q)^q C_2^{n-q} \varepsilon^q \quad \text{on} \quad Q(\varepsilon) \quad \text{for any } \varepsilon > 0.$$

On the other hand since  $P(\varepsilon) = \prod_{j=1}^{n} \lambda_{\varepsilon,j}$  is a polynomial in  $\varepsilon$  of degree n and E is nef, letting  $P(\varepsilon) = \sum_{i=0}^{n} a_i \varepsilon^i$  we obtain :  $a_i > 0$  if  $i \ge n - \kappa$  and  $a_i = 0$  if  $i < n - \kappa$  by the definition of  $\kappa = \kappa_*(E)$  and (3.13). This implies that

(3.15) 
$$a_{n-\kappa}\varepsilon^{n-\kappa} \leq \prod_{j=1}^n \lambda_{\varepsilon,j} \quad \text{on} \quad X.$$

By (3.14) and (3.15) we can get  $a_{n-\kappa}\varepsilon^{n-\kappa} \leq (2q)^q C_2^{n-q}\varepsilon^q$ , which is a contradiction as  $\varepsilon$  tends to zero because  $q > n - \kappa$ . The idea of this proof is due to Enoki [5]. This completes the proof of Theorem 3.10.

Next we show the following injectivity theorem.

#### Theorem 3.16.

(i) If the *j*-times tensor product  $E^{\otimes j}$  of E admits a non-trivial holomorphic section  $\sigma$  with

$$C(\sigma) := \operatorname{ess.}_{X} \sup |\sigma|_{E^{\otimes j}}^2 e^{-j\varphi_{\infty}} < \infty$$

then the homomorphism

n

$$\mathcal{H}^{n,q}_{\infty}(\sigma):\mathcal{H}^{n,q}_{\infty}(X, E^{\otimes i}\bigotimes F)\longrightarrow H^{n,q}_{\infty}(X, E^{\otimes (i+j)}\bigotimes F)$$

induced by the tensor product with  $\sigma$  is well defined and particularly injective for any  $q \ge 0$ , i and  $j \ge 1$ .

(ii) If the k-times tensor product  $F^{\otimes k}$  of F admits a non-trivial holomorphic section  $\theta$ , then

$$\mathcal{H}^{n,q}_{\infty}(\theta):\mathcal{H}^{n,q}_{\infty}(X,E\bigotimes F^{\otimes j})\longrightarrow H^{n,q}_{\infty}(X,E\bigotimes F^{\otimes (j+k)})$$

induced by the tensor product with  $\theta$  is well defined and particularly injective for any  $q \ge 0$ , j and  $k \ge 1$ .

Proof of (i). For  $u \in \mathcal{H}^{n,q}_{\infty}(X, E^{\otimes i} \otimes F)$ , setting  $v = \sigma \otimes u$  we have only to show  $(v, \bar{\partial}w)_{\infty} = 0$  for any  $w \in L^{n,q-1}_{\infty}(X, E^{\otimes (i+j)} \otimes F)$  with  $\bar{\partial}w \in L^{n,q}_{\infty}(X, E^{\otimes (i+j)} \otimes F)$ . Since  $\bar{\partial}v = \bar{\partial}v = 0$ , and  $\Theta_F$  is semi-positive, by Calabi-Nakano-Vesentini's formula, Lemma 3.3 and Proposition 3.8, we can conclude :

$$egin{aligned} \|artheta_{(k)}v\|_k^2 &= (\mathbf{e}((i+j)(\Theta_E+dd^carphi_k)+\Theta_F)\Lambda v,v)_k\ &\leq \left(rac{i+j}{i}
ight)(\mathbf{e}(i(\Theta_E+dd^carphi_k+arepsilon_k\omega_X)+\Theta_F)\Lambda v,v)_k\ &\leq arepsilon_kqC(\sigma)\left(rac{i+j}{i}
ight)\|u\|_\infty^2 o 0 \quad \mathrm{as} \quad k o\infty. \end{aligned}$$

Hence by Lebesgue's dominant convergence theorem we have

$$(v, \bar{\partial}w)_{\infty} = \lim_{k \to \infty} (v, \bar{\partial}w)_k = \lim_{k \to \infty} (\vartheta_{(k)}v, w)_k = 0.$$

Proof of (ii). Since the length of  $\theta$  is bounded, the proof can be done similarly. This completes the proof of Theorem 3.16.

REMARK. If the almost plurisubharmonic function  $\varphi_{\infty}$  is determined independently of the choice of  $\{\varepsilon_k\}$ , then from the above proof it can be verified that  $\mathcal{H}^{n,q}_{\infty}(\sigma): \mathcal{H}^{n,q}_{\infty}(X, E^{\otimes i} \bigotimes F) \longrightarrow \mathcal{H}^{n,q}_{\infty}(X, E^{\otimes (i+j)} \bigotimes F)$  is well defined.

Comment. In the situation of this section, setting F = the trivial line bundle, Enoki claims that  $H^{n,q}(X, E) = 0$  if  $q > n - \kappa_*(E)$ , which implies that  $H^q(X, \Omega_X^n(E)) = 0$  if  $q > n - \kappa_*(E)$  (cf. [5, Theorem 0.1]). His idea of the proof consists of two parts; i.e., an  $L^2$ -estimate for the harmonic forms in  $H^{n,q}(X, E)$  and the argument used to show Theorem 3.10. In fact he claims the following  $L^2$ -estimate (cf. [5, Proposition 3.1]):

Let E be a holomorphic line bundle provided with a smooth hermitian metric  $h_E$ on a compact Kähler manifold X of dimension n provided with a Kähler metric  $\omega_X$ . Then for any real-valued smooth function  $\varphi$  on X and  $u \in H^{n,q}(X, E)$  with  $q \ge 1$ , setting  $\eta := e^{\varphi}$  the following inequality holds

$$(\eta \mathbf{e}(\Theta_E + dd^c \varphi) \Lambda u, u) \le 0.$$

Here we should note that any specific condition for the curvature of  $(E, h_E)$  is not assumed to show the above inequality in his proof. However the sign of the left hand side can not be always determined in the following sense.

First for any *E*-valued smooth (n,q) form v on X we can obtain the following integral formula (cf. [17, §1, Proposition 1.11]) :

$$\|\sqrt{\eta}(\bar{\partial} + \mathbf{e}(\bar{\partial}\varphi))v\|^2 + \|\sqrt{\eta}\vartheta_h v\|^2 = \|\sqrt{\eta}(\bar{\vartheta} - \mathbf{e}(\partial\varphi)^*)v\|^2 + (\eta \mathbf{e}(\Theta_E + dd^c\varphi)\Lambda v, v).$$

Hence if  $u \in H^{n,q}(X, E)$ , by setting w = \*u and using  $\mathbf{e}(\partial \varphi)^* = *\mathbf{e}(\bar{\partial}\varphi)*$  we can verify the following from the above formula :

$$\begin{aligned} (\eta \mathbf{e}(\Theta_E + dd^c \varphi) \Lambda u, u) &= -\|\sqrt{\eta} (\bar{\vartheta} - \mathbf{e}(\partial \varphi)^*) u\|^2 + \|\sqrt{\eta} \mathbf{e}(\bar{\partial} \varphi) u\|^2 \\ &= -\|\sqrt{\eta} (\bar{\partial} + \mathbf{e}(\bar{\partial} \varphi)) w\|^2 + \|\sqrt{\eta} \mathbf{e}(\partial \varphi)^* w\|^2. \end{aligned}$$

Here we note that  $\bar{\partial}w$  is primitive; i.e.,  $A\bar{\partial}w = 0$  by  $\bar{\partial}u = 0$  and  $\bar{\vartheta} = -\sqrt{-1}[\bar{\partial}, \Lambda]$ . For any *E*-valued smooth (n-q, 1) form  $\alpha$ , let  $\alpha = \alpha_1 + \alpha_2$  be the primitive decomposition of the form; i.e.,  $A\alpha_1 = 0$  and  $\alpha_2 = 1/(q+1)LA\alpha$  (cf.[20, Chap.V, Theorem 1.8]). Here the coefficient 1/(q+1) of  $\alpha_2$  is crucial. Since  $\mathbf{e}(\partial\varphi)^* = \sqrt{-1}[\mathbf{e}(\bar{\partial}\varphi), \Lambda]$ , by applying the decomposition to  $\alpha := \mathbf{e}(\bar{\partial}\varphi)w$  and the above equality it can be verified that

$$(\eta \mathbf{e}(\Theta_E + dd^c \varphi) \Lambda u, u) = -\|\sqrt{\eta}(\bar{\partial}w + \alpha_1)\|^2 + q\|\sqrt{\eta}\alpha_2\|^2$$

and

$$\alpha_2 = 0$$
 if and only if  $\mathbf{e}(\bar{\partial}\varphi)u = 0$ 

Therefore if  $u \in H^{n,q}(X, E)$  satisfies the equality

$$(\eta \mathbf{e}(\Theta_E + dd^c \varphi) \Lambda u, u) = -\|\sqrt{\eta}(\bar{\partial}w + \alpha_1)\|^2 \le 0$$

for any real-valued smooth function  $\varphi$  on X as he claims (see the last line of his proof of Proposition 3.1 in [5]), then by the above observations an  $E^*$  (the dual of E)-valued harmonic (0, n - q) form  $\overline{*(hu)}$  satisfies the  $\overline{\partial}$ -Neumann condition on every open ball with smooth boundary contained in any local coordinate neighborhood of X. Hence such a form should vanish on it in view of the solvability for  $\overline{\partial}$  on open balls and its boundary condition (cf.[17, §4. Theorem 4.3, (iv)]), and so identically on X by a unique continuation property for harmonic forms, which implies  $H^q(X, \Omega_X^n(E)) = 0$ . However  $H^q(X, \Omega_X^n(E))$  does not vanish without any specific condition in general.

### 4. On cohomology groups of nef line bundles on compact Kähler manifolds

First we state the following Lefschetz type theorem (cf. [5, Theorem 0.3]).

**Theorem 4.1.** Let X be a connected compact Kähler manifold of dimension n provided with a Kähler metric  $\omega_X$ . Let E (resp. F) be a nef (resp. semi-positive) line bundle provided with a smooth metric  $h_E$  (resp.  $h_F$  with  $\Theta_F = dd^c(-\log h_F) \ge 0$ ) on X. Let  $\varphi_{\infty}$  be an almost plurisubharmonic function with  $\Theta_E + dd^c \varphi_{\infty} \gtrsim 0$  determined in Proposition 2.5 and let  $\mathcal{I}(\varphi_{\infty})$  be the multiplier ideal sheaf associated to  $\varphi_{\infty}$ . Then for any  $q \ge 1$  the homomorphism

$$L^{q}: \Gamma(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega_{X}^{n-q}(E \bigotimes F)) \longrightarrow \operatorname{Image}_{\iota}^{q}(\varphi_{\infty}) \subset H^{q}(X, \Omega_{X}^{n}(E \bigotimes F))$$

is surjective and the Hodge star operator relative to  $\omega_X$  yields a splitting homomorphism

$$\delta^q: \operatorname{Image} \iota^q(\varphi_\infty) \longrightarrow \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(E \otimes F))$$

with  $L^q \circ \delta^q = \text{id}$ , where  $\iota^q(\varphi_\infty) : H^q(X, \mathcal{I}(\varphi_\infty) \otimes \Omega^n_X(E \otimes F)) \longrightarrow H^q(X, \Omega^n_X(E \otimes F))$  is the canonical homomorphism induced by  $\iota : \mathcal{I}(\varphi_\infty) \otimes \Omega^n_X(E \otimes F)$  $\hookrightarrow \Omega^n_X(E \otimes F)$ .

Proof. The conclusion follows from Theorem 3.5 because the image of  $\iota^q(\varphi_{\infty})$  can be identified with  $\mathcal{H}^{n,q}_{\infty}(X, E \bigotimes F)$  by the commutative diagram in 1.7, Theorem.

We denote  $V(\varphi_{\infty})$  the compact analytic subset of X defined by the multiplier ideal sheaf  $\mathcal{I}(\varphi_{\infty})$  and define  $d(\varphi_{\infty}) := \max\{\dim_{\mathbb{C}} V(\varphi_{\infty})_{\alpha} : V(\varphi_{\infty})_{\alpha} \text{ is}$ any irreducible component of  $V(\varphi_{\infty})\}$  (we set  $d(\varphi_{\infty}) = -1$  if  $V(\varphi_{\infty}) = \phi$ ; i.e.,  $\mathcal{I}(\varphi_{\infty}) \cong \mathcal{O}_X$ ). It is clear that  $d(j\varphi_{\infty}) \le d(k\varphi_{\infty})$  if  $1 \le j < k$ , and  $\iota^q(\varphi_{\infty})$  is bijective (resp. surjective) if  $q > d(\varphi_{\infty}) + 1$  (resp.  $q > d(\varphi_{\infty})$ ). If the Lelong number of  $\varphi_{\infty}$  is less than one everywhere on X, then  $d(\varphi_{\infty}) = -1$  (cf. [3, (5.6) Lemma]). Under the hypothesis of Theorem 4.1, by Theorem 3.10 we can obtain the following vanishing theorem immediately (cf. [5], [9], [15], [19]).

**Theorem 4.2.** Suppose  $q > n - \max\{\kappa_*(E), \kappa_*(F)\}$ . Then

$$\iota^{q}(\varphi_{\infty}): H^{q}(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega^{n}_{X}(E \bigotimes F)) \longrightarrow H^{q}(X, \Omega^{n}_{X}(E \bigotimes F))$$

is the zero homomorphism. Especially the following assertions hold :

(i) If  $\iota^q(\varphi_{\infty})$  is surjective (resp. injective) and  $q > n - \max\{\kappa_*(E), \kappa_*(F)\}$ , then

$$H^{q}(X, \Omega^{n}_{X}(E\bigotimes F)) = 0 \quad (resp. \ H^{q}(X, \mathcal{I}(\varphi_{\infty})\bigotimes \Omega^{n}_{X}(E\bigotimes F)) = 0)$$

(ii) If  $q > \max\{n - \max\{\kappa_*(E), \kappa_*(F)\}, d(\varphi_\infty)\}$ , then

$$H^q(X, \Omega^n_X(E\bigotimes F)) = 0$$

where  $\kappa_*(E)$  (resp.  $\kappa_*(F)$ ) is the numerical Kodaira dimension of E (resp. F).

**REMARK** 1. The homomorphism  $\iota^q(\varphi_{\infty})$  is not always injective (cf. [4, Example 1.7]).

At last we can get the following theorem from Theorem 3.16 (cf. [5, Theorem 0.2] and [10, Theorem 2.2]).

**Theorem 4.3.** Under the hypothesis of Theorem 4.1 the following assertions hold :

(i) Suppose a non-trivial holomorphic section  $\sigma$  of  $E^{\otimes j}$  satisfies ess.  $\sup_X |\sigma|^2_{E^{\otimes j}} \times e^{-j\varphi_{\infty}} < \infty$  and  $q > d((i+j)\varphi_{\infty}) + 1$ . Then the homomorphism

$$H^{n,q}(\sigma): H^q(X, \Omega^n_X(E^{\otimes i}\bigotimes F)) \longrightarrow H^q(X, \Omega^n_X(E^{\otimes (i+j)}\bigotimes F))$$

induced by the tensor product with  $\sigma$  is injective for any i and  $j \ge 1$ .

(ii) Suppose  $\theta$  is a non-trivial holomorphic section of  $F^{\otimes j}$  and  $q > d(\varphi_{\infty}) + 1$ . Then the homomorphism

$$H^{n,q}(\theta): H^q(X, \Omega^n_X(E\bigotimes F^{\otimes i})) \longrightarrow H^q(X, \Omega^n_X(E\bigotimes F^{\otimes (i+j)}))$$

induced by the tensor product with  $\theta$  is injective for any *i* and  $j \ge 1$ .

REMARK 2. Theorems 4.2 and 4.3 yield us an indication about Kawamata-Viehweg type vanishing theorem for nef line bundles on compact Kähler manifolds ; i.e.,  $H^q(X, \Omega_X^n(L)) = 0$  if a holomorphic line bundle L on a compact Kähler manifold X with dim<sub>C</sub> X = n is *nef* and *good*; i.e.,  $\kappa(L) = \kappa_*(L)$  and  $q > n - \kappa_*(L)$ , where  $\kappa(L)$  is the Kodaira dimension of L. In this situation by replacing X by a bimeromorphic Kähler model of X there exist a surjective morphism  $\pi : X \to Y$ with connected fibres from X to a projective algebraic manifold Y with dim<sub>C</sub> Y =  $\kappa_*(L)$  and a nef-big Q-divisor B on Y such that (i)  $L = \pi^*B$ , (ii) kB = A + Dwith a very ample divisor A and an effective divisor D on Y for  $k \gg 0$  (cf. [13, §2, Proposition 2.14]). This implies that  $L^{\otimes k}$  is written by the tensor product of a semi-positive line bundle  $\pi^*[A]$  and a pseudo effective one  $\pi^*[D]$ , and admits a non-trivial section  $\theta$  which vanishes along  $\pi^*D$  (cf. Theorem 4.3 and [17, §6]).

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