



Title	On the exchange property in a directsum of indecomposable modules
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Citation	Osaka Journal of Mathematics. 1975, 12(3), p. 719-736
Version Type	VoR
URL	https://doi.org/10.18910/8566
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ON THE EXCHANGE PROPERTY IN A DIRECTSUM OF INDECOMPOSABLE MODULES

Dedicated to Professor Kiiti Morita on his 60th birthday

MANABU HARADA

(Received January 8, 1975)

Throughout R will denote a ring with identity and every R -modules considered in this note are unitary R -modules. Let M be an R -module. If $\text{End}_R(M)$ is a local ring, we call M a *completely indecomposable module*. We take a set of completely indecomposable modules $\{M_\alpha\}_I$ and put $M = \sum_I \oplus M_\alpha$. Then we know several properties of M with respect to this decomposition. For instance, let $M = \sum_J \oplus N_\beta$ be another decomposition and I' a finite subset of I , then $M = \sum_{I'} \oplus M_{\alpha'} \oplus \sum_{J - \varphi(I')} \oplus N_\beta$, where $\varphi: I' \rightarrow J$ is a one-to-one into mapping [1]. H. Kanbara [8] shows that the above fact is true for any subset I' of I if and only if $\{M_\alpha\}_I$ is a locally semi-T-nilpotent (see the definition below).

In this note, we fix a subset I' (not necessarily finite) and give criteria for $\sum_{I'} \oplus M_{\alpha'}$ to satisfy the above property. If $\{M_{\alpha'}\}_{I'}$ is locally semi-T-nilpotent, $\sum_{I'} \oplus M_{\alpha'}$ satisfies it, however the converse is not true [4]. When we fix the subset I' , the above property does depend not only on $\sum_{I'} \oplus M_{\alpha'}$ but also on $\sum_{I - I'} \oplus M_{\alpha''}$. On the other hand, the concept of semi-T-nilpotency of $\{M_{\alpha'}\}_{I'}$ does depend only on $\sum_{I'} \oplus M_{\alpha'}$. Hence, we shall define a new concept in this note, namely relative semi-T-nilpotency (see the definition below) and give a relation between relative semi-T-nilpotency and the property above.

In the final section (Appendix), we shall generalize [6], Lemma 5 as Theorem A.1 by virtue of K. Yamagata's idea [12], [13] and [14] (Lemma A.1). That theorem gives the complete proof of [5], Lemma 2 (Corollary 2) and a generalization of [14], Theorem (Theorem A.2).

1. Definition

Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules. We shall recall definitions of locally semi-T-nilpotency and the induced category \mathfrak{A} from

$\{M_\alpha\}_I$. Let $\{M_{\alpha_i}\}_1^\infty$ be a countable subset of $\{M_\alpha\}_I$ and $\{f_i: M_{\alpha_i} \rightarrow M_{\alpha_{i+1}}\}$ a set of non-isomorphisms. If for any m in M_{α_1} there exists a natural number n , which depends on m , such that $f_n f_{n-1} \cdots f_1(m) = 0$, then the set $\{f_i\}$ is called *locally T-nilpotent*. If every sets $\{f_i\}$ of non-isomorphisms for every countable subsets $\{M_{\alpha_i}\}$ of $\{M_\alpha\}_I$ are locally T-nilpotent, then the set $\{M_\alpha\}_I$ is called *locally semi-T-nilpotent* (cf. [3] and [4]). In the definition above, if we allow that $\alpha_i = \alpha_j$ for some $i \neq j$, we call $\{M_\alpha\}_I$ *locally T-nilpotent*. We shall generalize this concept as follows: Let $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ be sets of completely indecomposable modules. We take countable subsets $\{M_{\alpha_i}\}_1^\infty, \{N_{\beta_j}\}_1^\infty$ of $\{M_\alpha\}_I$ and $\{N_\beta\}_J$, respectively. We take sets $\{f_i\}$ and $\{g_i\}$ of non-isomorphisms, where $f_i: M_{\alpha_i} \rightarrow N_{\beta_i}$ and $g_i: N_{\beta_i} \rightarrow M_{\alpha_{i+1}}$. If for any element m in M_{α_1} there exists n , which depends on m , such that $g_n f_n \cdots g_1 f_1(m) = 0$, then we call $\{f_i, g_i\}$ (*locally and relatively T-nilpotent*). If for every countable subsets $\{M_{\alpha_i}\}_1^\infty$ and $\{N_{\beta_j}\}_1^\infty$, every sets $\{f_i, g_i\}$ of non-isomorphisms are relatively T-nilpotent, then we call $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ (*locally and relatively semi-T-nilpotent*) (see Remark 3 in §4). We note that if $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ are relatively semi-T-nilpotent, there exists n' for any element x in N_{β_1} , such that $f_{n'+1} g_{n'} f_{n'} \cdots f_2 g_1(x) = 0$. If we allow in the above that $\alpha_i = \alpha_{i'}$ ($\beta_j = \beta_{j'}$) for some $i \neq i'$ ($j \neq j'$), we call $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ *relatively T-nilpotent*. It is clear that if either $\{M_\alpha\}_I$ or $\{N_\beta\}_J$ is locally semi-T-nilpotent, then $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ are relatively semi-T-nilpotent, however the converse is not true. If *either I or J is finite*, we assume as a definition that $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ are *relatively semi-T-nilpotent*, ($\{M_\alpha\}_I$ or $\{N_\beta\}_J$ is *locally semi-T-nilpotent*).

Let $\{M_\alpha\}_I$ be as before and \mathfrak{M}_R the category of right R -modules. Let \mathfrak{A} be the full sub additive category of \mathfrak{M}_R , whose objects consist of all modules which are isomorphic to directsums of some modules in $\{M_\alpha\}_I$. We define the ideal \mathfrak{S}' in \mathfrak{A} as follows: $\mathfrak{S}' \cap [A, B] = \{f \in \text{Hom}_R(A, B) \mid p_\beta f i_\alpha \text{ are non-isomorphisms for all } \alpha \text{ and } \beta\}$, where $A \approx \sum_I \oplus M_{\alpha'} \in \mathfrak{A}$, $B \approx \sum_J \oplus M_{\beta'} \in \mathfrak{A}$ and $i_{\alpha'}: M_{\alpha'} \rightarrow A$ injection, $p_{\beta'}: B \rightarrow M_{\beta'}$ projection. By $\mathfrak{A}/\mathfrak{S}'$ we denote the factor category of \mathfrak{A} with respect to \mathfrak{S}' . Let A be in \mathfrak{A} and $A = \sum_I \oplus A_\alpha$ (A_α are not necessarily in \mathfrak{A}). Then there exist submodules $A_{\alpha'}$ in \mathfrak{A} of A_α for each α such that $\sum_I \oplus A_{\alpha'} = A$ in $\mathfrak{A}/\mathfrak{S}'$ and those $A_{\alpha'}$ are unique up to isomorphism. We call those $A_{\alpha'}$ *dense submodules* of A_α (see [4], [5] and [6] for detail).

2. Main theorem

We recall Krull-Remak-Schmidt-Azumaya theorem. Let $\{M_\alpha\}_I$ be as in §1 and $M = \sum_I \oplus M_\alpha$. We take any other decomposition of M by completely indecomposable modules N_β : $M = \sum_J \oplus N_\beta$. In the theorem above we consider the following two properties for M :

P,1 For a direct summand $\sum_{I'} \oplus M_{\alpha'}$ of M ($I' \subseteq I$), there exists a one-to-one mapping φ of I' into I such that $M_{\alpha'} \approx N_{\varphi(\alpha')}$ for all $\alpha' \in I'$ and $M = \sum_{I'} \oplus N_{\varphi(\alpha')} \oplus \sum_{I-I'} \oplus M_{\alpha''}$.

P,2 For a direct summand $\sum_{I'} \oplus M_{\alpha'}$ of M ($I' \subseteq I$), there exists a one-to-one mapping ψ of I' into J such that $M = \sum_{I'} \oplus M_{\alpha'} \oplus \sum_{J-\psi(I')} \oplus N_{\beta''}$.

Those two properties are general cases of the exchange property in M which is defined in [4] as follows: Let N be a direct summand of M and $M = \sum_K \oplus L_\varepsilon$ any decomposition of M (L_ε are not necessarily indecomposable). If $M = N \oplus \sum_K \oplus L'_\varepsilon$ for any decomposition above, where $L'_\varepsilon \subseteq L_\varepsilon$ for all $\varepsilon \in K$, we say N has the exchange property in M . If N has the above property only in case all L_ε are indecomposable, we say N has the exchange property in M with respect to indecomposable modules (briefly w. r. t. inde. modules).

It is clear that if N has the exchange property in M , N is in \mathfrak{A} and (P,2) is equivalent to the exchange property in M w. r. t. inde. modules. We have already known that (P,1) is true for any subset I' of I if and only if $\{M_\alpha\}_{I'}$ is locally semi-T-nilpotent [8]. Now we fix $\{M_\alpha\}_I$ and a subset I' of I and consider (P,1) and (P,2) for any decompositions of M . We shall show the following results and give proofs in the next section.

Theorem. *Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $M = \sum_I \oplus M_\alpha$. Let $M = S \oplus T$ and $S' = \sum_{I'} \oplus S_{\alpha'}$, $T' = \sum_{I-I'} \oplus T_{\alpha''}$ dense submodules of T and S , respectively. Then the following statements are equivalent.*

- 1) *S has the exchange property in M w. r. t. inde. modules.*
- 2) *T has the exchange property in M w. r. t. inde. modules.*
- 3) *$\{S_{\alpha'}\}_{I'}$ and $\{T_{\alpha''}\}_{I-I'}$ are relatively semi-T-nilpotent,*

where $S_{\alpha'}$ and $T_{\alpha''}$ are completely indecomposable modules.

In those cases S and T are direct sums of completely indecomposable modules.

Corollary 1. *Let M , S and T be as in Theorem. Then S has the exchange property in M if and only if 3) in Theorem is satisfied and any direct summand L of M has the following decomposition: $L = L_1 \oplus L_2$ and a dense submodule L_1' (resp. L_2') of L_1 (resp. L_2) is isomorphic to a summand of S (resp. T), (cf. Theorem A.2 in §4).*

Corollary 2. *Let $M = \sum_{i=1}^n \oplus M^{(i)}$ and $M^{(i)}$ in \mathfrak{A} for all i . Then the following statements are equivalent.*

- 1) *$M^{(1)}$ has the exchange property in M w. r. t. inde. modules.*
- 2) *$M^{(1)}$ has the exchange property in $M^{(1)} \oplus M^{(i)}$ w. r. t. inde. modules for all i .*

If $M^{(i)}$ and $M^{(j)}$ have the exchange property in M w. r. t. inde. modules, then so does $M^{(i)} \oplus M^{(j)}$. Conversely, if $M^{(i)} \oplus M^{(j)}$ has the exchange property in M , then $M^{(i)}$ has the exchange property in M if and only if so does $M^{(j)}$. (cf. [2], Lemma 3.11)

Corollary 3. Let $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ be sets of completely indecomposable modules and $\mathfrak{A}, \mathfrak{B}$ the induced categories from $\{M_\alpha\}_I$ and $\{N_\beta\}_J$, respectively. Then $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ are relatively T -nilpotent if and only if for any modules M and N in $\mathfrak{A}, \mathfrak{B}$ respectively, M (resp. N) has the exchange property in $M \oplus N$ w. r. t. inde. modules.

Corollary 4. Let $\{S_\alpha\}_{I'}$ (resp. $\{T_{\alpha''}\}_{I''}$) be a set of noetherian (resp. injective) and completely indecomposable modules. Then $\sum_{I'} \oplus S_{\alpha'}$ (and $\sum_{I''} \oplus T_{\alpha''}$) have the exchange property in $\sum_{I'} \oplus S_{\alpha'} \oplus \sum_{I''} \oplus T_{\alpha''}$ w. r. t. inde. modules.

Corollary 5. Let S, T and M be as in Theorem. If $\text{Hom}_R(S, T) = 0$ or $\text{Hom}_R(T, S) = 0$, then S and T have the exchange property in M w. r. t. inde. modules.

3. Proof of Theorem

Let $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ be as in §2. We shall rearrange them as follows: $\{M_\alpha\}_I = \{M_{kj}\}_{k \in K, j \in I_k}$ and $\{N_\beta\}_J = \{N_{kj}\}_{k \in K, j \in J_k}$, where $M_{kj} \approx M_{kj'} \approx N_{kj} \approx N_{kj'}$ and $M_{kj} \not\approx M_{k'j'}$, $N_{kj} \not\approx N_{k'j'}$ if $k \neq k'$.

Lemma 1. Let $\{M_{kj}\}_{k \in K, j \in I_k}$ and $\{N_{kj}\}_{k \in K, j \in J_k}$ be as above and K' a subset of K . We assume $\{M_{kj}\}$ and $\{N_{kj}\}$ are relatively semi- T -nilpotent. Then
 1) if I_k and J_k are infinite for all $k \in K'$, then $\{M_{kj}\}_{K', I_k}$ is locally T -nilpotent.
 2) If either $|I_k| \leq |J_k|$ or I_k is finite and $J_k \neq \emptyset$ for all $k \in K'$, then $\{M_{kj}\}_{K', I_k}$ is locally semi- T -nilpotent, where $|I|$ means the cardinal number of a set I .

Proof. 1) and the first part of 2) are clear. We assume I_k is finite and $J_k \neq \emptyset$. Let $\{M_i\}_1^\infty$ be a countable subset of $\{M_{kj}\}_{K', I_k}$ and $\{f_i: M_i \rightarrow M_{i+1}\}$ a set of non-isomorphisms. We assume $M_1 = M_{k_1 j_1}$. Since I_{k_1} is finite, there exists n_2 such that $M_{n'} \approx M_{k_1 j_1}$ for all $n' \geq n_2$. Next, we assume $M_{n_2} = M_{k_2 j_2}$ ($k_1 \neq k_2$). Again there exists $n_3 \geq n_2$ such that $M_{n'} \approx M_{k_2 j_2}$ for all $n' \geq n_3$. Repeating those arguments, we obtain a subset $\{M_{n_i}\}$ of $\{M_i\}$, ($n_1 = 1$) such that $M_{n_i} \approx M_{n_j}$ for all $i \neq j$ and a set $\{g_i = f_{n_{i+1}-1} f_{n_{i+1}-2} \cdots f_{n_i}: M_{n_i} \rightarrow M_{n_{i+1}}\}$. It is clear that no one of g_i is isomorphic by [3], Lemma 4. Furthermore, $M_{n_i} \approx N_{kj}$ for some k . Hence, $\{g_i\}$ is locally T -nilpotent from the assumption. Therefore, $\{f_i\}$ is locally T -nilpotent.

Lemma 2. If $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ are locally semi- T -nilpotent, then so is $\{M_\alpha, N_\beta\}_{I \cup J}$.

Proof. Let $\{T_i\}_1^\infty$ be a countable subset of $\{M_\alpha, N_\beta\}_{I \cup J}$ and $\{f_i: T_i \rightarrow T_{i+1}\}$ a set of non-isomorphisms. We assume $\{T_i\}_1^\infty$ contains an infinite number of modules in $\{M_\alpha\}_I$; say $\{T_i\} \supseteq \{M_\alpha\}_1^\infty$. Then $\{g_i = f_{\alpha_{i+1}-1} f_{\alpha_{i+1}-2} \cdots f_{\alpha_i}\}$ is locally T -nilpotent and hence so is $\{f_i\}$. If $\{T_i\}$ contains only a finite number of modules in $\{M_\alpha\}_I$, then there exists n such that $\{T_i\}_{i \geq n} \subseteq \{N_\beta\}_J$. Therefore, $\{f_i\}$ is also locally T -nilpotent.

Lemma 3 ([5], Theorem 2). *Let M be as before and N_1 a direct summand of M : $M = N_1 \oplus N_2$. We assume $N_1 = \sum_{I'} \oplus N_{\alpha'}$ and $\{N_{\alpha'}\}_{I'}$, a set of completely indecomposable modules, is locally semi- T -nilpotent. Then N_i has the exchange property in M for $i=1,2$.¹⁾*

Lemma 4. *Let $\{M_\alpha\}_I$ and M be as before. Let $M = T \oplus S$ and N a direct summand of M such that $N = \sum_J \oplus N_\beta$ and every N_β is completely indecomposable and not isomorphic to any summand of S . If $\{N_\beta\}_J$ is locally semi- T -nilpotent, $M = N \oplus T' \oplus S$, where $T' \subseteq T$.*

Proof. We have $M = N \oplus T' \oplus S'$, $T = T' \oplus T''$ and $S = S' \oplus S''$ by the assumption and Lemma 3. Since $T'' \oplus S'' \approx N$, $S'' = (0)$.

Lemma 5. *Let $M = M_1 \oplus M_2 \oplus N_1 \oplus N_2$ and $M_i = \sum_{\kappa_i \ni \alpha_i} \sum_{I \alpha_i} \oplus M_{\alpha_i j}$, $N_i = \sum_{I_i \ni \beta_i} \sum_{J \beta_i} \oplus N_{\beta_i k}$; the $M_{\alpha_i j}$, $N_{\beta_i k}$ are completely indecomposable. We assume $M_{\alpha_1 j} \approx M_{\alpha_2 j'}$ and $M_{\alpha_1 j} \approx N_{\alpha_2 j''}$, $N_{\beta_1 j} \approx N_{\beta_2 j'}$ and $N_{\beta_1 j} \approx M_{\beta_2 j''}$ for any j, j' and j'' . Let $M = T \oplus S$ and T', S' dense submodules of T and S , respectively. If $\{M_{\alpha_1}\}_{\kappa_1, I \alpha_1}$ is locally semi- T -nilpotent and $T' \approx M_1 \oplus M_2$, $S' \approx N_1 \oplus N_2$, then we have the following decompositions:*

$$M = (M_1 \oplus T_1'') \oplus M_2 \oplus N_1' \oplus N_2 = (M_1 \oplus T_1'') \oplus T'' \oplus S'',$$

where 1) $T = T_1' \oplus T''$, $T_1' = T_1'' \oplus T_1'''$ ($T = T_1'' \oplus T'' \oplus T_1'''$), 2) $S = S'' \oplus S'''$, 3) $N_1 = N_1' \oplus N_1''$, 4) T_1'' , T_1''' , S''' and N_1' are in \mathfrak{A} and 5) T'' (resp. S'') contains a dense submodule which is isomorphic to M_2 (resp. $N_1' \oplus N_2'$; N_2' is in \mathfrak{A} and is a summand of N_2).

Proof. Since $T' \approx M_1 \oplus M_2$ and $\{M_{\alpha_1}\}_{\kappa_1, I \alpha_1}$ is semi- T -nilpotent, $T' = T_1' \oplus$

1) Correction to the proof of [5], Theorem 2.

Replace the following words on the left side by ones on the right.
 "p" on 4, 5 and 6 th lines (from the bottom) $\Rightarrow p_\beta$. " $\sum \oplus T_\beta''$ " on 5th line $\Rightarrow T_\beta''$. " $\text{Ker } \bar{p} = \sum \oplus T_\beta''$ " on 4th line $\Rightarrow \text{Ker } \bar{p}_\beta = \bar{T}_\beta' \oplus \sum_{\gamma \neq \beta} \oplus T_\gamma$. " $L = \sum \oplus T_\beta^*$ and $\bar{L} = \text{Ker } \bar{p}$ " on 3rd line $\Rightarrow \bar{T}_\beta^* \subseteq \text{Ker } \bar{p}_\beta \cap \bar{T}_\beta = \bar{T}_\beta'$. "Therefore" on 2nd line \Rightarrow Finally, let p be the projection of M onto $\sum_K \oplus T_\beta''$ with $\text{Ker } p = \sum_K \oplus T_\beta^*$. Then

T_2' ; $T_i' \approx M_i$ and $T = T_1' \oplus T''$ by [4], Proposition 2. Now, we consider all modules in $\bar{\mathfrak{A}} = \mathfrak{A}/\mathfrak{S}'$. By \bar{A} we shall denote the residue class of A in $\bar{\mathfrak{A}}$. If B is a direct summand of A , \bar{B} means the image of \bar{e} in $\bar{\mathfrak{A}}$, where e is the projection of A to B . Then $\bar{T} = \bar{T}_1' \oplus \bar{T}'' = \bar{T}' = \bar{T}_1' \oplus \bar{T}_2'$ and hence $\bar{T}'' \approx \bar{T}_2' \approx \bar{M}_2$ in $\bar{\mathfrak{A}}$. Accordingly T'' contains a dense submodule isomorphic to M_2 . If we apply Lemma 4 to the decomposition $M = T_1' \oplus T'' \oplus S$, then $M = M_1 \oplus T_1'' \oplus S''$, where $T_1'' \subseteq T_1'$ and $S'' \subseteq S$. We put $T_1' = T_1'' \oplus T_1'''$, $S = S'' \oplus S'''$, and $\bar{M} = M/M_1$. Then $S''' \oplus T_1' \approx M/(S'' \oplus T'') \approx M_1 \oplus T_1'' \in \mathfrak{A}$ and hence, $S''' \in \mathfrak{A}$ by Lemma 3. Further

$$\bar{M} \approx T_1'' \oplus T'' \oplus S'' \approx M_2 \oplus N_1 \oplus N_2 \quad \dots\dots\dots (*).$$

We may assume those modules are equal to each other through isomorphisms. Since T_1'' is isomorphic to summand of M_1 , $\bar{M} = T_1'' \oplus M_2 \oplus N_1' \oplus N_2$ form (*) and Lemma 4, where $N_1 = N_1' \oplus N_1''$. $\bar{M}/T_1'' \approx M_2 \oplus N_1' \oplus N_2 \approx T'' \oplus S''$. On the other hand, $T_1'' \approx \bar{M}/(M_2 \oplus N_1' \oplus N_2) \approx N_1''$ and hence, N_1' is in \mathfrak{A} by Lemma 3. Since T'' contains a dense submodule isomorphic to M_2 and S'' is a direct summand of S , S'' contains a dense submodule isomorphic to $N_1' \oplus N_2'$, where N_2' is in \mathfrak{A} and a direct summand of N_2 . Therefore, $M = M_1 \oplus T_1'' \oplus M_2 \oplus N_1' \oplus N_2 = M_1 \oplus T_1'' \oplus S'' \oplus T''$ are the desired decompositions.

Lemma 6. Let $\{M_{\alpha'}\}_I$ and M be as in the theorem. We assume $M = N_1 \oplus N_2$, $N_1 = \sum_{I'} \oplus M_{\alpha'}$, $N_2 = \sum_{I''} \oplus M_{\alpha''}$ and $M_{\alpha'} \approx M_{\alpha''}$ for any pair $(\alpha', \alpha'') \in I' \times I''$, ($I = I' \cup I''$). We assume further that $M = T \oplus S$, $T = T_1 \oplus T_2$ and T (resp. S) contains a dense submodule which is isomorphic to N_1 (resp. N_2). Then for any element t in T_1 there exists a direct summand $\sum_1^n \oplus T_{i'}$ of T_1 such that $t \in \sum_1^n \oplus T_{i'}$ and $T_{i'} \approx M_{\alpha_{i'}}$ for some $\alpha_{i'} \in I'$.

Proof. Let $t \in (\sum_1^{n_1} \oplus M_{\alpha_{i'}} \oplus \sum_1^{n_2} \oplus M_{\alpha_{i''}}) = M'$. Then $M = M' \oplus T_1' \oplus T_2' \oplus S'$ by Lemma 3, where $T_i = T_i' \oplus T_i''$, $S = S' \oplus S''$ and $T_1'' = T_1 \cap (M' \oplus T_2' \oplus S') \ni t$. Since $T_1'' \oplus T_2'' \oplus S'' \approx M'$, $T_1'' = \sum_1^n \oplus T_{i'}$, $T_{i'} \approx M_{\alpha_{i'}}$ by the assumption and Lemma 3.

Lemma 7. Let $M = N_1 \oplus N_2$ be as in Lemma 6. Then $M = T \oplus N_2 = N_1 \oplus S$ for any decomposition $M = T \oplus S$ as in Lemma 6 if and only if $\{M_{\alpha'}\}_{I'}$ and $\{M_{\alpha''}\}_{I''}$ are relatively semi- T -nilpotent. In this case $T \approx N_1$ and $S \approx N_2$, (cf. Corollary 2 to Theorem A.1 in §4).

Proof. We assume $M = N_1 \oplus S = T \oplus N_2$ as in the lemma. We may assume that I' and I'' are infinite. Let $\{M_{2i-1}\}_1^\infty$ and $\{M_{2i}\}_1^\infty$ be countable subsets of

$\{M_{\alpha'}\}_{I'}$ and $\{M_{\alpha''}\}_{I''}$, respectively and let $\{f_n: M_n \rightarrow M_{n+1}\}$ be a set of non-isomorphisms. We shall make use of the same argument as the proof in [3], Lemma 9 and [5], Lemma 2. Put $M_n' = \{x + f_n(x) | x \in M_n\} \subseteq M_n \oplus M_{n+1} \subseteq M$. Then $M = \sum_1^{\infty} \oplus M_{2i-1}' \oplus N_1^* \oplus N_2 = N_1 \oplus \sum_1^{\infty} \oplus M_{2i}' \oplus N_2^*$, where $N_1^* = \sum_{I' - \aleph_0'} \oplus M_{\alpha'}$, $\aleph_0' = \{1, 3, \dots, 2i-1, \dots\}$ and $N_2^* = \sum_{I'' - \aleph_0''} \oplus M_{\alpha''}$, $\aleph_0'' = \{2, 4, \dots, 2i, \dots\}$. It is clear that $\sum_1^{\infty} \oplus M_{2i-1}' \approx \sum_1^{\infty} \oplus M_{2i-1}'$. We define an automorphism Φ of M by setting $\Phi = \varphi + I_{\sum_{I' - \aleph_0'} \oplus M_{\alpha'} \oplus N_2}$. Then $M = \Phi^{-1}(M) = \Phi^{-1}(N_1) \oplus \sum \oplus \Phi^{-1}(M_{2i}') \oplus \Phi^{-1}(N_2^*)$. Hence we have $M = N_1 \oplus \sum \oplus \Phi^{-1}(M_{2i}') \oplus \Phi^{-1}(N_2^*)$ from the assumption. Therefore, $M = \Phi(M) = \Phi(N_1) \oplus \sum \oplus M_{2i}' \oplus N_2^* = \sum_1^{\infty} \oplus M_{2i}' \oplus \sum_1^{\infty} \oplus M_{2i-1}' \oplus N_1^* \oplus N_2^*$. We can easily show from this decomposition that $\{f_i\}$ is locally T -nilpotent, (cf. [3], Lemma 9). Hence, $\{M_{\alpha'}\}_{I'}$ and $\{M_{\alpha''}\}_{I''}$ are relatively semi- T -nilpotent. We shall show the converse. Put $M^* = N_1 \cup S$. Let $s \in N_1 \cap S$, then there exists a direct summand S_1 of S (and hence of M) such that $s \in S_1$ and $S_1 \approx \sum_1^n \oplus M_{\alpha_i''}$ by Lemma 6. On the other hand, $M = S_1 \oplus N_1 \oplus N_2'$ from Lemma 4. Hence, $N_1 \cap S = (0)$. If either I' or I'' is finite, $M = N_1 \oplus S' \oplus T'$ by Lemma 3, where $S' \subseteq S$ and $T' \subseteq T$. Since $N_2 \approx M/N_1 \approx S' \oplus T'$, $T' = (0)$ and so $M = N_1 \oplus S' \subseteq M^* \subseteq M$. We assume I' are infinite and $M \neq M^*$. Since $M^* \supseteq N_1$, there exists $\alpha_1'' \in I''$ such that $M_{\alpha_1''} \not\subseteq M^*$. Let $m_{\alpha_1''} \in M_{\alpha_1''} - M^*$. From the decomposition

$$M = T \oplus S \quad \dots\dots\dots(1)$$

$m_{\alpha_1''} = t + s$; $t \in T$, $s \in S$. Since $s \in S \subseteq M^*$, $t \in T - M^*$. We obtain, from Lemma 6, a direct summand $\sum_1^n \oplus T_i$ of T (and hence of M) such that $T_i \approx M_{\alpha_i'}$ and $t \in \sum \oplus T_i$. Let $p_i^{(1)}$ be the projection of M to T_i in that decomposition. Then there exists $i (= \alpha_2')$ such that $p_{\alpha_2'}^{(1)}(m_{\alpha_1'}) = t_{\alpha_2'} \in T_{\alpha_2'} - M^*$. From Lemma 3 and the assumption, S contains a summand $S_{\alpha_2'}^{(1)}$ isomorphic to $M_{\alpha_1''}$. We obtain again from Lemma 3

$$M = N_1 \oplus \sum_{I' - \varepsilon_1} \oplus M_{\alpha''} \oplus S_{\alpha_2'}^{(1)} \quad \dots\dots\dots(2),$$

where, ε_1 is some index.

Let $p_{\alpha_1''}^{(2)}$ be the projection of M to $M_{\alpha_1''}$ in (2). Since $N_1 \oplus S_{\alpha_2'}^{(1)} \subseteq M^*$, there exists $\alpha_3'' \in I'' - \varepsilon_1$ such that $p_{\alpha_3''}^{(2)}(t_{\alpha_2'}) = m_{\alpha_3''} \in M_{\alpha_3''} - M^*$, (it is possible $\alpha_1'' = \alpha_3''$). In this case (2) implies M contains at least two summands isomorphic to $M_{\alpha_1''}$. We may assume $T_{\alpha_2'} \approx M_{\alpha_2'}^{(2)}$ for some $\alpha_2' \in I'$. Then again from Lemma 3 and (1) we have

$$M = M_{\alpha'_2} \oplus T' \oplus S \quad \dots\dots\dots (3),$$

where $T' \subseteq T$. Then $m_{\alpha_3}'' = x + t$; $x \in M^*$, $t \in T' - M^*$. Similarly to the argument after the decomposition (1), we obtain α'_4 and a homomorphism $p_{\alpha'_4}^{(3)}$ of M to $T_{\alpha'_4}$ such that $p_{\alpha'_4}^{(3)}(m_{\alpha_3}'') = t_{\alpha'_4} \in T_{\alpha'_4} - M^*$ and $T_{\alpha'_4}$ is a summand of T' (it is possible $T_{\alpha'_2} \approx T_{\alpha'_4}$). In this case (3) implies M contains at least two summands isomorphic to $M_{\alpha'_2}$. From the remark after (2) and Lemma 3, there exists $\alpha''(3)$ such that

$$M = N_1 \oplus \sum_{I'' - \{\varepsilon'_1, \varepsilon'_2\}} \oplus M_{\alpha''} \oplus S_{\alpha''(1)}' \oplus S_{\alpha''(3)}' \quad \dots\dots\dots (4),$$

where $S \supseteq S_{\alpha''(3)}' \approx M_{\alpha_3}''$ and $S_{\alpha''(1)}' \approx S_{\alpha''(1)}$. Hence, we obtain $\alpha_5'' \in I'' - \{\varepsilon'_1, \varepsilon'_2\}$ such that $p_{\alpha_5}''^{(4)}(t_{\alpha'_4}) = m_{\alpha_5}'' \in M_{\alpha_5}'' - M^*$. Similarly we have

$$M = M_{\alpha'_2} \oplus M_{\alpha'_4} \oplus T'' \oplus S \quad \dots\dots\dots (5),$$

where $M_{\alpha'_4} \approx T_{\alpha'_4}$, $M_{\alpha'_2} \approx M_{\alpha'_2}$ and $T'' \subseteq T$. Repeating those arguments, we have a series of indecomposable modules; M_{α_1}'' , T_{α_2}' , M_{α_3}'' , T_{α_4}' , \dots and a series of homomorphisms $p_1 = p_{\alpha_2}^{(1)} | M_{\alpha_1}''$, $p_2 = p_{\alpha_3}^{(2)} | T_{\alpha_2}'$, \dots (it is possible $\alpha_i'' = \alpha_j''$ (resp. $T_{\alpha_k}' \approx T_{\alpha_l}'$) for $i \neq j$ (resp. $k \neq l$)). Put $M^{(n)} = \sum_1^n \oplus M_{\alpha_{2i+1}}''$, $T^{(n)} = \sum_1^n \oplus T_{\alpha_{2i}'}'$ (external directsum). Then $M^{(n)}$ and $T^{(n)}$ are isomorphic to direct summands of M for all n from the decompositions (n). We now concentrate to find a contradiction to the assumption of relative semi-T-nilpotency and hence, after replacing M_{α_i}'' (resp. T_{α_k}') by another isomorphic summands when $\alpha_i'' = \alpha_j''$ (resp. $\alpha'(k) = \alpha'(l)$) for $i \neq j$ (resp. $k \neq l$), we may assume $\alpha_i'' \neq \alpha_j''$ (resp. $\alpha'(k) \neq \alpha'(l)$). It is clear that any p_i are non-isomorphic and $p_n p_{n-1} \dots p_1(m_{\alpha_1}'') \neq 0$ for all n . This is a contradiction to the relative semi-T-nilpotency. Therefore, $M = M^*$. Similarly, we have $M = T \oplus N_2$.

Lemma 8. Let M and $\{M_{\alpha}\}_I$ be as in the theorem and I' a subset of I . Put $M = (\sum_I \oplus M_{\alpha}) N_1 \oplus N_2$, where $N_1 = \sum_{I'} \oplus M_{\alpha}$ and $N_2 = \sum_{I-I'} \oplus M_{\alpha''}$. Then the following statements are equivalent.

- 1) N_1 satisfies (P,2), (equivalently N_2 satisfies (P,1)).
- 2) N_2 satisfies (P,2), (equivalently N_1 satisfies (P,1)).
- 3) $\{M_{\alpha'}\}_{I'}$ and $\{M_{\alpha''}\}_{I-I'}$ are relatively semi-T-nilpotent.

Proof. Since the condition in 3) is symmetric, we may show 1) is equivalent to 3). We know already from [5], Lemma 2 that 1) implies 3) (see Corollary 2 to Theorem A.1 in §4). Now we assume 3) and $\{M_{\alpha}\}_I = \{M_{kj}\}_{k \in K, j \in I_k}$ as in the beginning and $N_1 = \sum_K \sum_{I_k} \oplus M_{kj}$, $N_2 = \sum_K \sum_{J_k} \oplus N_{kj}$. We consider a partition of K as follows:

$$\begin{aligned} K_1 &= \{k \in K \mid I_k \text{ and } J_k \text{ are infinite}\}, \\ K_2 &= \{k \in K \mid I_k \neq \phi \text{ and } J_k \text{ is finite}\}, \\ K_3 &= \{k \in K \mid I_k \text{ is finite and } J_k \text{ is infinite}\}, \\ K_4 &= \{k \in K \mid I_k = \phi\} \text{ and} \\ K_5 &= \{k \in K \mid J_k = \phi\}. \end{aligned}$$

We put $M(i) = \sum_{K_i} \sum_{J_k} \oplus M_{kj}$ and $N(i) = \sum_{K_i} \sum_{J_k} \oplus N_{kj}$. $\{M_{kj}\}_{K_1, I_k}$, $\{N_{kj}\}_{K_1, J_k}$ are locally T -nilpotent and $\{M_{kj}\}_{K_3, I_k}$, $\{N_{kj}\}_{K_2, J_k}$ are locally semi- T -nilpotent by the assumption and Lemma 1. Hence, $\{M_{kj}\}_{K_1 \cup K_3, I_k}$ is locally semi- T -nilpotent from Lemma 2. Let $M = \sum_I \oplus L_e$ be any decomposition with L_e indecomposable. Then $M = M(1) \oplus M(3) \oplus \sum_{I'} \oplus L_{e'}$ for some $I' \subseteq I$ by Lemma 3. We put $\bar{M} = M / (M(1) \oplus M(3))$ then

$$\bar{M} \approx M(2) \oplus M(5) \oplus \sum_{i=1}^4 \oplus N(i) \approx \sum_{I'} \oplus L_{e'} \dots \dots \dots (**).$$

Since $\{N_{kj}\}_{K_2, J_k}$ is locally semi- T -nilpotent, we obtain from (**) and Lemma 3 $\bar{M} \approx M(2) \oplus M(5) \oplus \sum_{i=2} \oplus N(i) \oplus \sum_{I''} \oplus L_{e''}$ for some $I'' \subseteq I'$. Hence, $\bar{M} / (\sum_{I''} \oplus L_{e''}) \approx M(2) \oplus M(5) \oplus \sum_{i=2} \oplus N(i) \approx \sum_{I'''} \oplus L_{e'''}$, where $I''' = I' - I''$. We consider a partition of I''' as follows: $I_1''' = \{e''' \in I''' \mid L_{e'''} \approx M_{kj} \text{ for some } k \in K_2 \cup K_5\}$ and $I_2''' = I''' - I_1'''$. Then $\sum_{I_1'''} \oplus L_{e'''}$ is a dense submodule of itself, which satisfies the assumption in Lemma 7. Hence, $\bar{M} / (\sum_{I_1'''} \oplus L_{e'''}) \approx M(2) \oplus M(5) \oplus \sum_{I_2'''} \oplus L_{e'''}$. Therefore, $M = N_1 \oplus \sum_{I''} \oplus L_{e''} \oplus \sum_{I_2'''} \oplus L_{e'''}$.

Lemma 9. Let M and $\{M_\alpha\}_I$ be as in Theorem. We assume $M = T \oplus S$, and T', S' are dense submodules of T and S , respectively: $T' = \sum_{I'} \oplus T_{\alpha'}$, $S' = \sum_{I''} \oplus S_{\alpha''}$. If $\{T_{\alpha'}\}_{I'}$ and $\{S_{\alpha''}\}_{I''}$ are relatively semi- T -nilpotent, then T and S are in \mathfrak{A} , where $T_{\alpha'}$ and $S_{\alpha''}$ are completely indecomposable.

Proof. Let $M = T \oplus S$ and T', S' dense submodules of T and S , respectively. Then $M \stackrel{\mathcal{P}}{\approx} T' \oplus S'$. We shall use the same notations as in the proof of Lemma 8 and put $N_1 = \varphi^{-1}(T')$, $N_2 = \varphi^{-1}(S')$. Since $\{M_{kj}\}_{K_1 \cup K_3}$ is locally semi- T -nilpotent,

$$\begin{aligned} M &= M(1) \oplus M(3) \oplus T_1'' \oplus M(2) \oplus M(5) \oplus (N(1) \oplus N(3))' \oplus N(2) \oplus N(4) \\ &= M(1) \oplus M(3) \oplus T_1'' \oplus T'' \oplus S'', \quad T = T_1' \oplus T'' \oplus T_1'' \text{ and } S = S'' \oplus S''' \end{aligned}$$

by Lemma 5, where $(N(1) \oplus N(3))' \subseteq N(1) \oplus N(3)$ and T'' (resp. S'') contains a dense submodule isomorphic to $M(2) \oplus M(5)$ (resp. $(N(1) \oplus N(3))' \oplus N(2)' \oplus N(4)$), where $N(2)'$ is a summand of $N(2)$; $N(2) \approx \sum_{K_2} \sum_{J_{\beta_2}} \oplus N_{\beta_2 j}$, $(N(4))' = N(4)$

in this case). Then a dense submodule of T'' is isomorphic also to $M(2) \oplus M(5) \oplus \sum_{K_2} \sum_{J_{\beta_2}^{-J_{\beta_2}'}} \oplus N_{\beta_2 j}$. Put $N(2)'' = \sum_{K_2} \sum_{J_{\beta_2}^{-J_{\beta_2}'}} \oplus N_{\beta_2 j}$, and $\bar{M} = M/(M(1) \oplus M(3) \oplus T_1'')$ ($\approx M(2) \oplus M(5) \oplus (N(1) \oplus N(3))' \oplus N(2) \oplus N(4) \approx T'' \oplus S''$). $\{N_{kj}\}_{K_2, J_k}$ is locally semi-T-nilpotent. We apply Lemma 5 to a decomposition $(N(2)') \oplus ((N(1) \oplus N(3))' \oplus N(4)) \oplus (M(2) \oplus N(2)'') \oplus (M(5)) = S'' \oplus T''$, then $\bar{M} \approx N(2)' \oplus S_1^{(4)} \oplus (N(1) \oplus N(3))' \oplus N(4) \oplus (M(2) \oplus N(2)'')' \oplus M(5) = N(2)' \oplus S_1^{(4)} \oplus S^{(4)} \oplus T^{(4)}$, where $S^{(4)}$ contains a dense submodule isomorphic to $(N(1) \oplus N(3))' \oplus N(4)$ and $T^{(4)}$ does a dense submodule isomorphic to $(M(2) \oplus N(2)'')' \oplus M(5)$ from the structure of $M(i)$ and $N(j)$, and $S'' = S_1^{(4)} \oplus S^{(4)} \oplus S_1^{(5)}$, $T'' = T^{(4)} \oplus T^{(5)}$. Accordingly $(N(1) \oplus N(3))' \oplus N(4) \oplus (M(2) \oplus N(2)'')' \oplus M(5) \approx S^{(4)} \oplus T^{(4)}$. Since J_k is finite for $k \in K_2$ and $\{N_{kj}\}_{K_2, J_k}$ is locally semi-T-nilpotent, $(N(1) \oplus N(3))' \oplus N(4)$ and $(M(2) \oplus N(2)'')' \oplus M(5)$ satisfy the assumption in Lemma 7 (cf. the proof of Lemma 2). Hence, $S^{(4)} \approx (N(1) \oplus N(3))' \oplus N(4)$ and $T^{(4)} \approx (M(2) \oplus N(2)'')' \oplus M(5)$ by Lemma 7. Since $T = T_1'' \oplus T'' \oplus T_1'''$ and $T'' = T^{(4)} \oplus T^{(5)}$, T is in \mathfrak{A} from Lemma 5. Similarly, $S = S'' \oplus S'''$ and $S'' = S_1^{(4)} \oplus S^{(4)} \oplus S^{(5)}$.

Proof of Theorem. Since the condition 3) is symmetric, we may show that 1) is equivalent to 3). We assume 1). Then T and S are in \mathfrak{A} and hence, we obtain 3) from Lemma 8. We assume 3). Then T and S are again in \mathfrak{A} from Lemma 9. Hence, we have 1) from Lemma 8.

Proofs of Corollaries.

1: We assume S has the exchange property in M . Let $M = L \oplus L'$. Then $M = S \oplus L_1 \oplus L_1'$ and $L = L_1 \oplus L_2$, $L' = L_1' \oplus L_2'$. Since $L_1 \oplus L_1' \approx T$ and $L_2' \oplus L_2 \approx S$, L_1 (resp. L_2) contains a dense submodule isomorphic to a direct summand of T (resp. S). Conversely, we assume the above fact and 3) in Theorem. Then $S \approx \sum_{i'} \oplus M_{\alpha'}$ and $T \approx \sum_{i''} \oplus N_{\alpha''}$ from Lemma 9. We use the same notations as in the proof of Lemma 8, and put $S = N_1$ and $T = N_2$. Let $M = \sum_K \oplus L_e$ be any decomposition of M . Then $\bar{M} \approx M(2) \oplus M(5) \oplus \sum_{i=1}^4 \oplus N(i) \approx \sum_K \oplus L_e'$ by Lemma 3, where $L_e = L_e' \oplus L_e''$. Again from Lemmas 2 and 3 we have $\bar{M} \approx M(2) \oplus M(5) \oplus \sum_{i=3,4} \oplus N(i) \oplus \sum_K \oplus L_e'''$, where $L_e' = L_e''' \oplus L_e^{(4)}$. Hence,

$$\bar{\bar{M}} = \bar{M}/(\sum_K \oplus L_e''') \approx M(2) \oplus M(5) \oplus \sum_{i=3,4} \oplus N(i) \approx \sum_K \oplus L_e^{(4)} \dots \dots \dots (**).$$

Now we shall apply the assumption to $L_e^{(4)}$. Put $L_e^{(4)} = L$. Then $L = L_1 \oplus L_2$ and L_i contains a dense submodule L_i' which is isomorphic to a direct summand of N_i . Hence, $L_1' = M^*(2) \oplus M^*(5) \oplus N^*(3)$ and $L_2' = M^*(2)' \oplus N^*(3)' \oplus N^*(4)'$ from (**), where $M^*()$ and $M^*()'$ (resp. $N^*()$ and $N^*()'$) are isomorphic to direct summands of $M()$ (resp. $N()$). On the other hand, $N^*(3)$ is isomorphic to a direct summand of N_1 and hence of $M^*(3)$. Therefore, $N^*(3)$ has the

exchange property in M by Lemma 3. Similarly, $M^*(2)'$ has the same property. Therefore, $N^*(3)$ and $M^*(2)'$ are direct summands of M (and hence of L) by [4], Proposition 2: $L_1 = N^*(3) \oplus L_1''$, $L_2 = M^*(2)' \oplus L_2''$ and L_1'' (resp. L_2'') contains a dense submodule isomorphic to $M^*(2) \oplus M^*(5)$ (resp. $N^*(3)' \oplus N^*(4)'$), (see the proof of Lemma 5). Accordingly $M(2) \oplus M(5) \oplus N(3) \oplus N(4) = \sum_K L_e^{(4)} \oplus L_e^{(4)} = \sum_K (L_e^{(4)'} \oplus N^*(3)_e) \oplus \sum_K (L_e^{(4)''} \oplus M^*(2)_e') = \sum_K (M^*(2)_e' \oplus L_e^{(4)'} \oplus \sum_K (N^*(3)_e \oplus L_e^{(4)''}))$. Therefore, we obtain from Lemma 7 that $\bar{M} \approx M(2) \oplus M(5) \oplus \sum_K (N^*(3)_e \oplus L_e^{(4)''})$. Thus, $M = N_1 \oplus \sum_K (L_e^{(4)''} \oplus N^*(3)_e \oplus L_e^{(4)''})$ and $L_e^{(4)''} \oplus N^*(3)_e \oplus L_e^{(4)''}$ is a direct summand of L_e .

2 and 3: They are clear from Lemma 8.

4: Let $\{S_{2i-1}\}$ and $\{N_{2i}\}$ be countable subsets of $\{S_\omega\}'$ and $\{N_\omega\}''$, respectively and $\{f_{2i-1}: S_{2i-1} \rightarrow N_{2i}\}$, $\{g_{2i}: N_{2i} \rightarrow S_{2i+1}\}$ sets of non-isomorphisms. Since N_{2i} is injective, $\text{Ker } g_{2i} \neq 0$ is essential in N_{2i} . Hence, $\text{Ker } f_{2i-1}g_{2i-2}\cdots f_1 \subseteq \text{Ker } g_{2i}f_{2i-1}\cdots f_1$ and so $\{f_{2i-1}, g_{2i}\}$ is T-nilpotent.

5: Let T' and S' be dense submodules of T and S , respectively. We take indecomposable summands T_1 and S_1 of T' and S' . Then $T = T_1 \oplus T''$ and $S = S_1 \oplus S''$ by [4], Proposition 2. Hence, $\text{Hom}_R(S_1, T_1) = 0$ or $\text{Hom}_R(T_1, S_1) = 0$.

EXAMPLES. 1. Let Z be the ring of integers and p, q primes. Then $\{Z/p^i\}_1^\infty$ and $\{Z/q^i\}_1^\infty$ are relatively T-nilpotent, but $\{Z/p^i\}_1^\infty$ is not T-nilpotent. Put $N_1 = \sum_1^\infty \oplus Z/p^{2^i-1}$ and $N_2 = \sum_1^\infty \oplus Z/p^{2^i}$. Then all Z/p^* have finite composition series, but N_i does not have the exchange property in $\sum_1^\infty \oplus Z/p^i$.

2. Let K be a field and R the ring of lower tri-angular and column sum-mable matrices over K with degree. \aleph_0 . Let $\{e_{ij}\}$ be a set of matrix units in R . We put $N_1 = \sum_1^\infty \oplus e_{2i-1, 2i-1}R$ and $N_2 = \sum_1^\infty \oplus e_{2i, 2i}R$. Then all $e_{ii}R$ are projective and noetherian (artinian), but N_i does not have the exchange property in $N_1 \oplus N_2$.

4. Appendix (The finite exchange property)

In §3 we have used Lemma 2 in [5]. However, I gave, in [5], only an idea of the proof of this lemma. In this section we shall give its proof as a more general form for the sake of completeness. Making use of a remark by K. Yamagata [12], [13], and [14], we shall deal with a relation between the finite exchange property and the exchange property and give generalizations of [6], Lemma 5 and [14], Theorem.

Let M be an R -module. In §2 we have defined the exchange property in M for a direct summand N . If we consider only decompositions $M = \sum_K \oplus L_e$ with $|K| \leq m$ in that definition, we say N has the m -exchange property in M . In

[2] we have several properties on modules with m -exchange property (not necessarily in M), however they are not valid in our restricted case. Hence, we shall give proofs for some results in [2], if we are necessary to change some parts of proofs.

The following lemma is substantially due to K. Yamagata [9].

Lemma A.1. *Let T be an R -module and $T = A_1 \oplus A_2 = M \oplus N$. We assume $M = \sum_K \oplus M_\alpha$ and every M_α has the finite exchange property (in the usual sense) and $A_1 \cap M \neq (0)$. Then there exists a finite subset $\{1, 2, \dots, m\}$ in K such that $T = \sum_1^m \oplus M_i^* \oplus A_1^* \oplus A_2$, where $M_i^* \subseteq M_i$ ($M_j^* \neq (0)$ for some j) and $A_1^* \subseteq A_1$.*

Proof. There exists a finite subset $\{1, 2, \dots, m\}$ in K such that $A_1 \cap (\sum_1^m \oplus M_i) \neq (0)$. We put $M^\Delta = \sum_1^m \oplus M_i$, then M^Δ has the finite exchange property by [2], Lemma 3.10. Hence, $T = M^\Delta \oplus A_1' \oplus A_2'$, where $A_i = A_i' \oplus A_i''$. Since $M^\Delta \cap A_1 \neq (0)$, $A_1' \neq A_1$ and so $A_1'' \neq (0)$. Put $\bar{T} = T / (A_1' \oplus A_2') = \bar{A}_1'' \oplus \bar{A}_2'' = \bar{M}^\Delta$. By [2], Lemma 3.10 \bar{A}_1'' has the finite exchange property and hence $\bar{T} = \bar{A}_1'' \oplus \sum_1^m \oplus \bar{M}_i'$, where $M_i = M_i' \oplus M_i''$.²⁾ Then $\bar{T} = \bar{T} / \sum \oplus \bar{M}_i' = \bar{A}_1'' = \sum_1^m \oplus \bar{M}_i''$. We may assume $\bar{M}_1'' \neq (0)$. Then $A_1'' = A_1''' \oplus A_1^{iv}$ and $\bar{A}_1''' = \bar{M}_1'' \neq (0)$, $\bar{A}_1^{iv} = \sum_{i \geq 2}^m \oplus \bar{M}_i''$. Accordingly $\bar{T} = \sum_1^m \oplus \bar{M}_i' \oplus \bar{A}_1^{iv} \oplus \bar{M}_1''$. On the other hand, \bar{A}_2'' has also the finite exchange property. Hence, we have $\bar{T} = \bar{A}_2'' \oplus \bar{A}_1^{iv*} \oplus \sum_1^m \oplus \bar{M}_i^*$, where $A_1^{iv*} \subseteq A_1^{iv}$ and $M_i^* \subseteq M_i$. Since $A_1^{iv*} \subseteq A_1^{iv} \subseteq A_1''$, $\sum_1^m \oplus M_i^* \neq (0)$. Hence, $T = \sum_1^m \oplus M_i^* \oplus (A_1^{iv*} \oplus A_1') \oplus A_2$ is a desired decomposition.

Lemma A.2 ([5], Lemma 1). *Let T be an R -module and $T = N_1 \oplus N_2$. We assume that N_1 has the m -exchange property in M and $T = N_1' \oplus N_2'$; $N_i' \approx N_i$, $i = 1, 2$. Then N_1' has the m -exchange property in M .*

It is clear (cf. the proof of Lemma 7).

Lemma A.3 ([2], Lemma 3.10). *Let $T = B_1 \oplus B_2 \oplus B_3$ be R -modules. We assume B_1 has the m -exchange property in T and B_2 has the m -exchange property in $B_2 \oplus B_3$. Then $B_1 \oplus B_2$ has the m -exchange property in T .*

It is clear.

2) added in proof: Use \bar{A}_2'' instead of \bar{A}_1'' and we obtain $\bar{T} = \bar{A}_2'' \oplus \sum_1^m \oplus \bar{M}_i'$. Hence, $T = A_1' \oplus \sum_1^m \oplus M_i' \oplus A_2$.

Lemma A.4 ([2], Lemma 3.11). *Let T be an R -module and N a direct summand of T . If N has the 2-exchange property in T , then N has the finite exchange property in T .*

Proof. It is sufficient to show that N has the 3-exchange property in T .
Let $T = N \oplus N_1 = \sum_1^3 \oplus A_i$. Then

$$T = N \oplus A_1' \oplus (A_2 \oplus A_3)',$$

where $A_1 = A_1' \oplus A_1''$, $A_2 \oplus A_3 = (A_2 \oplus A_3)' \oplus (A_2 \oplus A_3)''$ and $(A_2 \oplus A_3)'' = (A_2 \oplus A_3) \cap (N \oplus A_1')$. On the other hand, $N \approx A_1'' \oplus (A_2 \oplus A_3)''$ and $N_1 \approx A_1' \oplus (A_2 \oplus A_3)'$. Hence, $A_1'' \oplus (A_2 \oplus A_3)''$ has the 2-exchange property in T by Lemma A.2. Accordingly $T = (A_1 \oplus A_2) \oplus A_3 = A_1'' \oplus (A_2 \oplus A_3)'' \oplus (A_1 \oplus A_2)' \oplus A_3'$, where $(A_1 \oplus A_2)' \subseteq (A_1 \oplus A_2)$ and $A_3' \subseteq A_3$. Hence, since $(A_2 \oplus A_3)'' \oplus A_3' \subseteq A_2 \oplus A_3$,

$$A_2 \oplus A_3 = (A_2 \oplus A_3)'' \oplus A_3' \oplus D,$$

where $D = (A_2 \oplus A_3) \cap (A_1'' \oplus (A_1 \oplus A_2)') \subseteq (A_2 \oplus A_3) \cap (A_1 \oplus A_2) = A_2$, namely D is a direct summand of A_2 . Put $N \oplus A_1' = (A_2 \oplus A_3)'' \oplus K$. Then $T = N \oplus A_1' \oplus (A_2 \oplus A_3)' = (A_2 \oplus A_3)'' \oplus K \oplus (A_2 \oplus A_3)' = (A_2 \oplus A_3) \oplus K = (A_2 \oplus A_3)'' \oplus A_3' \oplus D \oplus K = N \oplus A_1' \oplus D \oplus A_3'$.

The following theorem is a generalization of [6], Lemma 5.

Theorem A.1. *Let $\{P_\alpha\}_I$ be an infinite set of R -modules which have the finite exchange property and $P = \sum_I \oplus P_\alpha$. Let I' be an infinite subset of I with infinite complement $I - I'$. We assume $P_{I'} = \sum_{I'} \oplus P_\alpha$ has the 2 (finite)-exchange property in P . Then if we take any countable subsets $\{P_{2i-1}\}_1^\infty$ and $\{P_{2i}\}_1^\infty$ of $\{P_\alpha\}_{I'}$ and $\{P_\alpha\}_{I-I'}$, respectively and any sets of homomorphisms $f_i: P_i \rightarrow P_{i+1}$ such that for any direct summands X in P_{2i-1} (or Y in P_{2i}) $f_{2i-1}(X)$ (or $f_{2i-1}^{-1}(Y)$) is not a direct summand, provided $f_{2i-1}(X) \neq (0)$ (or $f_{2i-1}^{-1}(Y) \neq P_{2i-1}$) for all i (e.g. $\text{Im } f_{2i-1}$ is small in P_{2i} or $\text{Ker } f_{2i-1}$ is large in P_{2i-1}), then there exists n , depending on x in P_1 such that $f_n f_{n-1} \cdots f_1(x) = 0$.*

Proof. We can prove the theorem similarly to [6], Lemma 5 and so we shall give a sketch of the proof. We shall use the same notations as in the proof of Lemma 7, changing M_α by P_α . Put $P_i' = \{p_i + f_i(p_i) \mid p_i \in P_i\} \subseteq P_i \oplus P_{i+1}$. Then $P = \sum_1^\infty \oplus P_{2i-1}' \oplus P^{(1)} \oplus \sum_1^\infty \oplus P_{2i} \oplus P^{(2)} = \sum_1^\infty \oplus P_{2i-1} \oplus P^{(1)} \oplus \sum_1^\infty \oplus P_{2i}' \oplus P^{(2)}$. Since $\sum_1^\infty \oplus P_{2i-2}' \oplus P^{(1)}$ has the finite exchange property in P from the assumption and Lemma A.2, we obtain from the decomposition above

$$P = \sum_1^\infty \oplus P_{2i-1}' \oplus P^{(1)} \oplus X \oplus Y \oplus Z \quad \dots\dots\dots (1),$$

where $X \subseteq \sum_1^\infty \oplus P_{2i-1}$, $Y \subseteq \sum_1^\infty \oplus P_{2i}'$ and $Z \subseteq P^{(2)}$. We shall show $X=(0)$. We assume contrary $X \neq (0)$. Then we have from Lemma A.1

$$P = \sum_1^i \oplus P_{2i-1}^* \oplus \sum_1^\infty \oplus P_{2i-1}' \oplus P^{(1)} \oplus X' \oplus Y \oplus Z \quad \dots\dots\dots (2),$$

where $P_{2i-1}^* \subseteq P_{2i-1}$ ($P_{2j-1}^* \neq (0)$ for some j) and $X' \subseteq X$. We consider the following modules and a decomposition of P :

$$\begin{aligned} P_1^* \oplus \sum_1^\infty \oplus P_{2i-1}' \oplus P^{(1)} \quad \text{and} \quad P = P_1 \oplus (\sum_{i \geq 2} \oplus P_{2i-1}) \oplus P^{(1)} \\ \oplus P_2' \oplus (\sum_{i \geq 2} \oplus P_{2i}') \oplus P^{(2)} \quad \dots\dots\dots (3). \end{aligned}$$

Since the former module has the finite exchange property in P by Lemma A.3 and [2], Lemma 3.10, we obtain from (3)

$$P = (P_1^* \oplus \sum_1^\infty \oplus P_{2i-1}' \oplus P^{(1)}) \oplus P_1^{**} \oplus A \oplus P_2'^* \oplus B \oplus C,$$

where $P_1^{**} \subseteq P_1$, $A \subseteq \sum_{i \geq 2} \oplus P_{2i-1}$, $P_2'^* \subseteq P_2'$, $B \subseteq \sum_{i \geq 2} \oplus P_{2i}'$ and $C \subseteq P^{(2)}$. Hence

$$P = (P_1^\S \oplus P_1') \oplus (P_2'^* \oplus D) \quad \dots\dots\dots (4),$$

where $P_1^\S = P_1^* \oplus P_1^{**}$ and $D = \sum_{i \geq 2} \oplus P_{2i-1}' \oplus P^{(1)} \oplus A \oplus B \oplus C \subseteq \sum_{i \geq 3} \oplus P_i$. Using only a fact $D \subseteq \sum_{i \geq 3} \oplus P_i$ in (4), we shall show that $P_1^\S = (0)$. Let x be in P_1^\S . If $f_1(x) \in (P_2'^* \oplus D)$, $x=0$ from (4). Hence, $f_i|P_1^\S$ is monomorphic and

$$P_2 = f_1(P_1^\S) \oplus N \quad \dots\dots\dots (5),$$

where $N = \{x \in P_2 | f_2(x) \in D\}$, (see [6], Lemma 5). Furthermore

$$P_1 = P_1^\S \oplus f_1^{-1}(N) \quad \dots\dots\dots (6),$$

since $f_1|P_1^\S$ is monomorphic. Hence, $P_1^\S = (0)$ from (5), (6) and the assumptions. Therefore, $P_1^* = (0)$ in (2). Next, we consider similarly to (3)

$$\begin{aligned} P_3^* \oplus \sum_1^\infty \oplus P_{2i-1}' \oplus P^{(1)} \quad \text{and} \quad P = P_1 \oplus P_3 \oplus (\sum_{i \geq 3} \oplus P_{2i-1}) \\ \oplus P^{(1)} \oplus P_2' \oplus P_4' \oplus (\sum_{i \geq 3} \oplus P_{2i}') \oplus P^{(2)} \quad \dots\dots\dots (3'). \end{aligned}$$

Then $P = (P_3^* \oplus \sum_1^\infty \oplus P_{2i-1}' \oplus P^{(1)}) \oplus P_1^\Delta \oplus P_3^\Delta \oplus A' \oplus P_2'^* \oplus P_4'^* \oplus B' \oplus C'$, where $P_{2i}'^* \subseteq P_{2i}'$, $P_{2i-1}^\Delta \subseteq P_{2i-1}$, $A' \subseteq \sum_{i \geq 3} \oplus P_{2i-1}$, $B' \subseteq \sum_{i \geq 3} \oplus P_{2i}'$ and $C' \subseteq P^{(2)}$.

From the argument after (4) we know $P_1^\Delta = (0)$. Thus, we have

$$P = (P_1' \oplus P_2'^*) \oplus \{(P_3^\S \oplus P_3') + (P_4'^* \oplus D')\} \quad \dots\dots\dots (4'),$$

where $P_3^\S = P_3^* \oplus P_3^\Delta$ and $D' = \sum_{i \geq 3} \oplus P_{2i-1}' \oplus P^{(1)} \oplus A' \oplus B' \oplus C' \subseteq \sum_{i \geq 5} \oplus P_i$.

Applying the same arguments on $P_3^{\otimes} \oplus P_3'$ and P_4 in (4') as ones after (4), we obtain $P_3^{\otimes} = P_3^* = (0)$. Continuing those arguments, we have a contradiction to the assumption $X \neq (0)$ in (1). Therefore, we have from (1)

$$P = \sum_1^{\infty} \oplus P_{2i-1}' \oplus P^{(1)} \oplus Y \oplus P^{(2)} \quad \text{and} \quad Y \subseteq \sum_1^{\infty} \oplus P_{2i}' \dots\dots\dots (7).$$

Hence, $\{f_i\}$ is locally T-nilpotent.

From Theorem A.1 and [10], [11] we have

Corollary 1. *Let E be an injective module. If $\sum_1^{\infty} \oplus E_{2i}$ has the finite exchange property in $\sum_1^{\infty} \oplus E_i$; $E_i \approx E$ (e.g. E is Σ -injective), then the radical of $\text{End}_R(E)$ is locally T-nilpotent.*

Corollary 1'. *Let P be a projective module with finite exchange property. If $\sum_1^{\infty} \oplus P_{2i}$ has the finite exchange property in $\sum_1^{\infty} \oplus P_i$; $P_i \approx P$, then the radical of $\text{End}_R(P)$ is locally T-nilpotent.*

Corollary 2 ([5], Lemma 2). *Let $\{M_{\alpha}\}_I$ be a set of completely indecomposable modules and $M = \sum_I \oplus M_{\alpha}$. Put $N_i = \sum_{I_i} \oplus M_{\alpha'}$, where $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. If N_1 has the 2-exchange property in M , then $\{M_{\alpha'}\}_{I_1}$ and $\{M_{\alpha''}\}_{I_2}$ are relatively semi-T-nilpotent.*

Proof. We may assume that I_i are infinite. Let $\{M_{2i-1}\}_1^{\infty}$ and $\{M_{2i}\}_1^{\infty}$ be any countable subsets of $\{M_{\alpha'}\}_{I_1}$ and $\{M_{\alpha''}\}_{I_2}$, respectively and $\{f_r: M_n \rightarrow M_{n+1}\}$ a set of non-isomorphisms. We shall show that f_{2i-1} satisfies the assumptions in Theorem A.1. Since M_i is completely indecomposable, M_i has the (finite) exchange property by [9], Proposition 1. If $\text{Ker } f_{2i-1}$ is a direct summand of M_{2i-1} , $\text{Ker } f_{2i-1} = M_{2i-1}$ or $\text{Ker } f_{2i-1} = (0)$. The former case implies $f_{2i-1} = 0$. We assume $\text{Ker } f_{2i-1} = (0)$. If $\text{Im } f_{2i-1}$ is a direct summand of M_{2i} , then f_{2i-1} is isomorphic. Hence, $\text{Im } f_{2i-1}$ is not direct summand of M_{2i} . Therefore, f_{2i-1} satisfies the assumptions in Theorem A.1.

Corollary 3. *Let $\{M_{\alpha}\}_I$ and M be as in Corollary 2. For any subset I' of I we put $M_{I'} = \sum_{I'} \oplus M_{\alpha}$. Then the following statements are equivalent.*

- 1) $\{M_{\alpha}\}_I$ is locally semi-T-nilpotent.
- 2) $M_{I'}$ has the 2-exchange property in M for any $I' \subseteq I$.
- 3) $M_{I'}$ has the finite exchange property in M for any $I' \subseteq I$.
- 4) $M_{I'}$ has the exchange property in M for any $I' \subseteq I$, (cf. [14]).

Proof. It is clear from Lemma A.5 and [8], Theorem.

Theorem A.2. Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $M = \sum_I \oplus M_\alpha$. We put $M_{I'} = \sum_{I'} \oplus M_{\alpha'}$ for some $I' \subseteq I$. Then the following statements are equivalent.

- 1) $M_{I'}$ has the 2-exchange property in M .
- 2) $M_{I'}$ has the finite exchange property in M .
- 3) $M_{I'}$ has the exchange property in M .
- 4) $M_{I-I'}$ has the exchange property in M .

Proof. 3) \rightarrow 2) \rightarrow 1) are clear. We assume 1). Then $\{M_{\alpha'}\}_{I-I'}$ and $\{M_{\alpha''}\}_{I-I'}$ are relatively semi-T-nilpotent by Corollary 2 to Theorem A.1. Hence, $M_{I'}$ (resp. $M_{I-I'}$) satisfies conditions in Corollary 1 of Theorem (cf. its proof) and so $M_{I'}$ and $M_{I-I'}$ have the exchange property in M .

Corollary 1. Let M be as above and $M = T \oplus S$. Then T has the exchange property in M if and only if so does S .

Corollary 2. Let $M = \sum_I \oplus M_\alpha$ be as above. We assume $M = S \oplus T$ and any indecomposable direct summands of S are not isomorphic to direct summands of T . Then S has the 2-exchange property in M if and only if S has the exchange property in M .

Proof. Let S' and T' be dense submodules of S and T , respectively. Since $S' \oplus T' \approx M$, $M = \sum_{I'} \oplus M'_{\alpha'} \oplus \sum_{I''} \oplus M'_{\alpha''}$; $\sum_{I'} \oplus M'_{\alpha'} \approx S'$ and $\sum_{I''} \oplus M'_{\alpha''} \approx T'$. We assume S has the 2-exchange property in M . Then $M = S \oplus \sum_{I''} \oplus M'_{\alpha''}$ from the assumption, (cf. the proof of Lemma 6). Hence, $S \approx \sum_{I'} \oplus M'_{\alpha'}$ and $T \approx \sum_{I''} \oplus M'_{\alpha''}$. Therefore, S has the exchange property in M by Theorem A.2.

Corollary 3. Let $M = \sum \oplus M_\alpha$. We assume $M_\alpha \not\approx M_{\alpha'}$ if $\alpha \neq \alpha'$. Then a direct summand S of M has the 2-exchange property in M if and only if S has the exchange property in M .

Corollary 4. Let $M = \sum_I M_\alpha = S \oplus T$. We assume S has the exchange property in M . If $M = S_1 \oplus T_1$ and a dense submodule of S_1 (resp. T_1) is isomorphic to S (resp. T), then S_1 has the exchange property in M .

Proof. We may assume $S = M_{I'}$ and $T = M_{I''}$. Then $\{M_{\alpha'}\}_{I'}$ and $\{M_{\alpha''}\}_{I''}$ are relatively semi-T-nilpotent. Hence, S_1 and T_1 are in \mathfrak{A} by the assumption and Theorem. Therefore, $S_1 \approx S' \approx S$ ($T_1 \approx T' \approx T$).

REMARKS. 1. If every direct summand of M is in \mathfrak{A} (e.g. all M_α are countably generated), then Theorem A.2 shows that 2-exchange property in M of a direct summand is equal to the exchange property in M . Furthermore, it

is equivalent to a fact that $\{M_{\alpha'}\}_{I'}$ and $\{M_{\alpha''}\}_{I-I'}$ are relatively semi-T-nilpotent.

2. Let $M = \sum_I \oplus M_{\alpha} = S \oplus T$ be as before. We assume that S has the 2-exchange property in M . Then the proof of Theorem A.1 shows that for any direct summands $\sum_K \oplus M'_{\alpha'}$ and $\sum_{K'} \oplus M'_{\alpha''}$ of S and T , respectively $\{M'_{\alpha'}\}_K$ and $\{M'_{\alpha''}\}_{K'}$ are relatively semi-T-nilpotent.

3. In the definition of relative semi-T-nilpotency in §1, we took a set of non-isomorphisms $\{f_i, g_i\}$. However, this definition is equivalent to a stronger one in which we assume only $\{f_i\}$ or $\{g_i\}$ is a set of non-isomorphisms, (cf. Theorem A.1).

4. Let $\{M_{\alpha}\}_I$ be a set of completely indecomposable modules such that $\{M_{\alpha}\}_I$ is locally semi-T-nilpotent. We assume $M = \sum_I \oplus M_{\alpha}$ and $T = M \oplus N = \sum_1^{\infty} \oplus A_i$. Then we obtain, from [6], Lemma 8, decompositions $A_i = A'_i \oplus A''_i$ such that $(\sum_1^{\infty} \oplus A'_i) \cap M = (0)$ and $\sum_1^n \oplus A''_i$ is isomorphic to a direct summand of M . We further assume that N does not contain any direct summands isomorphic to some M_{α} in $\{M_{\alpha}\}_I$. Then if we make use of the same argument in the proof of Lemma 7, we can prove $T = M \oplus \sum \oplus A'_i$, namely M has the \aleph_0 -exchange property in T , because if $T \neq M \oplus \sum \oplus A'_i$, there exist a subset $\{M_{\alpha_i}\}$ of $\{M_{\alpha}\}_I$, an element $x \in M_{\alpha_1}$ and a set of homomorphisms $f_{2i-1}: M_{\alpha_{2i-1}} \rightarrow N$ and $f_{2i}: N \rightarrow M_{\alpha_{2i}}$ such that $f_{2i}f_{2i-1} \cdots f_1(x) \neq 0$ for all i .

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