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NATURAL MORITA EQUIVALENCES OF DEGREE n

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Abstract

Let G be a finite group, H a normal subgroup of G and b and c block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. Under the assumption that $C_H(R) \subset \mathcal{O}_{p',p}(H)$ for a Sylow p -subgroup R of $\mathcal{O}_{p',p}(H)$ and c is also a block idempotent of $\mathcal{O}\mathcal{O}_{p'}(H)$, we give two equivalent conditions about when $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are natural Morita equivalent of degree n (see Theorem 1.5).

1. Introduction

1.1. Fix a prime number p . Let \mathcal{O} be a complete discrete valuation ring with a residue field k of characteristic p . Let G be a finite group, H a subgroup of G and b and c block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. In terms of the terminology of A. Hida and S. Koshitani [5], $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are said to be naturally Morita equivalent of degree n for a positive integer number n if there exists an unitary \mathcal{O} -subalgebra S of $\mathcal{O}Gb$ such that S is a full matrix algebra over \mathcal{O} of degree n and the map

$$\mathcal{O}Hc \otimes_{\mathcal{O}} S \rightarrow \mathcal{O}Gb, \quad x \otimes y \mapsto xy$$

is an isomorphism of \mathcal{O} -algebras. When H is normal in G and $\mathcal{O} = k$, this definition is firstly due to B. Külshammer [6].

1.2. For our question below, now we make the additional assumption that the characteristic of \mathcal{O} is zero, the quotient field \mathcal{K} of \mathcal{O} is big enough for all algebras involved below, the residue field k is algebraically closed and H is normal in G ; the assumption will also be kept throughout this paper. As a consequence of [13, Theorems 2 and 3], we can easily conclude that the following three conditions are equivalent:

- 1.2.1. the map $\mathcal{O}Gb \rightarrow \mathcal{O}Hc$, $x \mapsto xc$ is an \mathcal{O} -algebra isomorphism;
- 1.2.2. the restriction from G to H induces a bijection between the sets of all non-isomorphic simple modules of $\mathcal{O}Gb$ and $\mathcal{O}Hc$ and the quotient group G/H is a p' -group;
- 1.2.3. the restriction from G to H induces a bijection between the sets of all non-isomorphic simple modules of $\mathcal{K}Gb$ and $\mathcal{K}Hc$.

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Noticing that Condition 1.2.1 is actually saying that $\mathcal{O}Gb$ and $\mathcal{O}Hc$ is naturally Morita equivalent of degree 1, we ask ourselves a question: can this statement above be generalized to natural Morita equivalences of degree n ? In this paper, we investigate the question.

1.3. Now we begin with some preparations in order to state our main theorem. Let M be an $\mathcal{O}G$ -module and N an $\mathcal{O}K$ -module. We denote by $\text{Res}_K^G(M)$ the restriction of M from G to K and by $\text{Ind}_K^G(N)$ the induction of N from K to G . Given a positive integer number n , we denote by nM the direct sum of n copies of M . Obviously the product $b \cdot M$ of b and M is an $\mathcal{O}G$ -submodule of M and b acts on $b \cdot M$ as the identity homomorphism. When $b \cdot M = M$, then we say that the $\mathcal{O}G$ -module M is associated to the block b of $\mathcal{O}G$. We denote by $\text{IBr}(b)$ the set of all non-isomorphic simple $\mathcal{O}G$ -modules associated to b . All notations above except $\text{IBr}(b)$ can be slightly modified to apply to $\mathcal{K}G$ -modules. In general, we denote by $\text{Irr}(b)$ the set of all non-isomorphic simple $\mathcal{K}G$ -modules associated to b . Given a positive integer number m , $v_p(m)$ denotes the largest non-negative integer t such that $p^t \mid m$.

1.4. Assume that $bc \neq 0$ and b and c have a common defect group P . Since $bc \neq 0$, it is well known (refer to [3]) that there exist block idempotents b_P and c_P of $kC_G(P)$ and $kC_H(P)$ such that $b_P \text{Br}_P^{\mathcal{O}G}(b) = b_P$, $c_P \text{Br}_P^{\mathcal{O}H}(c) = c_P$ and $b_P c_P \neq 0$. Since P is a defect group of b and c , b_P and c_P have defect group $Z(P)$, thus $kC_G(P)b_P$ and $kC_H(P)c_P$ are nilpotent (refer to [10]) and have only one simple module, say V_{b_P} and V_{c_P} . Since H is normal in G , so is $C_H(P)$ in $C_G(P)$; then by Clifford theory, we can conclude that the dimension $\dim_k(V_{c_P})$ of V_{c_P} over k divides the dimension $\dim_k(V_{b_P})$ of V_{b_P} over k . Note that (P, b_P) and (P, c_P) actually are maximal Brauer pairs of b and c , which are unique up to G - and H -conjugation (refer to [1]). Therefore the quotient $\dim_k(V_{b_P})/\dim_k(V_{c_P})$ is independent of the choices of b_P and c_P . We denote this quotient by $n(b, c)$. Note that by [10, 1.4.1], $n(b, c) = \sqrt{\dim_k(kC_G(P)b_P)/\dim_k(kC_H(P)c_P)}$; even in order to compute $n(b, c)$, it suffices for us to choose b_P and c_P of $kC_G(P)$ and $kC_H(P)$ such that $b_P \text{Br}_P^{\mathcal{O}G}(b) = b_P$ and $c_P \text{Br}_P^{\mathcal{O}H}(c) = c_P$.

Theorem 1.5. *Let G be a finite group and H be a normal subgroup of G such that $C_H(R) \subset O_{p',p}(H)$ for a Sylow p -subgroup R of $O_{p',p}(H)$. Let b and c be respective block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ and let n be a positive integer. If c is also a block idempotent of $\mathcal{O}O_p(H)$, then the following conditions are equivalent:*

1.5.1. $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are naturally Morita equivalent of degree n ;

1.5.2. for any simple $\mathcal{O}G$ -module S associated to b , there exists a unique simple $\mathcal{O}H$ -module S_H associated to c such that $\text{Res}_H^G(S) \cong nS_H$ and $b \cdot \text{Ind}_H^G(S_H) \cong nS$, the correspondence $\text{IBr}(b) \rightarrow \text{IBr}(c)$, $S \mapsto S_H$ is a bijection, and $n \leq n(b, c)$.

1.5.3. $v_p(|G : H|) = v_p(n)$, for any simple $\mathcal{K}G$ -module V associated to b , there exists a unique simple $\mathcal{K}H$ -module V_H associated to c such that $\text{Res}_H^G(V) \cong nV_H$, and the

correspondence $\text{Irr}(b) \rightarrow \text{Irr}(c)$, $V \mapsto V_H$ is a bijection, and $n \leq n(b, c)$.
 Moreover in this case, n is equal to $n(b, c)$.

REMARK 1.6. 1. Conditions 1.5.2 and 1.5.3 both imply that b and c have the same defect groups, so $n(b, c)$ makes sense. For details, refer to the proofs of Theorems 3.6 and 3.7.

2. When $n = 1$, by [4, Chapter IV, Theorem 4.5], it is easily checked that Conditions 1.5.2 and 1.5.3 both imply that the quotient group G/H is a p' -group; in addition $n \leq n(b, c)$ automatically holds. Therefore the theorem above covers the equivalences between Conditions 1.2.1, 1.2.2 and 1.2.3.

3. There are examples to explain why the condition $n \leq n(b, c)$ is necessary.

2. Fong's reduction

In this section, an \mathcal{O} -algebra A that is involved is always associative, unitary and \mathcal{O} -free of finite rank as an \mathcal{O} -module; A^* and $J(A)$ denote the multiplicative group of all invertible elements of A and the Jacobson radical of A respectively. Occasionally, in order to avoid confusion, we denote by 1_A of the identity element of A . A homomorphism $f: A \rightarrow B$ between \mathcal{O} -algebras is an embedding if f is injective and $f(A) = f(1_A)Bf(1_A)$.

2.1. Let K be a finite group and \hat{K} be a k^* -group with the k^* -quotient K endowed with the homomorphism $\rho: k^* \rightarrow \hat{K}$. By \hat{K} , we can construct two k^* -groups: the group \hat{K} endowed with the group homomorphism $k^* \rightarrow \hat{K}$ sending λ onto $\rho(\lambda^{-1})$ and the opposite group $(\hat{K})^\circ$ with the group homomorphism ρ ; in order to differ from the k^* -group \hat{K} , we denote the first k^* -group by \hat{K}° . But the two k^* -groups are isomorphic: there is an isomorphism of k^* -groups $(\hat{K}^\circ)^\circ \rightarrow \hat{K}^\circ$, $x \mapsto x^{-1}$ (refer to [9]). For any subgroup L of K , we denote by \hat{L} its inverse image in \hat{K} and for any element $x \in L$, by \hat{x} a lifting in \hat{K} of x . When L is a p -group, \hat{L} can be uniquely decomposed as the direct product $k^* \times L$ (refer to [9, Lemma 5.5]) and thus we always regard L as a subgroup of \hat{K} . Let \check{K} be another k^* -group with the k^* -quotient K . Then the central product of \hat{K} and \check{K} over k^* defines a k^* -group $\hat{K} \otimes \check{K}$ with the k^* -quotient isomorphic to $K \times K$ and we identify this k^* -quotient with $K \times K$. We also identify K with the diagonal subgroup in $K \times K$ and denote by $\hat{K} * \check{K}$ the inverse image in $\hat{K} \otimes \check{K}$ of K . Then $\hat{K} * \check{K}$ is a new k^* -group with the k^* -quotient K .

2.2. Obviously the surjective homomorphism $\mathcal{O} \rightarrow k$ induces a surjective group homomorphism $\mathcal{O}^* \rightarrow k^*$; since k is algebraically closed, k is perfect and thus by [14, Chapter II, Proposition 8], there exists a unique section $k^* \rightarrow \mathcal{O}^*$ of this group homomorphism. Through this section, we can regard \mathcal{O} as a right module over the group algebra of k^* over \mathcal{O} . Let K be a finite group and \hat{K} be a k^* -group with the k^* -quotient K . Obviously the inclusion $k^* \subset \hat{K}$ induces a left $\mathcal{O}k^*$ -module structure on the group

algebra $\mathcal{O}\hat{K}$ of \hat{K} over \mathcal{O} . Now we consider the tensor product $\mathcal{O} \otimes_{\mathcal{O}k^*} \mathcal{O}\hat{K}$ and define a distributive product on $\mathcal{O} \otimes_{\mathcal{O}k^*} \mathcal{O}\hat{K}$ by the equality

$$(a \otimes x)(b \otimes y) = ab \otimes xy$$

for $a, b \in \mathcal{O}$ and $x, y \in \hat{K}$. Then the tensor product $\mathcal{O} \otimes_{\mathcal{O}k^*} \mathcal{O}\hat{K}$ with the above product becomes an \mathcal{O} -algebra; we call it the twisted group algebra of \hat{K} over \mathcal{O} and denote it by $\mathcal{O}_*\hat{K}$. Obviously the k^* -group isomorphism $(\hat{K})^\circ \cong \hat{K}^\circ$, $x \mapsto x^{-1}$ induces an isomorphism of \mathcal{O} -algebras from the opposite ring $(\mathcal{O}_*\hat{K})^\circ$ to $\mathcal{O}_*\hat{K}^\circ$; moreover since the map $\mathcal{O}_*(\hat{K} \otimes \hat{K}^\circ) \rightarrow \mathcal{O}_*\hat{K} \otimes_{\mathcal{O}} \mathcal{O}_*\hat{K}^\circ$ sending $1 \otimes (x \otimes y)$ to $(1 \otimes x) \otimes (1 \otimes y)$ for $x \otimes y \in \hat{K} \otimes \hat{K}^\circ$ is an isomorphism, we can define a left $\mathcal{O}_*(\hat{K} \otimes \hat{K}^\circ)$ -module structure on $\mathcal{O}_*\hat{K}$ by the equality $(x \otimes y)a = xay^{-1}$ for $x, y \in \hat{K}$ and $a \in \mathcal{O}_*\hat{K}$. The tensor product $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}_*\hat{K}$ is also what we are concerned below and we denote it by $\mathcal{K}_*\hat{K}$.

2.3. Recall that an \mathcal{O} -algebra A is called a \hat{K} -interior algebra (see [9, 5.10]) if there exists a group homomorphism $\varphi: \hat{K} \rightarrow A^*$. For any $a \in A$ and liftings \hat{x}, \hat{y} in \hat{K} of $x, y \in K$, we will write $\varphi(\hat{x})a\varphi(\hat{y})$ as $\hat{x}a\hat{y}$ for convenience. Obviously when $\hat{y} = \hat{x}^{-1}$, the product $\hat{x}a\hat{x}^{-1}$ is independent of the choice of \hat{x} in \hat{K} and therefore we also often write it as $a^{x^{-1}}$. Moreover the map $\varphi_x: A \cong A$, $a \mapsto a^{x^{-1}}$ is an automorphism, the map $K \rightarrow \text{Aut}(A)$, $x \mapsto \varphi_x$ is a group homomorphism, thus A is a K -algebra. Let C be another \hat{K} -interior algebra; an \mathcal{O} -algebra homomorphism $f: A \rightarrow C$ is called a homomorphism of \hat{K} -interior algebras if $f(\hat{x}a\hat{y}) = \hat{x}f(a)\hat{y}$ for any $a \in A$ and liftings \hat{x}, \hat{y} in \hat{K} of $x, y \in K$. Let \check{K} be another k^* -group with the k^* -quotient K and A' be a \check{K} -interior algebra; then the \hat{K} -interior algebra structure on A and the \check{K} -interior algebra structure on A' determine a $\hat{K} \otimes \check{K}$ -interior algebra structure on the tensor product $A \otimes_{\mathcal{O}} A'$, which, by restriction, induces a $\hat{K} * \check{K}$ -interior algebra structure on $A \otimes_{\mathcal{O}} A'$.

2.4. Let A be a \hat{K} -interior algebra and P a p -subgroup of K . We denote by A^P the subalgebra consisting of all P -fixed elements of A . Clearly A^P is a $C_{\hat{K}}(P)$ -interior algebra with the homomorphism $C_{\hat{K}}(P) \rightarrow (A^P)^*$, $\hat{x} \mapsto \hat{x}1$, where $C_{\hat{K}}(P)$ is the centralizer of P in \hat{K} . For any subgroup Q of P , we denote by Tr_Q^P the relative trace map $A^Q \rightarrow A^P$ and by A_Q^P its image. We define $A(P)$ to be the Brauer quotient $k \otimes_{\mathcal{O}} (A^P / \sum_S A_S^P)$, where S runs over the set of proper subgroups of P , and denote by Br_P^A the Brauer homomorphism $A^P \rightarrow A(P)$. Note that $A(P) \neq 0$ forces P to be a p -group. When $A = \mathcal{O}_*\hat{K}$ and P is a p -subgroup of K , by [11, Proposition 2.2], Br_P^A induces an isomorphism $k_*C_{\hat{K}}(P) \cong A(P)$; in this case, we always identify $A(P)$ with $k_*C_{\hat{K}}(P)$ through this isomorphism.

2.5. In this paragraph, we generalize the definitions and notations in Introduction to twisted group algebras. Let L be a subgroup of K and e and g be block idempotents of $\mathcal{O}_*\hat{K}$ and $\mathcal{O}_*\hat{L}$ respectively. $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{L}g$ are said to be naturally Morita equiv-

alent of degree n for a positive integer number n if there exists a unitary \mathcal{O} -subalgebra S of $\mathcal{O}_* \hat{K} e$ such that S is a full matrix algebra over \mathcal{O} of degree n and the map

$$\mathcal{O}_* \hat{L} g \otimes_{\mathcal{O}} S \rightarrow \mathcal{O}_* \hat{K} e, \quad x \otimes y \mapsto xy$$

is an isomorphism of \mathcal{O} -algebras. Let M be an $\mathcal{O}_* \hat{K}$ -module and N an $\mathcal{O}_* \hat{L}$ -module. We denote by mM the direct sum of m copies of M for a positive integer number m , by $\text{Res}_{\hat{L}}^{\hat{K}}(M)$ the restriction of M from $\mathcal{O}_* \hat{K}$ to $\mathcal{O}_* \hat{L}$, and by $\text{Ind}_{\hat{L}}^{\hat{K}}(N)$ the induction of N from $\mathcal{O}_* \hat{L}$ to $\mathcal{O}_* \hat{K}$. Let i be an idempotent of $\mathcal{O}_* \hat{K}$. We denote by $i \cdot M$ the product of i and M . Note that if i commutes with a unitary subalgebra B of $\mathcal{O}_* \hat{K}$, then the $\mathcal{O}_* \hat{K}$ -module structure on M induces a B -module structure on $i \cdot M$. So $e \cdot M$ is an $\mathcal{O}_* \hat{K}$ -module structure and when $e \cdot M = M$, then we say that the $\mathcal{O}_* \hat{K}$ -module M is associated to the block e of $\mathcal{O}_* \hat{K}$. We denote by $\text{IBr}(e)$ the set of all non-isomorphic simple $\mathcal{O}_* \hat{K}$ -modules associated to e . All notations above except $\text{IBr}(e)$ can be slightly modified to apply to $\mathcal{K}_* \hat{G}$ -modules. We denote by $\text{Irr}(e)$ the set of all non-isomorphic simple $\mathcal{K}_* \hat{G}$ -modules associated to e .

2.6. Let K be a finite group, \hat{K} a k^* -group with the k^* -quotient K , L a normal p' -subgroup of K and f a K -stable block idempotent of $\mathcal{O}_* \hat{L}$. Then K acts on the full matrix algebra $\mathcal{O}_* \hat{L} f$ over \mathcal{O} and thus by the Skolem–Noether theorem, there exists a group homomorphism

$$\rho: K \rightarrow \text{Aut}(\mathcal{O}_* \hat{L} f) \cong (\mathcal{O}_* \hat{L} f)^* / \mathcal{O}^*.$$

We denote by \check{K} the set of all elements (c, x) such that $\rho(x)$ is the image of c in $(\mathcal{O}_* \hat{L} f)^* / \mathcal{O}^*$, where $c \in (\mathcal{O}_* \hat{L} f)^*$ and $x \in K$. Obviously \check{K} is an \mathcal{O}^* -group with the \mathcal{O}^* -quotient K with the homomorphism $\mathcal{O}^* \rightarrow \check{K}$, $\lambda \mapsto (\lambda, 1)$, the map $\hat{L} \rightarrow \check{K}$, $\hat{x} \mapsto (\hat{x}, x)$ is an injective group homomorphism and its image is normal in \check{K} ; in this sense, we identify \hat{L} with a normal subgroup of \check{K} .

2.7. Now we claim that there exists a subgroup \tilde{K} of \check{K} which is a k^* -group of k^* -quotient K and contains \hat{L} . Consider the quotient group \check{K} / \hat{L} . Obviously $\hat{L} \mathcal{O}^* / \hat{L}$ is a central subgroup of \check{K} / \hat{L} isomorphic to $1 + J(\mathcal{O})$ and $(\check{K} / \hat{L}) / (\hat{L} \mathcal{O}^* / \hat{L}) \cong K / L$, thus we can regard \check{K} / \hat{L} as a central extension of K / L by $1 + J(\mathcal{O})$. Let P be a Sylow p -subgroup of K . Since L is a p' -group, the image of P in K / L is isomorphic to P ; so we identify P with its image in K / L . Again since L is a p' -group, it is well known that $\mathcal{O}_* \hat{L} f$ is a full matrix algebra over \mathcal{O} and has the \mathcal{O} -rank prime to p , thus the action of P on $\mathcal{O}_* \hat{L} f$ can be lifted to a group homomorphism $P \rightarrow (\mathcal{O}_* \hat{L} f)^*$ (see [10, Paragraph 6.2]). This implies that there exists a group homomorphism $\theta: P \rightarrow \check{K} / \hat{L}$ such that for any $u \in P$, the image of $\theta(u)$ through the surjective homomorphism $\check{K} / \hat{L} \rightarrow K / L$ is u . Since $1 + J(\mathcal{O})$ is a p' -divisible group, the sur-

jective homomorphism $\check{K}/\hat{L} \rightarrow K/L$ splits and thus has a section $K/L \rightarrow \check{K}/\hat{L}$. Then the inverse image of the image of K/L in \check{K}/\hat{L} in \check{K} is just the desired k^* -group \check{K} .

2.8. Consequently we have a group homomorphism $\vartheta: \check{K} \rightarrow (\mathcal{O}_*\hat{L}f)^*$ and thus $\mathcal{O}_*\hat{L}f$ becomes a \check{K} -interior algebra. Consider the k^* -group $\check{K} = \hat{K} * \check{K}^\circ$. Obviously $\check{L} = \hat{L} * \hat{L}^\circ$ has a normal subgroup $\{\hat{x} \otimes \hat{x}^{-1} \mid x \in L\}$ isomorphic to L ; we still denote this group by L . We claim that L is normal in \check{K} . Indeed, for any $\hat{y} \otimes \tilde{y} \in \check{K}$ and $\hat{x} \otimes \hat{x}^{-1} \in L$, we have $(\hat{y} \otimes \tilde{y})(\hat{x} \otimes \hat{x}^{-1})(\hat{y} \otimes \tilde{y})^{-1} = (\hat{y} \otimes \tilde{y})(\hat{x} \otimes \hat{x}^{-1})(\hat{y}^{-1} \otimes \tilde{y}^{-1}) = \hat{y}\hat{x}\hat{y}^{-1} \otimes \tilde{y}\hat{x}^{-1}\tilde{y}^{-1} = \hat{x}^{\nu^{-1}} \otimes (\hat{x}^{\nu^{-1}})^{-1}$ since the \hat{K} - and \check{K} -conjugation induce the same action of K on \hat{L} . Set $\check{K} = \check{K}/L$. Then we obtain a k^* -group \check{K} with the k^* -quotient K/L . Through the surjective group homomorphism $\check{K} \rightarrow \check{K}$, we endow the twisted group algebra $\mathcal{O}_*\check{K}$ of \check{K} over \mathcal{O} with a \check{K} -interior algebra structure.

Theorem 2.9. *Keep the notations as in Paragraphs 2.6, 2.7 and 2.8. Then there exists an isomorphism of \hat{K} -interior algebras*

$$(2.9.1) \quad \mathcal{O}_*\hat{K}f \cong \mathcal{O}_*\hat{L}f \otimes_{\mathcal{O}} \mathcal{O}_*\check{K}.$$

In particular, the functors $U \mapsto i \cdot U$ and $V \mapsto \mathcal{O}_\hat{L}i \otimes_{\mathcal{O}} V$ are inverse isomorphisms between the categories of finitely generated $\mathcal{O}_*\hat{K}f$ - and $\mathcal{O}_*\check{K}$ -modules, where i is a primitive idempotent of $\mathcal{O}_*\hat{L}f$.*

The above theorem is also called the second Fong's reduction theorem.

Proof. Since $\mathcal{O}_*\hat{L}f$ is a full matrix algebra over \mathcal{O} , by [8, Proposition 2.1], the map

$$\mathcal{O}_*\hat{L}f \otimes_{\mathcal{O}} C_{\mathcal{O}_*\hat{K}f}(\mathcal{O}_*\hat{L}f) \cong \mathcal{O}_*\hat{K}f, \quad x \otimes y \mapsto xy$$

is an isomorphism of \mathcal{O} -algebras, where $C_{\mathcal{O}_*\hat{K}f}(\mathcal{O}_*\hat{L}f)$ is the centralizer of $\mathcal{O}_*\hat{L}f$ in $\mathcal{O}_*\hat{K}f$. Let R be a set of representatives of cosets of L in K and write $\mathcal{O}_*\hat{K}f$ as the direct sum $\bigoplus_{x \in R} (\mathcal{O}_*\hat{L}f)\hat{x}$. Since \hat{L} is normal in \hat{K} , it is easily computed that $C_{\mathcal{O}_*\hat{K}f}(\mathcal{O}_*\hat{L}f)$ is equal to the direct sum $\bigoplus_{x \in R} C_{(\mathcal{O}_*\hat{L}f)\hat{x}}(\mathcal{O}_*\hat{L}f)$. For any $x \in R$, since \hat{x} and $\vartheta(\tilde{x})$ have the same action on $\mathcal{O}_*\hat{L}f$ by conjugation, $\hat{x}\vartheta(\tilde{x}^{-1}) \in C_{(\mathcal{O}_*\hat{L}f)\hat{x}}(\mathcal{O}_*\hat{L}f)$; moreover by comparing the \mathcal{O} -ranks, it is not difficult to find $\mathcal{O}\hat{x}\vartheta(\tilde{x}^{-1}) = C_{(\mathcal{O}_*\hat{L}f)\hat{x}}(\mathcal{O}_*\hat{L}f)$ and thus $C_{\mathcal{O}_*\hat{K}f}(\mathcal{O}_*\hat{L}f) = \bigoplus_{x \in R} \mathcal{O}\hat{x}\vartheta(\tilde{x}^{-1})$. Finally it is easily checked that the map $\check{K} \rightarrow (C_{\mathcal{O}_*\hat{K}f}(\mathcal{O}_*\hat{L}f))^*$, $\hat{x} \otimes \tilde{x} \mapsto \hat{x}\vartheta(\tilde{x}^{-1})$ is a group homomorphism with the kernel L ; in particular, the group homomorphism induces an isomorphism $\mathcal{O}_*\check{K} \cong C_{\mathcal{O}_*\hat{K}f}(\mathcal{O}_*\hat{L}f)$. \square

2.10. Keep the notations in Theorem 2.9. Let N be a subgroup of K containing L , \bar{N} the quotient group of N in the quotient group $\bar{K} = K/L$, \hat{N} , \check{N} and \check{N} the

inverse images of N in \hat{K} , \check{K} and \check{K} respectively, and \check{N} the inverse image of \bar{N} in \check{K} . Consider $\mathcal{O}_*\hat{L}f$ as an \check{N} -interior algebra through the restriction of the structural homomorphism of the \check{K} -interior algebra $\mathcal{O}_*\hat{L}f$ to \check{N} and $\mathcal{O}_*\check{N}$ as an \check{N} -interior algebra through the homomorphism $\check{N} \rightarrow \check{N} \subset (\mathcal{O}_*\check{N})^*$. Then the isomorphism (2.9.1) induces an \hat{N} -interior algebra isomorphism

$$(2.10.1) \quad \mathcal{O}_*\hat{N}f \cong \mathcal{O}_*\hat{L}f \otimes_{\mathcal{O}} \mathcal{O}_*\check{N}.$$

In particular, the functors $X \mapsto i \cdot X$ and $Y \mapsto \mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} Y$ are inverse isomorphisms between the categories of finitely generated $\mathcal{O}_*\hat{N}f$ - and $\mathcal{O}_*\check{N}$ -modules. Let h be a block idempotent of $\mathcal{O}_*\hat{K}$ such that $hf \neq 0$, \bar{h} the corresponding block idempotent of $\mathcal{O}_*\check{K}$ determined by h through the isomorphism (2.9.1), l a block idempotent of $\mathcal{O}_*\hat{N}$ and \bar{l} the corresponding block idempotent of $\mathcal{O}_*\check{N}$ determined by l through the isomorphism (2.10.1). Then by the isomorphisms (2.9.1) and (2.10.1) and the definition of natural Morita equivalences of degree n , we can easily verify the following:

2.10.2. $\mathcal{O}_*\hat{K}h$ and $\mathcal{O}_*\hat{N}l$ are naturally Morita equivalent of degree n if and only if $\mathcal{O}_*\check{K}\bar{h}$ and $\mathcal{O}_*\check{N}\bar{l}$ are naturally Morita equivalent of degree n .

2.11. Finally we claim the following:

2.11.1. for any $\mathcal{O}_*\check{K}$ -module V , $\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} \text{Res}_{\check{N}}^{\check{K}}(V) \cong \text{Res}_{\check{N}}^{\hat{K}}(\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} V)$, and for any $\mathcal{O}_*\check{N}$ -module Y , $\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} \text{Ind}_{\check{N}}^{\check{K}}(Y) \cong \text{Ind}_{\check{N}}^{\hat{K}}(\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} Y)$.

The first isomorphism is obvious, so the rest is to prove the second equality. We consider $\mathcal{O}_*\check{K}$ as a subalgebra of $\mathcal{O}_*\hat{K}f$ through the isomorphism (2.9.1) and thus $\mathcal{O}_*\check{N}$ is also a subalgebra of $\mathcal{O}_*\hat{N}f$. We claim that the map

$$(2.11.2) \quad \mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} \text{Ind}_{\check{N}}^{\check{K}}(Y) \rightarrow \text{Ind}_{\check{N}}^{\hat{K}}(\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} Y)$$

sending $x \otimes (y \otimes z)$ to $y \otimes (x \otimes z)$ is an isomorphism of $\mathcal{O}_*\hat{K}$ -modules, where $x \in \mathcal{O}_*\hat{L}i$, $y \in \mathcal{O}_*\check{K}$ and $z \in Y$. Note that any element of $\text{Ind}_{\check{N}}^{\hat{K}}(\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} Y)$ can be written as a sum of elements like $y \otimes (x \otimes z)$, where $x \in \mathcal{O}_*\hat{L}i$, $y \in \mathcal{O}_*\check{K}$ and $z \in Y$; that implies that the homomorphism (2.11.2) is surjective. Then $\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} \text{Ind}_{\check{N}}^{\check{K}}(Y)$ and $\text{Ind}_{\check{N}}^{\hat{K}}(\mathcal{O}_*\hat{L}i \otimes_{\mathcal{O}} Y)$ having the same \mathcal{O} -rank forces (2.11.2) to be an isomorphism.

2.12. As consequences of Statement 2.11.1, we have the followings:

2.12.1. If S is a simple $\mathcal{O}_*\hat{K}h$ -module and $S_{\check{N}}$ is a simple $\mathcal{O}_*\hat{N}l$ -module such that

$$\text{Res}_{\check{N}}^{\hat{K}}(S) \cong nS_{\check{N}}$$

and $h \cdot \text{Ind}_{\hat{N}}^{\hat{K}}(S_{\hat{N}}) \cong nS$ for a positive integer number n , then $\text{Res}_{\hat{N}}^{\hat{K}}(i \cdot S) \cong n(i \cdot S_{\hat{N}})$ and $\bar{h} \cdot \text{Ind}_{\hat{N}}^{\check{K}}(i \cdot S_{\hat{N}}) \cong n(i \cdot S)$.

2.12.2. If W is a simple $\mathcal{K}_* \hat{K}h$ -module and $W_{\hat{N}}$ is a simple $\mathcal{K}_* \hat{N}l$ -module such that

$$\text{Res}_{\hat{N}}^{\hat{K}}(W) \cong nW_{\hat{N}}$$

for a positive integer number n , then $\text{Res}_{\hat{N}}^{\check{K}}(i \cdot W) \cong n(i \cdot W_{\hat{N}})$.

Lemma 2.13. *Keep notations as above. If $\mathcal{O}_* \hat{K}h$ covers $\mathcal{O}_* \hat{N}l$ and $\mathcal{O}_* \hat{K}h$ and $\mathcal{O}_* \hat{N}l$ have common defect groups, then $\mathcal{O}_* \check{K}\bar{h}$ covers $\mathcal{O}_* \check{N}\bar{l}$, $\mathcal{O}_* \check{K}\bar{h}$ and $\mathcal{O}_* \check{N}\bar{l}$ have common defect groups, and $n(h, l) = n(\bar{h}, \bar{l})$.*

Proof. By the choices of h and \bar{h} , the isomorphism (2.9.1) induces an isomorphism of \hat{K} -interior algebras $\mathcal{O}_* \hat{K}h \cong \mathcal{O}_* \hat{L}f \otimes_{\mathcal{O}} \mathcal{O}_* \check{K}\bar{h}$. Let P be a defect group of h . Then it follows from [12, Corollary 3.3] that the image of P in \check{K} , which is isomorphic to P and we still denote by P , is a defect group of \bar{h} , $\mathcal{O}_* \hat{L}f$ has a P -stable basis and $(\mathcal{O}_* \hat{L}f)(P) \neq 0$. So we can use [10, Proposition 5.6] to obtain the following $C_{\hat{K}}(P)$ -interior algebra isomorphism

$$(2.13.1) \quad k_* C_{\hat{K}}(P) \text{Br}_P^{\mathcal{O}_* \hat{K}}(h) \cong (\mathcal{O}_* \hat{L}f)(P) \otimes_k k_* C_{\check{K}}(P) \text{Br}_P^{\mathcal{O}_* \check{K}}(\bar{h}).$$

Fix a block idempotent h_P of $k_* C_{\hat{K}}(P)$ such that $\text{Br}_P^{\mathcal{O}_* \hat{K}}(h)h_P = h_P$. Since $(\mathcal{O}_* \hat{L}f)(P)$ is a full matrix algebra over k , there exists a block idempotent \bar{h}_P of $k_* C_{\check{K}}(P)$ such that $\text{Br}_P^{\mathcal{O}_* \check{K}}(\bar{h})\bar{h}_P = \bar{h}_P$ and the isomorphism (2.13.1) induces an isomorphism

$$(2.13.2) \quad k_* C_{\hat{K}}(P)h_P \cong (\mathcal{O}_* \hat{L}f)(P) \otimes_k k_* C_{\check{K}}(P)\bar{h}_P.$$

Since we are assuming that $\mathcal{O}_* \hat{K}h$ and $\mathcal{O}_* \hat{N}l$ have common defect groups, P is also a defect group of $\mathcal{O}_* \hat{N}l$. Then similarly, we can find block idempotents l_P and \bar{l}_P of $k_* C_{\hat{N}}(P)$ and $k_* C_{\check{N}}(P)$ respectively, such that $\text{Br}_P^{\mathcal{O}_* \hat{N}}(l)l_P = l_P$, $\text{Br}_P^{\mathcal{O}_* \check{N}}(\bar{l})\bar{l}_P = \bar{l}_P$ and there is an isomorphism

$$(2.13.3) \quad k_* C_{\hat{N}}(P)l_P \cong (\mathcal{O}_* \hat{L}f)(P) \otimes_k k_* C_{\check{N}}(P)\bar{l}_P.$$

Finally since we are also assuming that $\mathcal{O}_* \hat{K}h$ covers $\mathcal{O}_* \hat{N}l$, $\mathcal{O}_* \check{K}\bar{h}$ covers $\mathcal{O}_* \check{N}\bar{l}$ and

thus $n(h, l)$ and $n(\bar{h}, \bar{l})$ make sense; by isomorphisms (2.13.2) and (2.13.3), we can conclude that

$$\begin{aligned} n(h, l) &= \sqrt{\frac{\dim_k(k_*C_{\hat{K}}(P)h_P)}{\dim_k(k_*C_{\hat{N}}(P)l_P)}} \\ &= \sqrt{\frac{\dim_k(k_*C_{\hat{K}}(P)\bar{h}_P)}{\dim_k(k_*C_{\hat{N}}(P)\bar{l}_P)}} = n(\bar{h}, \bar{l}). \quad \square \end{aligned}$$

3. Proof of Theorem 1.5

Lemma 3.1. *Let K be a finite group and H a normal subgroup of K . Let \hat{K} be a k^* -group with the k^* -quotient K and e and f block idempotents of $\mathcal{O}_*\hat{K}$ and $\mathcal{O}_*\hat{H}$ respectively. If $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}f$ are naturally Morita equivalent of degree m , then for a common defect group P of e and f , there exists block idempotents e_P and f_P of $k_*C_{\hat{K}}(P)$ and $k_*C_{\hat{H}}(P)$ such that $\text{Br}_P^{\mathcal{O}_*\hat{K}}(e)e_P = e_P$, $\text{Br}_P^{\mathcal{O}_*\hat{H}}(f)f_P = f_P$ and $k_*C_{\hat{K}}(P)e_P$ and $k_*C_{\hat{H}}(P)f_P$ are naturally Morita equivalent of degree m too.*

Proof. Since $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}f$ are naturally Morita equivalent of degree m , by definitions, there exists a unitary subalgebra S of $\mathcal{O}_*\hat{K}e$, which is a full matrix algebra over \mathcal{O} of degree m , such that the product in $\mathcal{O}_*\hat{K}$ induces an isomorphism

$$\mathcal{O}_*\hat{K}e \cong S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{H}f.$$

This isomorphism implies that P acts trivially on S by conjugation and then by [10, Proposition 5.6], we obtain an isomorphism

$$k_*C_{\hat{K}}(P)\text{Br}_P^{\mathcal{O}_*\hat{K}}(e) \cong S(P) \otimes_k k_*C_{\hat{H}}(P)\text{Br}_P^{\mathcal{O}_*\hat{H}}(f).$$

Fix a block idempotent e_P of $k_*C_{\hat{K}}(P)$ such that $\text{Br}_P^{\mathcal{O}_*\hat{K}}(e)e_P = e_P$. Since $S(P) \cong k \otimes_{\mathcal{O}} S$, e_P determines a unique block idempotent f_P of $k_*C_{\hat{H}}(P)$ such that $\text{Br}_P^{\mathcal{O}_*\hat{H}}(f)f_P = f_P$ and $k_*C_{\hat{K}}(P)e_P \cong (k \otimes_{\mathcal{O}} S) \otimes_k k_*C_{\hat{H}}(P)f_P$. \square

3.2. Let H be a finite group and R a subgroup of H . We denote by $(\mathcal{O}H)^R$ the subalgebra of all R -fixed elements of $\mathcal{O}H$. Recall that a pointed group P_γ on $\mathcal{O}H$ is a pair (P, γ) consisting of a subgroup P of H and a $((\mathcal{O}H)^P)^*$ -conjugate class γ of primitive idempotents of $(\mathcal{O}H)^P$. Another pointed group R_ε is contained in P_γ if $R \leq P$ and there exists $j \in \varepsilon$ and $i \in \gamma$ such that $ji = ij = j$. P_γ is local if $\text{Br}_P^{\mathcal{O}H}(\gamma) \neq \{0\}$. Let c be a block idempotent of $\mathcal{O}H$. Then $\{c\}$ becomes a point of H on $\mathcal{O}H$. We say that P_γ is a defect pointed group of $\{c\}$ or simply c if P_γ is a maximal local pointed group contained in $H_{\{c\}}$ with respect inclusion. By [8, Theorem 1.2], H acts transitively on the set of all defect pointed groups of $H_{\{c\}}$. Fix $i \in \gamma$ and set $(\mathcal{O}H)_\gamma = i(\mathcal{O}H)i$. Then $(\mathcal{O}H)_\gamma$ is called a source algebra of $H_{\{c\}}$ or simply c .

3.3. Let P_γ be a defect pointed group of a block c of $\mathcal{O}H$ and denote by $N_H(P_\gamma)$ the stabilizer of P_γ in H and by $(\mathcal{O}H)(P_\gamma)$ the simple factor of $(\mathcal{O}H)^P$ such that the image of γ through the surjective homomorphism $(\mathcal{O}H)^P \rightarrow (\mathcal{O}H)(P_\gamma)$ is not zero. The obvious action of $N_H(P_\gamma)$ on $(\mathcal{O}H)^P$ induces an action of $N_H(P_\gamma)$ on $(\mathcal{O}H)(P_\gamma)$. By the Skolem–Noether theorem, we have a group homomorphism $\rho: N_H(P_\gamma) \rightarrow \text{Aut}((\mathcal{O}H)(P_\gamma)) \cong ((\mathcal{O}H)(P_\gamma))^*/k^*$. We denote by $\hat{N}_H(P_\gamma)$ the set of all elements (c, x) such that $\rho(x)$ is the image of c in $((\mathcal{O}H)(P_\gamma))^*/k^*$, where $c \in ((\mathcal{O}H)(P_\gamma))^*$ and $x \in N_H(P_\gamma)$. Then $\hat{N}_H(P_\gamma)$ is a k^* -group with the k^* -quotient $N_H(P_\gamma)$ with the homomorphism $k^* \rightarrow \hat{N}_H(P_\gamma)$, $\lambda \mapsto (\lambda, 1)$, and the map $PC_H(P) \rightarrow \hat{N}_H(P_\gamma)$, $x \mapsto (x, x)$ is an injective homomorphism, whose image is normal in $\hat{N}_H(P_\gamma)$ and intersects k^* trivially. We identify $PC_H(P)$ with a normal subgroup of $\hat{N}_H(P_\gamma)$ through the injective homomorphism and then the quotient $\hat{N}_H(P_\gamma)/PC_H(P)$ is a k^* -group with the k^* -quotient $N_H(P_\gamma)/PC_H(P)$. Let G be a finite group containing H as a normal subgroup and $C_G(P_\gamma)$ be the stabilizer of P_γ in $C_G(P)$. Then it is very obvious that the conjugation action of $C_G(P_\gamma)$ on H induces an action of $C_G(P_\gamma)$ on $N_H(P_\gamma)$ and actions of $C_G(P_\gamma)$ on $(\mathcal{O}H)(P_\gamma)$ and $((\mathcal{O}H)(P_\gamma))^*/k^*$ and that the homomorphism $\rho: N_H(P_\gamma) \rightarrow ((\mathcal{O}H)(P_\gamma))^*/k^*$ and the surjective homomorphism $((\mathcal{O}H)(P_\gamma))^* \rightarrow ((\mathcal{O}H)(P_\gamma))^*/k^*$ preserve the corresponding $C_G(P_\gamma)$ -actions. So $C_G(P_\gamma)$ acts on $\hat{N}_H(P_\gamma)/PC_H(P)$.

Lemma 3.4. *Let H be a finite group fulfilling that $C_H(O_p(H)) \subset O_p(H)$, P be a Sylow p -subgroup of H and \hat{H} be a k^* -group with the k^* -quotient H . Then the unit element 1 of $\mathcal{O}_*\hat{H}$ is the unique block idempotent of $\mathcal{O}_*\hat{H}$ and $P_{\{1\}}$ is a defect pointed group of $H_{\{1\}}$.*

Proof. Consider the Brauer homomorphism $\text{Br}_{O_p(H)}^{\mathcal{O}_*\hat{H}}: (\mathcal{O}_*\hat{H})^{O_p(H)} \rightarrow k_*C_{\hat{H}}(O_p(H))$. Since $C_H(O_p(H)) \subset O_p(H)$, $C_{\hat{H}}(O_p(H)) \cong k^* \times Z(O_p(H))$ and thus $k_*C_{\hat{H}}(O_p(H)) \cong kZ(O_p(H))$. On the other hand, since $O_p(H)$ is normal in H , $\text{Ker}(\text{Br}_{O_p(H)}^{\mathcal{O}_*\hat{H}}) \subset J(\mathcal{O}_*\hat{H}) \cap (\mathcal{O}_*\hat{H})^{O_p(H)} \subset J((\mathcal{O}_*\hat{H})^{O_p(H)})$. Thus $\{1\}$ is the unique local point of $O_p(H)$ on $\mathcal{O}_*\hat{H}$ and then the lemma follows. \square

Let G be a finite group, H a normal subgroup of G , \hat{G} a k^* -group of the k^* -group G and c a G -stable block idempotent of $\mathcal{O}_*\hat{H}$. We denote by $G[c]$ the group of all $g \in G$ such that there exists some $x_g \in (\mathcal{O}_*\hat{H}c)^*$ fulfilling $a^g = a^{x_g}$ for any $a \in \mathcal{O}_*\hat{H}c$. By [2, Proposition 2.7 and Theorem 3.5], $G[c]$ is normal in G and $b \in \mathcal{O}_*\widehat{G}[c]$.

Lemma 3.5. *Let G be a finite group, H a normal subgroup of G such that $C_H(O_p(H)) \leq O_p(H)$ and P a Sylow p -subgroup of H . Let \hat{G} be a k^* -group and assume that $\mathcal{O}_*\hat{G}$ has a block with P as a defect group. Then $G[1] = C_G(P)H$.*

Here 1 is the block idempotent of $\mathcal{O}_*\hat{H}$ (see Lemma 3.4).

Proof. We firstly prove $C_G(P)H \subset G[1]$. By [9, Lemma 5.5], there exists a finite subgroup G' of \hat{G} such that $\hat{G} = k^*G'$; moreover if we let Z' be the intersection of k^* and G' , H' the intersection of G' and \hat{H} and ι the central idempotent $1/|Z'| \sum_{z \in Z'} \lambda_z z^{-1}$ of $\mathcal{O}G'$, by [9, Theorem 5.15], the inclusion $G' \subset \hat{G}$ induces an isomorphism of \mathcal{O} -algebras

$$(3.5.1) \quad \mathcal{O}G'\iota \cong \mathcal{O}_* \hat{G},$$

whose restriction to H' induces an isomorphism

$$(3.5.2) \quad \mathcal{O}H'\iota \cong \mathcal{O}_* \hat{H}.$$

Since $C_H(\mathcal{O}_P(H)) \subset \mathcal{O}_P(H)$, by Lemma 3.4, $c = 1$ is the unique block idempotent of $\mathcal{O}_* \hat{H}$ and $\gamma = \{1\}$ is the unique local point of P on $\mathcal{O}_* \hat{H}$, thus ι is a block idempotent of $\mathcal{O}H'$, $\gamma' = \{\iota\}$ is the unique local point of P on $\mathcal{O}H'\iota$ and the P -interior algebra $\mathcal{O}H'\iota$ with the homomorphism $P \rightarrow (\mathcal{O}H'\iota)^*$, $u \mapsto u\iota$ is a source algebra of ι . For any $x \in C_{G'}(P)$, we consider the automorphism φ_x on the source algebra $\mathcal{O}H'\iota$ induced by x . Clearly $C_G(P)$ stabilizes P_γ , thus $C_{G'}(P)$ stabilizes $P_{\gamma'}$ and then $C_{G'}(P)$ acts on the k^* -group $\hat{N}_{H'}(P_{\gamma'})/PC_{H'}(P)$ (refer to Paragraph 3.3). But it follows from $C_H(\mathcal{O}_P(H)) \subset \mathcal{O}_P(H)$ that $(\mathcal{O}_* \hat{H})(P_\gamma) \cong k$, $(\mathcal{O}H')(P_{\gamma'}) \cong k$ and thus $\hat{N}_{H'}(P_{\gamma'})/PC_{H'}(P) \cong k^* \times N_{H'}(P_{\gamma'})/PC_{H'}(P)$; on the other hand, $C_{G'}(P)$ acts trivially on the group $N_{H'}(P_{\gamma'})/PC_{H'}(P)$. Consequently $C_{G'}(P)$ acts trivially on the k^* -group $\hat{N}_{H'}(P_{\gamma'})/PC_{H'}(P)$. Therefore by [9, Proposition 14.9], φ_x is induced by some element $a' \in (\mathcal{O}H'\iota)^*$; in particular, this shows that the automorphism on $\mathcal{O}_* \hat{H}$ induced by $x \in C_G(P)$ is induced by some $a \in (\mathcal{O}_* \hat{H})^*$. Thus $x \in G[1]$.

In order to prove $G[1] = C_G(P)H$, now we assume $G = G[1]$ without loss of generality. Set $K = C_G(P)H$ and let b be a block idempotent of $\mathcal{O}_* \hat{G}$ with P as a defect group and e be a block idempotent of $\mathcal{O}_* \hat{K}$ such that $be \neq 0$. Obviously e also covers the unique block 1 of $\mathcal{O}_* \hat{H}$ and thus P is also a defect group of e . By [6, Theorem 7], $\mathcal{O}_* \hat{G}b$ and $\mathcal{O}_* \hat{H}$ are naturally Morita equivalent of degree n for a positive integer and $\mathcal{O}_* \hat{K}e$ and $\mathcal{O}_* \hat{H}$ are naturally Morita equivalent of degree m for a positive integer. We claim that n is equal to m . Indeed, since $be \neq 0$ and $G \supset HC_G(P)$, $\mathcal{O}_* \hat{G}b$ and $\mathcal{O}_* \hat{K}e$ at least have a common block idempotent f of $k_* C_{\hat{G}}(P)$ such that $\text{Br}_P^{\mathcal{O}_* \hat{G}}(b)f \neq f$ and $\text{Br}_P^{\mathcal{O}_* \hat{K}}(e)f \neq f$. Then by Lemma 3.1, n is equal to m ; in particular, this shows that $\mathcal{O}_* \hat{K}e$ and $\mathcal{O}_* \hat{G}b$ have the same \mathcal{O} -rank. Since P is a Sylow p -subgroup of H , by Frattini argument, we have $G = N_G(P)H$. Thus K is normal in $G[1]$. Then by [6, Theorem 1], $k \otimes_{\mathcal{O}} \mathcal{O}_* \hat{K}e$ and $k \otimes_{\mathcal{O}} \mathcal{O}_* \hat{G}b$ are isomorphic. Finally by [5, Corollary 4.5], $G[1] = C_{G[1]}(P)K = C_G(P)H$. \square

Theorem 3.6. *Let G be a finite group and H a normal subgroup of G such that $C_H(R) \subset \mathcal{O}_{p',p}(H)$ for a Sylow p -subgroup R of $\mathcal{O}_{p',p}(H)$. Let \hat{G} be a k^* -group with*

the k^* -quotient G , b and c block idempotents of $\mathcal{O}_*\hat{G}$ and $\mathcal{O}_*\hat{H}$ respectively, and n a positive integer. If c is also a block idempotent of $\mathcal{O}_*\widehat{O_p(H)}$, then the following two conditions are equivalent:

3.6.1. $\mathcal{O}_*\hat{G}b$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree n ;

3.6.2. for any simple $\mathcal{O}_*\hat{G}$ -module S associated to b , there exists a unique simple $\mathcal{O}_*\hat{H}$ -module $S_{\hat{H}}$ associated to c such that $\text{Res}_{\hat{H}}^{\hat{G}}(S) \cong nS_{\hat{H}}$ and $b \cdot \text{Ind}_{\hat{H}}^{\hat{G}}(S_{\hat{H}}) \cong nS$, the correspondence $\text{IBr}(b) \rightarrow \text{IBr}(c)$, $S \mapsto S_{\hat{H}}$ is a bijection, and $n \leq n(b, c)$.

Moreover in this case, $n = n(b, c)$.

Proof. By [5, Proposition 2.6] and Lemma 3.1, Condition 3.6.1 implies Condition 3.6.2. Now we assume that Condition 3.6.2 holds. By the isomorphism (2.9.1) applied to $\mathcal{O}_*\hat{G}c$ and $\mathcal{O}_*\widehat{O_p(H)}c$, we can find a k^* -group \check{G} with the k^* -quotient $\check{G} = G/O_p(H)$ such that there exists an isomorphism of \hat{G} -interior algebras

$$(3.6.3) \quad \mathcal{O}_*\hat{G}c \cong \mathcal{O}_*\widehat{O_p(H)}c \otimes_{\mathcal{O}} \mathcal{O}_*\check{G}$$

which, by restriction to $\mathcal{O}_*\hat{H}c$, induces an isomorphism of \hat{H} -interior algebras

$$\mathcal{O}_*\hat{H}c \cong \mathcal{O}_*\widehat{O_p(H)}c \otimes_{\mathcal{O}} \mathcal{O}_*\check{H}$$

where \check{H} is the inverse image of $\bar{H} = H/O_p(H)$ in \check{G} .

Since $\mathcal{O}_*\widehat{O_p(H)}c$ is a full matrix algebra over \mathcal{O} and $bc = b$, b determines a unique block idempotent \bar{b} of $\mathcal{O}_*\check{G}$ through (3.6.3) such that

$$(3.6.4) \quad \mathcal{O}_*\hat{G}b \cong \mathcal{O}_*\widehat{O_p(H)}c \otimes_{\mathcal{O}} \mathcal{O}_*\check{G}\bar{b}.$$

But notice that 1 is the unique block idempotent of $\mathcal{O}_*\check{H}$ since we are assuming $C_H(R) \subset O_{p',p}(H)$ for a Sylow p -subgroup R of H and thus $C_{\bar{H}}(O_p(\bar{H})) \subset O_p(\bar{H})$ (see Lemma 3.4). Let i be a primitive idempotent of $\mathcal{O}_*\widehat{O_p(H)}c$. Since we are also assuming that there exists a unique simple $\mathcal{O}_*\hat{H}$ -module S_H associated to c such that $\text{Res}_{\hat{H}}^{\hat{G}}(S) \cong nS_{\hat{H}}$ and $b \cdot \text{Ind}_{\hat{H}}^{\hat{G}}(S_{\hat{H}}) \cong nS$ for any simple $\mathcal{O}_*\hat{G}$ -module S associated to b and that the correspondence $\text{IBr}(b) \rightarrow \text{IBr}(c)$, $S \mapsto S_H$ is a bijection, it follows from Statement 2.12.1 that we have equalities $\text{Res}_{\check{G}}^{\check{H}}(i \cdot S) \cong n(i \cdot S_{\hat{H}})$ and $\bar{b} \cdot \text{Ind}_{\check{H}}^{\check{G}}(i \cdot S_{\hat{H}}) \cong n(i \cdot S)$ and from Theorem 2.9 that the map the correspondence $\text{IBr}(\bar{b}) \rightarrow \text{IBr}(1)$, $i \cdot S \mapsto i \cdot (S_H)$ is a bijection; here in order to avoid confusion, we remind that $\text{IBr}(1)$ is the set of all simple $\mathcal{O}_*\check{H}$ -modules. Finally by our hypothesis, b and c have common defect groups (refer to [7, Chapter 4, Lemma 3.4] and [4, Chapter IV, Lemma 4.6]), so $n(b, c)$ makes sense and so does $n(\bar{b}, 1)$; by Lemma 2.13, we have $n(b, c) = n(\bar{b}, 1)$.

If we can prove that $\mathcal{O}_*\check{G}\bar{b}$ and $\mathcal{O}_*\check{H}$ are naturally Morita equivalent of degree n , by Lemma 2.10.2, so are $\mathcal{O}_*\hat{G}b$ and $\mathcal{O}_*\hat{H}c$. So in order to prove the theorem,

we can assume $C_H(O_p(H)) \subset O_p(H)$. Let P be a common defect group of b and c . Since H is normal in G and H and G act transitively on the sets of defect groups of c and b , by Frattini argument, we have $G = N_G(P)H$. Now consider the obvious normal subgroup $K = C_G(P)H$ of G and let e be a block idempotent of $\mathcal{O}_*\hat{K}$ such that $be \neq 0 \neq ce$. Then P has to be a defect group of e . By Lemma 3.5 and [6, Theorem 7], $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree m ; moreover by Lemma 3.1 and the definition of $n(b, c)$, $m = n(b, c) \geq n$. \square

Let S be a simple $\mathcal{O}_*\hat{G}b$ -module. Since $be \neq 0 \neq ce$ and $\text{Res}_{\hat{H}}^{\hat{G}}(S) = nS_{\hat{H}}$, by Clifford theorem, there exists a simple $\mathcal{O}_*\hat{K}e$ -module $S_{\hat{K}}$ such that $S_{\hat{K}}$ is a direct summand of $\text{Res}_{\hat{K}}^{\hat{G}}(S)$ and $S_{\hat{H}}$ is a direct summand of $\text{Res}_{\hat{H}}^{\hat{K}}(S_{\hat{K}})$. Since $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree m , by [5, Proposition 2.6], $\text{Res}_{\hat{H}}^{\hat{K}}(S_{\hat{K}}) = mS_{\hat{H}}$. Then the inequality $m \geq n$ shows that $\dim_k(S_{\hat{K}}) \geq \dim_k(S)$, thus $\text{Res}_{\hat{K}}^{\hat{G}}(S) = S_{\hat{K}}$ and $m = n$; in particular, this also implies that G stabilizes e and thus $be = b$. By Lemma 3.5 and [6, Corollary 4], $b \in \mathcal{O}_*\hat{K}$ and thus $be = e$. Therefore $b = e$. That $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree n also implies $b \cdot \text{Ind}_{\hat{H}}^{\hat{K}}(S_{\hat{H}}) = nS_{\hat{K}}$ (refer to [5, Proposition 2.6]). We rewrite $b \cdot \text{Ind}_{\hat{H}}^{\hat{G}}(S_{\hat{H}})$ as $\text{Ind}_{\hat{K}}^{\hat{G}}(b \cdot \text{Ind}_{\hat{H}}^{\hat{K}}(S_{\hat{H}})) = \text{Ind}_{\hat{K}}^{\hat{G}}(nS_{\hat{K}}) = n \text{Ind}_{\hat{K}}^{\hat{G}}(S_{\hat{K}})$. Then the equality $n \text{Ind}_{\hat{K}}^{\hat{G}}(S_{\hat{K}}) = nS$ forces $S = \text{Ind}_{\hat{K}}^{\hat{G}}(S_{\hat{K}})$. But we also have $\text{Res}_{\hat{K}}^{\hat{G}}(S) = S_{\hat{K}}$ and therefore G has to be equal to K .

Theorem 3.7. *Let G be a finite group and H a normal subgroup of G such that $C_H(R) \subset O_{p',p}(H)$ for a Sylow p -subgroup R of $O_{p',p}(H)$. Let \hat{G} be a k^* -group with the k^* -quotient G , b and c block idempotents of $\mathcal{O}_*\hat{G}$ and $\mathcal{O}_*\hat{H}$ respectively, and n be a positive integer. If c is also a block idempotent of $\mathcal{O}_*\widehat{O_p(H)}$, then the following two conditions are equivalent:*

3.7.1. $\mathcal{O}_*\hat{G}b$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree n ;

3.7.2. $v_p(|G : H|) = v_p(n)$, for any simple $\mathcal{K}_*\hat{G}$ -module V associated to b , there exists a unique simple $\mathcal{K}_*\hat{H}$ -module $V_{\hat{H}}$ associated to c such that $\text{Res}_{\hat{H}}^{\hat{G}}(V) \cong nV_{\hat{H}}$, the correspondence $\text{Irr}(b) \rightarrow \text{Irr}(c)$, $V \mapsto V_{\hat{H}}$ is a bijection, and $n(b, c) \geq n$.

Moreover in this case, $n = n(b, c)$.

Proof. By [5, Proposition 2.6] and Lemma 3.1, Condition 3.7.1 implies Condition 3.7.2. Now assume that Condition 3.7.2 holds. Note that the first three statements imply that b and c have common defect groups (refer to [4, Chapter IV, Theorem 4.5]). Then by the first and second paragraph in Theorem 3.6, in order to prove 3.7.1, we can assume $C_H(O_p(H)) \subset O_p(H)$ without loss of generality. Let P be a common defect group of b and c . Since H is normal in G and H and G act transitively on the sets of defect groups of c and b , by Frattini argument, we have $G = N_G(P)H$. Now

consider the obvious normal subgroup $K = C_G(P)H$ of G and let e be a block idempotent of $\mathcal{O}_*\hat{K}$ such that $be \neq 0 \neq ce$. Then P has to be a defect group of e . By Lemma 3.5 and [6, Theorem 7], $\mathcal{O}_*\hat{K}e$ and $\mathcal{O}_*\hat{H}c$ are naturally Morita equivalent of degree m and by Lemma 3.1 and the definition of $n(b, c)$, $m = n(b, c) \geq n$. \square

Let V be a simple $\mathcal{K}_*\hat{G}b$ -module. Since $be \neq 0 \neq ce$ and $\text{Res}_{\hat{H}}^{\hat{G}}(V) = nV_{\hat{H}}$, by Clifford theorem, there exists a simple $\mathcal{K}_*\hat{K}e$ -module $V_{\hat{K}}$ such that $V_{\hat{K}}$ is a direct summand of $\text{Res}_{\hat{K}}^{\hat{G}}(V)$ and $V_{\hat{H}}$ is a direct summand of $\text{Res}_{\hat{H}}^{\hat{K}}(V_{\hat{K}})$. Since $\mathcal{K}_*\hat{K}e$ and $\mathcal{K}_*\hat{H}c$ are naturally Morita equivalent of degree m , by [5, Proposition 2.6], $\text{Res}_{\hat{H}}^{\hat{K}}(V_{\hat{K}}) = mV_{\hat{H}}$. Then the inequality $m \geq n$ shows that $\dim_{\mathcal{K}}(V_{\hat{K}}) \geq \dim_{\mathcal{K}}(V)$ and then $\dim_{\mathcal{K}}(V_{\hat{K}}) = \dim_{\mathcal{K}}(V)$, thus $\text{Res}_{\hat{K}}^{\hat{G}}(V) = V_{\hat{K}}$ and $m = n$; in particular, this also implies that G stabilizes e and thus $be = b$. By Lemma 3.5 and [6, Corollary 4], $b \in \mathcal{O}_*\hat{K}$ and thus $be = e = b$. Moreover it is easily checked that the map $V \rightarrow V_{\hat{K}}$ is a bijection between the sets of all simple $\mathcal{K}_*\hat{G}b$ - and $\mathcal{K}_*\hat{K}e$ -modules; in particular, this implies that $\mathcal{O}_*\hat{G}b$ and $\mathcal{O}_*\hat{K}e$ have the same \mathcal{O} -rank. But obviously the \mathcal{O} -rank of $\mathcal{O}_*\hat{G}b$ is equal to the product of $|G : K|$ with the \mathcal{O} -rank of $\mathcal{O}_*\hat{K}e$ too. So G is forced to equal to K . We are done.

3.8. Proof of Theorem 1.5. It suffices for us to take \hat{G} and \hat{H} to be $G \times k^*$ and $H \times k^*$ and then Theorems 3.6 and 3.7 imply Theorem 1.5. \square

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