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NATURAL MORITA EQUIVALENCES OF DEGREE *n*

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Abstract

Let G be a finite group, H a normal subgroup of G and b and c block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. Under the assumption that $C_H(R) \subset O_{p',p}(H)$ for a Sylow p-subgroup R of $O_{p',p}(H)$ and c is also a block idempotent of $\mathcal{O}O_{p'}(H)$, we give two equivalent conditions about when $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are natural Morita equivalent of degree n (see Theorem 1.5).

1. Introduction

1.1. Fix a prime number p. Let $\mathcal{O}$ be a complete discrete valuation ring with a residue field k of characteristic p. Let G be a finite group, H a subgroup of G and b and c block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. In terms of the terminology of A. Hida and S. Koshitani [5], $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are said to be naturally Morita equivalent of degree n for a positive integer number n if there exists an unitary $\mathcal{O}$-subalgebra S of $\mathcal{O}Gb$ such that S is a full matrix algebra over $\mathcal{O}$ of degree n and the map

$$\mathcal{O}Hc \otimes_{\mathcal{O}} S \to \mathcal{O}Gb, \quad x \otimes y \mapsto xy$$

is an isomorphism of $\mathcal{O}$-algebras. When H is normal in G and $\mathcal{O} = k$, this definition is firstly due to B. Külshammer [6].

1.2. For our question below, now we make the additional assumption that the characteristic of $\mathcal{O}$ is zero, the quotient field $\mathcal{K}$ of $\mathcal{O}$ is big enough for all algebras involved below, the residue field k is algebraically closed and H is normal in G; the assumption will also be kept throughout this paper. As a consequence of [13, Theorems 2 and 3], we can easily conclude that the following three conditions are equivalent:
1.2.1. the map $\mathcal{O}Gb \to \mathcal{O}Hc$, $x \mapsto xc$ is an $\mathcal{O}$-algebra isomorphism;
1.2.2. the restriction from G to H induces a bijection between the sets of all non-isomorphic simple modules of $\mathcal{O}Gb$ and $\mathcal{O}Hc$ and the quotient group G/H is a p'-group;
1.2.3. the restriction from G to H induces a bijection between the sets of all non-isomorphic simple modules of $\mathcal{K}Gb$ and $\mathcal{K}Hc$.

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Noticing that Condition 1.2.1 is actually saying that $\mathcal{O}Gb$ and $\mathcal{O}Hc$ is naturally Morita equivalent of degree 1, we ask ourselves a question: can this statement above be generalized to natural Morita equivalences of degree n? In this paper, we investigate the question.

1.3. Now we begin with some preparations in order to state our main theorem. Let M be an $\mathcal{O}G$-module and N an $\mathcal{O}K$-module. We denote by $\mathrm{Res}_K^G(M)$ the restriction of M from G to K and by $\mathrm{Ind}_K^G(N)$ the induction of N from K to G. Given a positive integer number n, we denote by nM the direct sum of n copies of M. Obviously the product $b \cdot M$ of b and M is an $\mathcal{O}G$-submodule of M and b acts on $b \cdot M$ as the identity homomorphism. When $b \cdot M = M$, then we say that the $\mathcal{O}G$-module M is associated to the block b of $\mathcal{O}G$. We denote by $\mathrm{IBr}(b)$ the set of all non-isomorphic simple $\mathcal{O}G$-modules associated to b. All notations above except $\mathrm{IBr}(b)$ can be slightly modified to apply to $\mathcal{K}G$-modules. In general, we denote by $\mathrm{Irr}(b)$ the set of all non-isomorphic simple $\mathcal{K}G$-modules associated to b. Given a positive integer number m, $\upsilon_p(m)$ denotes the largest non-negative integer number t such that $p^t \mid m$.

1.4. Assume that $bc \neq 0$ and b and c have a common defect group P. Since $bc \neq 0$, it is well known (refer to [3]) that there exist block idempotents b_P and c_P of $kC_G(P)$ and $kC_H(P)$ such that $b_P\,\mathrm{Br}_P^{\mathcal{O}G}(b) = b_P$, $c_P\,\mathrm{Br}_P^{\mathcal{O}H}(c) = c_P$ and $b_Pc_P \neq 0$. Since P is a defect group of b and c, b_P and c_P have defect group $Z(P)$, thus $kC_G(P)b_P$ and $kC_H(P)c_P$ are nilpotent (refer to [10]) and have only one simple module, say V_{b_P} and V_{c_P}. Since H is normal in G, so is $C_H(P)$ in $C_G(P)$; then by Clifford theory, we can conclude that the dimension $\dim_k(V_{c_P})$ of V_{c_P} over k divides the dimension $\dim_k(V_{b_P})$ of V_{b_P} over k. Note that (P, b_P) and (P, c_P) actually are maximal Brauer pairs of b and c, which are unique up to G- and H-conjugation (refer to [1]). Therefore the quotient $\dim_k(V_{b_P})/\dim_k(V_{c_P})$ is independent of the choices of b_P and c_P. We denote this quotient by $n(b, c)$. Note that by [10, 1.4.1], $n(b, c) = \sqrt{\dim_k(kC_G(P)b_P)/\dim_k(kC_H(P)c_P)}$; even in order to compute $n(b, c)$, it suffices for us to choose b_P and c_P of $kC_G(P)$ and $kC_H(P)$ such that $b_P\,\mathrm{Br}_P^{\mathcal{O}G}(b) = b_P$ and $c_P\,\mathrm{Br}_P^{\mathcal{O}H}(c) = c_P$.

Theorem 1.5. *Let G be a finite group and H be a normal subgroup of G such that $C_H(R) \subset O_{p',p}(H)$ for a Sylow p-subgroup R of $O_{p',p}(H)$. Let b and c be respective block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ and let n be a positive integer. If c is also a block idempotent of $\mathcal{O}O_{p'}(H)$, then the following conditions are equivalent*:

1.5.1. *$\mathcal{O}Gb$ and $\mathcal{O}Hc$ are naturally Morita equivalent of degree n;*

1.5.2. *for any simple $\mathcal{O}G$-module S associated to b, there exists a unique simple $\mathcal{O}H$-module S_H associated to c such that $\mathrm{Res}_H^G(S) \cong nS_H$ and $b \cdot \mathrm{Ind}_H^G(S_H) \cong nS$, the correspondence $\mathrm{IBr}(b) \to \mathrm{IBr}(c)$, $S \mapsto S_H$ is a bijection, and $n \leq n(b, c)$.*

1.5.3. *$\upsilon_p(|G : H|) = \upsilon_p(n)$, for any simple $\mathcal{K}G$-module V associated to b, there exists a unique simple $\mathcal{K}H$-module V_H associated to c such that $\mathrm{Res}_H^G(V) \cong nV_H$, and the* *correspondence* $\mathrm{Irr}(b) \to \mathrm{Irr}(c)$, $V \mapsto V_H$ *is a bijection, and* $n \leq n(b, c)$. *Moreover in this case,* n *is equal to* $n(b, c)$.

REMARK 1.6. 1. Conditions 1.5.2 and 1.5.3 both imply that b and c have the same defect groups, so $n(b, c)$ makes sense. For details, refer to the proofs of Theorems 3.6 and 3.7.
2. When $n = 1$, by [4, Chapter IV, Theorem 4.5], it is easily checked that Conditions 1.5.2 and 1.5.3 both imply that the quotient group G/H is a p'-group; in addition $n \leq n(b, c)$ automatically holds. Therefore the theorem above covers the equivalences between Conditions 1.2.1, 1.2.2 and 1.2.3.
3. There are examples to explain why the condition $n \leq n(b, c)$ is necessary.

2. Fong's reduction

In this section, an $\mathcal{O}$-algebra A that is involved is always associative, unitary and $\mathcal{O}$-free of finite rank as an $\mathcal{O}$-module; A^* and $J(A)$ denote the multiplicative group of all invertible elements of A and the Jacobson radical of A respectively. Occasionally, in order to avoid confusion, we denote by 1_A of the identity element of A. A homomorphism $f: A \to B$ between $\mathcal{O}$-algebras is an embedding if f is injective and $f(A) = f(1_A)Bf(1_A)$.

2.1. Let K be a finite group and $\hat{K}$ be a k^*-group with the k^*-quotient K endowed with the homomorphism $\rho: k^* \to \hat{K}$. By $\hat{K}$, we can construct two k^*-groups: the group $\hat{K}$ endowed with the group homomorphism $k^* \to \hat{K}$ sending λ onto $\rho(\lambda^{-1})$ and the opposite group $(\hat{K})^\circ$ with the group homomorphism ρ; in order to differ from the k^*-group $\hat{K}$, we denote the first k^*-group by $\hat{K}^\circ$. But the two k^*-groups are isomorphic: there is an isomorphism of k^*-groups $(\hat{K})^\circ \to \hat{K}^\circ$, $x \mapsto x^{-1}$ (refer to [9]). For any subgroup L of K, we denote by $\hat{L}$ its inverse image in $\hat{K}$ and for any element $x \in L$, by $\hat{x}$ a lifting in $\hat{K}$ of x. When L is a p-group, $\hat{L}$ can be uniquely decomposed as the direct product $k^* \times L$ (refer to [9, Lemma 5.5]) and thus we always regard L as a subgroup of $\hat{K}$. Let $\check{K}$ be another k^*-group with the k^*-quotient K. Then the central product of $\hat{K}$ and $\check{K}$ over k^* defines a k^*-group $\hat{K} \otimes \check{K}$ with the k^*-quotient isomorphic to $K \times K$ and we identify this k^*-quotient with $K \times K$. We also identify K with the diagonal subgroup in $K \times K$ and denote by $\hat{K} * \check{K}$ the inverse image in $\hat{K} \otimes \check{K}$ of K. Then $\hat{K} * \check{K}$ is a new k^*-group with the k^*-quotient K.

2.2. Obviously the surjective homomorphism $\mathcal{O} \to k$ induces a surjective group homomorphism $\mathcal{O}^* \to k^*$; since k is algebraically closed, k is perfect and thus by [14, Chapter II, Proposition 8], there exists a unique section $k^* \to \mathcal{O}^*$ of this group homomorphism. Through this section, we can regard $\mathcal{O}$ as a right module over the group algebra of k^* over $\mathcal{O}$. Let K be a finite group and $\hat{K}$ be a k^*-group with the k^*-quotient K. Obviously the inclusion $k^* \subset \hat{K}$ induces a left $\mathcal{O}k^*$-module structure on the group algebra $\mathcal{O}\hat{K}$ of $\hat{K}$ over $\mathcal{O}$. Now we consider the tensor product $\mathcal{O}\otimes_{\mathcal{O}k^*}\mathcal{O}\hat{K}$ and define a distributive product on $\mathcal{O}\otimes_{\mathcal{O}k^*}\mathcal{O}\hat{K}$ by the equality

$$(a\otimes x)(b\otimes y)=ab\otimes xy$$

for $a,b\in\mathcal{O}$ and $x,y\in\mathcal{O}\hat{K}$. Then the tensor product $\mathcal{O}\otimes_{\mathcal{O}k^*}\mathcal{O}\hat{K}$ with the above product becomes an $\mathcal{O}$-algebra; we call it the twisted group algebra of $\hat{K}$ over $\mathcal{O}$ and denote it by $\mathcal{O}_*\hat{K}$. Obviously the k^*-group isomorphism $(\hat{K})^\circ\cong\hat{K}^\circ$, $x\mapsto x^{-1}$ induces an isomorphism of $\mathcal{O}$-algebras from the opposite ring $(\mathcal{O}_*\hat{K})^\circ$ to $\mathcal{O}_*\hat{K}^\circ$; moreover since the map $\mathcal{O}_*(\hat{K}\otimes\hat{K}^\circ)\to\mathcal{O}_*\hat{K}\otimes_{\mathcal{O}}\mathcal{O}_*\hat{K}^\circ$ sending $1\otimes(x\otimes y)$ to $(1\otimes x)\otimes(1\otimes y)$ for $x\otimes y\in\hat{K}\otimes\hat{K}^\circ$ is an isomorphism, we can define a left $\mathcal{O}_*(\hat{K}\otimes\hat{K}^\circ)$-module structure on $\mathcal{O}_*\hat{K}$ by the equality $(x\otimes y)a=xay^{-1}$ for $x,y\in\hat{K}$ and $a\in\mathcal{O}_*\hat{K}$. The tensor product $\mathcal{K}\otimes_{\mathcal{O}}\mathcal{O}_*\hat{K}$ is also what we are concerned below and we denote it by $\mathcal{K}_*\hat{K}$.

2.3. Recall that an $\mathcal{O}$-algebra A is called a $\hat{K}$-interior algebra (see [9, 5.10]) if there exists a group homomorphism φ: $\hat{K}\to A^*$. For any $a\in A$ and liftings $\hat{x},\hat{y}$ in $\hat{K}$ of $x,y\in K$, we will write $\varphi(\hat{x})a\varphi(\hat{y})$ as $\hat{x}a\hat{y}$ for convenience. Obviously when $\hat{y}=\hat{x}^{-1}$, the product $\hat{x}a\hat{x}^{-1}$ is independent of the choice of $\hat{x}$ in $\hat{K}$ and therefore we also often write it as $a^{x^{-1}}$. Moreover the map $\varphi_x\colon A\cong A$, $a\mapsto a^{x^{-1}}$ is an automorphism, the map $K\to\mathrm{Aut}(A)$, $x\mapsto\varphi_x$ is a group homomorphism, thus A is a K-algebra. Let C be another $\hat{K}$-interior algebra; an $\mathcal{O}$-algebra homomorphism $f\colon A\to C$ is called a homomorphism of $\hat{K}$-interior algebras if $f(\hat{x}a\hat{y})=\hat{x}f(a)\hat{y}$ for any $a\in A$ and liftings $\hat{x},\hat{y}$ in $\hat{K}$ of $x,y\in K$. Let $\check{K}$ be another k^*-group with the k^*-quotient K and A' be a $\check{K}$-interior algebra; then the $\hat{K}$-interior algebra structure on A and the $\check{K}$-interior algebra structure on A' determine a $\hat{K}\otimes\check{K}$-interior algebra structure on the tensor product $A\otimes_{\mathcal{O}}A'$, which, by restriction, induces a $\hat{K}*\check{K}$-interior algebra structure on $A\otimes_{\mathcal{O}}A'$.

2.4. Let A be a $\hat{K}$-interior algebra and P a p-subgroup of K. We denote by A^P the subalgebra consisting of all P-fixed elements of A. Clearly A^P is a $C_{\hat{K}}(P)$-interior algebra with the homomorphism $C_{\hat{K}}(P)\to(A^P)^*$, $\hat{x}\mapsto\hat{x}1$, where $C_{\hat{K}}(P)$ is the centralizer of P in $\hat{K}$. For any subgroup Q of P, we denote by Tr_Q^P the relative trace map $A^Q\to A^P$ and by A_Q^P its image. We define $A(P)$ to be the Brauer quotient $k\otimes_{\mathcal{O}}\left(A^P/\sum_S A_S^P\right)$, where S runs over the set of proper subgroups of P, and denote by Br_P^A the Brauer homomorphism $A^P\to A(P)$. Note that $A(P)\neq 0$ forces P to be a p-group. When $A=\mathcal{O}_*\hat{K}$ and P is a p-subgroup of K, by [11, Proposition 2.2], Br_P^A induces an isomorphism $k_*C_{\hat{K}}(P)\cong A(P)$; in this case, we always identify $A(P)$ with $k_*C_{\hat{K}}(P)$ through this isomorphism.

2.5. In this paragraph, we generalize the definitions and notations in Introduction to twisted group algebras. Let L be a subgroup of K and e and g be block idempotents of $\mathcal{O}_*\hat{K}$ and $\mathcal{O}_*\hat{L}$ respectively. $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{L}g$ are said to be naturally Morita equivalent of degree n for a positive integer number n if there exists a unitary $\mathcal{O}$-subalgebra S of $\mathcal{O}_*\hat{K}e$ such that S is a full matrix algebra over $\mathcal{O}$ of degree n and the map

$$\mathcal{O}_*\hat{L}g \otimes_{\mathcal{O}} S \to \mathcal{O}_*\hat{K}e, \quad x \otimes y \mapsto xy$$

is an isomorphism of $\mathcal{O}$-algebras. Let M be an $\mathcal{O}_*\hat{K}$-module and N an $\mathcal{O}_*\hat{L}$-module. We denote by mM the direct sum of m copies of M for a positive integer number m, by $\mathrm{Res}^{\hat{K}}_{\hat{L}}(M)$ the restriction of M from $\mathcal{O}_*\hat{K}$ to $\mathcal{O}_*\hat{L}$, and by $\mathrm{Ind}^{\hat{K}}_{\hat{L}}(N)$ the induction of N from $\mathcal{O}_*\hat{L}$ to $\mathcal{O}_*\hat{K}$. Let i be an idempotent of $\mathcal{O}_*\hat{K}$. We denote by $i \cdot M$ the product of i and M. Note that if i commutes with a unitary subalgebra B of $\mathcal{O}_*\hat{K}$, then the $\mathcal{O}_*\hat{K}$-module structure on M induces a B-module structure on $i \cdot M$. So $e \cdot M$ is an $\mathcal{O}_*\hat{K}$-module structure and when $e \cdot M = M$, then we say that the $\mathcal{O}_*\hat{K}$-module M is associated to the block e of $\mathcal{O}_*\hat{K}$. We denote by $\mathrm{IBr}(e)$ the set of all non-isomorphic simple $\mathcal{O}_*\hat{K}$-modules associated to e. All notations above except $\mathrm{IBr}(e)$ can be slightly modified to apply to $\mathcal{K}_*\hat{G}$-modules. We denote by $\mathrm{Irr}(e)$ the set of all non-isomorphic simple $\mathcal{K}_*\hat{K}$-modules associated to e.

2.6. Let K be a finite group, $\hat{K}$ a k^*-group with the k^*-quotient K, L a normal p'-subgroup of K and f a K-stable block idempotent of $\mathcal{O}_*\hat{L}$. Then K acts on the full matrix algebra $\mathcal{O}_*\hat{L}f$ over $\mathcal{O}$ and thus by the Skolem–Noether theorem, there exists a group homomorphism

$$\rho\colon K \to \mathrm{Aut}(\mathcal{O}_*\hat{L}f) \cong (\mathcal{O}_*\hat{L}f)^*/\mathcal{O}^*.$$

We denote by $\check{K}$ the set of all elements (c, x) such that $\rho(x)$ is the image of c in $(\mathcal{O}_*\hat{L}f)^*/\mathcal{O}^*$, where $c \in (\mathcal{O}_*\hat{L}f)^*$ and $x \in K$. Obviously $\check{K}$ is an $\mathcal{O}^*$-group with the $\mathcal{O}^*$-quotient K with the homomorphism $\mathcal{O}^* \to \check{K}$, $\lambda \mapsto (\lambda, 1)$, the map $\hat{L} \to \check{K}$, $\hat{x} \mapsto (\hat{x}, x)$ is an injective group homomorphism and its image is normal in $\check{K}$; in this sense, we identify $\hat{L}$ with a normal subgroup of $\check{K}$.

2.7. Now we claim that there exists a subgroup $\tilde{K}$ of $\check{K}$ which is a k^*-group of k^*-quotient K and contains $\hat{L}$. Consider the quotient group $\check{K}/\hat{L}$. Obviously $\hat{L}\mathcal{O}^*/\hat{L}$ is a central subgroup of $\check{K}/\hat{L}$ isomorphic to $1 + J(\mathcal{O})$ and $(\check{K}/\hat{L})/(\hat{L}\mathcal{O}^*/\hat{L}) \cong K/L$, thus we can regard $\check{K}/\hat{L}$ as a central extension of K/L by $1 + J(\mathcal{O})$. Let P be a Sylow p-subgroup of K. Since L is a p'-group, the image of P in K/L is isomorphic to P; so we identify P with its image in K/L. Again since L is a p'-group, it is well known that $\mathcal{O}_*\hat{L}f$ is a full matrix algebra over $\mathcal{O}$ and has the $\mathcal{O}$-rank prime to p, thus the action of P on $\mathcal{O}_*\hat{L}f$ can be lifted to a group homomorphism $P \to (\mathcal{O}_*\hat{L}f)^*$ (see [10, Paragraph 6.2]). This implies that there exists a group homomorphism $\theta\colon P \to \check{K}/\hat{L}$ such that for any $u \in P$, the image of $\theta(u)$ through the surjective homomorphism $\check{K}/\hat{L} \to K/L$ is u. Since $1 + J(\mathcal{O})$ is a p'-divisible group, the surjective homomorphism $\check{K}/\hat{L} \to K/L$ splits and thus has a section $K/L \to \check{K}/\hat{L}$. Then the inverse image of the image of K/L in $\check{K}/\hat{L}$ in $\check{K}$ is just the desired k^*-group $\tilde{K}$.

2.8. Consequently we have a group homomorphism $\vartheta : \tilde{K} \to (\mathcal{O}_*\hat{L} f)^*$ and thus $\mathcal{O}_*\hat{L} f$ becomes a $\tilde{K}$-interior algebra. Consider the k^*-group $\check{K} = \hat{K} * \tilde{K}^\circ$. Obviously $\check{L} = \hat{L} * \hat{L}^\circ$ has a normal subgroup $\{\hat{x} \otimes \hat{x}^{-1} \mid x \in L\}$ isomorphic to L; we still denote this group by L. We claim that L is normal in $\check{K}$. Indeed, for any $\hat{y} \otimes \tilde{y} \in \check{K}$ and $\hat{x} \otimes \hat{x}^{-1} \in L$, we have $(\hat{y} \otimes \tilde{y})(\hat{x} \otimes \hat{x}^{-1})(\hat{y} \otimes \tilde{y})^{-1} = (\hat{y} \otimes \tilde{y})(\hat{x} \otimes \hat{x}^{-1})(\hat{y}^{-1} \otimes \tilde{y}^{-1}) = \hat{y}\hat{x}\hat{y}^{-1} \otimes \tilde{y}\hat{x}^{-1}\tilde{y}^{-1} = \hat{x}^{y^{-1}} \otimes (\hat{x}^{y^{-1}})^{-1}$ since the $\hat{K}$- and $\tilde{K}$-conjugation induce the same action of K on $\hat{L}$. Set $\check{\check{K}} = \check{K}/L$. Then we obtain a k^*-group $\check{\check{K}}$ with the k^*-quotient K/L. Through the surjective group homomorphism $\check{K} \to \check{\check{K}}$, we endow the twisted group algebra $\mathcal{O}_*\check{\check{K}}$ of $\check{\check{K}}$ over $\mathcal{O}$ with a $\check{K}$-interior algebra structure.

Theorem 2.9. *Keep the notations as in* Paragraphs 2.6, 2.7 *and* 2.8. *Then there exists an isomorphism of $\hat{K}$-interior algebras*

$$\mathcal{O}_*\hat{K} f \cong \mathcal{O}_*\hat{L} f \otimes_{\mathcal{O}} \mathcal{O}_*\check{\check{K}}. \tag{2.9.1}$$

In particular, the functors $U \mapsto i \cdot U$ and $V \mapsto \mathcal{O}_*\hat{L} i \otimes_{\mathcal{O}} V$ are inverse isomorphisms between the categories of finitely generated $\mathcal{O}_*\hat{K} f$- and $\mathcal{O}_*\check{\check{K}}$-modules, where i is a primitive idempotent of $\mathcal{O}_*\hat{L} f$.

The above theorem is also called the second Fong's reduction theorem.

Proof. Since $\mathcal{O}_*\hat{L} f$ is a full matrix algebra over $\mathcal{O}$, by [8, Proposition 2.1], the map

$$\mathcal{O}_*\hat{L} f \otimes_{\mathcal{O}} C_{\mathcal{O}_*\hat{K} f}(\mathcal{O}_*\hat{L} f) \cong \mathcal{O}_*\hat{K} f, \quad x \otimes y \mapsto xy$$

is an isomorphism of $\mathcal{O}$-algebras, where $C_{\mathcal{O}_*\hat{K} f}(\mathcal{O}_*\hat{L} f)$ is the centralizer of $\mathcal{O}_*\hat{L} f$ in $\mathcal{O}_*\hat{K} f$. Let R be a set of representatives of cosets of L in K and write $\mathcal{O}_*\hat{K} f$ as the direct sum $\bigoplus_{x\in R}(\mathcal{O}_*\hat{L} f)\hat{x}$. Since $\hat{L}$ is normal in $\hat{K}$, it is easily computed that $C_{\mathcal{O}_*\hat{K} f}(\mathcal{O}_*\hat{L} f)$ is equal to the direct sum $\bigoplus_{x\in R} C_{(\mathcal{O}_*\hat{L} f)\hat{x}}(\mathcal{O}_*\hat{L} f)$. For any $x \in R$, since $\hat{x}$ and $\vartheta(\tilde{x})$ have the same action on $\mathcal{O}_*\hat{L} f$ by conjugation, $\hat{x}\vartheta(\tilde{x}^{-1}) \in C_{(\mathcal{O}_*\hat{L} f)\hat{x}}(\mathcal{O}_*\hat{L} f)$; moreover by comparing the $\mathcal{O}$-ranks, it is not difficult to find $\mathcal{O}\hat{x}\vartheta(\tilde{x}^{-1}) = C_{(\mathcal{O}_*\hat{L} f)\hat{x}}(\mathcal{O}_*\hat{L} f)$ and thus $C_{\mathcal{O}_*\hat{K} f}(\mathcal{O}_*\hat{L} f) = \bigoplus_{x\in R} \mathcal{O}\hat{x}\vartheta(\tilde{x}^{-1})$. Finally it is easily checked that the map $\check{K} \to (C_{\mathcal{O}_*\hat{K} f}(\mathcal{O}_*\hat{L} f))^*$, $\hat{x} \otimes \tilde{x} \mapsto \hat{x}\vartheta(\tilde{x}^{-1})$ is a group homomorphism with the kernel L; in particular, the group homomorphism induces an isomorphism $\mathcal{O}_*\check{\check{K}} \cong C_{\mathcal{O}_*\hat{K} f}(\mathcal{O}_*\hat{L} f)$. □

2.10. Keep the notations in Theorem 2.9. Let N be a subgroup of K containing L, $\bar{N}$ the quotient group of N in the quotient group $\bar{K} = K/L$, $\hat{N}$, $\tilde{N}$ and $\check{N}$ the inverse images of N in $\hat{K}$, $\tilde{K}$ and $\breve{K}$ respectively, and $\breve{\bar{N}}$ the inverse image of $\bar{N}$ in $\breve{\bar{K}}$. Consider $\mathcal{O}_*\hat{L}f$ as an $\tilde{N}$-interior algebra through the restriction of the structural homomorphism of the $\tilde{K}$-interior algebra $\mathcal{O}_*\hat{L}f$ to $\tilde{N}$ and $\mathcal{O}_*\breve{\bar{N}}$ as an $\breve{N}$-interior algebra through the homomorphism $\breve{N} \to \breve{\bar{N}} \subset (\mathcal{O}_*\breve{\bar{N}})^*$. Then the isomorphism (2.9.1) induces an $\hat{N}$-interior algebra isomorphism

$$\mathcal{O}_*\hat{N}f \cong \mathcal{O}_*\hat{L}f \otimes_{\mathcal{O}} \mathcal{O}_*\breve{\bar{N}}. \tag{2.10.1}$$

In particular, the functors $X \mapsto i \cdot X$ and $Y \mapsto \mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} Y$ are inverse isomorphisms between the categories of finitely generated $\mathcal{O}_*\hat{N}f$- and $\mathcal{O}_*\breve{\bar{N}}$-modules. Let h be a block idempotent of $\mathcal{O}_*\hat{K}$ such that $hf \neq 0$, $\bar{h}$ the corresponding block idempotent of $\mathcal{O}_*\breve{\bar{K}}$ determined by h through the isomorphism (2.9.1), l a block idempotent of $\mathcal{O}_*\hat{N}$ and $\bar{l}$ the corresponding block idempotent of $\mathcal{O}_*\breve{\bar{N}}$ determined by l through the isomorphism (2.10.1). Then by the isomorphisms (2.9.1) and (2.10.1) and the definition of natural Morita equivalences of degree n, we can easily verify the following:
2.10.2. $\mathcal{O}_*\hat{K}h$ and $\mathcal{O}_*\hat{N}l$ are naturally Morita equivalent of degree n if and only if $\mathcal{O}_*\breve{\bar{K}}\bar{h}$ and $\mathcal{O}_*\breve{\bar{N}}\bar{l}$ are naturally Morita equivalent of degree n.

2.11. Finally we claim the following:
2.11.1. for any $\mathcal{O}_*\breve{\bar{K}}$-module V, $\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} \mathrm{Res}^{\breve{\bar{K}}}_{\breve{\bar{N}}}(V) \cong \mathrm{Res}^{\hat{K}}_{\hat{N}}(\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} V)$, and for any $\mathcal{O}_*\breve{\bar{N}}$-module Y, $\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} \mathrm{Ind}^{\breve{\bar{K}}}_{\breve{\bar{N}}}(Y) \cong \mathrm{Ind}^{\hat{K}}_{\hat{N}}(\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} Y)$.
The first isomorphism is obvious, so the rest is to prove the second equality. We consider $\mathcal{O}_*\breve{\bar{K}}$ as a subalgebra of $\mathcal{O}_*\hat{K}f$ through the isomorphism (2.9.1) and thus $\mathcal{O}_*\breve{\bar{N}}$ is also a subalgebra of $\mathcal{O}_*\hat{N}f$. We claim that the map

$$\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} \mathrm{Ind}^{\breve{\bar{K}}}_{\breve{\bar{N}}}(Y) \to \mathrm{Ind}^{\hat{K}}_{\hat{N}}(\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} Y) \tag{2.11.2}$$

sending $x \otimes (y \otimes z)$ to $y \otimes (x \otimes z)$ is an isomorphism of $\mathcal{O}_*\hat{K}$-modules, where $x \in \mathcal{O}_*\hat{L}i$, $y \in \mathcal{O}_*\breve{\bar{K}}$ and $z \in Y$. Note that any element of $\mathrm{Ind}^{\hat{K}}_{\hat{N}}(\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} Y)$ can be written as a sum of elements like $y \otimes (x \otimes z)$, where $x \in \mathcal{O}_*\hat{L}i$, $y \in \mathcal{O}_*\breve{\bar{K}}$ and $z \in Y$; that implies that the homomorphism (2.11.2) is surjective. Then $\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} \mathrm{Ind}^{\breve{\bar{K}}}_{\breve{\bar{N}}}(Y)$ and $\mathrm{Ind}^{\hat{K}}_{\hat{N}}(\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} Y)$ having the same $\mathcal{O}$-rank forces (2.11.2) to be an isomorphism.

2.12. As consequences of Statement 2.11.1, we have the followings:
2.12.1. If S is a simple $\mathcal{O}_*\hat{K}h$-module and $S_{\hat{N}}$ is a simple $\mathcal{O}_*\hat{N}l$-module such that

$$\mathrm{Res}^{\hat{K}}_{\hat{N}}(S) \cong nS_{\hat{N}}$$

and $h\cdot\mathrm{Ind}_{\hat{N}}^{\hat{K}}(S_{\hat{N}})\cong nS$ for a positive integer number n, then $\mathrm{Res}_{\breve{\bar{N}}}^{\breve{\bar{K}}}(i\cdot S)\cong n(i\cdot S_{\hat{N}})$ and $\bar{h}\cdot\mathrm{Ind}_{\breve{\bar{N}}}^{\breve{\bar{K}}}(i\cdot S_{\hat{N}})\cong n(i\cdot S)$.

2.12.2. If W is a simple $\mathcal{K}_*\hat{K}h$-module and $W_{\hat{N}}$ is a simple $\mathcal{K}_*\hat{N}l$-module such that

$$\mathrm{Res}_{\hat{N}}^{\hat{K}}(W)\cong nW_{\hat{N}}$$

for a positive integer number n, then $\mathrm{Res}_{\breve{\bar{N}}}^{\breve{\bar{K}}}(i\cdot W)\cong n(i\cdot W_{\hat{N}})$.

Lemma 2.13. *Keep notations as above. If $\mathcal{O}_*\hat{K}h$ covers $\mathcal{O}_*\hat{N}l$ and $\mathcal{O}_*\hat{K}h$ and $\mathcal{O}_*\hat{N}l$ have common defect groups, then $\mathcal{O}_*\breve{\bar{K}}\bar{h}$ covers $\mathcal{O}_*\breve{\bar{N}}\bar{l}$, $\mathcal{O}_*\breve{\bar{K}}\bar{h}$ and $\mathcal{O}_*\breve{\bar{N}}\bar{l}$ have common defect groups, and $n(h,l)=n(\bar{h},\bar{l})$.*

Proof. By the choices of h and $\bar{h}$, the isomorphism (2.9.1) induces an isomorphism of $\hat{K}$-interior algebras $\mathcal{O}_*\hat{K}h\cong\mathcal{O}_*\hat{L}f\bigotimes_{\mathcal{O}}\mathcal{O}_*\breve{\bar{K}}\bar{h}$. Let P be a defect group of h. Then it follows from [12, Corollary 3.3] that the image of P in $\bar{K}$, which is isomorphic to P and we still denote by P, is a defect group of $\bar{h}$, $\mathcal{O}_*\hat{L}f$ has a P-stable basis and $(\mathcal{O}_*\hat{L}f)(P)\neq 0$. So we can use [10, Proposition 5.6] to obtain the following $C_{\hat{K}}(P)$-interior algebra isomorphism

$$k_*C_{\hat{K}}(P)\,\mathrm{Br}_P^{\mathcal{O}_*\hat{K}}(h)\cong(\mathcal{O}_*\hat{L}f)(P)\otimes_k k_*C_{\breve{\bar{K}}}(P)\,\mathrm{Br}_P^{\mathcal{O}_*\breve{\bar{K}}}(\bar{h}). \tag{2.13.1}$$

Fix a block idempotent h_P of $k_*C_{\hat{K}}(P)$ such that $\mathrm{Br}_P^{\mathcal{O}_*\hat{K}}(h)h_P=h_P$. Since $(\mathcal{O}_*\hat{L}f)(P)$ is a full matrix algebra over k, there exists a block idempotent $\bar{h}_P$ of $k_*C_{\breve{\bar{K}}}(P)$ such that $\mathrm{Br}_P^{\mathcal{O}_*\breve{\bar{K}}}(\bar{h})\bar{h}_P=\bar{h}_P$ and the isomorphism (2.13.1) induces an isomorphism

$$k_*C_{\hat{K}}(P)h_P\cong(\mathcal{O}_*\hat{L}f)(P)\otimes_k k_*C_{\breve{\bar{K}}}(P)\bar{h}_P. \tag{2.13.2}$$

Since we are assuming that $\mathcal{O}_*\hat{K}h$ and $\mathcal{O}_*\hat{N}l$ have common defect groups, P is also a defect group of $\mathcal{O}_*\hat{N}l$. Then similarly, we can find block idempotents l_P and $\bar{l}_P$ of $k_*C_{\hat{N}}(P)$ and $k_*C_{\breve{\bar{N}}}(P)$ respectively, such that $\mathrm{Br}_P^{\mathcal{O}_*\hat{N}}(l)l_P=l_P$, $\mathrm{Br}_P^{\mathcal{O}_*\breve{\bar{N}}}(\bar{l})\bar{l}_P=\bar{l}_P$ and there is an isomorphism

$$k_*C_{\hat{N}}(P)l_P\cong(\mathcal{O}_*\hat{L}f)(P)\otimes_k k_*C_{\breve{\bar{N}}}(P)\bar{l}_P. \tag{2.13.3}$$

Finally since we are also assuming that $\mathcal{O}_*\hat{K}h$ covers $\mathcal{O}_*\hat{N}l$, $\mathcal{O}_*\breve{\bar{K}}\bar{h}$ covers $\mathcal{O}_*\breve{\bar{N}}\bar{l}$ and thus $n(h,l)$ and $n(\bar{h},\bar{l})$ make sense; by isomorphisms (2.13.2) and (2.13.3), we can conclude that

$$\begin{aligned} n(h,l) &= \sqrt{\frac{\dim_k(k_* C_{\hat{K}}(P) h_P)}{\dim_k(k_* C_{\hat{N}}(P) l_P)}} \\ &= \sqrt{\frac{\dim_k(k_* C_{\check{\hat{K}}}(P) \bar{h}_P)}{\dim_k(k_* C_{\check{\hat{N}}}(P) \bar{l}_P)}} = n(\bar{h},\bar{l}). \end{aligned}$$ □

3. Proof of Theorem 1.5

Lemma 3.1. *Let K be a finite group and H a normal subgroup of K. Let $\hat{K}$ be a k^*-group with the k^*-quotient K and e and f block idempotents of $\mathcal{O}_*\hat{K}$ and $\mathcal{O}_*\hat{H}$ respectively. If $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}f$ are naturally Morita equivalent of degree m, then for a common defect group P of e and f, there exists block idempotents e_P and f_P of $k_*C_{\hat{K}}(P)$ and $k_*C_{\hat{H}}(P)$ such that $\mathrm{Br}_P^{\mathcal{O}_*\hat{K}}(e)e_P = e_P$, $\mathrm{Br}_P^{\mathcal{O}_*\hat{H}}(f)f_P = f_P$ and $k_*C_{\hat{K}}(P)e_P$ and $k_*C_{\hat{H}}(P)f_P$ are naturally Morita equivalent of degree m too.*

Proof. Since $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}f$ are naturally Morita equivalent of degree m, by definitions, there exists a unitary subalgebra S of $\mathcal{O}_*\hat{K}e$, which is a full matrix algebra over $\mathcal{O}$ of degree m, such that the product in $\mathcal{O}_*\hat{K}$ induces an isomorphism

$$\mathcal{O}_*\hat{K}e \cong S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{H}f.$$

This isomorphism implies that P acts trivially on S by conjugation and then by [10, Proposition 5.6], we obtain an isomorphism

$$k_*C_{\hat{K}}(P)\,\mathrm{Br}_P^{\mathcal{O}_*\hat{K}}(e) \cong S(P) \otimes_k k_*C_{\hat{H}}(P)\,\mathrm{Br}_P^{\mathcal{O}_*\hat{H}}(f).$$

Fix a block idempotent e_P of $k_*C_{\hat{K}}(P)$ such that $\mathrm{Br}_P^{\mathcal{O}_*\hat{K}}(e)e_P = e_P$. Since $S(P) \cong k \otimes_{\mathcal{O}} S$, e_P determines a unique block idempotent f_P of $k_*C_{\hat{H}}(P)$ such that $\mathrm{Br}_P^{\mathcal{O}_*\hat{H}}(f)f_P = f_P$ and $k_*C_{\hat{K}}(P)e_P \cong (k \otimes_{\mathcal{O}} S) \otimes_k k_*C_{\hat{H}}(P)f_P$. □

3.2. Let H be a finite group and R a subgroup of H. We denote by $(\mathcal{O}H)^R$ the subalgebra of all R-fixed elements of $\mathcal{O}H$. Recall that a pointed group P_γ on $\mathcal{O}H$ is a pair (P,γ) consisting of a subgroup P of H and a $((\mathcal{O}H)^P)^*$-conjugate class γ of primitive idempotents of $(\mathcal{O}H)^P$. Another pointed group R_ε is contained in P_γ if $R \le P$ and there exists $j \in \varepsilon$ and $i \in \gamma$ such that $ji = ij = j$. P_γ is local if $\mathrm{Br}_P^{\mathcal{O}H}(\gamma) \ne \{0\}$. Let c be a block idempotent of $\mathcal{O}H$. Then $\{c\}$ becomes a point of H on $\mathcal{O}H$. We say that P_γ is a defect pointed group of $\{c\}$ or simply c if P_γ is a maximal local pointed group contained in $H_{\{c\}}$ with respect inclusion. By [8, Theorem 1.2], H acts transitively on the set of all defect pointed groups of $H_{\{c\}}$. Fix $i \in \gamma$ and set $(\mathcal{O}H)_\gamma = i(\mathcal{O}H)i$. Then $(\mathcal{O}H)_\gamma$ is called a source algebra of $H_{\{c\}}$ or simply c.

3.3. Let P_γ be a defect pointed group of a block c of $\mathcal{O}H$ and denote by $N_H(P_\gamma)$ the stabilizer of P_γ in H and by $(\mathcal{O}H)(P_\gamma)$ the simple factor of $(\mathcal{O}H)^P$ such that the image of γ through the surjective homomorphism $(\mathcal{O}H)^P \to (\mathcal{O}H)(P_\gamma)$ is not zero. The obvious action of $N_H(P_\gamma)$ on $(\mathcal{O}H)^P$ induces an action of $N_H(P_\gamma)$ on $(\mathcal{O}H)(P_\gamma)$. By the Skolem–Noether theorem, we have a group homomorphism $\rho\colon N_H(P_\gamma) \to \mathrm{Aut}((\mathcal{O}H)(P_\gamma)) \cong ((\mathcal{O}H)(P_\gamma))^*/k^*$. We denote by $\hat{N}_H(P_\gamma)$ the set of all elements (c, x) such that $\rho(x)$ is the image of c in $((\mathcal{O}H)(P_\gamma))^*/k^*$, where $c \in ((\mathcal{O}H)(P_\gamma))^*$ and $x \in N_H(P_\gamma)$. Then $\hat{N}_H(P_\gamma)$ is a k^*-group with the k^*-quotient $N_H(P_\gamma)$ with the homomorphism $k^* \to \hat{N}_H(P_\gamma)$, $\lambda \mapsto (\lambda, 1)$, and the map $PC_H(P) \to \hat{N}_H(P_\gamma)$, $x \mapsto (x, x)$ is an injective homomorphism, whose image is normal in $\hat{N}_H(P_\gamma)$ and intersects k^* trivially. We identify $PC_H(P)$ with a normal subgroup of $\hat{N}_H(P_\gamma)$ through the injective homomorphism and then the quotient $\hat{N}_H(P_\gamma)/PC_H(P)$ is a k^*-group with the k^*-quotient $N_H(P_\gamma)/PC_H(P)$. Let G be a finite group containing H as a normal subgroup and $C_G(P_\gamma)$ be the stabilizer of P_γ in $C_G(P)$. Then it is very obvious that the conjugation action of $C_G(P_\gamma)$ on H induces an action of $C_G(P_\gamma)$ on $N_H(P_\gamma)$ and actions of $C_G(P_\gamma)$ on $(\mathcal{O}H)(P_\gamma)$ and $((\mathcal{O}H)(P_\gamma))^*/k^*$ and that the homomorphism ρ: $N_H(P_\gamma) \to ((\mathcal{O}H)(P_\gamma))^*/k^*$ and the surjective homomorphism $((\mathcal{O}H)(P_\gamma))^* \to ((\mathcal{O}H)(P_\gamma))^*/k^*$ preserve the corresponding $C_G(P_\gamma)$-actions. So $C_G(P_\gamma)$ acts on $\hat{N}_H(P_\gamma)/PC_H(P)$.

Lemma 3.4. *Let H be a finite group fulfilling that $C_H(O_p(H)) \subset O_p(H)$, P be a Sylow p-subgroup of H and $\hat{H}$ be a k^*-group with the k^*-quotient H. Then the unit element 1 of $\mathcal{O}_*\hat{H}$ is the unique block idempotent of $\mathcal{O}_*\hat{H}$ and $P_{\{1\}}$ is a defect pointed group of $H_{\{1\}}$.*

Proof. Consider the Brauer homomorphism $\mathrm{Br}_{O_p(H)}^{\mathcal{O}_*\hat{H}}$: $(\mathcal{O}_*\hat{H})^{O_p(H)} \to k_*C_{\hat{H}}(O_p(H))$. Since $C_H(O_p(H)) \subset O_p(H)$, $C_{\hat{H}}(O_p(H)) \cong k^* \times Z(O_p(H))$ and thus $k_*C_{\hat{H}}(O_p(H)) \cong kZ(O_p(H))$. On the other hand, since $O_p(H)$ is normal in H, $\mathrm{Ker}\big(\mathrm{Br}_{O_p(H)}^{\mathcal{O}_*\hat{H}}\big) \subset J(\mathcal{O}_*\hat{H}) \cap (\mathcal{O}_*\hat{H})^{O_p(H)} \subset J((\mathcal{O}_*\hat{H})^{O_p(H)})$. Thus $\{1\}$ is the unique local point of $O_p(H)$ on $\mathcal{O}_*\hat{H}$ and then the lemma follows. □

Let G be a finite group, H a normal subgroup of G, $\hat{G}$ a k^*-group of the k^*-group G and c a G-stable block idempotent of $\mathcal{O}_*\hat{H}$. We denote by $G[c]$ the group of all $g \in G$ such that there exists some $x_g \in (\mathcal{O}_*\hat{H}c)^*$ fulfilling $a^g = a^{x_g}$ for any $a \in \mathcal{O}_*\hat{H}c$. By [2, Proposition 2.7 and Theorem 3.5], $G[c]$ is normal in G and $b \in \mathcal{O}_*\widehat{G[c]}$.

Lemma 3.5. *Let G be a finite group, H a normal subgroup of G such that $C_H(O_p(H)) \le O_p(H)$ and P a Sylow p-subgroup of H. Let $\hat{G}$ be a k^*-group and assume that $\mathcal{O}_*\hat{G}$ has a block with P as a defect group. Then $G[1] = C_G(P)H$.*

Here 1 is the block idempotent of $\mathcal{O}_*\hat{H}$ (see Lemma 3.4).

Proof. We firstly prove $C_G(P)H \subset G[1]$. By [9, Lemma 5.5], there exists a finite subgroup G' of $\hat{G}$ such that $\hat{G} = k^*G'$; moreover if we let Z' be the intersection of k^* and G', H' the intersection of G' and $\hat{H}$ and ι the central idempotent $1/|Z'| \sum_{z\in Z'} \lambda_z z^{-1}$ of $\mathcal{O}G'$, by [9, Theorem 5.15], the inclusion $G' \subset \hat{G}$ induces an isomorphism of $\mathcal{O}$-algebras

$$\mathcal{O}G'\iota \cong \mathcal{O}_*\hat{G}, \tag{3.5.1}$$

whose restriction to H' induces an isomorphism

$$\mathcal{O}H'\iota \cong \mathcal{O}_*\hat{H}. \tag{3.5.2}$$

Since $C_H(O_P(H)) \subset O_P(H)$, by Lemma 3.4, $c = 1$ is the unique block idempotent of $\mathcal{O}_*\hat{H}$ and $\gamma = \{1\}$ is the unique local point of P on $\mathcal{O}_*\hat{H}$, thus ι is a block idempotent of $\mathcal{O}H'$, $\gamma' = \{\iota\}$ is the unique local point of P on $\mathcal{O}H'\iota$ and the P-interior algebra $\mathcal{O}H'\iota$ with the homomorphism $P \to (\mathcal{O}H'\iota)^*$, $u \mapsto u\iota$ is a source algebra of ι. For any $x \in C_{G'}(P)$, we consider the automorphism φ_x on the source algebra $\mathcal{O}H'\iota$ induced by x. Clearly $C_G(P)$ stabilizes P_γ, thus $C_{G'}(P)$ stabilizes $P_{\gamma'}$ and then $C_{G'}(P)$ acts on the k^*-group $\hat{N}_{H'}(P_{\gamma'})/PC_{H'}(P)$ (refer to Paragraph 3.3). But it follows from $C_H(O_p(H)) \subset O_p(H)$ that $(\mathcal{O}_*\hat{H})(P_\gamma) \cong k$, $(\mathcal{O}H')(P_{\gamma'}) \cong k$ and thus $\hat{N}_{H'}(P_{\gamma'})/PC_{H'}(P) \cong k^* \times N_{H'}(P_{\gamma'})/PC_{H'}(P)$; on the other hand, $C_{G'}(P)$ acts trivially on the group $N_{H'}(P_{\gamma'})/PC_{H'}(P)$. Consequently $C_{G'}(P)$ acts trivially on the k^*-group $\hat{N}_{H'}(P_{\gamma'})/PC_{H'}(P)$. Therefore by [9, Proposition 14.9], φ_x is induced by some element $a' \in (\mathcal{O}H'\iota)^*$; in particular, this shows that the automorphism on $\mathcal{O}_*\hat{H}$ induced by $x \in C_G(P)$ is induced by some $a \in (\mathcal{O}_*\hat{H})^*$. Thus $x \in G[1]$.

In order to prove $G[1] = C_G(P)H$, now we assume $G = G[1]$ without loss of generality. Set $K = C_G(P)H$ and let b be a block idempotent of $\mathcal{O}_*\hat{G}$ with P as a defect group and e be a block idempotent of $\mathcal{O}_*\hat{K}$ such that $be \neq 0$. Obviously e also covers the unique block 1 of $\mathcal{O}_*\hat{H}$ and thus P is also a defect group of e. By [6, Theorem 7], $\mathcal{O}_*\hat{G}b$ and $\mathcal{O}_*\hat{H}$ are naturally Morita equivalent of degree n for a positive integer and $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}$ are naturally Morita equivalent of degree m for a positive integer. We claim that n is equal to m. Indeed, since $be \neq 0$ and $G \supset HC_G(P)$, $\mathcal{O}_*\hat{G}b$ and $\mathcal{O}_*\hat{K}e$ at least have a common block idempotent f of $k_*C_{\hat{G}}(P)$ such that $\mathrm{Br}_P^{\mathcal{O}_*\hat{G}}(b)f \neq f$ and $\mathrm{Br}_P^{\mathcal{O}_*\hat{K}}(e)f \neq f$. Then by Lemma 3.1, n is equal to m; in particular, this shows that $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{G}b$ have the same $\mathcal{O}$-rank. Since P is a Sylow p-subgroup of H, by Frattini argument, we have $G = N_G(P)H$. Thus K is normal in $G[1]$. Then by [6, Theorem 1], $k \otimes_{\mathcal{O}} \mathcal{O}_*\hat{K}e$ and $k \otimes_{\mathcal{O}} \mathcal{O}_*\hat{G}b$ are isomorphic. Finally by [5, Corollary 4.5], $G[1] = C_{G[1]}(P)K = C_G(P)H$. □

Theorem 3.6. *Let G be a finite group and H a normal subgroup of G such that $C_H(R) \subset O_{p',p}(H)$ for a Sylow p-subgroup R of $O_{p',p}(H)$. Let $\hat{G}$ be a k^*-group with* *the k^*-quotient G, b and c block idempotents of $\mathcal{O}_*\hat{G}$ and $\mathcal{O}_*\hat{H}$ respectively, and n a positive integer. If c is also a block idempotent of $\mathcal{O}_*\widehat{O_{p'}(H)}$, then the following two conditions are equivalent*:

3.6.1. *$\mathcal{O}_*\hat{G}b$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree n*;

3.6.2. *for any simple $\mathcal{O}_*\hat{G}$-module S associated to b, there exists a unique simple $\mathcal{O}_*\hat{H}$-module $S_{\hat{H}}$ associated to c such that $\mathrm{Res}^{\hat{G}}_{\hat{H}}(S) \cong nS_{\hat{H}}$ and $b \cdot \mathrm{Ind}^{\hat{G}}_{\hat{H}}(S_{\hat{H}}) \cong nS$, the correspondence $\mathrm{IBr}(b) \to \mathrm{IBr}(c)$, $S \mapsto S_{\hat{H}}$ is a bijection, and $n \le n(b, c)$.*

Moreover in this case, $n = n(b, c)$.

Proof. By [5, Proposition 2.6] and Lemma 3.1, Condition 3.6.1 implies Condition 3.6.2. Now we assume that Condition 3.6.2 holds. By the isomorphism (2.9.1) applied to $\mathcal{O}_*\hat{G}c$ and $\mathcal{O}_*\widehat{O_{p'}(H)}c$, we can find a k^*-group $\breve{\bar{G}}$ with the k^*-quotient $\bar{G} = G/O_{p'}(H)$ such that there exists an isomorphism of $\hat{G}$-interior algebras

$$\mathcal{O}_*\hat{G}c \cong \mathcal{O}_*\widehat{O_{p'}(H)}c \otimes_{\mathcal{O}} \mathcal{O}_*\breve{\bar{G}} \tag{3.6.3}$$

which, by restriction to $\mathcal{O}_*\hat{H}c$, induces an isomorphism of $\hat{H}$-interior algebras

$$\mathcal{O}_*\hat{H}c \cong \mathcal{O}_*\widehat{O_{p'}(H)}c \otimes_{\mathcal{O}} \mathcal{O}_*\breve{\bar{H}}$$

where $\breve{\bar{H}}$ is the inverse image of $\bar{H} = H/O_{p'}(H)$ in $\breve{\bar{G}}$.

Since $\mathcal{O}_*\widehat{O_{p'}(H)}c$ is a full matrix algebra over $\mathcal{O}$ and $bc = b$, b determines a unique block idempotent $\bar{b}$ of $\mathcal{O}_*\breve{\bar{G}}$ through (3.6.3) such that

$$\mathcal{O}_*\hat{G}b \cong \mathcal{O}_*\widehat{O_{p'}(H)}c \otimes_{\mathcal{O}} \mathcal{O}_*\breve{\bar{G}}\bar{b}. \tag{3.6.4}$$

But notice that 1 is the unique block idempotent of $\mathcal{O}_*\breve{\bar{H}}$ since we are assuming $C_H(R) \subset O_{p',p}(H)$ for a Sylow p-subgroup R of H and thus $C_{\bar{H}}(O_p(\bar{H})) \subset O_p(\bar{H})$ (see Lemma 3.4). Let i be a primitive idempotent of $\mathcal{O}_*\widehat{O_{p'}(H)}c$. Since we are also assuming that there exists a unique simple $\mathcal{O}_*\hat{H}$-module S_H associated to c such that $\mathrm{Res}^{\hat{G}}_{\hat{H}}(S) \cong nS_{\hat{H}}$ and $b \cdot \mathrm{Ind}^{\hat{G}}_{\hat{H}}(S_{\hat{H}}) \cong nS$ for any simple $\mathcal{O}_*\hat{G}$-module S associated to b and that the correspondence $\mathrm{IBr}(b) \to \mathrm{IBr}(c)$, $S \mapsto S_H$ is a bijection, it follows from Statement 2.12.1 that we have equalities $\mathrm{Res}^{\breve{\bar{H}}}_{\breve{\bar{G}}}(i \cdot S) \cong n(i \cdot S_{\hat{H}})$ and $\bar{b} \cdot \mathrm{Ind}^{\breve{\bar{G}}}_{\breve{\bar{H}}}(i \cdot S_{\hat{H}}) \cong n(i \cdot S)$ and from Theorem 2.9 that the map the correspondence $\mathrm{IBr}(\bar{b}) \to \mathrm{IBr}(1)$, $i \cdot S \mapsto i \cdot (S_H)$ is a bijection; here in order to avoid confusion, we remind that $\mathrm{IBr}(1)$ is the set of all simple $\mathcal{O}_*\breve{\bar{H}}$-modules. Finally by our hypothesis, b and c have common defect groups (refer to [7, Chapter 4, Lemma 3.4] and [4, Chapter IV, Lemma 4.6]), so $n(b, c)$ makes sense and so does $n(\bar{b}, 1)$; by Lemma 2.13, we have $n(b, c) = n(\bar{b}, 1)$.

If we can prove that $\mathcal{O}_*\breve{\bar{G}}\bar{b}$ and $\mathcal{O}_*\breve{\bar{H}}$ are naturally Morita equivalent of degree n, by Lemma 2.10.2, so are $\mathcal{O}_*\hat{G}b$ and $\mathcal{O}_*\hat{H}c$. So in order to prove the theorem, we can assume $C_H(O_p(H)) \subset O_p(H)$. Let P be a common defect group of b and c. Since H is normal in G and H and G act transitively on the sets of defect groups of c and b, by Frattini argument, we have $G = N_G(P)H$. Now consider the obvious normal subgroup $K = C_G(P)H$ of G and let e be a block idempotent of $\mathcal{O}_*\hat{K}$ such that $be \neq 0 \neq ce$. Then P has to be a defect group of e. By Lemma 3.5 and [6, Theorem 7], $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree m; moreover by Lemma 3.1 and the definition of $n(b, c)$, $m = n(b, c) \geq n$. □

Let S be a simple $\mathcal{O}_*\hat{G}b$-module. Since $be \neq 0 \neq ce$ and $\mathrm{Res}^{\hat{G}}_{\hat{H}}(S) = nS_{\hat{H}}$, by Clifford theorem, there exists a simple $\mathcal{O}_*\hat{K}e$-module $S_{\hat{K}}$ such that $S_{\hat{K}}$ is a direct summand of $\mathrm{Res}^{\hat{G}}_{\hat{K}}(S)$ and $S_{\hat{H}}$ is a direct summand of $\mathrm{Res}^{\hat{K}}_{\hat{H}}(S_{\hat{K}})$. Since $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree m, by [5, Proposition 2.6], $\mathrm{Res}^{\hat{K}}_{\hat{H}}(S_{\hat{K}}) = mS_{\hat{H}}$. Then the inequality $m \geq n$ shows that $\dim_k(S_K) \geq \dim_k(S)$, thus $\mathrm{Res}^{\hat{G}}_{\hat{K}}(S) = S_{\hat{K}}$ and $m = n$; in particular, this also implies that G stabilizes e and thus $be = b$. By Lemma 3.5 and [6, Corollary 4], $b \in \mathcal{O}_*\hat{K}$ and thus $be = e$. Therefore $b = e$. That $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree n also implies $b \cdot \mathrm{Ind}^{\hat{K}}_{\hat{H}}(S_{\hat{H}}) = nS_{\hat{K}}$ (refer to [5, Proposition 2.6]). We rewrite $b \cdot \mathrm{Ind}^{\hat{G}}_{\hat{H}}(S_{\hat{H}})$ as $\mathrm{Ind}^{\hat{G}}_{\hat{K}}(b \cdot \mathrm{Ind}^{\hat{K}}_{\hat{H}}(S_{\hat{H}})) = \mathrm{Ind}^{\hat{G}}_{\hat{K}}(nS_{\hat{K}}) = n\,\mathrm{Ind}^{\hat{G}}_{\hat{K}}(S_{\hat{K}})$. Then the equality $n\,\mathrm{Ind}^{\hat{G}}_{\hat{K}}(S_{\hat{K}}) = nS$ forces $S = \mathrm{Ind}^{\hat{G}}_{\hat{K}}(S_K)$. But we also have $\mathrm{Res}^{\hat{G}}_{\hat{K}}(S) = S_{\hat{K}}$ and therefore G has to be equal to K.

Theorem 3.7. *Let G be a finite group and H a normal subgroup of G such that $C_H(R) \subset O_{p',p}(H)$ for a Sylow p-subgroup R of $O_{p',p}(H)$. Let $\hat{G}$ be a k^*-group with the k^*-quotient G, b and c block idempotents of $\mathcal{O}_*\hat{G}$ and $\mathcal{O}_*\hat{H}$ respectively, and n be a positive integer. If c is also a block idempotent of $\mathcal{O}_*\widehat{O_{p'}(H)}$, then the following two conditions are equivalent*:

3.7.1. *$\mathcal{O}_*\hat{G}b$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree n*;

3.7.2. *$v_p(|G : H|) = v_p(n)$, for any simple $\mathcal{K}_*\hat{G}$-module V associated to b, there exists a unique simple $\mathcal{K}_*\hat{H}$-module $V_{\hat{H}}$ associated to c such that $\mathrm{Res}^{\hat{G}}_{\hat{H}}(V) \cong nV_{\hat{H}}$, the correspondence $\mathrm{Irr}(b) \to \mathrm{Irr}(c)$, $V \mapsto V_{\hat{H}}$ is a bijection, and $n(b, c) \geq n$.*

Moreover in this case, $n = n(b, c)$.

Proof. By [5, Proposition 2.6] and Lemma 3.1, Condition 3.7.1 implies Condition 3.7.2. Now assume that Condition 3.7.2 holds. Note that the first three statements imply that b and c have common defect groups (refer to [4, Chapter IV, Theorem 4.5]). Then by the first and second paragraph in Theorem 3.6, in order to prove 3.7.1, we can assume $C_H(O_p(H)) \subset O_p(H)$ without loss of generality. Let P be a common defect group of b and c. Since H is normal in G and H and G act transitively on the sets of defect groups of c and b, by Frattini argument, we have $G = N_G(P)H$. Now consider the obvious normal subgroup $K = C_G(P)H$ of G and let e be a block idempotent of $\mathcal{O}_*\hat{K}$ such that $be \neq 0 \neq ce$. Then P has to be a defect group of e. By Lemma 3.5 and [6, Theorem 7], $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree m and by Lemma 3.1 and the definition of $n(b, c)$, $m = n(b, c) \geq n$. $\square$

Let V be a simple $\mathcal{K}_*\hat{G}b$-module. Since $be \neq 0 \neq ce$ and $\mathrm{Res}^{\hat{G}}_{\hat{H}}(V) = nV_{\hat{H}}$, by Clifford theorem, there exists a simple $\mathcal{K}_*\hat{K}e$-module $V_{\hat{K}}$ such that $V_{\hat{K}}$ is a direct summand of $\mathrm{Res}^{\hat{G}}_{\hat{K}}(V)$ and $V_{\hat{H}}$ is a direct summand of $\mathrm{Res}^{\hat{K}}_{\hat{H}}(V_{\hat{K}})$. Since $\mathcal{K}_*\hat{K}e$ and $\mathcal{K}_*\hat{H}c$ are naturally Morita equivalent of degree m, by [5, Proposition 2.6], $\mathrm{Res}^{\hat{K}}_{\hat{H}}(V_{\hat{K}}) = mV_{\hat{H}}$. Then the inequality $m \geq n$ shows that $\dim_{\mathcal{K}}(V_{\hat{K}}) \geq \dim_{\mathcal{K}}(V)$ and then $\dim_{\mathcal{K}}(V_{\hat{K}}) = \dim_{\mathcal{K}}(V)$, thus $\mathrm{Res}^{\hat{G}}_{\hat{K}}(V) = V_{\hat{K}}$ and $m = n$; in particular, this also implies that G stabilizes e and thus $be = b$. By Lemma 3.5 and [6, Corollary 4], $b \in \mathcal{O}_*\hat{K}$ and thus $be = e = b$. Moreover it is easily checked that the map $V \to V_{\hat{K}}$ is a bijection between the sets of all simple $\mathcal{K}_*\hat{G}b$- and $\mathcal{K}_*\hat{K}e$-modules; in particular, this implies that $\mathcal{O}_*\hat{G}b$ and $\mathcal{O}_*\hat{K}e$ have the same $\mathcal{O}$-rank. But obviously the $\mathcal{O}$-rank of $\mathcal{O}_*\hat{G}b$ is equal to the product of $|G : K|$ with the $\mathcal{O}$-rank of $\mathcal{O}_*\hat{K}e$ too. So G is forced to equal to K. We are done.

3.8. Proof of Theorem 1.5. It suffices for us to take $\hat{G}$ and $\hat{H}$ to be $G \times k^*$ and $H \times k^*$ and then Theorems 3.6 and 3.7 imply Theorem 1.5. $\square$

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