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NATURAL MORITA EQUIVALENCES OF DEGREE $n$

YUN FAN, QINQIN YANG and YUANYANG ZHOU

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Abstract

Let $G$ be a finite group, $H$ a normal subgroup of $G$ and $b$ and $c$ block idempotents of $OG$ and $OH$ respectively. Under the assumption that $C_H(R) \subseteq O_{p'}(H)$ for a Sylow $p$-subgroup $R$ of $O_{p'}(H)$ and $c$ is also a block idempotent of $O_{p'}(H)$, we give two equivalent conditions about when $OGb$ and $OHc$ are natural Morita equivalent of degree $n$ (see Theorem 1.5).

1. Introduction

1.1. Fix a prime number $p$. Let $O$ be a complete discrete valuation ring with a residue field $k$ of characteristic $p$. Let $G$ be a finite group, $H$ a subgroup of $G$ and $b$ and $c$ block idempotents of $OG$ and $OH$ respectively. In terms of the terminology of A. Hida and S. Koshitani [5], $OGb$ and $OHc$ are said to be naturally Morita equivalent of degree $n$ for a positive integer number $n$ if there exists an unitary $O$-subalgebra $S$ of $OGb$ such that $S$ is a full matrix algebra over $O$ of degree $n$ and the map

$$OHc \otimes O \rightarrow OGb, \quad x \otimes y \mapsto xy$$

is an isomorphism of $O$-algebras. When $H$ is normal in $G$ and $O = k$, this definition is firstly due to B. Külshammer [6].

1.2. For our question below, now we make the additional assumption that the characteristic of $O$ is zero, the quotient field $K$ of $O$ is big enough for all algebras involved below, the residue field $k$ is algebraically closed and $H$ is normal in $G$; the assumption will also be kept throughout this paper. As a consequence of [13, Theorems 2 and 3], we can easily conclude that the following three conditions are equivalent:

1.2.1. the map $OGb \rightarrow OHc$, $x \mapsto xc$ is an $O$-algebra isomorphism;
1.2.2. the restriction from $G$ to $H$ induces a bijection between the sets of all non-isomorphic simple modules of $OGb$ and $OHc$ and the quotient group $G/H$ is a $p'$-group;
1.2.3. the restriction from $G$ to $H$ induces a bijection between the sets of all non-isomorphic simple modules of $KGb$ and $KHc$.

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Noticing that Condition 1.2.1 is actually saying that $OGb$ and $OHc$ is naturally Morita equivalent of degree 1, we ask ourselves a question: can this statement above be generalized to natural Morita equivalences of degree $n$? In this paper, we investigate the question.

1.3. Now we begin with some preparations in order to state our main theorem. Let $M$ be an $OG$-module and $N$ an $OK$-module. We denote by $\text{Res}_G^H(M)$ the restriction of $M$ from $G$ to $K$ and by $\text{Ind}_K^G(N)$ the induction of $N$ from $K$ to $G$. Given a positive integer number $n$, we denote by $nM$ the direct sum of $n$ copies of $M$. Obviously the product $b \cdot M$ of $b$ and $M$ is an $OG$-submodule of $M$ and $b$ acts on $b \cdot M$ as the identity homomorphism. When $b \cdot M = M$, then we say that the $OG$-module $M$ is associated to the block $b$ of $OG$. We denote by $\text{IBr}(b)$ the set of all non-isomorphic simple $OG$-modules associated to $b$. All notations above except $\text{IBr}(b)$ can be slightly modified to apply to $KG$-modules. In general, we denote by $\text{Irr}(b)$ the set of all non-isomorphic simple $KG$-modules associated to $b$. Given a positive integer number $m$, $v_p(m)$ denotes the largest non-negative integer number $t$ such that $p^t | m$.

1.4. Assume that $bc \neq 0$ and $b$ and $c$ have a common defect group $P$. Since $bc \neq 0$, it is well known (refer to [3]) that there exist block idempotents $b_P$ and $c_P$ of $kC_G(P)$ and $kC_H(P)$ such that $b_P \text{ Br}^{OG}_P(b) = b_P$, $c_P \text{ Br}^{OH}_P(c) = c_P$ and $b_Pc_P \neq 0$. Since $P$ is a defect group of $b$ and $c$, $b_P$ and $c_P$ have defect group $Z(P)$, thus $kC_G(P)b_P$ and $kC_H(P)c_P$ are nilpotent (refer to [10]) and have only one simple module, say $V_{b_P}$ and $V_{c_P}$. Since $H$ is normal in $G$, so is $C_H(P)$ in $C_G(P)$; then by Clifford theory, we can conclude that the dimension $\dim_k(V_{b_P})$ of $V_{c_P}$ over $k$ divides the dimension $\dim_k(V_{b_P})$ of $V_{b_P}$ over $k$. Note that $(P, b_P)$ and $(P, c_P)$ actually are maximal Brauer pairs of $b$ and $c$, which are unique up to $G$- and $H$-conjugation (refer to [1]). Therefore the quotient $\dim_k(V_{b_P})/\dim_k(V_{c_P})$ is independent of the choices of $b_P$ and $c_P$. We denote this quotient by $n(b, c)$. Note that by [10, 1.4.1], $n(b, c) = \sqrt{\dim_k(kC_G(P)b_P)/\dim_k(kC_H(P)c_P)}$; even in order to compute $n(b, c)$, it suffices for us to choose $b_P$ and $c_P$ of $kC_G(P)$ and $kC_H(P)$ such that $b_P \text{ Br}^{OG}_P(b) = b_P$ and $c_P \text{ Br}^{OH}_P(c) = c_P$.

Theorem 1.5. Let $G$ be a finite group and $H$ be a normal subgroup of $G$ such that $C_H(R) \subset O_{P'}(H)$ for a Sylow $p'$-subgroup $R$ of $O_{P'}(H)$. Let $b$ and $c$ be respective block idempotents of $OG$ and $OH$ and let $n$ be a positive integer. If $c$ is also a block idempotent of $OO_{P'}(H)$, then the following conditions are equivalent:

1.5.1. $OGb$ and $OHc$ are naturally Morita equivalent of degree $n$;
1.5.2. for any simple $OG$-module $S$ associated to $b$, there exists a unique simple $OH$-module $S_H$ associated to $c$ such that $\text{Res}_H^G(S) \cong nS_H$ and $b \cdot \text{Ind}_H^G(S_H) \cong nS$, the correspondence $\text{IBr}(b) \rightarrow \text{IBr}(c)$, $S \mapsto S_H$ is a bijection, and $n \leq n(b, c)$;
1.5.3. $v_p([G : H]) = v_p(n)$, for any simple $KG$-module $V$ associated to $b$, there exists a unique simple $KH$-module $V_H$ associated to $c$ such that $\text{Res}_H^G(V) \cong nV_H$, and the
correspondence \( \text{Irr}(b) \to \text{Irr}(c) \), \( V \mapsto V_H \) is a bijection, and \( n \leq n(b, c) \).

Moreover in this case, \( n \) is equal to \( n(b, c) \).

**Remark 1.6.** 1. Conditions 1.5.2 and 1.5.3 both imply that \( b \) and \( c \) have the same defect groups, so \( n(b, c) \) makes sense. For details, refer to the proofs of Theorems 3.6 and 3.7.

2. When \( n = 1 \), by [4, Chapter IV, Theorem 4.5], it is easily checked that Conditions 1.5.2 and 1.5.3 both imply that the quotient group \( G/H \) is a \( p' \)-group; in addition \( n \leq n(b, c) \) automatically holds. Therefore the theorem above covers the equivalences between Conditions 1.2.1, 1.2.2 and 1.2.3.

3. There are examples to explain why the condition \( n \leq n(b, c) \) is necessary.

**2. Fong’s reduction**

In this section, an \( \mathcal{O} \)-algebra \( A \) that is involved is always associative, unitary and \( \mathcal{O} \)-free of finite rank as an \( \mathcal{O} \)-module; \( A^* \) and \( J(A) \) denote the multiplicative group of all invertible elements of \( A \) and the Jacobson radical of \( A \) respectively. Occasionally, in order to avoid confusion, we denote by \( 1_A \) of the identity element of \( A \). A homomorphism \( f: A \to B \) between \( \mathcal{O} \)-algebras is an embedding if \( f \) is injective and \( f(A) = f(1_A)Bf(1_A) \).

**2.1.** Let \( K \) be a finite group and \( \hat{K} \) be a \( k^* \)-group with the \( k^* \)-quotient \( K \) endowed with the homomorphism \( \rho: k^* \to \hat{K} \). By \( \hat{K} \), we can construct two \( k^* \)-groups: the group \( \hat{K} \) endowed with the group homomorphism \( k^* \to \hat{K} \) sending \( \lambda \) onto \( \rho(\lambda^{-1}) \) and the opposite group \((\hat{K})' \) with the group homomorphism \( \rho \); in order to differ from the \( k^* \)-group \( \hat{K} \), we denote the first \( k^* \)-group by \( \hat{K}^* \). But the two \( k^* \)-groups are isomorphic: there is an isomorphism of \( k^* \)-groups \((\hat{K})' \to \hat{K}^* , x \mapsto x^{-1} \) (refer to [9]). For any subgroup \( L \) of \( K \), we denote by \( \hat{L} \) its inverse image in \( \hat{K} \) and for any element \( x \in L \), by \( \hat{x} \) a lifting in \( \hat{K} \) of \( x \). When \( L \) is a \( p \)-group, \( \hat{L} \) can be uniquely decomposed as the direct product \( k^* \times L \) (refer to [9, Lemma 5.5]) and thus we always regard \( L \) as a subgroup of \( \hat{K} \). Let \( \hat{K} \) be another \( k^* \)-group with the \( k^* \)-quotient \( K \). Then the central product of \( \hat{K} \) and \( \hat{K} \) over \( k^* \) defines a \( k^* \)-group \( \hat{K} \otimes \hat{K} \) with the \( k^* \)-quotient isomorphic to \( K \times K \) and we identify this \( k^* \)-quotient with \( K \times K \). We also identify \( K \) with the diagonal subgroup in \( K \times K \) and denote by \( \hat{K} \ast \hat{K} \) the inverse image in \( \hat{K} \otimes \hat{K} \) of \( K \). Then \( \hat{K} \ast \hat{K} \) is a new \( k^* \)-group with the \( k^* \)-quotient \( K \).

**2.2.** Obviously the surjective homomorphism \( \mathcal{O} \to k \) induces a surjective group homomorphism \( \mathcal{O}^* \to k^* \); since \( k \) is algebraically closed, \( k \) is perfect and thus by [14, Chapter II, Proposition 8], there exists a unique section \( k^* \to \mathcal{O}^* \) of this group homomorphism. Through this section, we can regard \( \mathcal{O} \) as a right module over the group algebra of \( k^* \) over \( \mathcal{O} \). Let \( K \) be a finite group and \( \hat{K} \) be a \( k^* \)-group with the \( k^* \)-quotient \( K \). Obviously the inclusion \( k^* \subset \hat{K} \) induces a left \( \mathcal{O}k^* \)-module structure on the group.
algebra $O\hat{K}$ of $\hat{K}$ over $O$. Now we consider the tensor product $O \otimes_{O^e} O\hat{K}$ and define a distributive product on $O \otimes_{O^e} O\hat{K}$ by the equality
\[(a \otimes x)(b \otimes y) = ab \otimes xy\]
for $a, b \in O$ and $x, y \in O\hat{K}$. Then the tensor product $O \otimes_{O^e} O\hat{K}$ with the above product becomes an $O$-algebra; we call it the twisted group algebra of $\hat{K}$ over $O$ and denote it by $O\hat{K}$. Obviously the $k^*$-group isomorphism $(\hat{K})^* \cong \hat{K}^*$, $x \mapsto x^{-1}$ induces an isomorphism of $O$-algebras from the opposite ring $(O\hat{K})^*$ to $O\hat{K}^*$; moreover since the map $O\hat{K}(\hat{K} \otimes \hat{K}^*) \rightarrow O\hat{K} \otimes_{O^e} O\hat{K}^*$ sending $1 \otimes (x \otimes y)$ to $(1 \otimes x) \otimes (1 \otimes y)$ for $x \otimes y \in \hat{K} \otimes \hat{K}^*$ is an isomorphism, we can define a left $O\hat{K}(\hat{K} \otimes \hat{K}^*)$-module structure on $O\hat{K}$ by the equality $(x \otimes y)a = xay^{-1}$ for $x, y \in \hat{K}$ and $a \in O\hat{K}$. The tensor product $K \otimes_{O^e} O\hat{K}$ is also what we are concerned below and we denote it by $K\hat{K}$.

2.3. Recall that an $O$-algebra $A$ is called a $\hat{K}$-interior algebra (see [9, 5.10]) if there exists a group homomorphism $\varphi: \hat{K} \rightarrow A^*$. For any $a \in A$ and liftings $\hat{x}, \hat{y}$ in $\hat{K}$ of $x, y \in K$, we will write $\varphi(\hat{x})\varphi(\hat{y})$ as $\hat{x}\hat{y}$ for convenience. Obviously when $\hat{y} = \hat{x}^{-1}$, the product $\hat{x}\hat{y}^{-1}$ is independent of the choice of $\hat{x}$ in $\hat{K}$ and therefore we also often write it as $a^{-1}$. Moreover the map $\varphi_A: A \cong A, a \mapsto a^{-1}$ is an automorphism, the map $K \rightarrow \text{Aut}(A)$, $x \mapsto \varphi_A$ is a group homomorphism, thus $A$ is a $K$-algebra. Let $C$ be another $\hat{K}$-interior algebra; an $O$-algebra homomorphism $f: A \rightarrow C$ is called a homomorphism of $\hat{K}$-interior algebras if $f(\hat{x}\hat{y}) = \hat{x}f(a)\hat{y}$ for any $a \in A$ and liftings $\hat{x}, \hat{y}$ in $\hat{K}$ of $x, y \in K$. Let $\tilde{K}$ be another $k^*$-group with the $k^*$-quotient $K$ and $A'$ be a $\tilde{K}$-interior algebra; then the $\tilde{K}$-interior algebra structure on $A$ and the $\tilde{K}$-interior algebra structure on $A'$ determine a $\hat{K} \otimes \hat{K}$-interior algebra structure on the tensor product $A \otimes_{O^e} A'$, which, by restriction, induces a $\hat{K} \otimes \hat{K}$-interior algebra structure on $A \otimes_{O^e} A'$.

2.4. Let $A$ be a $\hat{K}$-interior algebra and $P$ a $p$-subgroup of $K$. We denote by $A^P$ the subalgebra consisting of all $P$-fixed elements of $A$. Clearly $A^P$ is a $C^P_\hat{K}$-interior algebra with the homomorphism $C_\hat{K}(P) \rightarrow (A^P)^*$, $\hat{x} \mapsto \hat{1}$, where $C_\hat{K}(P)$ is the centralizer of $P$ in $\hat{K}$. For any subgroup $Q$ of $P$, we denote by $\text{Tr}_{Q}^{P}$ the relative trace map $A^Q \rightarrow A^P$ and by $A^P_{Q}$ its image. We define $A(P)$ to be the Brauer quotient $k \otimes_{O} \left( A^P / \sum_{S} A^P_S \right)$, where $S$ runs over the set of proper subgroups of $P$, and denote by $\text{Br}^{A}_{Q}$ the Brauer homomorphism $A^P \rightarrow A(P)$. Note that $A(P) \neq 0$ forces $P$ to be a $p$-group. When $A = O\hat{K}$ and $P$ is a $p$-subgroup of $K$, by [11, Proposition 2.2], $\text{Br}^{A}_{Q}$ induces an isomorphism $k_s C_\hat{K}(P) \cong A(P)$; in this case, we always identify $A(P)$ with $k_s C_\hat{K}(P)$ through this isomorphism.

2.5. In this paragraph, we generalize the definitions and notations in Introduction to twisted group algebras. Let $L$ be a subgroup of $K$ and $e$ and $g$ be block idempotents of $O\hat{K}$ and $O\hat{L}$ respectively. $O\hat{K}e$ and $O\hat{L}g$ are said to be naturally Morita equiv-
alent of degree $n$ for a positive integer number $n$ if there exists a unitary $O$-subalgebra $S$ of $O_s\hat{K}e$ such that $S$ is a full matrix algebra over $O$ of degree $n$ and the map

$$O_s\hat{L}g \otimes_O S \to O_s\hat{K}e, \quad x \otimes y \mapsto xy$$

is an isomorphism of $O$-algebras. Let $M$ be an $O_s\hat{K}$-module and $N$ an $O_s\hat{L}$-module. We denote by $mM$ the direct sum of $m$ copies of $M$ for a positive integer number $m$, by $\text{Res}_L^K(M)$ the restriction of $M$ from $O_s\hat{K}$ to $O_s\hat{L}$, and by $\text{Ind}_L^K(N)$ the induction of $N$ from $O_s\hat{L}$ to $O_s\hat{K}$. Let $i$ be an idempotent of $O_s\hat{K}$. We denote by $i \cdot M$ the product of $i$ and $M$. Note that if $i$ commutes with a unitary subalgebra $B$ of $O_s\hat{K}$, then the $O_s\hat{K}$-module structure on $M$ induces a $B$-module structure on $i \cdot M$. So $i \cdot M$ is an $O_s\hat{K}$-module structure and when $i \cdot M = M$, then we say that the $O_s\hat{K}$-module $M$ is associated to the block $e$ of $O_s\hat{K}$. We denote by $\text{IBr}(e)$ the set of all non-isomorphic simple $O_s\hat{K}$-modules associated to $e$. All notations above except $\text{IBr}(e)$ can be slightly modified to apply to $K_s\hat{G}$-modules. We denote by $\text{Irr}(e)$ the set of all non-isomorphic simple $K_s\hat{G}$-modules associated to $e$.

2.6. Let $K$ be a finite group, $\hat{K}$ a $k^*$-group with the $k^*$-quotient $K$, $L$ a normal $p'$-subgroup of $K$ and $f$ a $K$-stable block idempotent of $O_s\hat{L}$. Then $K$ acts on the full matrix algebra $O_s\hat{L}f$ over $O$ and thus by the Skolem–Noether theorem, there exists a group homomorphism

$$\rho : K \to \text{Aut}(O_s\hat{L}f) \cong (O_s\hat{L}f)^*/O^*.$$  

We denote by $\tilde{K}$ the set of all elements $(c, x)$ such that $\rho(x)$ is the image of $c$ in $(O_s\hat{L}f)^*/O^*$, where $c \in (O_s\hat{L}f)^*$ and $x \in K$. Obviously $\tilde{K}$ is an $O^*$-group with the $O^*$-quotient $K$ with the homomorphism $O^* \to \tilde{K}, \lambda \mapsto (\lambda, 1)$, the map $\hat{L} \to \tilde{K}, \hat{x} \mapsto (\hat{x}, x)$ is an injective group homomorphism and its image is normal in $\tilde{K}$; in this sense, we identify $\hat{L}$ with a normal subgroup of $\tilde{K}$.

2.7. Now we claim that there exists a subgroup $\tilde{K}$ of $\tilde{K}$ which is a $k^*$-group of $k^*$-quotient $K$ and contains $\hat{L}$. Consider the quotient group $\tilde{K}/\hat{L}$. Obviously $\hat{L}O^*/\hat{L}$ is a central subgroup of $\tilde{K}/\hat{L}$ isomorphic to $1 + J(O)$ and $(\tilde{K}/\hat{L})/(\hat{L}O^*/\hat{L}) \cong K/L$, thus we can regard $\tilde{K}/\hat{L}$ as a central extension of $K/L$ by $1 + J(O)$. Let $P$ be a Sylow $p$-subgroup of $K$. Since $L$ is a $p'$-group, the image of $P$ in $K/L$ is isomorphic to $P$; so we identify $P$ with its image in $K/L$. Again since $L$ is a $p'$-group, it is well known that $O_s\hat{L}f$ is a full matrix algebra over $O$ and has the $O$-rank prime to $p$, thus the action of $P$ on $O_s\hat{L}f$ can be lifted to a group homomorphism $P \to (O_s\hat{L}f)^*$ (see [10, Paragraph 6.2]). This implies that there exists a group homomorphism $\theta : P \to \tilde{K}/\hat{L}$ such that for any $u \in P$, the image of $\theta(u)$ through the surjective homomorphism $\tilde{K}/\hat{L} \to K/L$ is $u$. Since $1 + J(O)$ is a $p'$-divisible group, the sur-
jective homomorphism $\hat{K} / \hat{L} \rightarrow K / L$ splits and thus has a section $K / L \rightarrow \hat{K} / \hat{L}$. Then the inverse image of the image of $K / L$ in $\hat{K} / \hat{L}$ in $\hat{K}$ is just the desired $k^\ast$-group $\hat{K}$.

2.8. Consequently we have a group homomorphism $\vartheta : \hat{K} \rightarrow (O_\ast \hat{L} f)^\ast$ and thus $O_\ast \hat{L} f$ becomes a $\hat{K}$-interior algebra. Consider the $k^\ast$-group $\hat{K} = \hat{K} * \hat{K}$. Obviously $\hat{L} = \hat{L} * \hat{L}$ has a normal subgroup $\{\hat{x} \otimes \hat{x}^{-1} | x \in L\}$ isomorphic to $L$; we still denote this group by $L$. We claim that $L$ is normal in $\hat{K}$. Indeed, for any $\hat{y} \otimes \hat{y} \in \hat{K}$ and $\hat{x} \otimes \hat{x}^{-1} \in L$, we have $(\hat{y} \otimes \hat{y})(\hat{x} \otimes \hat{x}^{-1})(\hat{y} \otimes \hat{y})^{-1} = (\hat{y} \otimes \hat{y})(\hat{x} \otimes \hat{x}^{-1})(\hat{x}^{-1} \otimes \hat{y}^{-1}) = \hat{y} \hat{x}^{-1} \otimes \hat{y} \hat{x}^{-1} = (\hat{x} \otimes \hat{x}^{-1})^{-1}$ since the $\hat{K}$- and $\hat{K}$-conjugation induce the same action of $K$ on $\hat{L}$. Set $\tilde{K} = \hat{K} / L$. Then we obtain a $k^\ast$-group $\tilde{K}$ with the $k^\ast$-quotient $K / L$. Through the surjective group homomorphism $\hat{K} \rightarrow \tilde{K}$, we endow the twisted group algebra $O_\ast \tilde{K}$ of $\tilde{K}$ over $O$ with a $\tilde{K}$-interior algebra structure.

Theorem 2.9. Keep the notations as in Paragraphs 2.6, 2.7 and 2.8. Then there exists an isomorphism of $\tilde{K}$-interior algebras

\begin{equation}
O_\ast \tilde{K} f \cong O_\ast \hat{L} f \otimes_\mathcal{O} O_\ast \hat{K}.
\end{equation}

In particular, the functors $U \mapsto i \cdot U$ and $V \mapsto O_\ast Li \otimes_\mathcal{O} V$ are inverse isomorphisms between the categories of finitely generated $O_\ast \tilde{K} f$- and $O_\ast \hat{K} f$-modules, where $i$ is a primitive idempotent of $O_\ast \hat{L} f$.

The above theorem is also called the second Fong’s reduction theorem.

Proof. Since $O_\ast \hat{L} f$ is a full matrix algebra over $O$, by [8, Proposition 2.1], the map

$$O_\ast \hat{L} f \otimes_\mathcal{O} C_{O_\ast \hat{K} f}(O_\ast \hat{L} f) \cong O_\ast \hat{K} f, \quad x \otimes y \mapsto xy$$

is an isomorphism of $\mathcal{O}$-algebras, where $C_{O_\ast \hat{K} f}(O_\ast \hat{L} f)$ is the centralizer of $O_\ast \hat{L} f$ in $O_\ast \hat{K} f$. Let $R$ be a set of representatives of cosets of $L$ in $K$ and write $O_\ast \hat{K} f$ as the direct sum $\bigoplus_{x \in R} (O_\ast \hat{L} f) x$. Since $\hat{L}$ is normal in $\hat{K}$, it is easily computed that $C_{O_\ast \hat{K} f}(O_\ast \hat{L} f)$ is equal to the direct sum $\bigoplus_{x \in R} C_{(O_\ast \hat{L} f) x}(O_\ast \hat{L} f)$. For any $x \in R$, since $\hat{x}$ and $\vartheta(\hat{x})$ have the same action on $O_\ast \hat{L} f$ by conjugation, $\hat{x} \vartheta(\hat{x}^{-1}) \in C_{(O_\ast \hat{L} f) x}(O_\ast \hat{L} f)$; moreover by comparing the $\mathcal{O}$-ranks, it is not difficult to find $\mathcal{O} \hat{x} \vartheta(\hat{x}^{-1}) = C_{(O_\ast \hat{L} f) x}(O_\ast \hat{L} f)$ and thus $C_{O_\ast \hat{K} f}(O_\ast \hat{L} f) = \bigoplus_{x \in R} \mathcal{O} \hat{x} \vartheta(\hat{x}^{-1})$. Finally it is easily checked that the map $\hat{K} \rightarrow (C_{O_\ast \hat{K} f}(O_\ast \hat{L} f))^\ast$, $\hat{x} \otimes \hat{x} \mapsto \hat{x} \vartheta(\hat{x}^{-1})$ is a group homomorphism with the kernel $L$; in particular, the group homomorphism induces an isomorphism $O_\ast \tilde{K} \cong C_{O_\ast \hat{K} f}(O_\ast \hat{L} f)$.

2.10. Keep the notations in Theorem 2.9. Let $N$ be a subgroup of $K$ containing $L$, $\tilde{N}$ the quotient group of $N$ in the quotient group $\hat{K} = K / L$, $\hat{N}$, $\tilde{N}$ and $\tilde{N}$ the
inverse images of \( N \) in \( \hat{K}, \tilde{K} \) and \( \check{K} \) respectively, and \( \check{N} \) the inverse image of \( \tilde{N} \) in \( \tilde{K} \). Consider \( O_s\hat{L}f \) as an \( \check{N} \)-interior algebra through the restriction of the structural homomorphism of the \( \check{K} \)-interior algebra \( O_s\hat{L}f \) to \( \check{N} \) and \( O_s\check{N} \) as an \( \tilde{N} \)-interior algebra through the homomorphism \( \tilde{N} \rightarrow \check{N} \subset (O_s\tilde{N})^* \). Then the isomorphism (2.9.1) induces an \( \check{N} \)-interior algebra isomorphism

\[
O_s\check{N} f \cong O_s\hat{L}f \otimes_O O_s\check{N}.
\]

(2.10.1)

In particular, the functors \( X \mapsto i \cdot X \) and \( Y \mapsto O_s\hat{L}i \otimes_O Y \) are inverse isomorphisms between the categories of finitely generated \( O_s\hat{N}f \)- and \( O_s\check{N} \)-modules. Let \( h \) be a block idempotent of \( O_s\hat{K} \) such that \( hf \neq 0 \), \( \tilde{h} \) the corresponding block idempotent of \( O_s\tilde{K} \) determined by \( h \) through the isomorphism (2.9.1), \( l \) a block idempotent of \( O_s\check{N} \) and \( \tilde{l} \) the corresponding block idempotent of \( O_s\tilde{N} \) determined by \( l \) through the isomorphism (2.10.1). Then by the isomorphisms (2.9.1) and (2.10.1) and the definition of natural Morita equivalences of degree \( n \), we can easily verify the following:

2.10.2. \( O_s\tilde{K}h \) and \( O_s\check{N}l \) are naturally Morita equivalent of degree \( n \) if and only if \( O_s\check{K}h \) and \( O_s\tilde{N}l \) are naturally Morita equivalent of degree \( n \).

2.11. Finally we claim the following:

2.11.1. for any \( O_s\check{K} \)-module \( V \), \( O_s\hat{L}i \otimes_O \text{Res}^\check{K}_\check{N}(V) \cong \text{Res}^\check{K}_\check{N}(O_s\hat{L}i \otimes_O V) \), and for any \( O_s\check{N} \)-module \( Y \), \( O_s\hat{L}i \otimes_O \text{Ind}^\check{K}_\check{N}(Y) \cong \text{Ind}^\check{K}_\check{N}(O_s\hat{L}i \otimes_O Y) \).

The first isomorphism is obvious, so the rest is to prove the second equality. We consider \( O_s\check{K} \) as a subalgebra of \( O_s\hat{K}f \) through the isomorphism (2.9.1) and thus \( O_s\check{N} \) is also a subalgebra of \( O_s\check{N} f \). We claim that the map

\[
O_s\hat{L}i \otimes_O \text{Ind}^\check{K}_\check{N}(Y) \rightarrow \text{Ind}^\check{K}_\check{N}(O_s\hat{L}i \otimes_O Y)
\]

(2.11.2)

sending \( x \otimes (y \otimes z) \) to \( y \otimes (x \otimes z) \) is an isomorphism of \( O_s\check{K} \)-modules, where \( x \in O_s\hat{L}i \), \( y \in O_s\check{K} \) and \( z \in Y \). Note that any element of \( \text{Ind}^\check{K}_\check{N}(O_s\hat{L}i \otimes_O Y) \) can be written as a sum of elements like \( y \otimes (x \otimes z) \), where \( x \in O_s\hat{L}i \), \( y \in O_s\check{K} \) and \( z \in Y \); that implies that the homomorphism (2.11.2) is surjective. Then \( O_s\hat{L}i \otimes_O \text{Ind}^\check{K}_\check{N}(Y) \) and \( \text{Ind}^\check{K}_\check{N}(O_s\hat{L}i \otimes_O Y) \) having the same \( O \)-rank forces (2.11.2) to be an isomorphism.

2.12. As consequences of Statement 2.11.1, we have the followings:

2.12.1. If \( S \) is a simple \( O_s\check{K}h \)-module and \( S_\check{N} \) is a simple \( O_s\check{N}l \)-module such that

\[
\text{Res}^\check{K}_\check{N}(S) \cong nS_\check{N}
\]
and \( h \cdot \text{Ind}_{N}^{\hat{K}}(S_{N}) \cong nS \) for a positive integer number \( n \), then \( \text{Res}_{N}^{\hat{K}}(i \cdot S) \cong n(i \cdot S) \) and \( \hat{h} \cdot \text{Ind}_{\hat{N}}^{\hat{K}}(i \cdot S_{\hat{N}}) \cong n(i \cdot S) \).

2.12.2. If \( W \) is a simple \( K \hat{h} \)-module and \( W_{\hat{N}} \) is a simple \( K \hat{N}l \)-module such that

\[
\text{Res}_{N}^{\hat{K}}(W) \cong nW_{\hat{N}}
\]

for a positive integer number \( n \), then \( \text{Res}_{N}^{\hat{K}}(i \cdot W) \cong n(i \cdot W_{\hat{N}}) \).

**Lemma 2.13.** Keep notations as above. If \( O_{s} \hat{K}h \) covers \( O_{s} \hat{N}l \) and \( O_{s} \hat{K}h \) and \( O_{s} \hat{N}l \) have common defect groups, then \( O_{s} \hat{K}f \) covers \( O_{s} \hat{N}l \), \( O_{s} \hat{K}h \) and \( O_{s} \hat{N}l \) have common defect groups, and \( n(h, l) = n(\hat{h}, \hat{l}) \).

Proof. By the choices of \( h \) and \( \hat{h} \), the isomorphism (2.9.1) induces an isomorphism of \( \hat{K} \)-interior algebras \( O_{s} \hat{K}h \cong O_{s} \hat{L} f \otimes \hat{O}_{s} \hat{K}\hat{h} \). Let \( P \) be a defect group of \( h \). Then it follows from [12, Corollary 3.3] that the image of \( P \) in \( \hat{K} \), which is isomorphic to \( P \) and we still denote by \( P \), is a defect group of \( \hat{h} \), \( O_{s} \hat{L} f \) has a \( P \)-stable basis and \((O_{s} \hat{L} f)(P) \neq 0 \). So we can use [10, Proposition 5.6] to obtain the following \( C_{\hat{K}}(P) \)-interior algebra isomorphism

\[
(2.13.1) \quad k_{s} C_{\hat{K}}(P) \text{Br}_{\hat{P}}^{O_{s} \hat{K}}(h) \cong (O_{s} \hat{L} f)(P) \otimes k_{s} C_{\hat{K}}(P) \text{Br}_{\hat{P}}^{O_{s} \hat{K}}(\hat{h}).
\]

Fix a block idempotent \( h_{P} \) of \( k_{s} C_{\hat{K}}(P) \) such that \( \text{Br}_{\hat{P}}^{O_{s} \hat{K}}(h)h_{P} = h_{P} \). Since \((O_{s} \hat{L} f)(P) \) is a full matrix algebra over \( k \), there exists a block idempotent \( \hat{h}_{P} \) of \( k_{s} C_{\hat{K}}(P) \) such that \( \text{Br}_{\hat{P}}^{O_{s} \hat{K}}(\hat{h})\hat{h}_{P} = \hat{h}_{P} \) and the isomorphism (2.13.1) induces an isomorphism

\[
(2.13.2) \quad k_{s} C_{\hat{K}}(P)h_{P} \cong (O_{s} \hat{L} f)(P) \otimes k_{s} C_{\hat{K}}(P)\hat{h}_{P}.
\]

Since we are assuming that \( O_{s} \hat{K}h \) and \( O_{s} \hat{N}l \) have common defect groups, \( P \) is also a defect group of \( O_{s} \hat{N}l \). Then similarly, we can find block idempotents \( l_{P} \) and \( \hat{l}_{P} \) of \( k_{s} C_{\hat{N}}(P) \) and \( k_{s} C_{\hat{N}}(P) \) respectively, such that \( \text{Br}_{\hat{P}}^{O_{s} \hat{N}}(l)l_{P} = l_{P} \), \( \text{Br}_{\hat{P}}^{O_{s} \hat{N}}(\hat{l})\hat{l}_{P} = \hat{l}_{P} \) and there is an isomorphism

\[
(2.13.3) \quad k_{s} C_{\hat{N}}(P)l_{P} \cong (O_{s} \hat{L} f)(P) \otimes k_{s} C_{\hat{N}}(P)\hat{l}_{P}.
\]

Finally since we are also assuming that \( O_{s} \hat{K}h \) covers \( O_{s} \hat{N}l \), \( O_{s} \hat{K}h \) covers \( O_{s} \hat{N}l \) and
thus \( n(h, l) \) and \( n(\overline{h}, \overline{l}) \) make sense; by isomorphisms (2.13.2) and (2.13.3), we can conclude that

\[
n(h, l) = \sqrt{\frac{\dim_k(k_s C_{\bar{K}}(P)_{hP})}{\dim_k(k_s C_{\bar{H}}(P)_{lP})}} = \sqrt{\frac{\dim_k(k_s C_{\bar{K}}(\bar{P})_{\overline{h}P})}{\dim_k(k_s C_{\bar{H}}(\bar{P})_{\overline{l}P})}} = n(\overline{h}, \overline{l}).
\]

\[\square\]

3. Proof of Theorem 1.5

**Lemma 3.1.** Let \( K \) be a finite group and \( H \) a normal subgroup of \( K \). Let \( \hat{K} \) be a \( k^* \)-group with the \( k^* \)-quotient \( K \) and \( e \) and \( f \) block idempotents of \( O_s \hat{K} \) and \( O_s \hat{H} \) respectively. If \( O_s \hat{K} e \) and \( O_s \hat{H} f \) are naturally Morita equivalent of degree \( m \), then for a common defect group \( P \) of \( e \) and \( f \), there exists a block idempotent \( e_P \) and \( f_P \) of \( k_s C_{\hat{K}}(P) \) and \( k_s C_{\hat{H}}(P) \) such that \( Br_p^{O_s \hat{K}}(e) e_P = e_P \), \( Br_p^{O_s \hat{H}}(f) f_P = f_P \), and \( k_s C_{\hat{K}}(P) e_P \) and \( k_s C_{\hat{H}}(P) f_P \) are naturally Morita equivalent of degree \( m \) too.

Proof. Since \( O_s \hat{K} e \) and \( O_s \hat{H} f \) are naturally Morita equivalent of degree \( m \), by definitions, there exists a unitary subalgebra \( S \) of \( O_s \hat{K} e \), which is a full matrix algebra over \( O \) of degree \( m \), such that the product in \( O_s \hat{K} \) induces an isomorphism

\[
O_s \hat{K} e \cong S \otimes O_s \hat{H} f.
\]

This isomorphism implies that \( P \) acts trivially on \( S \) by conjugation and then by [10, Proposition 5.6], we obtain an isomorphism

\[
k_s C_{\hat{K}}(P) Br_p^{O_s \hat{K}}(e) \cong S(P) \otimes_k k_s C_{\hat{H}}(P) Br_p^{O_s \hat{H}}(f).
\]

Fix a block idempotent \( e_P \) of \( k_s C_{\hat{K}}(P) \) such that \( Br_p^{O_s \hat{K}}(e) e_P = e_P \). Since \( S(P) \cong k \otimes_O S \), \( e_P \) determines a unique block idempotent \( f_P \) of \( k_s C_{\hat{H}}(P) \) such that \( Br_p^{O_s \hat{H}}(f) f_P = f_P \) and \( k_s C_{\hat{K}}(P) e_P \cong (k \otimes_O S) \otimes_k k_s C_{\hat{H}}(P) f_P \). \[\square\]

3.2. Let \( H \) be a finite group and \( R \) a subgroup of \( H \). We denote by \((O H)^R\) the subalgebra of all \( R \)-fixed elements of \( O H \). Recall that a pointed group \( P \) on \( O H \) is a pair \((\gamma, P)\) consisting of a subgroup \( P \) of \( H \) and a \((O H)^P\)-conjugate class \( \gamma \) of primitive idempotents of \((O H)^P\). Another pointed group \( R\gamma \) is contained in \( P \) if \( R \leq P \) and there exists \( j \in e \) and \( i \in \gamma \) such that \( ji = ij = j \). \( P \) is local if \( Br_{Br_p}^{O_s \hat{H}}(\gamma) \neq \{0\} \). Let \( c \) be a block idempotent of \( O H \). Then \( [c] \) becomes a point of \( H \) on \( O H \). We say that \( P \) is a defect pointed group of \([c]\) or simply \( c \) if \( P \) is a maximal local pointed group contained in \( H(c) \) with respect inclusion. By [8, Theorem 1.2], \( H \) acts transitively on the set of all defect pointed groups of \( H(c) \). Fix \( i \in \gamma \) and set \( (O H)_\gamma = i(O H)i \). Then \( (O H)_\gamma \) is called a source algebra of \( H(c) \) or simply \( c \).
Let \( P_\gamma \) be a defect pointed group of a block \( c \) of \( \mathcal{O} H \) and denote by \( N_H(P_\gamma) \) the stabilizer of \( P_\gamma \) in \( H \) and by \( (\mathcal{O}H)(P_\gamma) \) the simple factor of \( (\mathcal{O}H)^P \) such that the image of \( \gamma \) through the surjective homomorphism \( (\mathcal{O}H)^P \rightarrow (\mathcal{O}H)(P_\gamma) \) is not zero. The obvious action of \( N_H(P_\gamma) \) on \( (\mathcal{O}H)^P \) induces an action of \( N_H(P_\gamma) \) on \( (\mathcal{O}H)(P_\gamma) \). By the Skolem–Noether theorem, we have a group homomorphism \( \rho: N_H(P_\gamma) \rightarrow \text{Aut}( (\mathcal{O}H)(P_\gamma) ) \cong \mathbb{F}_k((\mathcal{O}H)(P_\gamma))^*/k^* \). We denote by \( \tilde{N}_H(P_\gamma) \) the set of all elements \((c, x)\) such that \( \rho(x) \) is the image of \( c \) in \( ((\mathcal{O}H)(P_\gamma))^*/k^* \), where \( c \in ((\mathcal{O}H)(P_\gamma))^* \) and \( x \in N_H(P_\gamma) \). Then \( \tilde{N}_H(P_\gamma) \) is a \( k^* \)-group with the \( k^* \)-quotient \( N_H(P_\gamma) \) with the homomorphism \( k^* \rightarrow \tilde{N}_H(P_\gamma), \lambda \mapsto (\lambda, 1) \), and the map \( PC_H(P) \rightarrow \tilde{N}_H(P_\gamma), x \mapsto (x, x) \) is an injective homomorphism, whose image is normal in \( \tilde{N}_H(P_\gamma) \) and intersects \( k^* \) trivially. We identify \( PC_H(P) \) with a normal subgroup of \( \tilde{N}_H(P_\gamma) \) through the injective homomorphism and then the quotient \( \tilde{N}_H(P_\gamma)/PC_H(P) \) is a \( k^* \)-group with the \( k^* \)-quotient \( N_H(P_\gamma)/PC_H(P) \). Let \( G \) be a finite group containing \( H \) as a normal subgroup and \( C_G(P_\gamma) \) be the stabilizer of \( P_\gamma \) in \( C_G(P) \). Then it is very obvious that the conjugation action of \( C_G(P_\gamma) \) on \( H \) induces an action of \( C_G(P_\gamma) \) on \( N_H(P_\gamma) \) and actions of \( C_G(P_\gamma) \) on \( (\mathcal{O}H)(P_\gamma) \) and \( ((\mathcal{O}H)(P_\gamma))^*/k^* \) and that the homomorphism \( \rho: N_H(P_\gamma) \rightarrow ((\mathcal{O}H)(P_\gamma))^*/k^* \) and the surjective homomorphism \( ((\mathcal{O}H)(P_\gamma))^*/k^* \rightarrow ((\mathcal{O}H)(P_\gamma))^*/k^* \) preserve the corresponding \( C_G(P_\gamma) \)-actions. So \( C_G(P_\gamma) \) acts on \( \tilde{N}_H(P_\gamma)/PC_H(P) \).

**Lemma 3.4.** Let \( H \) be a finite group fulfilling that \( C_H(\mathcal{O}_p(H)) \subset \mathcal{O}_p(H) \), \( P \) be a Sylow \( p \)-subgroup of \( H \) and \( \hat{H} \) be a \( k^* \)-group with the \( k^* \)-quotient \( H \). Then the unit element \( 1 \) of \( \mathcal{O}_s\hat{H} \) is the unique block idempotent of \( \mathcal{O}_s\hat{H} \) and \( P_{[1]} \) is a defect pointed group of \( H_{[1]} \).

Proof. Consider the Brauer homomorphism \( Br_{\mathcal{O}_s\hat{H}}: (\mathcal{O}_s\hat{H})^{\mathcal{O}_p(H)} \rightarrow k_sC_{\hat{H}}(\mathcal{O}_p(H)) \).
Since \( C_H(\mathcal{O}_p(H)) \subset \mathcal{O}_p(H) \), \( C_{\hat{H}}(\mathcal{O}_p(H)) \cong k^* \times Z(\mathcal{O}_p(H)) \) and thus \( k_sC_{\hat{H}}(\mathcal{O}_p(H)) \cong kZ(\mathcal{O}_p(H)) \). On the other hand, since \( \mathcal{O}_p(H) \) is normal in \( H \), \( \text{Ker}(Br_{\mathcal{O}_s\hat{H}}) \subset J(\mathcal{O}_s\hat{H}) \cap (\mathcal{O}_s\hat{H})^{\mathcal{O}_p(H)} \subset J((\mathcal{O}_s\hat{H})^{\mathcal{O}_p(H)}). \) Thus \( [1] \) is the unique local point of \( \mathcal{O}_p(H) \) on \( \mathcal{O}_s\hat{H} \) and then the lemma follows.

Let \( G \) be a finite group, \( H \) a normal subgroup of \( G \), \( \hat{G} \) a \( k^* \)-group of the \( k^* \)-group \( G \) and \( c \) a \( G \)-stable block idempotent of \( \mathcal{O}_s\hat{H} \). We denote by \( G[c] \) the group of all \( g \in G \) such that there exists some \( x_g \in (\mathcal{O}_s\hat{H}c)^* \) fulfilling \( a^g = a^{x_g} \) for any \( a \in \mathcal{O}_s\hat{H}c \). By [2, Proposition 2.7 and Theorem 3.5], \( G[c] \) is normal in \( G \) and \( b \in \mathcal{O}_s\hat{G}[c] \).

**Lemma 3.5.** Let \( G \) be a finite group, \( H \) a normal subgroup of \( G \) such that \( C_H(\mathcal{O}_p(H)) \leq \mathcal{O}_p(H) \) and \( P \) a Sylow \( p \)-subgroup of \( H \). Let \( \hat{G} \) be a \( k^* \)-group and assume that \( \mathcal{O}_s\hat{G} \) has a block with \( P \) as a defect group. Then \( G[1] = C_G(P)H \).

Here \( 1 \) is the block idempotent of \( \mathcal{O}_s\hat{H} \) (see Lemma 3.4).
Proof. We firstly prove \( C_G(P)H \subset G[1] \). By [9, Lemma 5.5], there exists a finite subgroup \( G' \) of \( \hat{G} \) such that \( \hat{G} = k^*G' \); moreover if we let \( Z' \) be the intersection of \( k^* \) and \( G' \), \( H' \) the intersection of \( G' \) and \( \hat{G} \) and \( i \) the central idempotent \( 1/|Z'| \sum_{z \in Z'} z^{-1} \) of \( \mathcal{O}G' \), by [9, Theorem 5.15], the inclusion \( G' \subset \hat{G} \) induces an isomorphism of \( \mathcal{O} \)-algebras

\[
\mathcal{O}G'i \cong \mathcal{O}_s\hat{G},
\]

whose restriction to \( H' \) induces an isomorphism

\[
\mathcal{O}H'i \cong \mathcal{O}_s\hat{H}.
\]

Since \( C_H(O_{P}(H)) \subset O_{P}(H) \), by Lemma 3.4, \( c = 1 \) is the unique block idempotent of \( \mathcal{O}_s\hat{H} \) and \( \gamma = [1] \) is the unique local point of \( P \) on \( \mathcal{O}_s\hat{H} \), thus \( i \) is a block idempotent of \( \mathcal{O}H' \), \( \gamma' = [i] \) is the unique local point of \( P \) on \( \mathcal{O}H'i \) and the \( P \)-interior algebra \( \mathcal{O}H'i \) with the homomorphism \( P \rightarrow (\mathcal{O}H'i)^* \), \( u \mapsto ui \) is a source algebra of \( i \). For any \( x \in C_G(P) \), we consider the automorphism \( \varphi_x \) on the source algebra \( \mathcal{O}H'i \) induced by \( x \). Clearly \( C_G(P) \) stabilizes \( P_{\gamma} \), thus \( C_G(P) \) stabilizes \( P_{\gamma'} \) and then \( C_G(P) \) acts on the \( k^* \)-group \( \hat{N}_{H'}(P_{\gamma'})/PC_{H'}(P) \) (refer to Paragraph 3.3). But it follows from \( C_H(O_{P}(H)) \subset O_{P}(H) \) that \( (\mathcal{O}_s\hat{H})(P_{\gamma'}) \cong k \), \( (\mathcal{O}H')(P_{\gamma'}) \cong k \) and thus \( \hat{N}_{H'}(P_{\gamma'})/PC_{H'}(P) \cong k^* \times N_{H'}(P_{\gamma'})/PC_{H'}(P) \); on the other hand, \( C_G(P) \) acts trivially on the group \( N_{H'}(P_{\gamma'})/PC_{H'}(P) \). Consequently \( C_G(P) \) acts trivially on the \( k^* \)-group \( \hat{N}_{H'}(P_{\gamma'})/PC_{H'}(P) \). Therefore by [9, Proposition 14.9], \( \varphi_x \) is induced by some element \( a' \in (\mathcal{O}H'i)^* \); in particular, this shows that the automorphism on \( \mathcal{O}_s\hat{H} \) induced by \( x \in C_G(P) \) is induced by some \( a \in (\mathcal{O}_s\hat{H})^* \). Thus \( x \in G[1] \).

In order to prove \( G[1] = C_G(P)H \), now we assume \( G = G[1] \) without loss of generality. Set \( K = C_G(P)H \) and let \( b \) be a block idempotent of \( \mathcal{O}_s\hat{H} \) with \( P \) as a defect group and \( e \) be a block idempotent of \( \mathcal{O}_s\hat{K} \) such that \( be \neq 0 \). Obviously \( e \) also covers the unique block 1 of \( \mathcal{O}_s\hat{H} \) and thus \( P \) is also a defect group of \( e \). By [6, Theorem 7], \( \mathcal{O}_s\hat{Gb} \) and \( \mathcal{O}_s\hat{H} \) are naturally Morita equivalent of degree \( n \) for a positive integer and \( \mathcal{O}_s\hat{Ke} \) and \( \mathcal{O}_s\hat{H} \) are naturally Morita equivalent of degree \( m \) for a positive integer. We claim that \( n \) is equal to \( m \). Indeed, since \( be \neq 0 \) and \( \mathcal{O}_s\hat{Ke} \) and \( \mathcal{O}_s\hat{Gb} \) at least have a common block idempotent \( f \) of \( k \mathcal{O}_sC_G(P) \) such that \( Br_p^{\mathcal{O}_s\hat{G}}(b)f \neq f \) and \( Br_p^{\mathcal{O}_s\hat{K}}(e)f \neq f \). Then by Lemma 3.1, \( n \) is equal to \( m \); in particular, this shows that \( \mathcal{O}_s\hat{Ke} \) and \( \mathcal{O}_s\hat{Gb} \) have the same \( \mathcal{O} \)-rank. Since \( P \) is a Sylow \( p \)-subgroup of \( H \), by Frattini argument, we have \( G = N_G(P)H \). Thus \( K \) is normal in \( G[1] \). Then by [6, Theorem 1], \( k \otimes \mathcal{O}_s\hat{Ke} \) and \( k \otimes \mathcal{O}_s\hat{Gb} \) are isomorphic. Finally by [5, Corollary 4.5], \( G[1] = C_{G[1]}(P)K = C_G(P)H \).

**Theorem 3.6.** Let \( G \) be a finite group and \( H \) a normal subgroup of \( G \) such that \( C_H(R) \subset O_{\mu',\rho}(H) \) for a Sylow \( p \)-subgroup \( R \) of \( O_{\mu',\rho}(H) \). Let \( \hat{G} \) be a \( k^* \)-group with
the \( k^* \)-quotient \( G \), \( b \) and \( c \) block idempotents of \( \mathcal{O}_s \hat{G} \) and \( \mathcal{O}_s \hat{H} \) respectively, and \( n \) a positive integer. If \( c \) is also a block idempotent of \( \mathcal{O}_s \mathcal{O}_{p'}(H) \), then the following two conditions are equivalent:

3.6.1. \( \mathcal{O}_s \hat{G}b \) and \( \mathcal{O}_s \hat{H}c \) are naturally Morita equivalent of degree \( n \);

3.6.2. for any simple \( \mathcal{O}_s \hat{G} \)-module \( S \) associated to \( b \), there exists a unique simple \( \mathcal{O}_s \hat{H} \)-module \( S_{\hat{H}} \) associated to \( c \) such that \( \text{Res}^{\hat{H}}_{\hat{H}}(S) \cong nS_{\hat{H}} \) and \( b \cdot \text{Ind}^{\hat{H}}_{\hat{H}}(S_{\hat{H}}) \cong nS \), the correspondence \( \text{IBr}(b) \rightarrow \text{IBr}(c) \), \( S \mapsto S_{\hat{H}} \) is a bijection, and \( n \leq n(b, c) \).

Moreover in this case, \( n = n(b, c) \).

Proof. By [5, Proposition 2.6] and Lemma 3.1, Condition 3.6.1 implies Condition 3.6.2. Now we assume that Condition 3.6.2 holds. By the isomorphism (2.9.1) applied to \( \mathcal{O}_s \hat{G}c \) and \( \mathcal{O}_s \mathcal{O}_{p'}(H)c \), we can find a \( k^* \)-group \( \tilde{G} \) with the \( k^* \)-quotient \( \tilde{G} = G/O_{p'}(H) \) such that there exists an isomorphism of \( \tilde{G} \)-interior algebras

\[
\mathcal{O}_s \hat{G}c \cong \mathcal{O}_s \mathcal{O}_{p'}(H)c \otimes_{\mathcal{O}} \mathcal{O}_s \tilde{G}
\]

which, by restriction to \( \mathcal{O}_s \hat{H}c \), induces an isomorphism of \( \hat{H} \)-interior algebras

\[
\mathcal{O}_s \hat{H}c \cong \mathcal{O}_s \mathcal{O}_{p'}(H)c \otimes_{\mathcal{O}} \mathcal{O}_s \tilde{H}
\]

where \( \tilde{H} \) is the inverse image of \( \hat{H} = H/O_{p'}(H) \) in \( \tilde{G} \).

Since \( \mathcal{O}_s \mathcal{O}_{p'}(H)c \) is a full matrix algebra over \( \mathcal{O} \) and \( bc = b \), \( b \) determines a unique block idempotent \( \tilde{b} \) of \( \mathcal{O}_s \tilde{G} \) through (3.6.3) such that

\[
\mathcal{O}_s \hat{G}b \cong \mathcal{O}_s \mathcal{O}_{p'}(H)c \otimes_{\mathcal{O}} \mathcal{O}_s \tilde{G}\tilde{b}.
\]

But notice that \( 1 \) is the unique block idempotent of \( \mathcal{O}_s \hat{H} \) since we are assuming \( C_H(R) \subset O_{p', p}(H) \) for a Sylow \( p \)-subgroup \( R \) of \( H \) and thus \( C_{\hat{H}}(O_{p'}(H)) \subset O_{p}(\hat{H}) \) (see Lemma 3.4). Let \( i \) be a primitive idempotent of \( \mathcal{O}_s \mathcal{O}_{p'}(H)c \). Since we are also assuming that there exists a unique simple \( \mathcal{O}_s \hat{H} \)-module \( S_{\hat{H}} \) associated to \( c \) such that \( \text{Res}^{\hat{H}}_{\hat{H}}(S) \cong nS_{\hat{H}} \) and \( b \cdot \text{Ind}^{\hat{H}}_{\hat{H}}(S_{\hat{H}}) \cong nS \) for any simple \( \mathcal{O}_s \hat{G} \)-module \( S \) associated to \( b \) and that the correspondence \( \text{IBr}(b) \rightarrow \text{IBr}(c) \), \( S \mapsto S_{\hat{H}} \) is a bijection, it follows from Statement 2.12.1 that we have equalities \( \text{Res}^{\hat{H}}_{\hat{G}}(i \cdot S) \cong n(i \cdot S_{\hat{H}}) \) and \( \tilde{b} \cdot \text{Ind}^{\hat{G}}_{\hat{H}}(i \cdot S_{\hat{H}}) \cong n(i \cdot S) \) and from Theorem 2.9 that the map the correspondence \( \text{IBr}(\tilde{b}) \rightarrow \text{IBr}(1) \), \( i \cdot S \mapsto i \cdot (S_{\hat{H}}) \) is a bijection; here in order to avoid confusion, we remind that \( \text{IBr}(1) \) is the set of all simple \( \mathcal{O}_s \hat{H} \)-modules. Finally by our hypothesis, \( b \) and \( c \) have common defect groups (refer to [7, Chapter 4, Lemma 3.4] and [4, Chapter IV, Lemma 4.6]), so \( n(b, c) \) makes sense and so does \( n(\tilde{b}, 1) \); by Lemma 2.13, we have \( n(b, c) = n(\tilde{b}, 1) \).

If we can prove that \( \mathcal{O}_s \tilde{G}\tilde{b} \) and \( \mathcal{O}_s \hat{H}c \) are naturally Morita equivalent of degree \( n \), by Lemma 2.10.2, so are \( \mathcal{O}_s \hat{G}b \) and \( \mathcal{O}_s \hat{H}c \). So in order to prove the theorem,
we can assume $C_H(O_p(H)) \subset O_p(H)$. Let $P$ be a common defect group of $b$ and $c$. Since $H$ is normal in $G$ and $H$ and $G$ act transitively on the sets of defect groups of $c$ and $b$, by Frattini argument, we have $G = N_G(P)H$. Now consider the obvious normal subgroup $K = C_G(P)H$ of $G$ and let $e$ be a block idempotent of $O_\ast \hat{K}$ such that $be \neq 0 \neq ce$. Then $P$ has to be a defect group of $e$. By Lemma 3.5 and [6, Theorem 7], $O_\ast \hat{K}e$ and $O_\ast \hat{H}c$ are naturally Morita equivalent of degree $m$; moreover by Lemma 3.1 and the definition of $(n(b, c), m = n(b, c) \geq n$.

Let $S$ be a simple $O_\ast \hat{b}$-module. Since $be \neq 0 \neq ce$ and $\text{Res}^G_H(S) = nS_{\hat{H}}$, by Clifford theorem, there exists a simple $O_\ast \hat{e}$-module $S_K$ such that $S_K$ is a direct summand of $\text{Res}^G_H(S)$ and $S_{\hat{H}}$ is a direct summand of $\text{Res}^K_H(S_K)$. Since $O_\ast \hat{K}e$ and $O_\ast \hat{H}c$ are naturally Morita equivalent of degree $m$, by [5, Proposition 2.6], $\text{Res}^G_H(S_K) = mS_{\hat{H}}$. Then the inequality $m \geq n$ shows that $\dim_K(S_K) \geq \dim_K(S)$, thus $\text{Res}^G_H(S) = S_K$ and $m = n$; in particular, this also implies that $G$ stabilizes $e$ and thus $be = b$. By Lemma 3.5 and [6, Corollary 4], $b \in O_\ast \hat{K}$ and thus $be = e$. Therefore $b = e$. That $O_\ast \hat{K}e$ and $O_\ast \hat{H}c$ are naturally Morita equivalent of degree $n$ also implies $b \cdot \text{Ind}^K_H(S_{\hat{H}}) = nS_K$ (refer to [5, Proposition 2.6]). We rewrite $b \cdot \text{Ind}^G_H(S_{\hat{H}})$ as $\text{Ind}^G_K(b \cdot \text{Ind}^H_H(S_{\hat{H}})) = \text{Ind}^G_K(nS_K) = n \text{Ind}^G_K(S_K)$. Then the equality $n \text{Ind}^G_K(S_K) = nS$ forces $S = \text{Ind}^G_K(S_K)$. But we also have $\text{Res}^G_K(S) = S_K$ and therefore $G$ has to be equal to $K$.

**Theorem 3.7.** Let $G$ be a finite group and $H$ a normal subgroup of $G$ such that $C_H(R) \subset O_{p'}(H)$ for a Sylow $p$-subgroup $R$ of $O_{p'}(H)$. Let $\hat{G}$ be a $k^*$-group with the $k^*$-quotient $G$, $b$ and $c$ block idempotents of $O_\ast \hat{G}$ and $O_\ast \hat{H}$ respectively, and $n$ be a positive integer. If $c$ is also a block idempotent of $O_\ast \hat{O}_{p'}(H)$, then the following two conditions are equivalent:

3.7.1. $O_\ast \hat{b}$ and $O_\ast \hat{c}$ are naturally Morita equivalent of degree $n$;

3.7.2. $v_p([G : H]) = v_p(n)$, for any simple $K_\ast \hat{G}$-module $V$ associated to $b$, there exists a unique simple $K_\ast \hat{H}$-module $V_{\hat{H}}$ associated to $c$ such that $\text{Res}^G_H(V) \cong nV_{\hat{H}}$, the correspondence $\text{Irr}(b) \mapsto \text{Irr}(c), V \mapsto V_{\hat{H}}$ is a bijection, and $n(b, c) \geq n$.

Moreover in this case, $n = n(b, c)$.

Proof. By [5, Proposition 2.6] and Lemma 3.1, Condition 3.7.1 implies Condition 3.7.2. Now assume that Condition 3.7.2 holds. Note that the first three statements imply that $b$ and $c$ have common defect groups (refer to [4, Chapter IV, Theorem 4.5]). Then by the first and second paragraph in Theorem 3.6, in order to prove 3.7.1, we can assume $C_H(O_p(H)) \subset O_p(H)$ without loss of generality. Let $P$ be a common defect group of $b$ and $c$. Since $H$ is normal in $G$ and $H$ and $G$ act transitively on the sets of defect groups of $c$ and $b$, by Frattini argument, we have $G = N_G(P)H$. Now
consider the obvious normal subgroup $K = C_G(P)H$ of $G$ and let $e$ be a block idempotent of $O_\ast \hat{K}$ such that $be \neq 0 \neq ce$. Then $P$ has to be a defect group of $e$. By Lemma 3.5 and [6, Theorem 7], $O_\ast \hat{K}e$ and $O_\ast \hat{H}c$ are naturally Morita equivalent of degree $m$ and by Lemma 3.1 and the definition of $n(b, c)$, $m = n(b, c) \geq n$.

Let $V$ be a simple $K \ast \hat{G} b$-module. Since $be \neq 0 \neq ce$ and $\text{Res}_{\hat{H}}^G(V) = nV_{\hat{H}}$, by Clifford theorem, there exists a simple $K \ast \hat{K} e$-module $V_{\hat{K}}$ such that $V_{\hat{K}}$ is a direct summand of $\text{Res}_{\hat{K}}^G(V)$ and $V_{\hat{H}}$ is a direct summand of $\text{Res}_{\hat{H}}^K(V_{\hat{K}})$. Since $K \ast \hat{K} e$ and $K \ast \hat{H} c$ are naturally Morita equivalent of degree $m$, by [5, Proposition 2.6], $\text{Res}_{\hat{H}}^K(V_{\hat{K}}) = mV_{\hat{H}}$. Then the inequality $m \geq n$ shows that $\dim_K(V_{\hat{K}}) \geq \dim_K(V)$ and then $\dim_K(V_{\hat{K}}) = \dim_K(V)$, thus $\text{Res}_{\hat{K}}^G(V) = V_{\hat{K}}$ and $m = n$; in particular, this also implies that $G$ stabilizes $e$ and thus $be = ce = b$. Moreover it is easily checked that the map $V \to V_{\hat{K}}$ is a bijection between the sets of all simple $K \ast \hat{G} b$- and $K \ast \hat{K} e$-modules; in particular, this implies that $O_\ast \hat{G} b$ and $O_\ast \hat{K} e$ have the same $O$-rank. But obviously the $O$-rank of $O_\ast \hat{G} b$ is equal to the product of $|G : K|$ with the $O$-rank of $O_\ast \hat{K} e$ too. So $G$ is forced to equal to $K$. We are done.

3.8. Proof of Theorem 1.5. It suffices for us to take $\hat{G}$ and $\hat{H}$ to be $G \times k^*$ and $H \times k^*$ and then Theorems 3.6 and 3.7 imply Theorem 1.5. □

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