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NATURAL MORITA EQUIVALENCES OF DEGREE $n$

YUN FAN, QINQIN YANG and YUANYANG ZHOU

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Abstract

Let $G$ be a finite group, $H$ a normal subgroup of $G$ and $b$ and $c$ block idempotents of $OG$ and $OH$ respectively. Under the assumption that $C_H(R) \subset O_{p', p}(H)$ for a Sylow $p$-subgroup $R$ of $O_{p', p}(H)$ and $c$ is also a block idempotent of $OOG$ and $OOG$, we give two equivalent conditions about when $OGb$ and $OHC$ are natural Morita equivalent of degree $n$ (see Theorem 1.5).

1. Introduction

1.1. Fix a prime number $p$. Let $O$ be a complete discrete valuation ring with a residue field $k$ of characteristic $p$. Let $G$ be a finite group, $H$ a subgroup of $G$ and $b$ and $c$ block idempotents of $OG$ and $OH$ respectively. In terms of the terminology of A. Hida and S. Koshitani [5], $OGb$ and $OHC$ are said to be naturally Morita equivalent of degree $n$ for a positive integer number $n$ if there exists an unitary $O$-subalgebra $S$ of $OGb$ such that $S$ is a full matrix algebra over $O$ of degree $n$ and the map

$$OHC \otimes_O S \rightarrow OGb, \quad x \otimes y \mapsto xy$$

is an isomorphism of $O$-algebras. When $H$ is normal in $G$ and $O = k$, this definition is firstly due to B. Külshammer [6].

1.2. For our question below, now we make the additional assumption that the characteristic of $O$ is zero, the quotient field $K$ of $O$ is big enough for all algebras involved below, the residue field $k$ is algebraically closed and $H$ is normal in $G$; the assumption will also be kept throughout this paper. As a consequence of [13, Theorems 2 and 3], we can easily conclude that the following three conditions are equivalent:

1.2.1. the map $OGb \rightarrow OHC$, $x \mapsto xc$ is an $O$-algebra isomorphism;
1.2.2. the restriction from $G$ to $H$ induces a bijection between the sets of all non-isomorphic simple modules of $OGb$ and $OHC$ and the quotient group $G/H$ is a $p'$-group;
1.2.3. the restriction from $G$ to $H$ induces a bijection between the sets of all non-isomorphic simple modules of $KGb$ and $KHc$.

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Noticing that Condition 1.2.1 is actually saying that $O\Gamma b$ and $O\Omega c$ is naturally Morita equivalent of degree 1, we ask ourselves a question: can this statement above be generalized to natural Morita equivalences of degree $n$? In this paper, we investigate the question.

1.3. Now we begin with some preparations in order to state our main theorem. Let $M$ be an $O\Gamma G$-module and $N$ an $O\Omega K$-module. We denote by $\text{Res}_k^G(M)$ the restriction of $M$ from $G$ to $K$ and by $\text{Ind}_k^G(N)$ the induction of $N$ from $K$ to $G$. Given a positive integer number $n$, we denote by $nM$ the direct sum of $n$ copies of $M$. Obviously the product $b \cdot M$ of $b$ and $M$ is an $O\Gamma G$-submodule of $M$ and $b$ acts on $b \cdot M$ as the identity homomorphism. When $b \cdot M = M$, then we say that the $O\Gamma G$-module $M$ is associated to the block $b$ of $O\Gamma G$. We denote by $\text{IBr}(b)$ the set of all non-isomorphic simple $O\Gamma G$-modules associated to $b$. All notations above except $\text{IBr}(b)$ can be slightly modified to apply to $K\Omega G$-modules. In general, we denote by $\text{Irr}(b)$ the set of all non-isomorphic simple $K\Omega G$-modules associated to $b$. Given a positive integer number $m$, $v_p(m)$ denotes the largest non-negative integer number $t$ such that $p^t | m$.

1.4. Assume that $bc \neq 0$ and $b$ and $c$ have a common defect group $P$. Since $bc \neq 0$, it is well known (refer to [3]) that there exist block idempotents $b_P$ and $c_P$ of $kC_G(P)$ and $kC_H(P)$ such that $b_P \text{ Br}^G_P(b) = b_P$, $c_P \text{ Br}^H_P(c) = c_P$ and $b_P c_P \neq 0$. Since $P$ is a defect group of $b$ and $c$, $b_P$ and $c_P$ have defect group $Z(P)$, thus $kC_G(P) b_P$ and $kC_H(P) c_P$ are nilpotent (refer to [10]) and have only one simple module, say $V_{b_P}$ and $V_{c_P}$. Since $H$ is normal in $G$, so is $C_H(P)$ in $C_G(P)$; then by Clifford theory, we can conclude that the dimension $\dim_k(V_P)$ of $V_P$ over $k$ divides the dimension $\dim_k(V_{b_P})$ of $V_{b_P}$ over $k$. Note that $(P, b_P)$ and $(P, c_P)$ actually are maximal Brauer pairs of $b$ and $c$, which are unique up to $G$- and $H$-conjugation (refer to [1]). Therefore the quotient $\dim_k(V_{b_P})/\dim_k(V_{c_P})$ is independent of the choices of $b_P$ and $c_P$. We denote this quotient by $n(b, c)$. Note that by [10, 1.4.1], $n(b, c) = \sqrt{\dim_k(kC_G(P) b_P)/\dim_k(kC_H(P) c_P)}$; even in order to compute $n(b, c)$, it suffices for us to choose $b_P$ and $c_P$ of $kC_G(P)$ and $kC_H(P)$ such that $b_P \text{ Br}^G_P(b) = b_P$ and $c_P \text{ Br}^H_P(c) = c_P$.

Theorem 1.5. Let $G$ be a finite group and $H$ be a normal subgroup of $G$ such that $C_H(R) \subset O_{P'}(H)$ for a Sylow $p'$-subgroup $R$ of $O_{P'}(H)$. Let $b$ and $c$ be respective block idempotents of $O\Gamma G$ and $O\Omega H$ and let $n$ be a positive integer. If $c$ is also a block idempotent of $O\Omega O_{P'}(H)$, then the following conditions are equivalent:

1.5.1. $O\Gamma b$ and $O\Omega c$ are naturally Morita equivalent of degree $n$;

1.5.2. for any simple $O\Gamma G$-module $S$ associated to $b$, there exists a unique simple $O\Omega H$-module $S_H$ associated to $c$ such that $\text{Res}_H^G(S) \cong nS_H$ and $b \cdot \text{Ind}_H^G(S_H) \cong nS$, the correspondence $\text{IBr}(b) \rightarrow \text{IBr}(c)$, $S \mapsto S_H$ is a bijection, and $n \leq n(b, c)$.

1.5.3. $v_p([G : H]) = v_p(n)$, for any simple $K\Omega G$-module $V$ associated to $b$, there exists a unique simple $K\Omega H$-module $V_H$ associated to $c$ such that $\text{Res}_H^G(V) \cong nV_H$, and the
correspondence \( \text{Irr}(b) \mapsto \text{Irr}(c) \), \( V \mapsto V_H \) is a bijection, and \( n \leq n(b, c) \).

Moreover in this case, \( n \) is equal to \( n(b, c) \).

**Remark 1.6.**

1. Conditions 1.5.2 and 1.5.3 both imply that \( b \) and \( c \) have the same defect groups, so \( n(b, c) \) makes sense. For details, refer to the proofs of Theorems 3.6 and 3.7.

2. When \( n = 1 \), by [4, Chapter IV, Theorem 4.5], it is easily checked that Conditions 1.5.2 and 1.5.3 both imply that the quotient group \( G/H \) is a \( p' \)-group; in addition \( n \leq n(b, c) \) automatically holds. Therefore the theorem above covers the equivalences between Conditions 1.2.1, 1.2.2 and 1.2.3.

3. There are examples to explain why the condition \( n \leq n(b, c) \) is necessary.

2. **Fong’s reduction**

In this section, an \( \mathcal{O} \)-algebra \( A \) that is involved is always associative, unitary and \( \mathcal{O} \)-free of finite rank as an \( \mathcal{O} \)-module; \( A^* \) and \( J(A) \) denote the multiplicative group of all invertible elements of \( A \) and the Jacobson radical of \( A \) respectively. Occasionally, in order to avoid confusion, we denote by \( 1_A \) of the identity element of \( A \). A homomorphism \( f : A \to B \) between \( \mathcal{O} \)-algebras is an embedding if \( f \) is injective and \( f(A) = f(1_A)Bf(1_A) \).

2.1. Let \( K \) be a finite group and \( \hat{K} \) be a \( k^* \)-group with the \( k^* \)-quotient \( K \) endowed with the homomorphism \( \rho : k^* \to \hat{K} \). By \( \hat{K} \), we can construct two \( k^* \)-groups: the group \( \hat{K} \) endowed with the group homomorphism \( k^* \to \hat{K} \) sending \( \lambda \) onto \( \rho(\lambda^{-1}) \) and the opposite group \( (\hat{K})^* \) with the group homomorphism \( \rho \); in order to differ from the \( k^* \)-group \( \hat{K} \), we denote the first \( k^* \)-group by \( \hat{K}^* \). But the two \( k^* \)-groups are isomorphic: there is an isomorphism of \( k^* \)-groups \( (\hat{K})^* \to \hat{K}^* \), \( x \mapsto x^{-1} \) (refer to [9]). For any subgroup \( L \) of \( K \), we denote by \( \hat{L} \) its inverse image in \( \hat{K} \) and for any element \( x \in L \), by \( \hat{x} \) a lifting in \( \hat{K} \) of \( x \). When \( L \) is a \( p \)-group, \( \hat{L} \) can be uniquely decomposed as the direct product \( k^* \times L \) (refer to [9, Lemma 5.5]) and thus we always regard \( L \) as a subgroup of \( \hat{K} \). Let \( \hat{K} \) be another \( k^* \)-group with the \( k^* \)-quotient \( K \). Then the central product of \( \hat{K} \) and \( \hat{K} \) over \( k^* \) defines a \( k^* \)-group \( \hat{K} \otimes \hat{K} \) with the \( k^* \)-quotient isomorphic to \( K \times K \) and we identify this \( k^* \)-quotient with \( K \times K \). We also identify \( K \) with the diagonal subgroup in \( K \times K \) and denote by \( \hat{K} \ast \hat{K} \) the inverse image in \( \hat{K} \otimes \hat{K} \) of \( K \). Then \( \hat{K} \ast \hat{K} \) is a new \( k^* \)-group with the \( k^* \)-quotient \( K \).

2.2. Obviously the surjective homomorphism \( \mathcal{O} \to k \) induces a surjective group homomorphism \( \mathcal{O}^* \to k^* \); since \( k \) is algebraically closed, \( k \) is perfect and thus by [14, Chapter II, Proposition 8], there exists a unique section \( k^* \to \mathcal{O}^* \) of this group homomorphism. Through this section, we can regard \( \mathcal{O} \) as a right module over the group algebra of \( k^* \) over \( \mathcal{O} \). Let \( K \) be a finite group and \( \hat{K} \) be a \( k^* \)-group with the \( k^* \)-quotient \( K \). Obviously the inclusion \( k^* \subset \hat{K} \) induces a left \( \mathcal{O}k^* \)-module structure on the group.
algebra $\mathcal{O}\hat{K}$ of $\hat{K}$ over $\mathcal{O}$. Now we consider the tensor product $\mathcal{O} \otimes_{\mathcal{O} \hat{K}} \mathcal{O}\hat{K}$ and define a distributive product on $\mathcal{O} \otimes_{\mathcal{O} \hat{K}} \mathcal{O}\hat{K}$ by the equality

$$(a \otimes x)(b \otimes y) = ab \otimes xy$$

for $a, b \in \mathcal{O}$ and $x, y \in \mathcal{O}\hat{K}$. Then the tensor product $\mathcal{O} \otimes_{\mathcal{O} \hat{K}} \mathcal{O}\hat{K}$ with the above product becomes an $\mathcal{O}$-algebra; we call it the twisted group algebra of $\hat{K}$ over $\mathcal{O}$ and denote it by $\mathcal{O}_\ast \hat{K}$. Obviously the $k^\ast$-group isomorphism $(\hat{K})^\ast \cong \hat{K}^\ast$, $x \mapsto x^{-1}$ induces an isomorphism of $\mathcal{O}$-algebras from the opposite ring $(\mathcal{O}_\ast \hat{K})^\ast$ to $\mathcal{O}_\ast \hat{K}^\ast$; moreover since the map $\mathcal{O}_\ast (\hat{K} \otimes \hat{K}^\ast) \to \mathcal{O}_\ast \hat{K} \otimes \mathcal{O}_\ast \hat{K}^\ast$ sending $1 \otimes (x \otimes y)$ to $(1 \otimes x) \otimes (1 \otimes y)$ for $x \otimes y \in \hat{K} \otimes \hat{K}^\ast$ is an isomorphism, we can define a left $\mathcal{O}_\ast (\hat{K} \otimes \hat{K}^\ast)$-module structure on $\mathcal{O}_\ast \hat{K}$ by the equality $(x \otimes y)a = xay^{-1}$ for $x, y \in \hat{K}$ and $a \in \mathcal{O}_\ast \hat{K}$. The tensor product $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}_\ast \hat{K}$ is also what we are concerned below and we denote it by $\mathcal{K}_\ast \hat{K}$.

2.3. Recall that an $\mathcal{O}$-algebra $A$ is called a $\hat{K}$-interior algebra (see [9, 5.10]) if there exists a group homomorphism $\varphi: \hat{K} \to A^\ast$. For any $a \in A$ and liftings $\hat{x}, \hat{y}$ in $\hat{K}$ of $x, y \in K$, we will write $\varphi(\hat{x})a \varphi(\hat{y})$ as $\hat{x}a\hat{y}$ for convenience. Obviously when $\hat{y} = \hat{x}^{-1}$, the product $\hat{x}a\hat{x}^{-1}$ is independent of the choice of $\hat{x}$ in $\hat{K}$ and therefore we also often write it as $a^{x^{-1}}$. Moreover the map $\varphi_t: A \cong A$, $a \mapsto a^{x^{-1}}$ is an automorphism, the map $K \to \text{Aut}(A)$, $x \mapsto \varphi_t$ is a group homomorphism, thus $A$ is a $K$-algebra. Let $C$ be another $\hat{K}$-interior algebra; an $\mathcal{O}$-algebra homomorphism $f: A \to C$ is called a homomorphism of $\hat{K}$-interior algebras if $f(\hat{x}a\hat{y}) = \hat{x}f(a)\hat{y}$ for any $a \in A$ and liftings $\hat{x}, \hat{y}$ in $\hat{K}$ of $x, y \in K$. Let $\hat{K}$ be another $k^\ast$-group with the $k^\ast$-quotient $K$ and $A'$ be a $\hat{K}$-interior algebra; then the $\hat{K}$-interior algebra structure on $A$ and the $\hat{K}$-interior algebra structure on $A'$ determine a $\hat{K} \otimes \hat{K}$-interior algebra structure on the tensor product $A \otimes_{\mathcal{O}} A'$, which, by restriction, induces a $\hat{K} \ast \hat{K}$-interior algebra structure on $A \otimes_{\mathcal{O}} A'$.

2.4. Let $A$ be a $\hat{K}$-interior algebra and $P$ a $p$-subgroup of $K$. We denote by $A^p$ the subalgebra consisting of all $P$-fixed elements of $A$. Clearly $A^p$ is a $C_{\hat{K}}(P)$-interior algebra with the homomorphism $C_{\hat{K}}(P) \to (A^p)^\ast$, $\hat{x} \mapsto \hat{x}1$, where $C_{\hat{K}}(P)$ is the centralizer of $P$ in $\hat{K}$. For any subgroup $Q$ of $P$, we denote by $\text{Tr}^P_Q$ the relative trace map $A^Q \to A^p$ and by $A^p_Q$ its image. We define $A(P)$ to be the Brauer quotient $k \otimes_{\mathcal{O}} (A^p / \sum S A^p)$, where $S$ runs over the set of proper subgroups of $P$, and denote by $\text{Br}^A_P$ the Brauer homomorphism $A^p \to A(P)$. Note that $A(P) \neq 0$ forces $P$ to be a $p$-group. When $A = \mathcal{O}_\ast \hat{K}$ and $P$ is a $p$-subgroup of $K$, by [11, Proposition 2.2], $\text{Br}^A_P$ induces an isomorphism $k_\ast C_{\hat{K}}(P) \cong A(P)$; in this case, we always identify $A(P)$ with $k_\ast C_{\hat{K}}(P)$ through this isomorphism.

2.5. In this paragraph, we generalize the definitions and notations in Introduction to twisted group algebras. Let $L$ be a subgroup of $K$ and $e$ and $g$ be block idempotents of $\mathcal{O}_\ast \hat{K}$ and $\mathcal{O}_\ast \hat{L}$ respectively. $\mathcal{O}_\ast \hat{K}e$ and $\mathcal{O}_\ast \hat{L}g$ are said to be naturally Morita equiv-
alent of degree \( n \) for a positive integer number \( n \) if there exists a unitary \( \mathcal{O} \)-subalgebra \( S \) of \( \mathcal{O}_s \hat{K} e \) such that \( S \) is a full matrix algebra over \( \mathcal{O} \) of degree \( n \) and the map

\[
\mathcal{O}_s \hat{L} g \otimes \mathcal{O} S \rightarrow \mathcal{O}_s \hat{K} e, \quad x \otimes y \mapsto xy
\]
is an isomorphism of \( \mathcal{O} \)-algebras. Let \( M \) be an \( \mathcal{O}_s \hat{K} \)-module and \( N \) an \( \mathcal{O}_s \hat{L} \)-module. We denote by \( mM \) the direct sum of \( m \) copies of \( M \) for a positive integer number \( m \), by \( \text{Res}^\hat{K}_M(M) \) the restriction of \( M \) from \( \mathcal{O}_s \hat{K} \) to \( \mathcal{O}_s \hat{L} \), and by \( \text{Ind}^\hat{K}_{\hat{L}}(N) \) the induction of \( N \) from \( \mathcal{O}_s \hat{L} \) to \( \mathcal{O}_s \hat{K} \). Let \( i \) be an idempotent of \( \mathcal{O}_s \hat{K} \). We denote by \( i \cdot M \) the product of \( i \) and \( M \). Note that if \( i \) commutes with a unitary subalgebra \( B \) of \( \mathcal{O}_s \hat{K} \), then the \( \mathcal{O}_s \hat{K} \)-module structure on \( M \) induces a \( B \)-module structure on \( i \cdot M \). So \( e \cdot M \) is an \( \mathcal{O}_s \hat{K} \)-module structure and when \( e \cdot M = M \), then we say that the \( \mathcal{O}_s \hat{K} \)-module \( M \) is associated to the block \( e \) of \( \mathcal{O}_s \hat{K} \). We denote by \( \text{IBr}(e) \) the set of all non-isomorphic simple \( \mathcal{O}_s \hat{K} \)-modules associated to \( e \) and by \( \text{Irr}(e) \) the set of all non-isomorphic simple \( \mathcal{O}_s \hat{K} \)-modules associated to \( e \).

2.6. Let \( K \) be a finite group, \( \hat{K} \) a \( k^* \)-group with the \( k^* \)-quotient \( K \), \( L \) a normal \( p' \)-subgroup of \( K \) and \( f \) a \( K \)-stable block idempotent of \( \mathcal{O}_s \hat{L} \). Then \( K \) acts on the full matrix algebra \( \mathcal{O}_s \hat{L} f \) over \( \mathcal{O} \) and thus by the Skolem–Noether theorem, there exists a group homomorphism

\[
\rho: K \rightarrow \text{Aut}(\mathcal{O}_s \hat{L} f) \cong (\mathcal{O}_s \hat{L} f)^*/\mathcal{O}^*.
\]

We denote by \( \tilde{K} \) the set of all elements \((c, x)\) such that \( \rho(x) \) is the image of \( c \) in \((\mathcal{O}_s \hat{L} f)^*/\mathcal{O}^*\), where \( c \in (\mathcal{O}_s \hat{L} f)^* \) and \( x \in K \). Obviously \( \tilde{K} \) is an \( \mathcal{O}^* \)-group with the \( \mathcal{O}^* \)-quotient \( K \) with the homomorphism \( \mathcal{O}^* \rightarrow \tilde{K} \), \( \lambda \mapsto (\lambda, 1) \), the map \( \hat{L} \rightarrow \tilde{K} \), \( \hat{x} \mapsto (\hat{x}, x) \) is an injective group homomorphism and its image is normal in \( \tilde{K} \); in this sense, we identify \( \hat{L} \) with a normal subgroup of \( \tilde{K} \).

2.7. Now we claim that there exists a subgroup \( \tilde{K} \) of \( \tilde{K} \) which is a \( k^* \)-group of \( k^* \)-quotient \( K \) and contains \( \hat{L} \). Consider the quotient group \( \hat{K}/\hat{L} \). Obviously \( \hat{L}\mathcal{O}^*/\hat{L} \) is a central subgroup of \( \hat{K}/\hat{L} \) isomorphic to \( 1 + J(\mathcal{O}) \) and \( (\hat{K}/\hat{L})/(\hat{L}\mathcal{O}^*/\hat{L}) \cong K/L \), thus we can regard \( \hat{K}/\hat{L} \) as a central extension of \( K/L \) by \( 1 + J(\mathcal{O}) \). Let \( P \) be a Sylow \( p' \)-subgroup of \( K \). Since \( L \) is a \( p' \)-group, the image of \( P \) in \( K/L \) is isomorphic to \( P \); so we identify \( P \) with its image in \( K/L \). Again since \( L \) is a \( p' \)-group, it is well known that \( \mathcal{O}_s \hat{L} f \) is a full matrix algebra over \( \mathcal{O} \) and has the \( \mathcal{O} \)-rank prime to \( p \), thus the action of \( P \) on \( \mathcal{O}_s \hat{L} f \) can be lifted to a group homomorphism \( P \rightarrow (\mathcal{O}_s \hat{L} f)^* \) (see [10, Paragraph 6.2]). This implies that there exists a group homomorphism \( \theta: P \rightarrow \hat{K}/\hat{L} \) such that for any \( u \in P \), the image of \( \theta(u) \) through the surjective homomorphism \( \hat{K}/\hat{L} \rightarrow K/L \) is \( u \). Since \( 1 + J(\mathcal{O}) \) is a \( p' \)-divisible group, the sur-
jective homomorphism $\tilde{K}/\hat{L} \to K/L$ splits and thus has a section $K/L \to \tilde{K}/\hat{L}$. Then the inverse image of the image of $K/L$ in $\tilde{K}/\hat{L}$ in $\bar{K}$ is just the desired $k^*$-group $\bar{K}$.

2.8. Consequently we have a group homomorphism $\vartheta: \tilde{K} \to (\mathcal{O}_s\hat{K}f)^*$ and thus $\mathcal{O}_s\hat{L}f$ becomes a $\bar{K}$-interior algebra. Consider the $k^*$-group $\bar{K} = \hat{K} \ast \hat{K}$. Obviously $\hat{L} = \hat{L} \ast \hat{L}$ has a normal subgroup $\{\hat{x} \otimes \hat{x}^{-1} \mid x \in L\}$ isomorphic to $L$; we still denote this group by $L$. We claim that $L$ is normal in $\bar{K}$. Indeed, for any $\hat{y} \otimes \bar{y} \in \bar{K}$ and $\hat{x} \otimes \hat{x}^{-1} \in L$, we have $(\hat{y} \otimes \bar{y})(\hat{x} \otimes \hat{x}^{-1})(\hat{y} \otimes \bar{y})^{-1} = (\hat{y} \otimes \bar{y})(\hat{x} \otimes \hat{x}^{-1})(\hat{x}^{-1} \otimes \bar{y}^{-1}) = \hat{y}\hat{x}\hat{y}^{-1} \otimes \bar{y}\hat{x}^{-1} = \hat{x}\hat{y}\hat{x}^{-1} = (\hat{x} \otimes \hat{x}^{-1})^{-1}$ since the $\hat{K}$- and $\bar{K}$-conjugation induce the same action of $K$ on $\hat{L}$. Set $\tilde{K} = \bar{K}/L$. Then we obtain a $k^*$-group $\tilde{K}$ with the $k^*$-quotient $K/L$. Through the surjective group homomorphism $\tilde{K} \to \tilde{K}$, we endow the twisted group algebra $\mathcal{O}_s\tilde{K}$ of $\tilde{K}$ over $\mathcal{O}$ with a $\tilde{K}$-interior algebra structure.

**Theorem 2.9.** Keep the notations as in Paragraphs 2.6, 2.7 and 2.8. Then there exists an isomorphism of $\tilde{K}$-interior algebras

$$(2.9.1) \quad \mathcal{O}_s\hat{K}f \cong \mathcal{O}_s\hat{L}f \otimes_{\mathcal{O}} \mathcal{O}_s\tilde{K}.$$ 

In particular, the functors $U \mapsto i \cdot U$ and $V \mapsto \mathcal{O}_s\hat{L}i \otimes_{\mathcal{O}} V$ are inverse isomorphisms between the categories of finitely generated $\mathcal{O}_s\hat{K}f$- and $\mathcal{O}_s\tilde{K}$-modules, where $i$ is a primitive idempotent of $\mathcal{O}_s\hat{L}f$.

The above theorem is also called the second Fong’s reduction theorem.

Proof. Since $\mathcal{O}_s\hat{L}f$ is a full matrix algebra over $\mathcal{O}$, by [8, Proposition 2.1], the map

$\mathcal{O}_s\hat{L}f \otimes_{\mathcal{O}} C_{\mathcal{O},\hat{K}f}(\mathcal{O}_s\hat{L}f) \cong \mathcal{O}_s\hat{K}f, \quad x \otimes y \mapsto xy$

is an isomorphism of $\mathcal{O}$-algebras, where $C_{\mathcal{O},\hat{K}f}(\mathcal{O}_s\hat{L}f)$ is the centralizer of $\mathcal{O}_s\hat{L}f$ in $\mathcal{O}_s\hat{K}f$. Let $R$ be a set of representatives of cosets of $L$ in $K$ and write $\mathcal{O}_s\hat{K}f$ as the direct sum $\bigoplus_{x \in R}(\mathcal{O}_s\hat{L}f)x$. Since $\hat{L}$ is normal in $\bar{K}$, it is easily computed that $C_{\mathcal{O},\hat{K}f}(\mathcal{O}_s\hat{L}f)$ is equal to the direct sum $\bigoplus_{x \in R} C_{\mathcal{O},\hat{L}f}x(\mathcal{O}_s\hat{L}f)$. For any $x \in R$, since $\hat{x}$ and $\theta(\hat{x})$ have the same action on $\mathcal{O}_s\hat{L}f$ by conjugation, $\hat{x}\theta(\hat{x}^{-1}) \in C_{\mathcal{O},\hat{L}f}x(\mathcal{O}_s\hat{L}f)$; moreover by comparing the $\mathcal{O}$-ranks, it is not difficult to find $\mathcal{O}_s\hat{K}f(\mathcal{O}_s\hat{L}f)$ and thus $C_{\mathcal{O},\hat{K}f}(\mathcal{O}_s\hat{L}f) = \bigoplus_{x \in R} \mathcal{O}_s\hat{K}f(\mathcal{O}_s\hat{L}f)$ and finally it is easily checked that the map $\tilde{K} \to (\mathcal{O}_s\hat{K}f(\mathcal{O}_s\hat{L}f))^*$, $\hat{x} \otimes \hat{x} \mapsto \hat{x}\theta(\hat{x}^{-1})$ is a group homomorphism with the kernel $L$; in particular, the group homomorphism induces an isomorphism $\mathcal{O}_s\tilde{K} \cong C_{\mathcal{O},\hat{K}f}(\mathcal{O}_s\hat{L}f)$.  

**2.10.** Keep the notations in Theorem 2.9. Let $N$ be a subgroup of $K$ containing $L$, $\tilde{N}$ the quotient group of $N$ in the quotient group $\tilde{K} = K/L$, $\hat{N}$, $\bar{N}$ and $\tilde{N}$ the
inverse images of $N$ in $\hat{K}$, $\tilde{K}$ and $\bar{K}$ respectively, and $\tilde{N}$ the inverse image of $\hat{N}$ in $\tilde{K}$. Consider $O_s\hat{L}f$ as an $\tilde{N}$-interior algebra through the restriction of the structural homomorphism of the $\bar{K}$-interior algebra $O_s\hat{L}f$ to $\tilde{N}$ and $O_s\tilde{N}$ as an $\tilde{N}$-interior algebra through the homomorphism $\tilde{N} \to \bar{N} \subset (O_s\tilde{N})^*$. Then the isomorphism (2.10.1) induces an $\tilde{N}$-interior algebra isomorphism

\begin{equation}
O_s\tilde{N}f \cong O_s\hat{L}f \otimes O_s\bar{N}.
\end{equation}

In particular, the functors $X \mapsto i \cdot X$ and $Y \mapsto O_s\hat{L}i \otimes Y$ are inverse isomorphisms between the categories of finitely generated $O_s\hat{N}f$- and $O_s\tilde{N}$-modules. Let $h$ be a block idempotent of $O_s\hat{K}$ such that $hf \neq 0$, $\bar{h}$ the corresponding block idempotent of $O_s\bar{K}$ determined by $h$ through the isomorphism (2.9.1), $l$ a block idempotent of $O_s\tilde{N}$ and $\bar{l}$ the corresponding block idempotent of $O_s\bar{N}$ determined by $l$ through the isomorphism (2.10.1). Then by the isomorphisms (2.9.1) and (2.10.1) and the definition of natural Morita equivalences of degree $n$, we can easily verify the following:

2.10.2. $O_s\hat{K}h$ and $O_s\tilde{N}l$ are naturally Morita equivalent of degree $n$ if and only if $O_s\tilde{K}h$ and $O_s\tilde{N}l$ are naturally Morita equivalent of degree $n$.

2.11. Finally we claim the following:

2.11.1. for any $O_s\tilde{K}$-module $V$, $O_s\hat{L}i \otimes \text{Res}_{\tilde{N}}^\hat{K}(V) \cong \text{Res}_{\tilde{N}}^\hat{K}(O_s\hat{L}i \otimes V)$, and for any $O_s\tilde{N}$-module $Y$, $O_s\hat{L}i \otimes \text{Ind}_{\tilde{N}}^\hat{K}(Y) \cong \text{Ind}_{\tilde{N}}^\hat{K}(O_s\hat{L}i \otimes Y)$.

The first isomorphism is obvious, so the rest is to prove the second equality. We consider $O_s\tilde{K}$ as a subalgebra of $O_s\hat{K}f$ through the isomorphism (2.9.1) and thus $O_s\tilde{N}$ is also a subalgebra of $O_s\tilde{N}f$. We claim that the map

\begin{equation}
O_s\hat{L}i \otimes \text{Ind}_{\tilde{N}}^\hat{K}(Y) \to \text{Ind}_{\tilde{N}}^\hat{K}(O_s\hat{L}i \otimes Y)
\end{equation}

sending $x \otimes (y \otimes z)$ to $y \otimes (x \otimes z)$ is an isomorphism of $O_s\tilde{K}$-modules, where $x \in O_s\hat{L}i$, $y \in O_s\tilde{K}$ and $z \in Y$. Note that any element of $\text{Ind}_{\tilde{N}}^\hat{K}(O_s\hat{L}i \otimes Y)$ can be written as a sum of elements like $y \otimes (x \otimes z)$, where $x \in O_s\hat{L}i$, $y \in O_s\tilde{K}$ and $z \in Y$; that implies that the homomorphism (2.11.2) is surjective. Then $O_s\hat{L}i \otimes \text{Ind}_{\tilde{N}}^\hat{K}(Y)$ and $\text{Ind}_{\tilde{N}}^\hat{K}(O_s\hat{L}i \otimes Y)$ having the same $O$-rank forces (2.11.2) to be an isomorphism.

2.12. As consequences of Statement 2.11.1, we have the followings:

2.12.1. If $S$ is a simple $O_s\tilde{K}h$-module and $S_N$ is a simple $O_s\tilde{N}l$-module such that

\[
\text{Res}_{\tilde{N}}^\hat{K}(S) \cong nS_N
\]
and \( h \cdot \text{Ind}_{N}^{k}(S_{N}) \cong nS \) for a positive integer number \( n \), then \( \text{Res}_{N}^{k}(i \cdot S) \cong n(i \cdot S_{N}) \) and \( \tilde{h} \cdot \text{Ind}_{N}^{k}(i \cdot S_{N}) \cong n(i \cdot S) \).

2.12.2. If \( W \) is a simple \( K_{n} \tilde{h} \)-module and \( W_{N} \) is a simple \( K_{n} \tilde{N} l \)-module such that

\[
\text{Res}_{N}^{k}(W) \cong nW_{N}
\]

for a positive integer number \( n \), then \( \text{Res}_{N}^{k}(i \cdot W) \cong n(i \cdot W_{N}) \).

**Lemma 2.13.** Keep notations as above. If \( O_{1} \tilde{h} \) covers \( O_{1} \tilde{N} l \) and \( O_{1} \tilde{K} h \) and \( O_{1} \tilde{N} l \) have common defect groups, then \( O_{1} \tilde{K} h \) covers \( O_{1} \tilde{N} l \), \( O_{1} \tilde{K} h \) and \( O_{1} \tilde{N} l \) have common defect groups, and \( n(h, l) = n(\tilde{h}, \tilde{l}) \).

Proof. By the choices of \( h \) and \( \tilde{h} \), the isomorphism (2.9.1) induces an isomorphism of \( \tilde{h} \)-interior algebras \( O_{1} \tilde{K} h \cong O_{1} \tilde{L} f \otimes_{O_{1} \tilde{K} h} O_{1} \tilde{K} \tilde{h} \). Let \( P \) be a defect group of \( h \). Then it follows from [12, Corollary 3.3] that the image of \( P \) in \( \tilde{K} \), which is isomorphic to \( P \) and we still denote by \( P \), is a defect group of \( \tilde{h} \), \( O_{1} \tilde{L} f \) has a \( P \)-stable basis and \((O_{1} \tilde{L} f)(P) \neq 0 \). So we can use [10, Proposition 5.6] to obtain the following \( C_{\tilde{K}}(P) \)-interior algebra isomorphism

\[
(2.13.1) \quad k_{s}C_{\tilde{K}}(P) \text{Br}_{P}^{O_{1} \tilde{K}}(h) \cong (O_{1} \tilde{L} f)(P) \otimes_{k} k_{s}C_{\tilde{K}}(P) \text{Br}_{P}^{O_{1} \tilde{K}}(\tilde{h}).
\]

Fix a block idempotent \( h_{P} \) of \( k_{s}C_{\tilde{K}}(P) \) such that \( \text{Br}_{P}^{O_{1} \tilde{K}}(h)h_{P} = h_{P} \). Since \((O_{1} \tilde{L} f)(P)\) is a full matrix algebra over \( k \), there exists a block idempotent \( \tilde{h}_{P} \) of \( k_{s}C_{\tilde{K}}(P) \) such that \( \text{Br}_{P}^{O_{1} \tilde{K}}(\tilde{h})\tilde{h}_{P} = \tilde{h}_{P} \) and the isomorphism (2.13.1) induces an isomorphism

\[
(2.13.2) \quad k_{s}C_{\tilde{K}}(P)h_{P} \cong (O_{1} \tilde{L} f)(P) \otimes_{k} k_{s}C_{\tilde{K}}(P)\tilde{h}_{P}.
\]

Since we are assuming that \( O_{1} \tilde{K} h \) and \( O_{1} \tilde{N} l \) have common defect groups, \( P \) is also a defect group of \( O_{1} \tilde{N} l \). Then similarly, we can find block idempotents \( l_{P} \) and \( \tilde{l}_{P} \) of \( k_{s}C_{\tilde{K}}(P) \) and \( k_{s}C_{\tilde{N}}(P) \) respectively, such that \( \text{Br}_{P}^{O_{1} \tilde{N}}(l)l_{P} = l_{P}, \text{Br}_{P}^{O_{1} \tilde{N}}(\tilde{l})\tilde{l}_{P} = \tilde{l}_{P} \) and there is an isomorphism

\[
(2.13.3) \quad k_{s}C_{\tilde{K}}(P)l_{P} \cong (O_{1} \tilde{L} f)(P) \otimes_{k} k_{s}C_{\tilde{N}}(P)\tilde{l}_{P}.
\]

Finally since we are also assuming that \( O_{1} \tilde{K} h \) covers \( O_{1} \tilde{N} l \), \( O_{1} \tilde{K} h \) covers \( O_{1} \tilde{N} l \) and
thus \( n(h, l) \) and \( n(\tilde{h}, \tilde{l}) \) make sense; by isomorphisms (2.13.2) and (2.13.3), we can conclude that

\[
n(h, l) = \sqrt{\frac{\dim_k(k_sC_K(P)h_P)}{\dim_k(k_sC_{\tilde{K}}(P)l_P)}} = \sqrt{\frac{\dim_k(k_sC_{\tilde{K}}(P)\tilde{h}_P)}{\dim_k(k_sC_{\tilde{K}}(P)\tilde{l}_P)}} = n(\tilde{h}, \tilde{l}).
\]

3. Proof of Theorem 1.5

**Lemma 3.1.** Let \( K \) be a finite group and \( H \) a normal subgroup of \( K \). Let \( \hat{K} \) be a \( k^* \)-group with the \( k^* \)-quotient \( K \) and \( e \) and \( f \) block idempotents of \( O_s\hat{K} \) and \( O_s\hat{H} \) respectively. If \( O_s\hat{K}e \) and \( O_s\hat{H}f \) are naturally Morita equivalent of degree \( m \), then for a common defect group \( P \) of \( e \) and \( f \), there exists a block idempotent \( e_P \) and \( f_P \) of \( k_sC_{\hat{K}}(P) \) and \( k_sC_{\hat{H}}(P) \) such that \( Br^\mathcal{O}_{\hat{K}}(e)e_P = e_P \), \( Br^\mathcal{O}_{\hat{H}}(f)f_P = f_P \) and \( k_sC_{\hat{K}}(P)e_P \) and \( k_sC_{\hat{H}}(P)f_P \) are naturally Morita equivalent of degree \( m \) too.

Proof. Since \( O_s\hat{K}e \) and \( O_s\hat{H}f \) are naturally Morita equivalent of degree \( m \), by definitions, there exists a unitary subalgebra \( S \) of \( O_s\hat{K}e \), which is a full matrix algebra over \( O \) of degree \( m \), such that the product in \( O_s\hat{K} \) induces an isomorphism

\[
O_s\hat{K}e \cong S \otimes_O O_s\hat{H}f.
\]

This isomorphism implies that \( P \) acts trivially on \( S \) by conjugation and then by [10, Proposition 5.6], we obtain an isomorphism

\[
k_sC_{\hat{K}}(P) Br^\mathcal{O}_{\hat{K}}(e) \cong S(P) \otimes_k k_sC_{\hat{H}}(P) Br^\mathcal{O}_{\hat{H}}(f).
\]

Fix a block idempotent \( e_P \) of \( k_sC_{\hat{K}}(P) \) such that \( Br^\mathcal{O}_{\hat{K}}(e)e_P = e_P \). Since \( S(P) \cong k \otimes_O S \), \( e_P \) determines a unique block idempotent \( f_P \) of \( k_sC_{\hat{H}}(P) \) such that \( Br^\mathcal{O}_{\hat{H}}(f)f_P = f_P \) and \( k_sC_{\hat{K}}(P)e_P \cong (k \otimes_O S) \otimes_k k_sC_{\hat{H}}(P)f_P \). □

3.2. Let \( H \) be a finite group and \( R \) a subgroup of \( H \). We denote by \((\mathcal{O}H)^R\) the subalgebra of all \( R \)-fixed elements of \( \mathcal{O}H \). Recall that a pointed group \( P_\gamma \) on \( \mathcal{O}H \) is a pair \((P, \gamma)\) consisting of a subgroup \( P \) of \( H \) and a \(((\mathcal{O}H)^R)^*\)-conjugate class \( \gamma \) of primitive idempotents of \((\mathcal{O}H)^R\). Another pointed group \( R_\gamma \) is contained in \( P_\gamma \) if \( R \leq P \) and there exists \( j \in \varepsilon \) and \( i \in \gamma \) such that \( ji = ij = j \). \( P_\gamma \) is local if \( Br^\mathcal{O}(\gamma) \neq \{0\} \). Let \( c \) be a block idempotent of \( \mathcal{O}H \). Then \([c]\) becomes a point of \( H \) on \( \mathcal{O}H \). We say that \( P_\gamma \) is a defect pointed group of \([c]\) or simply \( c \) if \( P_\gamma \) is a maximal local pointed group contained in \( H[c] \) with respect inclusion. By [8, Theorem 1.2], \( H \) acts transitively on the set of all defect pointed groups of \( H[c] \). Fix \( i \in \gamma \) and set \((\mathcal{O}H)_\gamma = i(\mathcal{O}H)i \). Then \((\mathcal{O}H)_\gamma \) is called a source algebra of \( H[c] \) or simply \( c \).
Let $P_\gamma$ be a defect pointed group of a block $c$ of $OH$ and denote by $N_H(P_\gamma)$ the stabilizer of $P_\gamma$ in $H$ and by $(OH)(P_\gamma)$ the simple factor of $(OH)^c$ such that the image of $\gamma$ through the surjective homomorphism $(OH)^c \rightarrow (OH)(P_\gamma)$ is not zero. The obvious action of $N_H(P_\gamma)$ on $(OH)^c$ induces an action of $N_H(P_\gamma)$ on $(OH)(P_\gamma)$. By the Skolem–Noether theorem, we have a group homomorphism $\rho: N_H(P_\gamma) \rightarrow \text{Aut}((OH)(P_\gamma)) \cong ((OH)(P_\gamma))^*/k^*$. We denote by $\hat{N}_H(P_\gamma)$ the set of all elements $(c, x)$ such that $\rho(x)$ is the image of $c$ in $((OH)(P_\gamma))^*/k^*$, where $c \in ((OH)(P_\gamma))^*$ and $x \in N_H(P_\gamma)$. Then $\hat{N}_H(P_\gamma)$ is a $k^*$-group with the $k^*$-quotient $N_H(P_\gamma)$ with the homomorphism $k^* \rightarrow \hat{N}_H(P_\gamma)$, $\lambda \mapsto (\lambda, 1)$, and the map $PC_H(P) \rightarrow \hat{N}_H(P_\gamma)$, $x \mapsto (x, x)$ is an injective homomorphism, whose image is normal in $\hat{N}_H(P_\gamma)$ and intersects $k^*$ trivially. We identify $PC_H(P)$ with a normal subgroup of $\hat{N}_H(P_\gamma)$ through the injective homomorphism and then the quotient $\hat{N}_H(P_\gamma)/PC_H(P)$ is a $k^*$-group with the $k^*$-quotient $N_H(P_\gamma)/PC_H(P)$. Let $G$ be a finite group containing $H$ as a normal subgroup and $C_G(P_\gamma)$ be the stabilizer of $P_\gamma$ in $C_G(P)$. Then it is very obvious that the conjugation action of $C_G(P_\gamma)$ on $H$ induces an action of $C_G(P_\gamma)$ on $N_H(P_\gamma)$ and actions of $C_G(P_\gamma)$ on $(OH)(P_\gamma)$ and $((OH)(P_\gamma))^*/k^*$ and that the homomorphism $\rho: N_H(P_\gamma) \rightarrow ((OH)(P_\gamma))^*/k^*$ and the surjective homomorphism $((OH)(P_\gamma))^*/k^* \rightarrow ((OH)(P_\gamma))^*/k^*$ preserve the corresponding $C_G(P_\gamma)$-actions. So $C_G(P_\gamma)$ acts on $\hat{N}_H(P_\gamma)/PC_H(P)$.

**Lemma 3.4.** Let $H$ be a finite group fulfilling that $C_H(O_p(H)) \subset O_p(H)$, $P$ be a Sylow $p$-subgroup of $H$ and $\hat{H}$ be a $k^*$-group with the $k^*$-quotient $H$. Then the unit element $1$ of $O_s\hat{H}$ is the unique block idempotent of $O_s\hat{H}$ and $P[1]$ is a defect pointed group of $H[1]$.

**Proof.** Consider the Brauer homomorphism $\text{Br}^{O_s\hat{H}}_{O_p(H)}: (O_s\hat{H})^{O_s(H)} \rightarrow k_sC_H(O_p(H))$. Since $C_H(O_p(H)) \subset O_p(H)$, $C_H(O_p(H)) \cong k^* \times Z(O_p(H))$ and thus $k_sC_H(O_p(H)) \cong kZ(O_p(H))$. On the other hand, since $O_p(H)$ is normal in $H$, $\text{Ker}(\text{Br}^{O_s\hat{H}}_{O_p(H)}) \subset J(O_s\hat{H}) \cap (O_s\hat{H})^{O_s(H)} \subset J((O_s\hat{H})^{O_s(H)})$. Thus $[1]$ is the unique local point of $O_p(H)$ on $O_s\hat{H}$ and then the lemma follows.

Let $G$ be a finite group, $H$ a normal subgroup of $G$, $\hat{G}$ a $k^*$-group of the $k^*$-group $G$ and $c$ a $G$-stable block idempotent of $O_s\hat{H}$. We denote by $G[c]$ the group of all $g \in G$ such that there exists some $x_g \in (O_s\hat{H}c)^*$ fulfilling $a^g = a^{x_g}$ for any $a \in O_s\hat{H}c$. By [2, Proposition 2.7 and Theorem 3.5], $G[c]$ is normal in $G$ and $b \in O_s\hat{G}[c]$.

**Lemma 3.5.** Let $G$ be a finite group, $H$ a normal subgroup of $G$ such that $C_H(O_p(H)) \leq O_p(H)$ and $P$ a Sylow $p$-subgroup of $H$. Let $\hat{G}$ be a $k^*$-group and assume that $O_s\hat{G}$ has a block with $P$ as a defect group. Then $G[1] = C_G(P)H$.

Here $1$ is the block idempotent of $O_s\hat{H}$ (see Lemma 3.4).
Proof. We firstly prove \( C_G(P)H \subset G[1] \). By [9, Lemma 5.5], there exists a finite subgroup \( G' \) of \( \hat{G} \) such that \( \hat{G} = k^*G' \); moreover if we let \( Z' \) be the intersection of \( k^* \) and \( G' \), \( H' \) the intersection of \( G' \) and \( \hat{H} \) and \( \iota \) the central idempotent \( 1/Z' \sum_{z \in Z} \lambda_z z^{-1} \) of \( OG' \), by [9, Theorem 5.15], the inclusion \( G' \subset \hat{G} \) induces an isomorphism of \( \mathcal{O} \)-algebras

\[(3.5.1) \quad OG' \cong \mathcal{O}_s \hat{G}, \]

whose restriction to \( H' \) induces an isomorphism

\[(3.5.2) \quad OH' \cong \mathcal{O}_s \hat{H}. \]

Since \( C_H(O_P(H)) \subset O_P(H) \), by Lemma 3.4, \( c = 1 \) is the unique block idempotent of \( \mathcal{O}_s \hat{H} \) and \( \gamma = \{1\} \) is the unique local point of \( P \) on \( \mathcal{O}_s \hat{H} \), thus \( \iota \) is a block idempotent of \( \mathcal{O}H' \), \( \gamma' = \{\iota\} \) is the unique local point of \( P \) on \( \mathcal{O}H' \) and the \( P \)-interior algebra \( \mathcal{O}H' \) with the homomorphism \( P \rightarrow (\mathcal{O}H')^\ast \), \( u \mapsto uu^\ast \) is a source algebra of \( \iota \). For any \( x \in C_G(P) \), we consider the automorphism \( \varphi(x) \) on the \( \mathcal{O} \)-algebra \( \mathcal{O}H' \) induced by \( x \). Clearly \( C_G(P) \) stabilizes \( P \), thus \( C_G(P) \) stabilizes \( P \) and then \( C_G(P) \) acts on the \( k^* \)-group \( \hat{\mathcal{N}}_H(P_{\varphi})/P\mathcal{C}_H(P) \) (refer to Paragraph 3.3). But it follows from \( \mathcal{H}_H(O_p(H)) \subset O_p(H) \) that \( (\mathcal{O}_s \hat{H})(x) \cong \mathcal{O}H' \) and thus \( \hat{\mathcal{N}}_H(P_{\varphi})/P\mathcal{C}_H(P) \cong k^* \times \mathcal{N}_H(P_{\varphi})/P\mathcal{C}_H(P) \); on the other hand, \( C_G(P) \) acts trivially on the group \( \mathcal{N}_H(P_{\varphi})/P\mathcal{C}_H(P) \). Consequently \( C_G(P) \) acts trivially on the \( k^* \)-group \( \hat{\mathcal{N}}_{H}(P_{\varphi})/P\mathcal{C}_H(P) \). Therefore by [9, Proposition 14.9], \( \varphi(x) \) is induced by some element \( a' \in (\mathcal{O}H')^\ast \); in particular, this shows that the automorphism on \( \mathcal{O}_s \hat{H} \) induced by \( x \in C_G(P) \) is induced by some \( a \in (\mathcal{O}_s \hat{H})^\ast \). Thus \( x \in G[1] \).

In order to prove \( G[1] = C_G(P)H \), now we assume \( G = G[1] \) without loss of generality. Set \( K = C_G(P)H \) and let \( b \) be a block idempotent of \( \mathcal{O}_s \hat{G} \) with \( P \) as a defect group and \( e \) be a block idempotent of \( \mathcal{O}_s \hat{K} \) such that \( be \neq 0 \). Obviously \( e \) also covers the unique block 1 of \( \mathcal{O}_s \hat{K} \) and thus \( P \) is also a defect group of \( e \). By [6, Theorem 7], \( \mathcal{O}_s \hat{G}b \) and \( \mathcal{O}_s \hat{K}e \) are naturally Morita equivalent of degree \( n \) for a positive integer and \( \mathcal{O}_s \hat{G}b \) and \( \mathcal{O}_s \hat{K}e \) are naturally Morita equivalent of degree \( m \) for a positive integer. We claim that \( n \) is equal to \( m \). Indeed, since \( be \neq 0 \) and \( G \supset H\mathcal{C}_G(P) \), \( \mathcal{O}_s \hat{G}b \) and \( \mathcal{O}_s \hat{K}e \) at least have a common block idempotent \( f \) of \( k_s C_G(P) \) such that \( Br_p \mathcal{O}_s \hat{G}b f \neq f \) and \( Br_p \mathcal{O}_s \hat{K}e f \neq f \). Then by Lemma 3.1, \( n \) is equal to \( m \); in particular, this shows that \( \mathcal{O}_s \hat{K}e \) and \( \mathcal{O}_s \hat{G}b \) have the same \( \mathcal{O} \)-rank. Since \( P \) is a Sylow \( p \)-subgroup of \( H \), by Frattini argument, we have \( G = N_G(P)H \). Thus \( K \) is normal in \( G[1] \). Then by [6, Theorem 1], \( k \otimes \mathcal{O}_s \hat{K}e \) and \( k \otimes \mathcal{O}_s \hat{G}b \) are isomorphic. Finally by [5, Corollary 4.5], \( G[1] = C_{G[1]}(P)K = C_G(P)H \).}

**Theorem 3.6.** Let \( G \) be a finite group and \( H \) a normal subgroup of \( G \) such that \( C_H(R) \subset O_{\varphi', p}(H) \) for a Sylow \( p \)-subgroup \( R \) of \( O_{\varphi', p}(H) \). Let \( \hat{G} \) be a \( k^* \)-group with
the $k^s$-quotient $G$, $b$ and $c$ block idempotents of $\mathcal{O}_s \hat{G}$ and $\mathcal{O}_s \hat{H}$ respectively, and $n$ a positive integer. If $c$ is also a block idempotent of $\mathcal{O}_s \overline{O}(H)$, then the following two conditions are equivalent:

3.6.1. $\mathcal{O}_s \hat{G}b$ and $\mathcal{O}_s \hat{H}c$ are naturally Morita equivalent of degree $n$;

3.6.2. for any simple $\mathcal{O}_s \hat{G}$-module $S$ associated to $b$, there exists a unique simple $\mathcal{O}_s \hat{H}$-module $S_{\hat{H}}$ associated to $c$ such that $\text{Res}_{\hat{H}}^\hat{G}(S) \cong nS_{\hat{H}}$ and $b \cdot \text{Ind}_{\hat{H}}^\hat{G}(S_{\hat{H}}) \cong nS$, the correspondence $\text{IBr}(b) \rightarrow \text{IBr}(c)$, $S \mapsto S_{\hat{H}}$ is a bijection, and $n \leq n(b, c)$.

Moreover in this case, $n = n(b, c)$.

Proof. By [5, Proposition 2.6] and Lemma 3.1, Condition 3.6.1 implies Condition 3.6.2. Now we assume that Condition 3.6.2 holds. By the isomorphism (2.9.1) applied to $\mathcal{O}_s \hat{G}c$ and $\mathcal{O}_s \overline{O}(H)c$, we can find a $k^s$-group $\tilde{G}$ with the $k^s$-quotient $\tilde{G} = G/O_p(H)$ such that there exists an isomorphism of $\tilde{G}$-interior algebras

\begin{equation}
\mathcal{O}_s \hat{G}c \cong \mathcal{O}_s \overline{O}(H)c \otimes_O \mathcal{O}_s \tilde{G}c
\end{equation}

which, by restriction to $\mathcal{O}_s \hat{H}c$, induces an isomorphism of $\hat{H}$-interior algebras

\begin{equation}
\mathcal{O}_s \hat{H}c \cong \mathcal{O}_s \overline{O}(H)c \otimes_O \mathcal{O}_s \hat{H}
\end{equation}

where $\hat{H}$ is the inverse image of $\hat{H} = H/O_p(H)$ in $\tilde{G}$.

Since $\mathcal{O}_s \overline{O}(H)c$ is a full matrix algebra over $O$ and $bc = b$, $b$ determines a unique block idempotent $\tilde{b}$ of $\mathcal{O}_s \tilde{G}$ through (3.6.3) such that

\begin{equation}
\mathcal{O}_s \hat{G}b \cong \mathcal{O}_s \overline{O}(H)c \otimes_O \mathcal{O}_s \tilde{G}b.
\end{equation}

But notice that $1$ is the unique block idempotent of $\mathcal{O}_s \hat{H}$ since we are assuming $C_H(R) \subset O_p(H)$ for a Sylow $p$-subgroup $R$ of $H$ and thus $\mathcal{C}_H(O_p(\hat{H})) \subset O_p(\hat{H})$ (see Lemma 3.4). Let $i$ be a primitive idempotent of $\mathcal{O}_s \overline{O}(H)c$. Since we are also assuming that there exists a unique simple $\mathcal{O}_s \hat{H}$-module $S_H$ associated to $c$ such that $\text{Res}_{\hat{H}}^\hat{G}(S) \cong nS_H$ and $b \cdot \text{Ind}_{\hat{H}}^\hat{G}(S_H) \cong nS$ for any simple $\mathcal{O}_s \hat{G}$-module $S$ associated to $b$ and that the correspondence $\text{IBr}(b) \rightarrow \text{IBr}(c)$, $S \mapsto S_{\hat{H}}$ is a bijection, it follows from Statement 2.12.1 that we have equalities $\text{Res}_{\hat{G}}^\hat{H}(i \cdot S) \cong n(i \cdot S_H)$ and $\tilde{b} \cdot \text{Ind}_{\hat{H}}^\hat{G}(i \cdot S_H) \cong n(i \cdot S)$ and from Theorem 2.9 that the map the correspondence $\text{IBr}(\tilde{b}) \rightarrow \text{IBr}(1)$, $i \cdot S \mapsto i \cdot (S_H)$ is a bijection; here in order to avoid confusion, we remind that $\text{IBr}(1)$ is the set of all simple $\mathcal{O}_s \hat{H}$-modules. Finally by our hypothesis, $b$ and $c$ have common defect groups (refer to [7, Chapter 4, Lemma 3.4] and [4, Chapter IV, Lemma 4.6]), so $n(b, c)$ makes sense and so does $n(\tilde{b}, 1)$; by Lemma 2.13, we have $n(b, c) = n(\tilde{b}, 1)$.

If we can prove that $\mathcal{O}_s \tilde{G}b$ and $\mathcal{O}_s \hat{H}$ are naturally Morita equivalent of degree $n$, by Lemma 2.10.2, so are $\mathcal{O}_s \hat{G}b$ and $\mathcal{O}_s \hat{H}c$. So in order to prove the theorem,
Since $H$ is normal in $G$ and $H$ and $G$ act transitively on the sets of defect groups of $c$ and $b$, by Frattini argument, we have $G = N_G(P)H$. Now consider the obvious normal subgroup $K = C_G(P)H$ of $G$ and let $e$ be a block idempotent of $O_s\hat{K}$ such that $be \neq 0 \neq ce$. Then $P$ has to be a defect group of $e$. By Lemma 3.5 and [6, Theorem 7], $O_s\hat{K}e$ and $O_s\hat{H}c$ are naturally Morita equivalent of degree $m$; moreover by Lemma 3.1 and the definition of $n(b, c)$, $m = n(b, c) \geq n$.

Let $S$ be a simple $O_s\hat{G}b$-module. Since $be \neq 0 \neq ce$ and $\text{Res}_{\hat{H}}^G(S) = nS_{\hat{H}}$, by Clifford theorem, there exists a simple $O_s\hat{K}e$-module $S_{\hat{K}}$ such that $S_{\hat{K}}$ is a direct summand of $\text{Res}_{\hat{K}}^G(S)$ and $S_{\hat{H}}$ is a direct summand of $\text{Res}_{\hat{K}}^G(S_{\hat{K}})$. Since $O_s\hat{K}e$ and $O_s\hat{H}c$ are naturally Morita equivalent of degree $m$, by [5, Proposition 2.6], $\text{Res}_{\hat{H}}^G(S_{\hat{K}}) = mS_{\hat{H}}$. Then the inequality $m \geq n$ shows that $\dim_b(S_{\hat{K}}) \geq \dim_b(S)$, thus $\text{Res}_{\hat{K}}^G(S) = S_{\hat{K}}$ and $m = n$; in particular, this also implies that $G$ stabilizes $e$ and thus $be = b$. By Lemma 3.5 and [6, Corollary 4], $b \in O_s\hat{K}$ and thus $be = e$. Therefore $b = e$. That $O_s\hat{K}e$ and $O_s\hat{H}c$ are naturally Morita equivalent of degree $n$ also implies $b \cdot \text{Ind}_{\hat{H}}^G(S_{\hat{H}}) = nS_{\hat{K}}$ (refer to [5, Proposition 2.6]). We rewrite $b \cdot \text{Ind}_{\hat{H}}^G(S_{\hat{H}})$ as $\text{Ind}_{\hat{K}}^G(b \cdot \text{Ind}_{\hat{H}}^G(S_{\hat{H}})) = \text{Ind}_{\hat{K}}^G(nS_{\hat{K}}) = n \text{Ind}_{\hat{K}}^G(S_{\hat{K}})$. Then the equality $n \text{Ind}_{\hat{K}}^G(S_{\hat{K}}) = nS$ forces $S = \text{Ind}_{\hat{K}}^G(S_{\hat{K}})$. But we also have $\text{Res}_{\hat{K}}^G(S) = S_{\hat{K}}$ and therefore $G$ has to be equal to $K$.

**Theorem 3.7.** Let $G$ be a finite group and $H$ a normal subgroup of $G$ such that $C_H(R) \subset O_{p', p}(H)$ for a Sylow $p$-subgroup $R$ of $O_{p', p}(H)$. Let $\hat{G}$ be a $k^*$-group with the $k^*$-quotient $G$, $b$ and $c$ block idempotents of $O_s\hat{G}$ and $O_s\hat{H}$ respectively, and $n$ be a positive integer. If $c$ is also a block idempotent of $O_s\hat{O}_{p'}(H)$, then the following two conditions are equivalent:

3.7.1. $O_s\hat{G}b$ and $O_s\hat{H}c$ are naturally Morita equivalent of degree $n$;

3.7.2. $\nu_p([G : H]) = \nu_p(n)$, for any simple $K_s\hat{G}$-module $V$ associated to $b$, there exists a unique simple $K_s\hat{H}$-module $V_{\hat{H}}$ associated to $c$ such that $\text{Res}_{\hat{H}}^G(V) \cong nV_{\hat{H}}$, the correspondence $\text{Irr}(b) \mapsto \text{Irr}(c), V \mapsto V_{\hat{H}}$ is a bijection, and $n(b, c) \geq n$.

Moreover in this case, $n = n(b, c)$.

Proof. By [5, Proposition 2.6] and Lemma 3.1, Condition 3.7.1 implies Condition 3.7.2. Now assume that Condition 3.7.2 holds. Note that the first three statements imply that $b$ and $c$ have common defect groups (refer to [4, Chapter IV, Theorem 4.5]). Then by the first and second paragraph in Theorem 3.6, in order to prove 3.7.1, we can assume $C_H(O_p(H)) \subset O_p(H)$ without loss of generality. Let $P$ be a common defect group of $b$ and $c$. Since $H$ is normal in $G$ and $H$ and $G$ act transitively on the sets of defect groups of $c$ and $b$, by Frattini argument, we have $G = N_G(P)H$. Now
consider the obvious normal subgroup $K = C_G(P)H$ of $G$ and let $e$ be a block idempotent of $O_s \hat{K}$ such that $be \neq 0 \neq ce$. Then $P$ has to be a defect group of $e$. By Lemma 3.5 and [6, Theorem 7], $O_s \hat{K}e$ and $O_s \hat{He}$ are naturally Morita equivalent of degree $m$ and by Lemma 3.1 and the definition of $n(b, c)$, $m = n(b, c) \geq n$. 

Let $V$ be a simple $K_s \hat{Gb}$-module. Since $be \neq 0 \neq ce$ and $\text{Res}^\hat{G}_H(V) = nV_H$, by Clifford theorem, there exists a simple $K_s \hat{Ke}$-module $V_H$ such that $V_H$ is a direct summand of $\text{Res}^\hat{G}_H(V)$ and $V_G$ is a direct summand of $\text{Res}^\hat{K}_e(V)$. Since $K_s \hat{Ke}$ and $K_s \hat{He}$ are naturally Morita equivalent of degree $m$, by [5, Proposition 2.6], $\text{Res}^\hat{K}_e(V) = mV_H$. Then the inequality $m \geq n$ shows that $\dim_K(V_H) \geq \dim_K(V)$ and then $\dim_K(V_G) = \dim_K(V)$, thus $\text{Res}^\hat{K}_e(V) = V_G$ and $m = n$; in particular, this also implies that $G$ stabilizes $e$ and thus $be = e = b$. By Lemma 3.5 and [6, Corollary 4], $b \in O_s \hat{K}$ and thus $be = e = b$. Moreover it is easily checked that the map $V \to V_G$ is a bijection between the sets of all simple $K_s \hat{Gb}$- and $K_s \hat{Ke}$-modules; in particular, this implies that $O_s \hat{Gb}$ and $O_s \hat{Ke}$ have the same $O$-rank. But obviously the $O$-rank of $O_s \hat{Gb}$ is equal to the product of $|G : K|$ with the $O$-rank of $O_s \hat{Ke}$ too. So $G$ is forced to equal to $K$. We are done.

3.8. Proof of Theorem 1.5. It suffices for us to take $\hat{G}$ and $\hat{H}$ to be $G \times k^*$ and $H \times k^*$ and then Theorems 3.6 and 3.7 imply Theorem 1.5. 

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