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STEFAN PROBLEMS WITH THE UNILATERAL BOUNDARY CONDITION ON THE FIXED BOUNDARY I

SHOJI YOTSUTANI

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Contents

0. Introduction

This paper is concerned with the following one dimensional one phase Stefan problems with the unilateral boundary condition on the fixed boundary: Given the data, ϕ and l, find two functions $s=s(t)$ and $u=u(x, t)$ such that the pair (s,u) satisfies

$$
(S)\quad \begin{cases}
(0.1)\quad Lu\equiv u_{xx}-u_t=0\,, & 0<x<s(t)\,,\quad 0<t\leqq T\,,\\
(0.2)\quad u_x(0, t)\in\gamma(u(0, t))\,, & 0<t\leqq T\,,\\
(0.3)\quad u(s(t), t)=0\,, & 0<t\leqq T\,,
\end{cases}
$$

$$\begin{cases} (0.4) \quad u(x, 0) = \phi(x) \geqq 0\,, \qquad 0 \leqq x \leqq l\,, \quad s(0) = l \geqq 0\,, \\ \text{and the free boundary condition} \\ (0.5) \quad \dot{s}(t) = -u_x(s(t), t)\,, \qquad\qquad 0 < t \leqq T\,. \end{cases}$$

Here $T>0$ and $l\geqq 0$ are the given constants. γ is a maximal monotone graph in $\boldsymbol{R}^2$ with $\gamma(H)\ni 0$, where H is a non-negative constant. (0.2) is the unilateral boundary condition, which appeared in the theory of nonlinear semigroups as the typical example in [1]. (0.5) is the so-called Stefan's condition.

The above problem arises as a mathematical model for melting of solid. The function $u(x,t)$ represents the temperature distribution in the liquid, and the curve $x=s(t)$ represents the position of the interface, which varies with the time t as the solid melts. The unilateral boundary condition (0.2) models serveral physical situations, including the temperature control through the boundary [7, Ch. 1] and the heat flow subject to the nonlinear cooling by the radiation on the boundary [9, Ch. 7].

For the sake of simplicity, in writing down (0.1)–(0.5) we choose a system of variables such that the thermal coefficients (conductivity, heat capacity, density, latent heat) disappear.

In this paper we prove the global existence and uniqueness of the classical solution (s,u). The problem of this type with the linear boundary condition on the fixed boundary have been considered by many authors [3], [4], [5], [6], [9], [12], [13], [15], [17] and [18]. The problem with a nonlinear boundary condition has been considered by Fasano and Primicerio [24]. However the problems with the unilateral boundary condition have not yet been studied. As is well known if $s(t)$ is given, the problem (0.1)–(0.4) is a unilateral problem which has been considered, using the theory of nonlinear semigroups in the Hilbert space L^2. (See [1], [2], [8] and [19].) Thus, there are two difficulties. One is the fact that $s(t)$ is unknown and the other is how to obtain a classical solution.

We construct a solution for good data by using a primitive implicit difference sheme with only a device of capturing a free boundary explicitly through step-by-step process in time. This Difference scheme is a modification of the Nogi's scheme [13] for the linear boundary condition on the fixed boundary. When we estimate the difference solutions, we use the ideas of Petrovskii [14], Nogi [13], Brézis [1,2] and Yotsutani [19]. For the general data we obtain a solution as a limit of solutions for good data. Uniqueness is based upon the maximum principle, its strong form [11], a parabolic version of Hopf's lemma [9] and the comparison theorem for the unilateral problem.

The plan of the paper is as follows. In § 1 we state main results. § 2–8 are devoted to prove the existence of a solution under the slightly stringent conditions on the data. The method consists of:

(i) Introducing a difference scheme.
(ii) Proving that the difference scheme has a unique solution.
(iii) Proving the convergence of the difference scheme.

§2 deals with item (i). In §3 we prepare the several comparison theorems and collect some known properties of the maximal monotone graphs in $\boldsymbol{R}^2$. These are used in §4, 5, 6, 7. §4 establishes item (ii). In §5, 6 and 7 we derive estimates for solutions of the difference scheme. These are used in §8 to establish item (iii). In §9 we prove the existence of the solution of the moving boundary problem which is auxiliary for the original one. In §10, 11 and 12 we prove the uniqueness theorems. In §13 we prove the existence theorem for the case $l>0$, and in §14 we give the proof of Corollary. In §15 we prove the existence theorem for the case $l=0$. In §16 we state the modification of the estimates of Bernstein and prove Lemma 8.1 and Lemma 8.2 (i).

We consider the asymptotic behavior of the solutions, and the two phase problem of this type in Yotsutani [21] and [20] respectively. As for the applications to mechanics, see Chung & Yeh [28] and Murao & Yotsutani [29].

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1. Statements of main results

As for the definition of the maximal monotone graphs, see §3 if necessary. The assumptions required on the Stefan data are as follows.

(A) $\phi(x)$ is non-negative, bounded and continuous for *a.e.* $x\in[0, l]$.

REMARK 1.1. The assumption $\phi\geqq 0$ results from the physical background.

REMARK 1.2. If $l=0$ there is no ϕ. We do not need (A).

We introduce the notations,

(1.1) $D=\{(x, t);\ 0<x<s(t),\ 0<t\leqq T\}$, $\bar{D}=$ the closure of D in $\boldsymbol{R}^2$,

$D^s=\{x, t);\ 0<x\leqq s(t),\ 0<t\leqq T\}$,

(1.2) $Z=\{x\in[0, l];\ x$ is a discontinuous point of $\phi\}\times\{0\}$.

Definition 1.1. The pair (s, u) is a solution of the Stefan problem (S) if

i) $s(0)=l.\ s(t)>0$ for $t>0$, $s\in C([0, T])\cap C^\infty([0, T])$.

ii) u is bounded on $\bar{D}$, $u\in C^\infty(D^s)\cap C(\bar{D}-Z)$,

$$\int_\tau^T\int_0^{s(t)} u_{xx}(x, t)^2 dxdt<+\infty \qquad \text{for each } \tau\in]0, T].$$

iii) (0,1), (0.3), (0.4) and (0.5) hold.

iv) For *a.e.* $t\in[0,T[$, $u_x(0, t)$ exists and satisfies (0.2).

REMARK 1.3. If $l=0$, we omit (0.4).

We can now state the existence and uniqueness theorems.

Theorem 1. *If $l>0$ and the data ϕ satisfies (A), then there exists a solution (s, u) of the Stefan problem satisfying*

$$(*)\qquad \int_0^T \int_0^{s(t)} t u_{xx}^2 dx\, dt + T \int_0^{s(T)} u_x^2 dx < +\infty .$$

Theorem 2. *Under the same assumption of Theorem 1, the solution (s, u) of the Stefan problem is unique.*

Corollary. *In particular, if γ is a single valued maximal monotone function, then it follows that $u_x(x, t) \in C(\bar{D} - \{t=0\})$ and*

$$(1.3)\qquad u_x(0, t) = \gamma(u(0, t)) \qquad \textit{for all } t \in]0, T] .$$

Theorem 3. *If $l=0$ and γ satisfies the following assumption,*

$$(\mathrm{B})\qquad D(\gamma) \supset [0, H] \quad \textit{and} \quad \gamma(0) \subset]-\infty, 0[,$$

then there exists a solution (s,u) of the Stefan problem satisfying $(*)$.

Theorem 4. *Under the same assumption of Theorem 3, the solution (s, u) of the Stefan problem is unique.*

REMARK 1.4. The assumption (B) guarantees that the solid melts. For example, if $\gamma \equiv 0$, then the solid could not melt.

REMARK 1.5. We will extend the results of Theorem 3 and 4 in [21].

REMARK 1.6. We know that the free boundary $x=s(t)$ is a monotone increasing function in § 13.

2. Difference scheme

Let $l>0$. We use a net of rectangular meshes with the uniform space width h and the variable time steps $\{k_n\}$ $(n=1, 2, \cdots)$. Time steps $\{k_n\}$ are assumed to be unknown and they are determined by the rule that h/k_n gives gradient of a desired free boundary at each time $t=t_n$, so that the free boundary crosses each mesh line just at each corresponding mesh point. Let's introduce discrete coordinates.

$$(2.1)\qquad x_j = j \cdot h , \qquad j = 0, 1, 2, \cdots,$$

$$(2.2)\qquad t_n = \textstyle\sum_{p=1}^n k_p , \qquad n = 1, 2, \cdots, \qquad t_0 = 0 ,$$

where h varies in such a way that $l/h=J_0$ is an integer, and net functions s_n and u_j^n which correspond to $s(t_n)$ and $u(x_j, t_n)$ respectively. By our rule we can put

(2.3) $$s_n = J_n \cdot h \qquad (J_n\text{: integer, } n=0, 1, 2, \cdots).$$

Further we introduce usual divided differences:

(2.4) $$u^n_{jx} = \frac{1}{h}(u^n_{j+1} - u^n_j), \quad u^n_{j\bar{x}} = \frac{1}{h}(u^n_j - u^n_{j-1}),$$

(2.5) $$u^n_{jx\bar{x}} = \frac{1}{h^2}(u^n_{j+1} - 2u^n_j + u^n_{j-1}), \quad u^n_{j\bar{t}} = \frac{1}{k_n}(u^n_j - u^{n-1}_j), \quad \text{etc.}$$

In our scheme the heat equation is replaced by the pure implicit difference equation,

(2.6) $$L_h u^n_j \equiv u^n_{jx\bar{x}} - u^n_{j\bar{t}} = 0, \qquad 1 \leqq j \leqq J_n - 1.$$

The boundary and initial conditions are put in the following forms,

(2.7) $$u^n_{0x} \in \gamma(u^n_0),$$

(2.8) $$u^n_{J_n} = 0,$$

(2.9) $$u^0_j = \phi_j \equiv \phi(x_j), \qquad 1 \leqq j \leqq J_0.$$

The Stefan's condition is replaced by an explicit formula

(2.10) $$\frac{h}{k_n} = -u^{n-1}_{J_{n-1}\bar{x}}.$$

Our algorism is the following. β: a positive constant.

1° $u^j_0 = \phi_j$ $(1 \leqq j \leqq J_0)$, $s_0 = J_0 \cdot h = l$.

For $n = 1, 2, 3, \cdots$ successively,

2.1° if $-u^{n-1}_{J_{n-1}\bar{x}} > \beta\sqrt{h}$, then we take $J_n = J_{n-1} + 1$ and get k_n from (2.10),

2.2° if $-u^{n-1}_{J_{n-1}\bar{x}} \leqq \beta\sqrt{h}$, then we take $J_n = J_{n-1}$ and $k_n = \sqrt{h}/\beta$,

3° solve the difference equation (2.6) for $\{u^n_j\}_j$ under the boundary conditions (2.7) and (2.8) with the initial condition $\{u^{n-1}_j\}_j$.

REMARK 2.1. We prove the well-definedness of 3° in §4.

3. Preliminaries

In this section we state several comparison theorems and recall some properties of maximal monotone graphs in $\boldsymbol{R}^2$.

Let us denote by $\bar{D}_h$ the set of mesh points of

$$\{(x_j, t_n);\ 0 \leqq x_j \leqq s_n,\ 0 \leqq t_n \leqq T\}.$$

We denote by Γ^1_h, Γ^2_h, Γ^3_h, Γ_h and D_h respectively

$$\Gamma^1_h = \bar{D}_h \cap \{(0, t_n);\ 0 \leqq t_n \leqq T\},$$
$$\Gamma^2_h = \bar{D}_h \cap \{(x_j, 0);\ 0 \leqq x_j \leqq l\},$$

$$\Gamma_h^3 = \bar{D}_h \cap \{(x_j, t_n);\ x_j = s_n\}\ ,$$
$$\Gamma_h = \Gamma_h^1 \cup \Gamma_h^2 \cup \Gamma_h^3\ ,$$
$$D_h = \bar{D}_h - \Gamma_h\ .$$

The following two lemmas are well-known. (See [14, p. 355–356].)

Lemma 3.1 (The maximum principle). *Let a function u_j^n be defined on $\bar{D}_h$ and satisfy the equation*

$$(3.1) \qquad u_{jx\bar{x}}^n - u_{j\bar{t}}^n \geqq 0 \qquad (\text{resp. } u_{jx\bar{x}}^n - u_{j\bar{t}}^n \leqq 0)$$

on D_h. Then it has the maximum (resp. minimum) value at the mesh points of Γ_h.

Lemma 3.2 (The strong maximum principle). *Under the assumptions of the previous lemma, let u_j^n have the maximum (resp. minimum) value at a mesh point $(x_P, t_N) \in D_h$. Then $u_j^n \equiv u_P^N$ on $\{(x_j, t_n);\ 0 \leqq x_j \leqq s_n,\ 0 \leqq t_n \leqq t_N\}$.*

The following lemma is essential in obtaining necessary estimates.

Lemma 3.3 (The Neumann-type comparison theorem). *Let a function w_j^n be defined on $\bar{D}_h$ and satisfies*

$$(3.2) \qquad L_h w_j^n = 0 \qquad \text{in } D_h\ ,$$
$$(3.3) \qquad w_j^n \geqq 0 \qquad \text{on } \Gamma_h^2 \cup \Gamma_h^3\ ,$$
$$(3.4) \qquad -w_{0x}^n \geqq 0 \qquad \text{on } \Gamma_h^1 - \{(0, 0)\}\ .$$

Then

$$w_j^n \geqq 0 \qquad \text{in } \bar{D}_h\ .$$

Proof. Assuming the contrary, w_j^n has the negative minimum. By the strong maximum principle and (3.3), the minimum is attained at a mesh point $(0, t_N)$, $N \geqq 1$, and we have $w_1^N > w_0^N$. Thus we get $w_{0x}^N > 0$, which contradicts (3.4). q.e.d.

Next, we collect some known properties of the maximal monotone graphs $\boldsymbol{R}^2$ in stated in Brézis [2].

Let γ be a mapping from $\boldsymbol{R}$ into $\boldsymbol{R}$ which could eventually be multivalued, i.e., to every $u \in \boldsymbol{R}$ we associate a subset $\gamma(u) \subset \boldsymbol{R}$ (which may be empty). We set $D(\gamma) = \{u \in \boldsymbol{R};\ \gamma(u) \text{ is not empty}\}$, $R(\gamma) = \cup \{\gamma(u);\ u \in \boldsymbol{R}\}$ and $(I+\gamma)(u) = \{u+f;\ f \in \gamma(u)\}$.

Definition 3.1. One says that γ is a monotone graph in $\boldsymbol{R}^2$ if it satisfies

$$(f_1 - f_2)(u_1 - u_2) \geqq 0 \qquad \text{for any } u_1, u_2 \in D(\gamma), f_1 \in \gamma(u_1), f_2 \in \gamma(u_2)\ .$$

Definition 3.2. One says that a monotone graph in $\boldsymbol{R}^2$ is maximal monotone if it is maximal in the sense of the inclusion of graphs, i.e., it admits no proper monotone extension.

Lemma 3.4. *Let γ be a maximal monotone graph in $\boldsymbol{R}^2$. Then for every $h>0$, $R(I+h\gamma)=\boldsymbol{R}$ and $(I+h\gamma)^{-1}$: $\boldsymbol{R}\to\boldsymbol{R}$ is a singlevalued contraction mapping, i.e.,*

$$|(I+h\gamma)^{-1}(g_1)-(I+h\gamma)^{-1}(g_2)|\leqq|g_1-g_2| \qquad \textit{for any } g_1, g_2\in\boldsymbol{R}\,.$$

Proof. See Brézis [2, Proposition 2.2].

Lemma 3.5. *Let γ be a maximal monotone graph in $\boldsymbol{R}^2$ with $\gamma(H)\ni 0$. Then there exists a lower semicontinuous convex function θ from $\boldsymbol{R}$ into $]-\infty, +\infty]$ such that $\theta\not\equiv+\infty$, $\theta\geqq 0$, $\theta(H)=0$ and $\partial\theta=\gamma$, where*

$$\partial\theta(u)=\{f\in\boldsymbol{R};\ \theta(v)-\theta(u)\geqq f(v-u) \qquad \textit{for any } v\in D(\theta)\}\,,$$
$$D(\theta)=\{u\in\boldsymbol{R};\ \theta(u)<+\infty\}\,.$$

Proof. See Brézis [2, p. 43].

4. Existence of the unique solution for the difference equation

We prove the following existence and uniqueness lemma.

Lemma 4.1. *Let $\{u_j^{n-1}\}_{1\leqq j\leqq J_{n-1}}$ and k_n be given. Then there exists a unique solution $\{u_j^n\}_{1\leqq j\leqq J_n}$ for the following difference equation*

$$\frac{u_j^n-u_j^{n-1}}{k_n}=\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}, \qquad 1\leqq j\leqq J_n-1\,, \tag{4.2}$$

$$\frac{u_1^n-u_0^n}{h}\in\gamma(u_0^n)\,, \tag{4.2}$$

$$u_{J_n}^n=0\,. \tag{4.3}$$

Proof. Since γ is maximal monotone, $(I+h\gamma)^{-1}$ is a contraction mapping from $\boldsymbol{R}$ to itself with $D((I+h\gamma)^{-1})=\boldsymbol{R}$ by Lemma 3.4. So (4.2) is equivalent to $u_0^n=(I+h\gamma)^{-1}(u_1^n)$. We consider the following auxiliary problem

$$\frac{v_j^n-u_j^{n-1}}{k_n}=\frac{v_{j+1}^n-2v_j^n+v_{j-1}^n}{h^2}, \qquad 1\leqq j\leqq J_n-1\,, \tag{4.4}$$

$$v_0^n=\xi\,, \tag{4.5}$$

$$v_{J_n}^n=0\,. \tag{4.6}$$

It is well-known that this problem has a unique solution (see [14]). We denote by $\{v_j^n(\xi)\}_j$ the solution of the problem. We define the mapping

$$\Pi:\ \boldsymbol{R}\ni\xi\mapsto(I+h\gamma)^{-1}(v_1^n(\xi))\in\boldsymbol{R}\,.$$

Thus the problem is reduced to the following problem: Prove that Π has a unique fixed point.

We shall show that Π is a strict contraction mapping from $\boldsymbol{R}$ to itself. Since $D((I+h\gamma^{-1}))=\boldsymbol{R}$ and $(I+h\gamma^{-1})$ is a contraction mapping, we have

$$D(\Pi)=\boldsymbol{R}, \tag{4.7}$$

$$|\Pi(\xi)-\Pi(\eta)|\leqq|v_1^n(\xi)-v_1^n(\eta)|. \tag{4.8}$$

Applying the maximum principle to $v_j^n(\xi)-v_j^n(\eta)$, we have

$$|v_j^n(\xi)-v_j^n(\eta)|\leqq|\xi-\eta|, \qquad 0\leqq j\leqq J_n. \tag{4.9}$$

Taking $j=1$ in (4.4), we have

$$v_1^n=\frac{\lambda}{1+2\lambda}(v_2^n+v_0^n)+\frac{1}{1+2\lambda}u_1^{n-1}, \quad \lambda=\frac{k_n}{h^2}.$$

Therefore we have

$$\begin{aligned} &|v_1^n(\xi)-v_1^n(\eta)| \\ &\leqq\frac{\lambda}{1+2\lambda}\{|v_2^n(\xi)-v_2^n(\eta)|+|v_0^n(\xi)-v_0^n(\eta)|\} \\ &\leqq\frac{2\lambda}{1+2\lambda}|\xi-\eta| \end{aligned} \tag{4.10}$$

by (4.9). Returning to (4.8) and using (4.10), we see that Π is the strict contraction mapping. q.e.d.

5. Estimates of the solutions under the condition (A.1)

Let us show some properties of the solutions of our scheme under the slightly stringent conditions on the data, which are the following:

(A.1) $l>0$, $\phi(x)$ is non-negative, bounded, continuous for *a.e.* $x\in[0, l]$ and further there exists a positive constant K such that

$$\phi(x)\leqq K(l-x), \qquad \text{for any } x\in[0, l].$$

REMARK 5.1. (A.1) implies (A).

Lemma 5.1. *Let ϕ satisfy (A), then we have*

$$0\leqq u_j^n\leqq\max\{\|\phi\|_{L^\infty(0,l)}, H\}, \qquad 0\leqq j\leqq J_n, \ t_n\leqq T. \tag{5.1}$$

Proof. Assume that u_j^n has the negative minimum. By the strong maximum principle, (2.8) and (2.9), the minimum is attained at a mesh point $(0, t_N)$, $N\geqq1$, and we have $u_1^N>u_0^N$ and $u_0^N<0$.

Thus we get $u_{0x}^N>0$ and $\gamma(u_0^N)\subset]-\infty, 0]$ by virtue of $\gamma(H)\ni 0$ with $H\geqq 0$. This is a contradiction. So we have $0\leqq u_j^n$. The other inequality follows from a similar argument. q.e.d.

REMARK 5.2. From this lemma, we have $-u_{J_n\bar{x}}^n\geqq 0$.

Lemma 5.2. *Let ϕ satisfy (A.1), then we have*

$$0\leqq -u_{J_n\bar{x}}^n\leqq C_l\,, \qquad t_n\leqq T\,, \tag{5.2}$$

where $C_l=\max\{K, \max\{\|\phi\|_{L^\infty(0,l)}, H\}/l\}$.

Proof. We put $L=C_l$ and consider the function

$$w_{n_0}(x_j, t_n)=L(s_{n_0}-x_j)\,, \qquad 0\leqq j\leqq J_n\,,\ n\leqq n_0\,.$$

It satisfies the difference equation (2.6) and the inequalities

$$w_{n_0}(s_n, t_n)=L(s_{n_0}-s_n)\geqq 0\,, \qquad n\leqq n_0\,;$$
$$w_{n_0}(x_j, 0)\geqq K(l-x_j)\geqq\phi(x_j)\,, \qquad 1\leqq j\leqq J_0\,,$$

by (A.1), and

$$w_{n_0}(0, t_n)\geqq Ll\geqq\max\{\|\phi\|_{L^\infty(0,l)}, H\}\geqq u_0^n\,, \qquad t_n\leqq T\,,$$

from Lemma 5.1. By the maximum principle, we get

$$w_{n_0}(x_j, t_n)\geqq u_j^n\,, \qquad 0\leqq j\leqq J_n,\ n\leqq n_0\,,$$

and especially for $n=n_0$ and $j=J_{n_0}-1$, we have

$$L(s_{n_0}-x_{J_{n_0}-1})\geqq u_{J_{n_0}-1}^{n_0}\,.$$

Thus we obtain

$$Lh\geqq -(u_{J_{n_0}}^{n_0}-u_{J_{n_0}-1}^{n_0})\,.$$

This implies (5.2). q.e.d.

Lemma 5.3. *Let ϕ satisfy (A.1), then we have*

$$0<\frac{h}{k_n}\leqq C_l \qquad \textit{for } 0<h\leqq C_l^2/\beta^2\,, \tag{5.3}$$

$$|s_{n_1}-s_{n_2}|\leqq C_l|t_{n_1}-t_{n_2}| \qquad \textit{for } 0\leqq t_{n_1},\ t_{n_2}\leqq T\,. \tag{5.4}$$

Proof. By Lemma 5.2 and the difference scheme, (5.3) is obvious. (5.3) implies (5.4). q.e.d.

Lemma 5.4. *Let ϕ satisfy (A.1), then we have*

$$l\leqq s_n\leqq l+C_lT\,, \qquad t_n\leqq T\,. \tag{5.5}$$

Proof. By Lemma 5.2, we have

$$\begin{aligned} l \leqq s_n &= l + {\sum}'^{n}_{p=1} k_p \frac{h}{k_p} \\ &= l + {\sum}'^{n}_{p=1} k_p(-u^{p-1}_{J_{p-1}\bar{x}}) \\ &\leqq l + {\sum}'^{n}_{p=1} k_p \cdot C_l \leqq l + C_l T , \end{aligned}$$

where $\sum'$ means summation except for numbers p such that $J_p = J_{p-1}$. q.e.d.

Lemma 5.5. *Let ϕ satisfy $(A.1)$, then we have*

$$(5.6) \qquad |u^n_{J_n-1,x\bar{x}}| \leqq C^2_l \qquad \text{for } 0 \leqq h \leqq C^2_l/\beta^2 .$$

Proof. We note $u^n_{J_n-1,x\bar{x}} = u^n_{J_n-1,\bar{t}}$.
Consider the case $J_n = J_{n-1}+1$. By $u^n_{J_n} = u^{n-1}_{J_{n-1}} = 0$ and Lemma 5.2, we have

$$\begin{aligned} |u^n_{J_n-1,\bar{t}}| = |u^n_{J_{n-1},\bar{t}}| &= \left| \frac{u^n_{J_{n-1}} - u^n_{J_n}}{h} \cdot \frac{h}{k_n} \right| \\ &= |(-u^n_{J_n\bar{x}})(-u^{n-1}_{J_{n-1}\bar{x}})| \leqq C^2_l . \end{aligned}$$

Now consider the case $J_n = J_{n-1}$. We see

$$\begin{aligned} |u^n_{J_n-1,\bar{t}}| &= \left| \frac{u^n_{J_n-1} - u^n_{J_n}}{h} \cdot \frac{h}{k_n} + \frac{u^{n-1}_{J_{n-1}} - u^{n-1}_{J_{n-1}-1}}{h} \cdot \frac{h}{k_n} \right| \\ &= \left| \frac{h}{k_n} \{ -u^n_{J_n\bar{x}} + u^{n-1}_{J_{n-1}\bar{x}} \} \right| \leqq C^2_l \end{aligned}$$

in the same way. q.e.d.

The next two lemmas are obtained by Lemma 5.1, 5.2, 5.4, 5.5 and the Petrovskii's technique (see Lemma 16.1, 16.2 and 16.3).

Lemma 5.6. *Let ϕ satisfy $(A.1)$ and $\phi_{|[l',l]} \in C^2([l', l])$, then there exists a positive constant $C_{l,d}$ depending on l and d such that*

$$(5.7) \qquad |u^n_{j\bar{x}}| \leqq C_{l,d} , \qquad l' < d \leqq x_j \leqq s_n , \ 0 \leqq t_n \leqq T ,$$

$$(5.8) \qquad |u^n_{jx\bar{x}}| \leqq C_{l,d} , \qquad l' < d \leqq x_j < s_n , \ 0 \leqq t_n \leqq T .$$

Lemma 5.7. *Let ϕ satisfy $(A.1)$ and $\phi_{|[l',l]} \in C^3([l', l])$, then there exists a positive constant C_{d_1,d_2} depending on d_1 and d_2 such that*

$$(5.9) \qquad |u^n_{j\bar{x}\bar{t}}| = |u^n_{jx\bar{x}\bar{x}}| \leqq C_{d_1,d_2} , \qquad l' < d_1 \leqq x_j \leqq d_2 < s_n , \ 0 \leqq t_n \leqq T .$$

6. Estimates of the solutions under the condition (A.2)

In this section we obtain the estimates which are independent of l under the slightly stringent conditions on the data. These estimates are necessary for the proof of Theorem 3 and the two phase problem [20]. We introduce the following condition:

(A.2) $l>0$, $\phi(x)$ is non-negative, bounded, continuous for *a.e.* $x\in[0, l]$ and further there exist constants $K\geqq 0$, $d'\in D(\gamma)$ and $d\in D(\gamma)$ such that

$$\phi(x)\leqq K(l-x) \qquad \text{for any } x\in]0, l]\,,$$
$$-Kx+d'\leqq\phi(x)\leqq Kx+d \qquad \text{for any } x\in]0, l[\,.$$

REMARK 6.1. (A.2) implies (A.1), and (A.1) implies (A).

Lemma 6.1. *Let ϕ satsify (A.2), then there exists a positive constant $C(>K)$ independent of l such that*

$$-C\leqq u^n_{0x}\leqq C\,, \qquad t_n\leqq T\,. \tag{6.1}$$

Proof. Since γ is maximal monotone and $\gamma(H)\ni 0$, we see

$$D(\gamma)\cap[H, \infty[\,=\begin{cases} \text{(i)} \quad [H, P[\,, \text{ where } H<P<\infty\,, \\ \text{(ii)} \quad [H, \infty[\,, \qquad \text{or} \\ \text{(iii)} \quad [H, P]\,, \text{ where } H\leqq P<\infty\,. \end{cases}$$

We shall obtain the upper bound for above three cases respectively.

Case (i). $D(\gamma)\cap[H, \infty[=[H, P[$, where $H<P<\infty$. There exists e such that

$$d<e<P, \ \min\{\xi;\ \xi\in\gamma(e)\}>K\,. \tag{6.2}$$

We shall show

$$u^n_0\leqq e\,, \qquad t_n\leqq T\,. \tag{6.3}$$

Consider first the case $n=1$. Assuming $u^1_0>e$, then the function $w^n_j=Kx_j+e-u^n_j$ satisfies

$$L_h w^1_j=0\,,$$
$$w^0_j=Kx_j+e-\phi(x_j)\geqq Kx_j+d-\phi(x_j)\geqq 0\,,$$
$$w^1_{J_1}=Ks_1+e\geqq 0\,,$$
$$-w^1_{0x}=-K+u^1_{0x}(\in -K+\gamma(u^1_0))\geqq -K+\min\{\xi;\ \xi\in\gamma(e)\}\geqq 0\,,$$

by virtue of (A.2) and (6.2). By Lemma 3.3, we have $w^1_j\geqq 0$. For $j=0$, we get $w^1_0=e-u^1_0\geqq 0$ which is a contradiction. Thus we obtain $u^1_0\leqq e$. Next, applying the maximum principle, we have $w^1_j\geqq 0$, that is, $Kx_j+e\geqq u^1_j$. Therefore we see $u^2_0\leqq e$ in the same way. Repeating this argument we obtain (6.3). From $u^n_{0x}\in\gamma(u^n_0)$ and (6.3), we have

$$u^n_{0x}\leqq\max\{\sigma;\ \sigma\in\gamma(e)\}\,, \qquad t_n\leqq T\,. \tag{6.4}$$

Case (ii). $D(\gamma)\cap[H, \infty[=[H, \infty[$.
If $\sup\{\eta; (\xi, \eta)\in\gamma\}=\infty$, then (6.4) follows from the argument similar to Case (i). If $\sup\{\eta; (\xi, \eta)\in\gamma\}<\infty$, we have $u_{0x}^n\leqq\sup\{\eta; (\xi, \eta)\in\gamma\}$, $t_n\leqq T$.

Case (iii). $D(\gamma)\cap[H, \infty[=[H, P]$, where $H\leqq P<\infty$.
The function $W(x_j, t_n)=Kx_j+P$ satisfies

$$\begin{aligned} &L_h W_j^n = 0\,, \\ &W_{J_n}^n\geqq 0\,, \\ &W_j^0 = P+Kx_j\geqq d+Kx_j\geqq\phi(x_j)\,, \\ &W_0^n = P\geqq u_0^n\,, \end{aligned}$$

by (A.2) and $u_0^n\in D(\gamma)$. Hence by the maximum principle, we have $W_j^n\geqq u_j^n$. Especially for $j=1$, we see

$$K\geqq\frac{u_1^n-P}{h}, \qquad t_n\leqq T\,.$$

Therefore we have

$$\begin{cases} \dfrac{u_1^n-u_0^n}{h}\leqq K & \text{for } n \text{ with } u_0^n=P\,, \\ \gamma(u_0^n)\ni\dfrac{u_1^n-u_0^n}{h}\leqq\min\{\xi;\ \xi\in\gamma(P)\} & \text{for } n \text{ with } u_0^n<P\,. \end{cases}$$

We can obtain the lower bound similarly if we replace K by $\max\{K, (\inf D(\gamma)+1)/l\}$. q.e.d.

Next, we obtain the estimate of the free boundary by using the previous lemma.

Lemma 6.2. *Let ϕ satisfy (A.2), then we have*

$$0\leqq -u_{J_n\bar{x}}^n\leqq C\,, \qquad t_n\leqq T\,. \tag{6.5}$$

Proof. The inequality $-u_{J_n\bar{x}}^n\geqq 0$ is obvious by Lemma 5.1. We show the other inequality. Consider the function

$$w_{n_0}(x_j, t_n) = C(s_{n_0}-x_j)\,, \qquad 0\leqq j\leqq J_n\,,\ n\leqq n_0\,,$$

which satisfies the difference equation (2.6) and the inequalities

$$\begin{aligned} &w_{n_0}(s_n, t_n) = C(s_{n_0}-s_n)\geqq 0 = u_{J_n}^n\,, && n\leqq n_0\,, \\ &w_{n_0}(x_j, 0)\geqq K(l-x_j)\geqq\phi(x_j)\,, && 1\leqq j\leqq J_0, \end{aligned}$$

by (A.2), and

$$-w_{n_0}(0, t_n)_x = C\geqq -u_{0x}^n\,, \qquad n\leqq n_0\,,$$

by the previous lemma. By Lemma 3.3 we get

$$w_{n_0}(x_j, t_n) \geqq u_j^n, \qquad 0 \leqq j \leqq J_n, \; n \leqq n_0,$$

and especially for $n = n_0$ and $j = J_{n_0} - 1$, we have

$$C(s_{n_0} - x_{J_{n_0}-1}) \geqq u_{J_{n_0}-1}^{n_0}.$$

This implies (6.5) by virtue of $u_{J_{n_0}}^{n_0} = 0$. q.e.d.

Noting that Lemma 5.3, 5.4 and 5.5 are derived from Lemma 5.2, we obtain the following lemmas.

Lemma 6.3. *Let* ϕ *satisfy* (A.2). *then we have*

$$0 < \frac{h}{k_n} \leqq C \qquad \text{for } 0 < h \leqq C^2/\beta^2. \tag{6.6}$$

Lemma 6.4. *Let* ϕ *satisfy* (*A*.2), *then we have*

$$l \leqq s_n \leqq l + CT, \qquad t_n \leqq T. \tag{6.7}$$

Lemma 6.5. *Let* ϕ *satisfy* (*A*.2), *then we have*

$$|u_{J_n-1,x\bar{x}}^n| \leqq C^2 \qquad \text{for } 0 < h \leqq C^2/\beta^2. \tag{6.8}$$

7. L^2-estimates of the solutions

In this section we get the L^2-estimates of the difference solutions. We employ the ideas of the nonlinear semigroups (see Brézis [1, 2, 23] and Yotsutani [19]).

Since γ is a maximal monotone group in $\boldsymbol{R}^2$ with $\gamma(H) \ni 0$, there exists a lower semicontinuous convex function θ from $\boldsymbol{R}$ into $]-\infty, \infty]$ such that $\theta \not\equiv +\infty$, $\theta \geqq 0$, $\theta(H) = 0$ and $\partial\theta = \gamma$ by Lemma 3.5.

The following inequality is a so-called variational inequality.

Lemma 7.1. *Let* u_j^n *satisfy* (2.7) *and* (2.8). *Then we have*

$$\sum_{j=0}^{J_n-1} u_{jx}^n (w_{jx}^n - u_{jx}^n) h + \theta(w_0^n) - \theta(u_0^n) \geqq -\sum_{j=1}^{J_n-1} u_{jx\bar{x}}^n (w_j^n - u_j^n) h, \tag{7.1}$$

$$\frac{1}{2} \sum_{j=0}^{J_n-1} (w_{jx}^{n2} - u_{jx}^{n2}) h + \theta(w_0^n) - \theta(u_0^n) \geqq -\sum_{j=1}^{J_n-1} u_{jx\bar{x}}^n (w_j^n - u_j^n) h, \tag{7.2}$$

for w_j^n *such that* $w_{J_n}^n = 0$ *and* $w_0^n \in D(\theta)$.

Proof. We first show (7.1). For the sake of simplicity we drop n.

$$\sum_{j=1}^{J-1} u_{jx}(w_{jx} - u_{jx}) h + \theta(w_0) - \theta(u_0)$$
$$= u_{J\bar{x}}(w_J - u_J) - \sum_{j=1}^{J-1} (u_{jx} - u_{j-1,x})(w_j - u_j) - u_{0x}(w_0 - u_0) + \theta(w_0) - \theta(u_0)$$

$$\geqq -\sum_{j=1}^{J-1}(u_{jx}-u_{j-1,x})(w_j-u_j) = -\sum_{j=1}^{J-1} u_{jx\bar{x}}(w_j-u_j)h$$

by $w_J=u_J=0$ and $u_{0x}\in\partial\theta(u_0)$. (7.2) follows (7.1) by the inequality $ab\leqq (a^2+b^2)/2$. q.e.d.

The next lemma is useful for further estimates. Let us fix the function $v(x)$

$$v(x)=\begin{cases} -\dfrac{H}{l}(x-l) & \text{for } 0\leqq x\leqq l\,, \\ 0 & \text{for } l\leqq x\,. \end{cases} \tag{7.3}$$

Lemma 7.2. *Let u_j^n satisfy* (2.6)–(2.8). *Then we have*

$$\frac{1}{2}\sum_{p=1}^{n}\sum_{j=0}^{J_p-1} u_{jx}^{p2}hk_p+\sum_{p=1}^{n}\theta(u_0^p)k_p+\frac{1}{2}\sum_{j=1}^{J_n-1}(u_j^n-v_j)^2h \tag{7.4}$$

$$\leqq\frac{1}{2}\sum_{p=1}^{n}\sum_{j=0}^{J_p-1} v_{jx}^2hk_p+\sum_{p=1}^{n}\theta(v_0)k_p+\frac{1}{2}\sum_{j=1}^{J_0-1}(u_j^0-v_j)^2h\,,$$

where $v_j=v(x_j)$.

Proof. Substituting v_j for w_j^n in (7.2) and noting $u_{jx\bar{x}}^n=u_{j\bar{t}}^n$, we see

$$\begin{aligned} &\frac{1}{2}\sum_{j=0}^{J_n-1} v_{jx}^2h-\frac{1}{2}\sum_{j=0}^{J_n-1} u_{jx}^{n2}h+\theta(v_0)-\theta(u_0^n) \\ &\geqq -\sum_{j=1}^{J_n-1} u_{j\bar{t}}^n(v_j-u_j^n)h \\ &=\frac{1}{k_n}\sum_{j=1}^{J_n-1}\{(u_j^n-v_j)-(u_j^{n-1}-v_j)\}(u_j^n-v_j)h \\ &\geqq\frac{1}{k_n}\left\{\frac{1}{2}\sum_{j=1}^{J_n-1}(u_j^n-v_j)^2h-\frac{1}{2}\sum_{j=1}^{J_n-1}(u_j^{n-1}-v_j)^2h\right\}. \end{aligned} \tag{7.5}$$

Now we observe that

$$\frac{1}{2}\sum_{j=1}^{J_n-1}(u_j^{n-1}-v_j)^2h=\frac{1}{2}\sum_{j=1}^{J_{n-1}-1}(u_j^{n-1}-v_j)^2h \tag{7.6}$$

by $u_{J_{n-1}}^{n-1}=v_{J_{n-1}}=0$, since $J_n=J_{n-1}$ or $J_{n-1}+1$. Multiplying (7.5) k_n and noting (7.6), we have

$$\begin{aligned} &\frac{1}{2}\sum_{j=0}^{J_n-1} v_{jx}{}^2hk_n-\frac{1}{2}\sum_{j=0}^{J_n-1} u_{jx}^n{}^2hk_n+\theta(v_0)k_n-\theta(u_0^n)k_n \\ &\geqq\frac{1}{2}\sum_{j=1}^{J_n-1}(u_j^n-v_j)^2h-\frac{1}{2}\sum_{j=1}^{J_{n-1}-1}(u_j^{n-1}-v_j)^2h \end{aligned}$$

Therefore we obtain (7.4) by summing up. q.e.d.

The following lemma is the most important L^2-estimate.

Lemma 7.3. *Let u_j^n satisfy* (2.6)–(2.8). *Then we have*

$$(7.7)\qquad \sum_{p=2}^{n}\sum_{j=1}^{J_p-1} t_{p-1}(u_{jx\bar{x}}^p)^2 hk_p+\frac{1}{2}t_{n-1}\sum_{j=0}^{J_n-1} u_{jx}^{n\,2}h+t_{n-1}\theta(u_0^n)$$

$$\leqq \frac{1}{2}\sum_{p=1}^{n-1}\sum_{j=0}^{J_p-1} u_{jx}^{p\,2}hk_p+\sum_{p=1}^{n-1}\theta(u_0^p)k_p\,.$$

Proof. We set

$$\tilde{u}_j^{n-1}=\begin{cases} u_j^{n-1} & \text{for } 0\leqq j\leqq J_{n-1}\,,\\ 0 & \text{for } J_{n-1}\leqq j\,.\end{cases}$$

Taking $\tilde{u}^{n-1}$ as w_j^n in (7.2), we have

$$\frac{1}{2}\sum_{j=0}^{J_n-1}\tilde{u}_{j\ x}^{n-1\,2}h-\frac{1}{2}\sum_{j=0}^{J_n-1} u_{jx}^{n\,2}h+\theta(\tilde{u}_0^{n-1})-\theta(u_0^n)$$

$$\geqq -\sum_{j=1}^{J_n-1} u_{jx\bar{x}}^n(\tilde{u}_j^{n-1}-u_j^n)h\,.$$

By $\tilde{u}_j^{n-1}=0$ $(j\geqq J_{n-1})$ and $u_{jx\bar{x}}^n=u_{j\bar{t}}^n$ $(1\leqq j\leqq J_n-1)$, we have

$$(7.8)\qquad \frac{1}{2}\sum_{j=0}^{J_{n-1}-1} u_{j\ x}^{n-1\,2}h-\frac{1}{2}\sum_{j=1}^{J_n-1} u_{jx}^{n\,2}h+\theta(u_0^{n-1})-\theta(u_0^n)$$

$$\geqq\sum_{j=1}^{J_n-1} u_{jx\bar{x}}^{n\,2}hk_n\,.$$

Multiplying (7.8) t_{n-1}, and noting $k_{n-1}=t_{n-1}-t_{n-2}$, we get

$$\frac{1}{2}k_{n-1}\sum_{j=0}^{J_{n-1}-1} u_{j\ x}^{n-1\,2}h+\frac{1}{2}t_{n-2}\sum_{j=0}^{J_{n-1}-1} u_{j\ x}^{n-1\,2}h$$

$$-\frac{1}{2}t_{n-1}\sum_{j=0}^{J_n-1} u_{jx}^{n\,2}h+k_{n-1}\theta(u_0^{n-1})+t_{n-2}\theta(u_0^{n-1})-t_{n-1}\theta(u_0^n)$$

$$\geqq\sum_{j=1}^{J_n-1} t_{n-1}(u_{jx\bar{x}}^n)^2hk_n\,.$$

Therefore we obtain (7.7) by summing up. q.e.d.

Combining Lemma 7.2 and 7.3, we have the following lemma.

Lemma 7.4. *Let u_j^n satisfy* (2.6)–(2.8). *Then we have*

$$(7.9)\qquad \sum_{p=2}^{n}\sum_{j=1}^{J_p-1} t_{p-1}(u_{jx\bar{x}}^p)^2hk_p+\frac{1}{2}t_{n-1}\sum_{j=1}^{J_n-1} u_{jx}^{n\,2}h+t_{n-1}\theta(u_0^n)$$

$$\leqq\frac{1}{2}\sum_{p=1}^{n-1}\sum_{j=0}^{J_p-1} v_{jx}^{\ 2}hk_p+\sum_{p=1}^{n-1}\theta(v_0)k_p+\frac{1}{2}\sum_{j=1}^{J_0-1}(u_j^0-v_j)^2h\,.$$

Now we get the necessary estimate.

Lemma 7.5. *Let ϕ satisfy* $(A.1)$. *Then there is a positive constant $\tilde{C}_l$ such that*

(7.10) $\sum_{p=2}^{n}\sum_{j=1}^{J_p-1} t_{p-1}(u^p_{jx\bar{x}})^2hk_p+\frac{1}{2}t_{n-1}\sum_{j=0}^{J_n-1} u^n_{jx}{}^2h+t_{n-1}\theta(u^n_0)\leqq\tilde{C}_l\,.$

Proof. We have

(7.11) $$l\leqq\sum_{j=0}^{J_p-1} h = s_p\leqq l+C_lT$$

by Lemma 5.4, and

(7.12) $$|v_{jx}|\leqq H/l\,,\qquad \theta(v_0)=\theta(H)=0\,,$$

by the definition. Combining Lemma 7.4, (7.11) and (7.12), we get (7.10).

8. Convergence of the scheme

In this section we prove the convergence of the scheme when $\phi(x)$ satisfies the following condition (H.1).

(H.1) $\phi(x)$ satisfies (A.1) and $\phi_{|[l',l]}\in C^3([l',l])$ for some l',

$$0\leqq l'<l\,.$$

Now we take a sequence $\{h\}$ ($h\to 0$) such that $h=h_q=l/2^q$. Define a continuous function $s_h(t)$ connecting the adjacent points (s_n, t_n) and (s_{n+1}, t_{n+1}) by the straight line for each interval $[t_n, t_{n+1}]$. It follows from Lemma 5.3 and Ascoli-Arzela's theorem that a subsequence of $s_h(t)$ converges uniformly on $[0, T]$ to a continuous function $s(t)$.

We shall show that the net functions u^n_j can be extended to the region $\bar{G}=[0, \infty[\times[0, T]$ in such a way that the family of extended functions $\{u_h(x, t)\}_h$ will be uniformly bounded on $\bar{G}$ and equicontinuous in any region G^* whose closure is contained in $G=]0, \infty[\times]0, T]$. First we extend u^n_j to all the mesh points of $\bar{G}$ by defining $u^n_j=0$ ($j\geqq J_n$). u^j_n denotes extended u^n_j again. We devide each rectangle $[x_j, x_{j+1}]\times[t_n, t_{n+1}]$ into two triangles by a straight line connecting (x_j, t_n) and (x_{j+1}, t_{n+1}). We define $u_h(x, t)$ as a piecewise linear function which equals to the value of net function u^n_j at the corner of the triangles. It is easy to see that the function $u_h(x, t)$ constructed in this way is continuous in $\bar{G}$, and that, it has the maximum at a mesh point. Hence we have by Lemma 5.1,

(8.1) $$0\leqq u_h(x, t)\leqq\max\{\|\phi\|_{L^\infty}, H\}\qquad \text{in } \bar{G}\,.$$

Further we have the following lemma. For a proof, see § 16.

Lemma 8.1. *Let ϕ satisfy (A.1). Then there exists a constant C_τ depending on τ such that*

$$|u_h(x, t)-u_h(\tilde{x}, \tilde{t})|\leqq C_\tau\{|x-\tilde{x}|^{1/2}+|t-\tilde{t}|^{1/4}\}\qquad (x, \tilde{x}>0,\ t, \tilde{t}>\tau>0)\,.$$

By (8.1), Lemma 8.1 and Ascoli-Arzelà's theorem it follows that a subsequence of $u_h(x, t)$ converges uniformly in G to a function $u(x, t)\in C(G)$. Since $u_h(s_h(t), t)=0$, it is easily shown that

$$u(s(t), t) = 0 \qquad (0<t\leqq T)\,. \tag{0.3}$$

By using the Petrovskii's technique [14, p. 357–358] we have

$$\lim_{t\downarrow 0} u(x, t) = \phi(x) \tag{0.4}$$

for any continuous point of ϕ.

We prepare some lemmas to investigate the unilateral boundary condition (0.2).

Lemma 8.2. *Let ϕ satisfy (A.1). Then we have*

(i) $u(x, t)\in C^\infty(D)\cap C(\bar{D}-Z)$ *and*

$$u_{xx}-u_t = 0 \qquad in\ D\,. \tag{0.1}$$

(ii) $u(\cdot, t)\in C[([0, s(t)])$ *for any* $t\in]0, T]$.

(iii) $u(0, t)$ *is bounded on* $]0, T]$ *and* $u(0, t)\in C(]0, T])$.

(iv) *For any* $t\in]0, T]$ *and* $\varepsilon>0$, *there exist* $\delta>0$ *and* $h_0>0$ *such that*

$$|u_h(0, s)-u_h(0, t)|<\varepsilon \qquad for\ |s-t|<\delta\ and\ h\leqq h_0\,.$$

In particular $u_h(0, t)\to u(0, t)$ *as the subsequence* $h\to 0$.

(v)
$$\int_0^t\int_0^{s(\tau)} \tau u_{xx}^2 dx d\tau+\frac{1}{2} t \int_0^{s(t)} u_x^2 dx+t\theta(u(0, t))$$
$$\leqq \tilde{C}_l \qquad for\ t\in]0, T]\,.$$

Proof. We prove that $u\in C^\infty(D)$ and $u_{xx}-u_t=0$ in § 16. We have

$$|u(x, t)-u(\tilde{x}, \tilde{t})|\leqq C_\tau\{|x-\tilde{x}|^{1/2}+|t-\tilde{t}|^{1/4}\} \qquad (x, \tilde{x}>0,\ t, \tilde{t}>\tau) \tag{8.2}$$

by Lemma 8.1. Hence we obtain (i), (ii) and (iii) by (8.1) and (8.2). We observe that

$$|u(x, t)-u(0, t)|\leqq x^{1/2}\left(\int_0^x u_\xi^2 d\xi\right)^{1/2}\leqq\{2x\tilde{C}_l/t\}^{1/2} \qquad (x>0) \tag{8.3}$$

by using Lemma 7.5. We have

$$\begin{aligned}
&|u_h(0, s)-u(0, t)|\\
&\leqq |u_h(0, s)-u_h(x, s)|+|u_h(x, s)-u(x, s)|\\
&\quad +|u(x, s)-u(0, s)|+|u(0, s)-u(0, t)|\\
&\leqq \{4x\tilde{C}_l/t\}^{1/2}+|u_h(x, s)-u(x, s))|\\
&\quad +\{4x\tilde{C}_l/t\}^{1/2}+|u(0, s)-u(0, t)|, \qquad \text{for } s>\frac{t}{2}\,,
\end{aligned}$$

by Lemma 7.5 and (8.3). For any $\varepsilon>0$, there exists x such that $\{4x\tilde{C}_l/t\}^{1/2}<\varepsilon/4$. Let fix x. Then there exist $h_0>0$ such that

$$|u_h(x, s)-u(x, s)|<\varepsilon/4 \qquad \text{for } h\leqq h_0,$$

and $\delta>0$ such that

$$|u(0, s)-u(0, t)|<\varepsilon/4 \qquad \text{for } s \text{ with } |s-t|<\delta.$$

Thus (iv) holds. It is easy to show (v) by using Lemma 7.5 and (iv). q.e.d..

Lemma 8.3. *Let ϕ satisfy (A.1). Then a.e. $t\in[0, T]$ $u_x(0, t)$ exists, and $u_x(0, t)\in L^2_{\mathrm{loc}}(]0, T])$.*

Proof. By Lemma 8.2 (v), we have

$$\int_0^T\int_0^{s(t)} t u_{xx}{}^2 dxdt \leqq \tilde{C}_l. \tag{8.4}$$

Therefore, for *a.e.* $t\in]0, T[$, there exist $C_t>0$ such that $\int_0^{s(t)} u_{xx}{}^2 dx\leqq C_t$. Hence, for *a.e.* $t\in]0, T[$, $u_x(0, t)$ exists and

$$u_x(0, t) = \lim_{x\to 0} u_x(x, t),$$

$$u_x(0, t) = u_x(l/2, t)+\int_{l/2}^0 u_{xx}(x, t)dx.$$

Thus we have

$$|u_x(0, t)|\leqq |u_x(l/2, t)|+\int_0^{s(t)}|u_{xx}|dx, \qquad a.e.\ t\in]0, T[.$$

This implies $u_x(0, t)\in L^2_{\mathrm{loc}}(]0, T])$ by Lemma 8.2 (i) and (8.4). q.e.d.

Now we investigate the unilateral boundary condition.

Lemma 8.4. *Let ϕ satisfy (A.1). Then for a.e. $t\in[0, T]$, $u_x(0, t)$ exists and satisfy*

$$u_x(0, t)\in\gamma(u(0, t)). \tag{0.2}$$

Proof. Fix $0<\tau_1<\tau_2\leqq T$. Let $t_{m-1}<\tau_1\leqq t_m\leqq t_n<\tau_2\leqq t_{n+1}$. By Lemma 7.1, we have

$$\begin{aligned} &\textstyle\sum_{p=m+1}^n \sum_{j=0}^{J_p-1} u^p_{jx}(w^p_{jx}-u^p_{jx})hk_p+\sum_{p=m+1}^n \theta(w^p_0)k_p-\sum_{p=m+1}^n \theta(u^p_0)k_p \\ &\qquad \geqq -\textstyle\sum_{p=m+1}^n \sum_{j=1}^{J_p-1} u^p_{jx\bar{x}}(w^p_j-u^p_j)hk_p, \end{aligned} \tag{8.6}$$

where $\eta\in D(\theta)$ and

$$w_j^p = \begin{cases} -\frac{\eta}{l}(x_j - l) & \text{for } 0 \leqq x_j \leqq l, \\ 0 & \text{for } l \leqq x_j. \end{cases}$$

In view of Lemma 7.5, it follows that

$$\sum_{p=m+1}^{n} \sum_{j=0}^{J_p-1} u_{jx\bar{x}}^{p\,2} hk_p \leqq \tilde{C}_l/\tau_1. \tag{8.7}$$

With the aid of (8.7), the weak compactness of $L^2(D)$ and the fact $u_h \to u$ pointwise in D, we have

$$\int_{\tau_1}^{\tau_2}\int_0^{s(t)} u_{xx}{}^2 dxdt \leqq \tilde{C}_l/\tau_1, \tag{8.8}$$

$$\sum_{p=m+1}^{n} \sum_{j=0}^{J_p-1} u_{jx\bar{x}}^p (w_j^p - u_j^p) hk_p \to \int_{\tau_1}^{\tau_2}\int_0^{s(t)} u_{xx}(w-u)dxdt, \tag{8.9}$$

as $h \to 0$ (through the subsequence of $\{h\}$). Noting (8.9) and Lemma 8.2 (iv), and applying Lebesgue's convergence theorem and Fatou's lemma to (8.6), we get

$$\begin{aligned} &\int_{\tau_1}^{\tau_2}\int_0^{s(t)} u_x(w-u)_x dxdt + \int_{\tau_1}^{\tau_2} \theta(\eta)dt - \int_{\tau_1}^{\tau_2} \theta(u(0, t))dt \\ &\geqq -\int_{\tau_1}^{\tau_2}\int_0^{s(t)} u_{xx}(w-u)dxdt \end{aligned} \tag{8.10}$$

Since we can integrate by parts in view of Lemma 8.3, it follows that

$$\begin{aligned} &\int_0^{s(t)} u_{xx}(w-u)dx \\ &= u_x(s(t), t)(w(s(t), t) - u(s(t), t)) - u_x(0, t)(w(0, t) - u(0, t)) \\ &\quad - \int_0^{s(t)} u_x(w-u)_x dx \\ &= -u_x(0, t)(\eta - u(0, t)) - \int_0^{s(t)} u_x(w-u)_x dx \end{aligned} \tag{8.11}$$

for *a.e.* $t \in [0, T]$. Thus it is easily seen that

$$\int_{\tau_1}^{\tau_2} \theta(\eta)dt - \int_{\tau_1}^{\tau_2} \theta(u(0, t))dt \geqq \int_{\tau_1}^{\tau_2} u_x(0, t)(\eta - (0, t))\,dt$$

by (8.10) and (8.11). Hence at all Lebesgue's points of $\theta(u(0, t))$ and $u_x(0, t)$, we have

$$\theta(\eta) - \theta(u(0, t)) \geqq u_x(0, t)(\eta - u(0, t))$$

for all $\eta \in D(\theta)$, which implies

$$u_x(0, t) \in \partial\theta(u(0, t)) = \gamma(u(0, t)) \qquad \text{for } a.e.\ t \in [0, T]$$

by the definition of $\partial\theta$ in Lemma 3.5. q.e.d.

We shall investigate the Stefan's condition (0.5).

Lemma 8.5. *Let ϕ satisfy $(H, 1)$. Then $s(t) \in C^1(]0, T])$ and*

$$\dot{s}(t) = -u_x(s(t), t), \qquad 0 < t \leqq T. \tag{0.5}$$

Proof. We see that $\lim_{\varepsilon \downarrow 0} u_x(s(t)-\varepsilon, t) = u_x(s(t), t)$ exists and $u_x(s(t), t) \in C(]0, T])$ using [10] or [3, Lemma 1] since $s(t)$ is Lipshitz continuous on $[0, T]$. We shall show

$$\lim_{h \to 0} \sum_{n=0}^{m-1} u^n_{J_n\bar{x}} k_{n+1} = \int_0^t u_x(s(\tau), \tau) d\tau \qquad (t_m \leqq t < t_{m+1}). \tag{8.12}$$

We put $l_0 = (l_1 + l_2)/2$, $l_1 = l' + (l-l')/4$, $l_2 = l - (l-l')/4$. We define $r = r(h) = \min\{j;\ x_j = jh \geqq l_0\}$. We have

$$\begin{aligned} \sum_{n=0}^{m-1} u^n_{J_n\bar{x}} k_{n+1} &= \sum_{n=0}^{m-1} u^n_{r\bar{x}} k_{n+1} + \sum_{n=0}^{m-1} \sum_{j=r}^{J_n-1} u^n_{jx\bar{x}} h k_{n+1} \\ &= \sum_{n=0}^{m-1} u_{\bar{x},h}(x_r, t_n) k_{n+1} + \int_0^{t_m} \int_{x_r}^{\infty} u^*_{x\bar{x},h}(x, \tau) dx d\tau, \end{aligned} \tag{8.13}$$

where $u_{\bar{x},h}$ is a piecewise linear function naturally extended from the net function $u^n_{j\bar{x}}$, and $u^*_{x\bar{x},h}$ is a step function naturally extended from the net function $u^n_{jx\bar{x}}$ by defining $u^*_{x\bar{x},h}(\xi, \tau) = 0$ if $(\xi, \tau) \notin \bar{D}_h$. We observe that $\{u_{\bar{x},h}\}_h$ are uniformly bounded and equicontinuous on $[l_1, l_2] \times [0, T]$ by Lemma 5.6 and 5.7, $\{u^*_{x\bar{x},h}\}_h$ are uniformly bounded on $[l_1, \infty[\times [0, T]$ by (5.8), and $u_{\bar{x},h} \to u_x$, $u^*_{x\bar{x},h} \to u_{xx}$ hold in the distribution's sense by the argument similar to the proof of Lemma 8.2 (i). Hence we have

$$\lim_{h \to 0} \sum_{n=0}^{m-1} u^n_{J_n\bar{x}} k_{n+1} = \int_0^t u_x(l_0, \tau) d\tau + \int_0^t \int_0^{s(\tau)} u_{xx} dx d\tau = \int_0^t u_x(s(\tau), \tau) d\tau$$

by (8.13), Ascoli-Arzela's theorem and the weak compactness of L^2.

Now we have

$$\begin{aligned} s_h(t) &= l + \int_0^t \dot{s}_h(\tau) d\tau \qquad (t_m \leqq t < t_{m+1}) \\ &= l + \sum_{n=0}^{m-1} \{ \operatorname*{sign}_{t_n \leqq \tau < t_{n+1}} \dot{s}_h(\tau)\} (h/k_{n+1}) k_{n+1} + \int_{t_m}^t \dot{s}_h(\tau) d\tau, \end{aligned}$$

where

$$\operatorname*{sign}_{t_u \leqq \tau < t_{n+1}} \dot{s}_h(\tau) = \begin{cases} 1 & (\text{if } \dot{s}_h(\tau) > 0 \text{ on } [t_n, t_{n+1}[), \\ 0 & (\text{if } \dot{s}_h(\tau) = 0 \text{ on } [t_n, t_{n+1}[). \end{cases}$$

This can be put in the form

$$s_h(t) = l - {\sum}'^{m-1}_{n=0} u^n_{J_n\bar{x}} k_{n+1} + \int_{t_m}^t \dot{s}_h(\tau) d\tau ,$$

where $\sum'$ means the summation except for number n such that $\dot{s}_h(t_n+0)=0$, which occurs if $|u^n_{J_n\bar{x}}| \leqq \beta\sqrt{h}$. Hence

$$s_h(t) = l - \sum^{m-1}_{n=0} u^n_{J_n\bar{x}} k_{n+1} + \int_{t_m}^t \dot{s}_h(\tau) d\tau + O(\sqrt{h}) , \qquad 0 < t \leqq T .$$

Take $h \to 0$ through the subsequence. Then we get

$$s(t) = l - \int_0^t u_x(s(\tau), \tau) d\tau$$

by (8.12), Lemma 5.2 and Lebesgue's convergence theorem. This means $s(t) \in C^1(]0, T])$ and $\dot{s}(t) = -u_x(s(t), t)$. q.e.d.

REMARK 8.1. It can be shown that $s(t) \in C^\infty(]0, T])$ and $u \in C^\infty(D^s)$ by virtue of Schaeffer [16] and Lemma 8.5.

REMARK 8.2. We will show the uniqueness of the solution in § 12. Therefore the full sequence of $(s_h(t), u_h(x, t))$ converges to the true solution.

We have proved the existence of a solution (s,u) under the condition (H.1).

9. Moving boundary problem

Consider the following moving boundary problem: Given the data ϕ and the given non-decreasing function $s(t) \in C([0, T]) \cap C^{0,1}(]0, T])$, that is positive for $t > 0$, find a function $u = u(x, t)$ such that

$$\text{(M)} \quad \begin{cases} (9.1) \quad Lu \equiv u_{xx} - u_t = 0 , & 0 < x < s(t) ,\ 0 < t \leqq T , \\ (9.2) \quad u_x(0, t) \in \gamma(u(0, t)) , & 0 < t \leqq T , \\ (9.3) \quad u(s(t), t) = 0 , & 0 < t \leqq T , \\ (9.4) \quad u(x, 0) = \phi(x) , & 0 < x < s(0) \equiv l . \end{cases}$$

Here T is a fixed but arbitrary positive number. γ is a maximal monotone graph in $\boldsymbol{R}^2$ with $\gamma(H) \ni 0$ where H is a constant.

REMARK 9.1. $C^{0,1}(]0, T])$ denotes the space of Lipshitz continuous functions on $]0, T]$. We need not assume the non-negativity of ϕ and H.

DEFINITION 9.1. $u = u(x, t)$ is a solution of the moving boundary problem (M) if

i) u is bounded on $\bar{D}$. $u \in C^\infty(D) \cap C(\bar{D} - Z)$,

$$\int_0^T \int_0^{s(t)} t u_{xx}(x, t)^2 dx dt < +\infty ,$$

where D and Z are the sets defined by (1.1) and (1.2) respectively.

ii) (9.1), (9.3) and (9.4) hold.

iii) For *a.e.* $t \in [0, T]$, $u_x(0, t)$ exists and satisfies (9.2).

REMARK 9.2. If $l=0$, there is no ϕ, and we omit (9.4).

Proposition 9.1. *If the data ϕ is bounded and continuous for a.e. $x \in [0, l]$ when $l>0$, or we suppose further that $\gamma(0)$ is not empty when $l=0$, then there exists a solution u of the moving boundary problem* (M).

REMARK 9.3. The uniqueness will be shown in Proposition 10.1.

We shall show Proof. We use the difference method and modify the difference scheme in § 2.

First consider the case $l>0$. We use a net of rectangular meshes with the uniform space width h and variable time steps $\{k_n\}$ $(n=1, 2, \cdots)$. Here h varies in such a way that $l/h \equiv J_0$ is an integer. We follow the notations from (2.1) to (2.9) in § 2.

Our algorism is the following.

1° $u_j^0 = \phi_j$ $(1 \leqq j \leqq J_0)$, $s_0 = J_0 \cdot h = l$.

For $n=1, 2, \cdots$ successively,

2.1° if $s(t_{n-1}+\sqrt{h})-s_{n-1}>h$, then we take $J_n=J_{n-1}+1$ and $k_n=\min\{t \geqq t_{n-1}; s(t)-s_{n-1} \geqq h\}-t_{n-1}$,

2.2° if $s(t_{n-1}+\sqrt{h})-s_{n-1} \leqq h$, then we take $J_n=J_{n-1}$ and $k_n=\sqrt{h}$,

3° solve the difference equation (2.6) for $\{u_j^n\}_j$ under the boundaıy condition (2.7) and (2.8) with the initial condition $\{u_j^{n-1}\}$.

The step 3° is well-defined by virtue of Lemma 4.1. By the proof of Lemma 5.1 we have

Lemma 9.1. *u_j^n is uniformly bounded with respect to h.*

In view of Lemma 7.4 and the proof of Lemma 7.5 we obtain

Lemma 9.2. *There is a positive constant $\tilde{C}_l$ such that*

$$\sum_{p=2}^{n} \sum_{j=1}^{J_p-1} t_{p-1}(u_{jx\bar{x}}^p)^2 h k_p + \frac{1}{2} t_{n-1} \sum_{j=0}^{J_n-1} u_{jx}^{n\,2} h + t_{n-1}\theta(u_0^n) \leqq \tilde{C}_l . \tag{9.5}$$

Noting that Lemma 8.1–8.4 holds in view of Lemma 7.5, the conclusions of Lemma 8.1–8.4 hold in this case, too. Hence we can easily show the convergence of the scheme as in § 8.

Now consider the case $l=0$. We change the scheme only for $n=0, 1, 2$. Define $t_0=0$, $t_1=\min\{t>0; s(t)=h\}$, $t_2=\min\{t>0; s(t)=2h\}$, $u_j^2=0$ $(0 \leqq j \leqq 2)$.

We use the same algorism as the case $l>0$ for $n>2$. Then noting that we can use $v\equiv 0$ in place of v defined in (7.3) by virtue of the assumption that $\gamma(0)$ is not empty, we can show the convergence of the scheme in the case $l=0$.

10. Comparison theorems for the moving boundary problem

We will show the uniqueness of the solution of the moving boundary problem (M) stated in § 9 and the comparison theorems. We need not assume that $s(t)$ is a non-decreasing function in this section.

First we prepare the fundamental lemma.

Lemma 10.1. *For a given function $s(t)\in C([0, T])\cap C^{0,1}(]0, T])$, let $p(x, t)$ and $q(x,t)$ be functions satisfying*

$$(10.1)\qquad p,\ q\in C^{\infty}(D)\cap C(\bar{D}-Z'\times\{0\})\cap L^{\infty}(\bar{D})$$

$$(10.2)\qquad \int_{\tau}^{T}\int_{0}^{s(t)}(p_{xx}^{2}+q_{xx}^{2})dxdt<+\infty \qquad \textit{for each } \tau\in]0, T],$$

$$(10.3)\qquad p_{xx}-p_{t}=0 \quad \textit{in } D,\ q_{xx}-q_{t}=0 \quad \textit{in } D,$$

$$(10.4)\qquad p(x, 0)\geqq q(x, 0) \qquad \textit{a.e. } x\in]0, l[,$$

$$(10.5)\qquad p(s(t), t)\geqq q(s(t), t) \qquad \textit{for all } t\in]0, T],$$

$$(10.6)\qquad (q_{x}(0, t)-p_{x}(0, t))(q(0, t)-p(0, t))^{+}\geqq 0 \qquad \textit{a.e } t\in]0, T],$$

where $\alpha^{+}=\max\{\alpha, 0\}$, $s(0)=l\geqq 0$,

$D=\{(x, t);\ 0<x<s(t),\ 0<t\leqq T\}$,

$Z'=\{x\in[0, l];\ x$ *is a discontinuous point of p or q*$\}$, *and Z' is a set of zero measure. Then we have*

$$p(x, t)\geqq q(x, t) \qquad \textit{in } D.$$

REMARK 10.1. If $s(0)=l=0$, there are no $p(x, 0)$ and $q(x, 0)$. We omit (10.4) and define $Z'=\{0\}$.

Proof of Lemma 10.1. We employ the idea of Brézis [1, p. 109–110]. Multiply the first equality of (10.3) by $v-p$ and integrate over $[\delta, s(t)-\delta]$, then

$$\int_{\delta}^{s(t)-\delta}p_{t}(v-p)\,dx-\int_{\delta}^{s(t)-\delta}p_{xx}(v-p)dx=0 \qquad \text{for } 0<t\leqq T.$$

With the aid of the integration by parts,

$$\int_{\delta}^{s(t)-\delta}p_{t}(v-p)dx-[p_{x}(v-p)]_{\delta}^{s(t)-\delta}+\int_{\delta}^{s(t)-\delta}p_{x}(v-p)_{x}dx=0.$$

Taking $v=\max\{p, q\}=p+(q-p)^{+}$, we obtain

$$(10.7)\qquad \int_{\delta}^{s(t)-\delta}p_{t}(q-p)^{+}dx-[p_{x}(q-p)^{+}]_{\delta}^{s(t)-\delta}+\int_{\delta}^{s(t)-\delta}p_{x}\{(q-p)^{+}\}_{x}dx=0.$$

Similarly we see that

$$\int_{\delta}^{s(t)-\delta} q_t(w-q)dx-[q_x(w-q)]_{\delta}^{s(t)-\delta}+\int_{\delta}^{s(t)-\delta} q_x(w-q)_x dx=0\,.$$

Taking $w=\min\{p, q\}=q-(q-p)^+$, we obtain

$$(10.8)\quad -\int_{\delta}^{s(t)-\delta} q_t(q-p)^+dx+[q_x(q-p)^+]_{\delta}^{s(t)-\delta}-\int_{\delta}^{s(t)-\delta} q_x\{(q-p)^+\}_x dx=0\,.$$

Adding (10.7) to (10.8), we get

$$(10.9)\quad \int_{\delta}^{s(t)-\delta}(q-p)_t(q-p)^+dt+\int_{\delta}^{s(t)-\delta}(q-p)_x\{(q-p)^+\}_x dx = [(q-p)_x(q-p)^+]_{\delta}^{s(t)-\delta}\,.$$

We note

$$(10.10)\quad \frac{1}{2}\frac{d}{dt}\int_{\delta}^{s(t)-\delta}\{(q-p)^+\}^2dx = \int_{\delta}^{s(t)-\delta}(q-p)_t(q-p)^+dx+\frac{1}{2}\dot{s}(t)\{(q(s(t)-\delta, t)-p(s(t)-\delta, t))^+\}^2$$

and

$$(10.11)\quad \int_{\delta}^{s(t)-\delta}(q-p)_x\{(q-p)^+\}_x dx=\int_{\delta}^{s(t)-\delta}\{(q-p)^+\}_x^2dx\geqq 0\,.$$

Consequently using (10.10) and (10.11) in (10.9), and integrate over $[\varepsilon, t]$, we get

$$(10.12)\quad \frac{1}{2}\int_{\delta}^{s(t)-\delta}\{(q-p)^+\}^2dx-\frac{1}{2}\int_{\delta}^{s(\varepsilon)-\delta}\{(q-p)^+\}^2dx \leqq \int_{\varepsilon}^{t}[(q-p)_x(q-p)^+]_{\delta}^{s(\tau)-\delta}d\tau +\frac{1}{2}\int_{\varepsilon}^{t}\dot{s}(\tau)\{(q(s(\tau)-\delta, \tau)-p(s(\tau)-\delta, \tau))^+\}^2d\tau\,.$$

With the aid of (10.2) and the proof of Lemma 8.3, we can use Lebesgue's convergence theorem. Letting $\delta\to 0$ and using (10.5), we obtain

$$(10.13)\quad \frac{1}{2}\int_{0}^{s(t)}\{(q-p)^+\}^2dx-\frac{1}{2}\int_{0}^{s(\varepsilon)}\{(q-p)^+\}^2dx \leqq -\int_{\varepsilon}^{t}(q_x(0, \tau)-p_x(0, \tau))(q(0, \tau)-p(0, \tau))^+d\tau\,.$$

Applying (10.6) for (10.13) and letting $\varepsilon\to 0$, we get

$$\frac{1}{2}\int_{0}^{s(t)}\{(q-p)^+\}^2dx\leqq\frac{1}{2}\int_{0}^{l}\{(q(x, 0)-p(x, 0))^+\}^2dx=0\,,\qquad 0<t\leqq T\,,$$

in view of (10.4), which implies $q \leqq p$ in D. q.e.d.

REMARK 10.2. We will also use Lemma 10.1 in [20] and [21].

As a consequence of Lemma 10.1 we have the following proposition.

Proposition 10.1. *Let u and $\tilde{u}$ be solutions of* (M) *corresponding, respectively, to the data* $\{s(t), u(x, 0)\}$ *and* $\{\tilde{s}(t), \tilde{u}(x, 0)\}$ *which satisfy the assumptions stated in Proposition* 9.1, $s(t) \leqq \tilde{s}(t)$, $u(x, 0) \leqq \tilde{u}(x, 0)$ *and* $\tilde{u}(s(t), t) \geqq u(s(t), t) = 0$. *Then we have*

$$0 \leqq \tilde{u} - u \leqq \max \{||\tilde{u}(\cdot, 0) - u(\cdot, 0)||_{L^\infty}, ||\tilde{u}(s(t), t)||_{L^\infty(0,T)}\} \tag{10.14}$$

in $\bar{D} = \{(x, t); 0 \leqq x \leqq s(t), 0 \leqq t \leqq T\}$, *where* $L^\infty = L^\infty(0, l)$, $l = s(0)$ *and we define* $||\tilde{u}(\cdot, 0) - u(\cdot, 0)||_{L^\infty} = 0$ *when* $l = 0$.

Proof. Noting the boundary condition (9.2) and the monotonicity of γ, and applying Lemma 10.1 in $\bar{D}$, for $p(x, t) = \tilde{u}(x, t)$ and $q(x, t) = u(x, t)$, we get $\tilde{u}(x, t) \geqq u(x, t)$. Similarly applying Lemma 10.1 in $\bar{D}$, for $p(x, t) = u(x, t) + \max \{||\tilde{u}(\cdot, 0) - u(\cdot, 0)||_{L^\infty}, ||u(s(t), t)||_{L^\infty}\}$ and $q(x, t) = \tilde{u}(x, t)$, we have the right part of the inequality (10.14). q.e.d.

REMARK 10.3. Proposition 10.1 implies the uniqueness of the solution of the moving boundary problem (M).

Proposition 10.2. *Let u be a solution of (M). If we suppose*

$$u(x, 0) = \phi(x) \geqq 0, \qquad 0 < x < s(0) = l,$$

especially when $l > 0$, and $H \geqq 0$. Then we have

$$0 \leqq u(x, t) \leqq \max \{H, ||\phi||_{L^\infty(0,l)}\},$$

where we define $||\phi||_{L^\infty(0,l)} = 0$ *when* $l = 0$.

Proof. Putting $k = \max \{H, ||\phi||_{L^\infty(0,l)}\}$, we obtain

$$(u(0, t)_x - k)(u(0, t) - k)^+ \geqq 0, \qquad \text{a.e. } t \in]0, T],$$

by virtue of the assumption $\gamma(H) \ni 0$ with $H \geqq 0$, the monotonicity of γ and the boundary condition (9.2). Hence applying Lemma 10.1 for $p(x, t) \equiv k$ and $q(x, t) = u(x, t)$, we have $k \geqq u(x, t)$. Similarly applying Lemma 10.1 for $p(x, t) = u(x, t)$ and $q(x, t) \equiv 0$, we get $u(x, t) \geqq 0$.

11. Reformation of the free boundary condition

We now return to the general situation of the Stefan problem (S). We state useful results concerning the reformation of the free boundary condition (0.5).

Lemma 11.1. *Let* (s, u) *be a solution of* (S), *then it follows that for any* $\sigma, t \in]0, T]$

$$(11.1) \qquad s(t) = s(\sigma) + \int_0^{s(\sigma)} u(x, \sigma)dx - \int_0^{s(t)} u(x, t)dx - \int_\sigma^t u_x(0, \tau)d\tau .$$

Conversely if $s(t) \in C([0, T]) \cap C^{0,1}(]0, T])$, u *satisfies* (M) *and* (11.1), *then we have* $\dot{s}(t) = -u_x(s(t), t)$, $t \in]0, T]$.

Proof. Let (s, u) be a solution of (S). Integrate (0.1) over its domain of validity, (11.1) follows in view of (0.3) and (0.5). Conversely suppose that (s, u) satisfies (M) and (11.1). By differentiating (11.1), $\dot{s}(t) = -u_x(s(t), t)$ follows in view of (0.3) if $u_x(s(t), t)$ exists and is continuous for $0 < t \leqq T$. But this last assumption is guaranteed by Geverey [10] or Lemma 1 of Cannon & Hill [3]. q.e.d.

Lemma 11.2. *Let* (s, u) *be a solution of* (S), *then it follows that for any* $\sigma, t \in]0, T]$

$$(11.2) \quad s(t)^2 = s(\sigma)^2 + 2\int_0^{s(\sigma)} xu(x, \sigma)dx - 2\int_0^{s(t)} xu(x, t)dx + 2\int_\sigma^t u(0, \tau)d\tau .$$

Conversely if $s(t) \in C([0, T]) \cap C^{0,1}(]0, T])$, u *satisfies* (M) *and* (11.2), *then we have* $\dot{s}(t) = -u_x(s(t), t)$, $t \in]0, T]$.

Proof. Let (s, u) be a solution of (S). We set $Lu=0$ and $v(x) = -x$ in Stoke's theorem

$$(11.3) \qquad \int_{\partial G} \{(vu_x - uv_x)d\tau + uv\,dx\} = \int_G \{vLu - uL^*v\}dx\,d\tau = 0$$

where L^* is the adjoint of L, and $G = \{(x, \tau); 0 < x < s(\tau), \sigma < \tau < t\}$. Thus (11.2) follows in view of (0.3) and (0.5). The latter part will be handled in the way we used in the latter part of the proof of Lemma 11.1. q.e.d.

12. Proof of Theorem 2 and Theorem 4

In what follows we use the fact that we have already proved the existence of solution of (S) under the assumption (H.1) in § 8.

Consider two sets $\{l_i, \phi_i\}$, $i=1, 2$, of Stefan data. If $l_i > 0$ we require that ϕ_i satisfies (A), and if $l_i = 0$ theie is no ϕ_i. Let (s_i, u_i) be a solution of (S) corresponding to the data $\{l_i, \phi_i\}$.

Lemma 12.1. *Under the above assumptions, let* $\phi_1 \leqq \phi_2$ *and* $0 \leqq l_1 \leqq l_2$. *Then, for* $0 \leqq t \leqq T$,

$$s_1(t) \leqq s_2(t) .$$

Proof. First we prove that, in the case $0 \leqq l_1 < l_2$, $s_1(t) < s_2(t)$. Assuming the contrary, set $t_0 = \min\{t; s_1(t) = s_2(t)\}$. Clearly $\dot{s}_1(t_0) \geqq \dot{s}_2(t_0)$ and $t_0 > 0$. We may have that $u_2(s(t), t) > 0$ for $0 < t < t_0$ by virtue of Proposition 10.1, Proposition 10.2 and the strong maximum principle [11], ruling out the trivial case $u_2 \equiv u_1 \equiv 0$, $s_1(t) \equiv l_1$, $s_2(t) \equiv l_2$. Hence $u_2 - u_1 > 0$ in the region, $0 < x < s_1(t)$, $0 < t \leqq t_0$, by Proposition 10.1 and the strong maximum pinciple. Since $u_2 - u_1$ vanishes at the point $(s_1(t_0), t_0)$, it follows from the parabolic version of Hopf's lemma [9] that

$$\dot{s}_1(t_0) = -u_{1,x}(s_1(t_0), t_0) < -u_{2,x}(s_1(t_0), t_0) = \dot{s}_2(t_0),$$

which is a contradiction.

Now we treat the case $l_1 = l_2$. Let (s_2^δ, u_2^δ) be a solution of (S) corresponding to the Stefan data $\{l_2 + \delta, \phi_2^\delta\}$, where

$$\phi_2^\delta(x) = \begin{cases} \phi_2(x), & 0 \leqq x \leqq l_1, \\ 0, & l_1 \leqq x \leqq l_1 + \delta. \end{cases}$$

Since ϕ_2 satisfies the assumption (A), ϕ_2^δ satisfies the assumption (H.1). Hence the above definition is well-defined. By the previous paragraph, $s_1(t) < s_2^\delta(t)$ for all $\delta > 0$. Now (s_2, u_2) and (s_2^δ, u_2^δ), being the solution of (S), must satisfy their versions of (11.1) respectively. Subtracting them and noting Proposition 10.2, we obtain

$$\begin{aligned} (12.1) \qquad & s_2^\delta(t) - s_2(t) \\ & \leqq s_2^\delta(\sigma) - s_2(\sigma) + \int_0^{s_2^\delta(\sigma)} u_2^\delta(x, \sigma)\, dx - \int_0^{s_2(\sigma)} u_2(x, \sigma) dx \\ & \quad - \int_0^{s_2(t)} \{u_2^\delta(x, t) - u_2(x, t)\}\, dx - \int_\sigma^t \{(u_{2,x}^\delta(0, \tau) - u_{2,x}(0, \tau)\} d\tau. \end{aligned}$$

But by Proposition 10.1 we get $u_2^\delta - u_2 \geqq 0$. We estimate the integrand of the last term. We have $u_2^\delta(0, \tau) \geqq u_2(0, \tau)$ for $\tau \in]0, T]$. Consider first the case $u_2^\delta(0, \tau) > u_2(0, \tau)$. We have $u_{2,x}^\delta(0, \tau) - u_{2,x}(0, \tau) \geqq 0$ by virtue of the unilateral boundary condition (0.2) and the monotonicity of γ, if the differential quotients of them exist. For the case $u_2^\delta(0, \tau) = u_2(0, \tau)$, we obtain $u_{2,x}^\delta(0, \tau) - u_{2,x}(0, \tau) \geqq 0$ by the fact $u_2^\delta - u_2 \geqq 0$, if the differential quotients of them exist. So we have $u_{2,x}^\delta(0, \tau) - u_{2,x}(0, \tau) \geqq 0$ a.e. $\tau \in]0, T]$. Hence letting σ tend to zero, we obtain $s_2^\delta(t) - s_2(t) \leqq (l + \delta) - l = \delta$ by (12.1) and the definition of ϕ_2^δ. Therefore $s_1(t) < s_2^\delta(t) \leqq s_2(t) + \delta$. Since $\delta > 0$ can be picked as small as desired, we obtain $s_1(t) \leqq s_2(t)$. q.e.d.

Now we give the proof of the uniqueness theorem.

Proof of Theorem 2 and Theorem 4. If (s_1, u_1) and (s_2, u_2) are two solutions of the Stefan problem (S), then $s_1 \equiv s_2$ by Lemma 12.1. Hence $u_1 \equiv u_2$ by

Propsiotion 10.1.

13. Proof of Theorem 1

We give the proof of the existence theorem for the case $l>0$ under the assumption (A).

Proof of Theorem 1. We may assume $\phi\not\equiv 0$. We begin the proof by defining

$$\phi^{\delta}(x)=\begin{cases}\phi(x)\,, & 0\leqq x<l-\delta\,,\\ 0\,, & l-\delta\leqq x<l\,,\end{cases}$$

for each δ satisfying $0<\delta<l$. Since ϕ satisfies the assumption (A), it follows that for each δ with $0<\delta<l$ there exists a $K=K(\delta)$ such that the assumption (H.1) is satisfied, that is, $0\leqq\phi^{\delta}(x)\leqq K(\delta)(l-x)$. Hence, for each $\delta_n=2^{-n}l$, $n=1, 2, \cdots$, there exists a unique solution (s^n, u^n) of the Stefan problem (S) corresponding to the data $\{l, \phi^{\delta_n}\}$ by the result in § 8 and Theorem 2. From Lemma 12.1 the sequence $\{s_n\}$ is a monotone increasing sequence of functions which are bounded above by the solution of the Stefan problem (S) corresponding to the data $\{l+1, \phi\}$ where ϕ is extended as zero over $l<x\leqq l+1$. Consequently there exists a function $s(t)=\lim_{n\to\infty} s^n(t)$. From Proposition 10.2, the strong maximum principle, the parabolic version of Hopf's lemma and the fact that $\phi\not\equiv 0$ it follows that each s^n, $n=1, 2, \cdots$, is a strictly monotone increasing function of t. Hence, for $\sigma>0$, $s^1(\sigma)-l>0$. Putting

$$K_{\tau}=\max\frac{M}{s^1(\tau)-l}\quad M=\max\{||\phi||_{L^{\infty}(0,l)}, H\}\,,$$

for each τ satisfying $0<\sigma\leqq\tau\leqq T$, we have from Proposition 10.1 and 10.2 that

$$0\leqq u^n(x, t)\leqq K_{t_0}(s^n(t_0)-x)$$

for $0\leqq x\leqq s^n(t)$ and $\sigma\leqq t\leqq t_0$. Since $u^n(s^n(t_0), t_0)=0$, it follows that

$$0<-u^n_x(s^n(t_0), t_0)\leqq K_{t_0}\,. \tag{13.1}$$

Hence, for all $n=1, 2, \cdots$,

$$0<\dot{s}^n(t)\leqq K_{\sigma} \tag{13.2}$$

for $\sigma\leqq t\leqq T$. Thus, it follows that the limit functon s is Lipshitz continuous with Lipshitz constant K_{σ} for $\sigma\leqq t\leqq T$. In order to demonstrate the continuity of s at $t=0$, consider the solution (ρ, v) of the Stefan problem

$$\begin{cases} Lv = 0\,, & l<x<\rho(t),\ 0<t\leqq T\,, \\ v(l,t) = M\,, & 0<t\leqq T\,, \\ v(\rho(t),t) = 0\,, & 0<t\leqq T\,, \\ \dot{\rho}(t) = -v_x(\rho(t),t)\,, & 0<t\leqq T\,, \\ \rho(0) = l\,. \end{cases}$$

The existence and uniqueness of the solution of the above problem is known in Cannon & Hill [4]. It follows from Proposition 10.2 and Result 2 in Cannon, Hill & Primicerio [5] that $s^1(t)\leqq s(t)\leqq\rho(t)$ for $0\leqq t\leqq T$. Since s^1 and ρ are continuous at $t=0$, so is s.

Let u denote the solution of (M) with the moving boundary s and the data ϕ. This is well-defined by Proposition 9.1 and 10.1. We shall show that for any $\sigma>0$

$$(13.3)\qquad u^n \to u \qquad \text{uniformly in } \bar{D}\cap\{t\geqq\sigma\}$$

as $n\to\infty$, where $D=\{(x,t);\ 0<x<s(t),\ 0<t\leqq T\}$ and it is understood that $\{u^n\}$ are extended as zero over their original domains of definition. By Proposition 9.1 we have $u\in C^\infty(D)\cap C(\bar{D}-Z)$ and

$$\int_0^T\int_0^{s(t)} t\,u_{xx}^2\,dxdt<\infty\,.$$

Since $0\leqq u^n\leqq M$, u^n is uniformly bounded in $\bar{D}$. With the aid of the proof of Lemma 7.5 and the assumption $l>0$, there exists a positive constant C independent of n such that

$$\int_0^T\int_0^{s^n(t)} t\,{u^n_t}^2\,dxdt\leqq C\,,$$

$$t\int_0^{s^n(t)} u^n_x(x,t)^2\,dx\leqq C\,,\qquad t\in]0,T]\,.$$

Hence it follows from Lemma 16.3 and Ascoli-Arzelà's theorem that there exists a function $\bar{u}\in C(\bar{D}-Z)$ such that

$$u^n\to\bar{u}\qquad \text{in } \bar{D}\cap\{t\geqq\sigma\}\ \text{ as } n=n_k\to\infty$$

for all $\sigma\in]0,T]$.

Now we wish to show that $u=\bar{u}$ in D. Repeating the calculations in the proof of Lemma 10.1 for $p=u^n$ and $q=u$, we see that for all $\varepsilon>0$

$$(13.4)\quad \frac{1}{2}\int_0^{s^n(t)}\{(u-u^n)^+\}^2dx-\frac{1}{2}\int_0^{s^n(\varepsilon)}\{(u-u^n)^+\}^2dx$$

$$\leqq\int_\varepsilon^t[(u-u^n)_x(u-u^n)^+]_n^{s^n(\tau)}d\tau+\frac{1}{2}\int_\varepsilon^t \dot{s}^n(\tau)\{(u(s^n(\tau),\tau)-u^n(s^n(\tau),\tau))^+\}^2d\tau$$

by using (10.12). Here we observe that

$$(13.5)\qquad \begin{aligned}&[(u-u^n)_x(u-u^n)^+]_0^{s^n(\tau)}\\ &=(u_x(s^n(\tau),\tau)-u^n{}_x(s^n(\tau),\tau))(u(s^n(\tau),\tau)-u^n(s^n(\tau),\tau))^+\\ &\quad-(u_x((0,\tau)-u^n{}_x(0,\tau))(u(0,\tau)-u^n(0,\tau))^+\\ &\leqq(u_x(s^n(\tau),\tau)-u^n{}_x(s^n(\tau),\tau))(u(s^n(\tau),\tau))^+\end{aligned}$$

by the unilateral boundary condition on $x=0$ and $u^n(s^n(\tau),\tau)=0$. Using (13.5) in (13.4), and noting that u_x is continuous to the moving boundary by Lemma 1 in Cannon & Hill [3], $\{u^n\}_n$ is a monotone increasing sequence of functions which are bounded above by u in view of Proposition 10.1, and (9.3) holds, we obtain that $u\geqq\bar{u}$ and

$$(13.6)\qquad \begin{aligned}&\frac{1}{2}\int_0^{s(t)}\{(u-\bar{u})^+\}^2dx\leq\frac{1}{2}\int_0^{s(\varepsilon)}\{(u-\bar{u})^+\}^2dx\\ &\leqq\frac{1}{2}\int_0^{s(\varepsilon)}\{(u-u^m)\}^2dx\end{aligned}$$

for any natural number m by letting $n\to\infty$ in (13.4). Letting $\varepsilon\to 0$ in (13.6), we get

$$\begin{aligned}&\frac{1}{2}\int_0^{s(t)}\{(u-\bar{u})^+\}^2dx\\ &\leqq\frac{1}{2}\int_0^{l}\{(\phi-\phi^{\delta_m})^+\}^2dx\leqq\delta_m\cdot\|\phi\|^2_{L^\infty(0,l)}\,.\end{aligned}$$

Since $\delta_m=l/2^m$ is arbitrary, we obtain

$$\int_0^{s(t)}\{(u-\bar{u})^+\}^2dx=0\,,$$

which implies $(u-\bar{u})^+=0$, i.e., $u\leqq\bar{u}$. Hence $u=\bar{u}$.

Consequently as $n\to\infty$

$$u^n\to u\qquad\text{uniformly in }\bar{D}\cap\{t\geqq\sigma\}$$

for all $\sigma\in]0,T]$.

Finally we investigate that (s,u) satisfies the Stefan condition (0.5). Since each (s^n,u^n) is a solution of the Stefan problem (S) with the data ϕ^{δ_n}, we have

$$s^n(t)^2=s^n(\sigma)^2+2\int_0^{s^n(\sigma)}xu^n(x,\sigma)\,dx-2\int_0^{s^n(t)}xu^n(x,t)dx+2\int_\sigma^t u^n(0,\tau)d\tau$$

for σ, $t\in]0,T]$ by Lemma 11.2. Taking the limit as $n\to\infty$ we obtain

$$s(t)^2=s(\sigma)^2+2\int_0^{s(\sigma)}xu(x,\sigma)\,dx-2\int_0^{s(t)}xu(x,t)\,dx+2\int_\sigma^t u(0,\tau)d\tau$$

for σ, $t\in]0, T]$, which implies that (s, u) satisfies the Stefan condition (0.5) in in view of Lemma 11.2. Further it can be shown that $s(t)\in C^{\infty}(]0, T])$ and $u\in C^{\infty}(D^{s})$ in view of Schaeffer [16]. q.e.d.

14. Proof of Corollary

We prepare the fundamental lemma concerning the maximal monotone graphs in $\boldsymbol{R}^2$.

Lemma 14.1. *Let γ be a maximal montone group in $\boldsymbol{R}^2$. Suppose that $\gamma(u)$ is single valued at $u\in D(\gamma)$. Then $\gamma(\cdot)$ is continuous at u.*

Proof. Let $D(\gamma)\ni u_n\to u\in D(\gamma)$, $\gamma(u_n)\ni f_n$ and $\gamma(u)=\{f\}$. We shall show that $f_n\to f$. If $u\notin \operatorname{int} D(\gamma)$, then $\gamma(u)$ could not be single valued by the assumption $\gamma\subset\boldsymbol{R}^2$. So we may suppose $u\in\operatorname{int} D(\gamma)$. Hence $\{f_n\}_n$ is bounded sequence in $\boldsymbol{R}$. There exist $\tilde{f}\in\boldsymbol{R}$ and a subsequence $\{f_{n_j}\}_j\subset\boldsymbol{R}$ such that $f_{n_j}\to f$. Thus $\{[u_{n_j}, f_{n_j}]\}_j\subset\gamma$, $u_{n_j}\to u, f_{n_j}\to\tilde{f}$. By Brézis [2, Proposition 2.5], we have $[u, \tilde{f}]\in\gamma$. Hence $\tilde{f}\in\gamma(u)$, that is $f=\tilde{f}$ in view of $\gamma(u)=\{f\}$. Therefore we obtain $f_n\to f$ without taking a subsequence. q.e.d.

Proof of Corollary. Since $u\in C^{\infty}(D^{s})\cap C(\bar{D}-Z)$ holds by Theorem 1 and 2, it follows that for any $t_0\in]0, T]$ $u(\cdot, t_0)\in C^{\infty}(]0, s(t_0)])\cap C([0, s(t_0)])$. Hence, without loss of generality, we may assume the assumption (H.1) for the data ϕ. We use the fact that u is constructed as the limit of the sequence $\{(s_h, u_h)\}_h$ of the solutions of the difference equations under the assumption (H.1) for the data ϕ. Therefore we can apply the results in § 8. Since γ is a single valued maximal monotone graph in $\boldsymbol{R}^2$, $D(\gamma)$ is an open interval and $\gamma(\cdot)$ is a continuous function on $D(\gamma)$ by Lemma 14.1. Using this fact and Lemma 8.2 (iv), for any $t\in]0, T]$ and $\varepsilon>0$, there are $\delta>0$ and $h_0>0$ such that

$$|\gamma(u_h(0, s))-\gamma(u(0, t))|<\varepsilon \tag{14.1}$$

for $|s-t|<\delta$ and $h\leqq h_0$. Set $z_j^n=u_{jx}^n$. It follows that $L_h z_j^n=0$ and $z_0^n=\gamma(u_0^n)$. Combining these with (14.1) and applying the Petrovskii's technique [14, p. 364–368] we observe that u_x is continuous to the boundary $x=0$ and $u_x(0, t)=\gamma(u(0, t))$. The continuity to the free boundary of u_x is known. Hence $u_x\in C(\bar{D}-\{t=0\})$. q.e.d.

15. Proof of Theorem 3

We give the proof of the existence theorem for the case $l=0$ under the hypothesis

$$D(\gamma)\supset[0, H] \quad\text{and}\quad \gamma(0)\subset]-\infty, 0[\,. \tag{B}$$

Proof of Theorem 3. Consider the sequence $\{(s^m, u^m)\}_m$ of the solutions of the Stefan problem (S) for the data $\{\{1/m, 0\}\}_m$. The sequence $\{s^m\}$ is a monotonically decreasing sequence of increasing continuous functions in t by Lemma 12.1. Set $s(t)=\lim_{m\to\infty} s^m(t)$. We note that each data $\{1/m, 0\}$ satisfies the assumption (A.2) and (H.1) by the hypothesis (B), so each solution (s^m, u^m) is constructed as the limit of the sequence $\{(s^m{}_h, u^m{}_h)_h\}$ of solutions of the difference equations. We can apply the estimates in § 6. By using Lemma 6.2, it follows that there exists a positive constant C independent of m such that

$$|\dot{s}^m(t)|\leqq C \qquad \text{for } 0\leqq t\leqq T.$$

Here, in view of Ascoli-Arzelà's theorem, we see that $s^m(t)\to s(t)$ uniformly on $[0, T]$ as $m\to\infty$ and $s(t)$ is an increasing continuous function on $[0, T]$. First we shall show that $s(t)>0$ for $t>0$. Assuming the contrary, let

$$0<t_0=\max\{t\in]0, T]; s(t)=0\}. \tag{15.1}$$

It follows that there exists a positive constant C independent of m such that

$$|u^m{}_x(x, t)|\leqq C, \qquad 0<x<s^m(t),\ 0<t\leqq T, \tag{15.2}$$

by using Lemma 6.1, Lemma 6.2, $u^m{}_x(x, 0)\equiv 0$ and the maximum principle. Hence

$$0\leqq u^m(x, t)\leqq C\cdot s^m(t), \qquad 0\leqq x\leqq s(t),\ 0\leqq t\leqq T. \tag{15.3}$$

Since $s^m(t)\to s(t)\equiv 0$ uniformly on $[0, t_0]$, we obtain

$$u^m(x, t)\to 0 \text{ uniformly}, \qquad 0\leqq x\leqq s(t),\ 0\leqq t\leqq t_0, \tag{15.4}$$

as $m\to\infty$. Let $\tilde{\gamma}$ be a maximal monotone in $L^2(0, t_0)$, which is defined as the natural extension of γ (see Brézis [2, p. 25]). Consequently we get

$$u^m{}_x(0, \cdot)\in\tilde{\gamma}(u^m(0, \cdot)) \qquad \text{in } L^2(0, t_0) \tag{15.5}$$

in view of $u^m{}_x(0, t)\in\gamma(u^m(0, t))$ a.e. $t\in]0, T]$. By using (15.2), $\{u^m{}_x(0, \cdot)\}_m$ is bounded in $L^2(0, t_0)$. Hence there exists a $v(\cdot)\in L^2(0, t_0)$ and a subsequence $\{u^{m_j}{}_x\}_j$ such that

$$u^{m_j}{}_x(0, \cdot)\to v(\cdot) \qquad \text{weakly in } L^2(0, t_0) \tag{15.6}$$

as $m_j\to\infty$. From (15.4), (15.5), (15.6) and Brézis [2, Proposition 2.5], we have $v(\cdot)\in\tilde{\gamma}(0)$ which implies

$$v(\tau)\in\gamma(0) \qquad \text{a.e. } \tau\in]0, t_0].$$

Using the hypothesis (B), we obtain

$$\int_0^{t_0} v(\tau)d\tau<0. \tag{15.7}$$

Meanwhile we have

$$s^m(t) = l/m - \int_0^{s^m(t)} u^m(x, t)dx - \int_0^t u^m{}_x(0, \tau)d\tau \tag{15.8}$$

by using Lemma 11.1 and the fact $u^m(x, 0)=0$, $0\leqq x\leqq 1/m$. Taking $t=t_0$ and letting $m=m_j\to\infty$, we obtain

$$0 = -\int_0^{t_0} v(\tau)d\tau > 0$$

with the aid of (15.1), (15.4), (15.6) and (15.7). This is a contradiction. Thus $s(t)>0$ for $t>0$.

Let u denote the solution of (M) with the moving boundary s and the data ϕ. This is well-defined by Proposition 9.1. We can show that for any $\sigma>0$

$$u^m \to u \qquad \text{uniformly in } \bar{D}\cap\{t\geqq\sigma\} \text{ as } m\to\infty \tag{15.9}$$

by the similar argument used in the proof of Theorem 1.

Hence, following the argument in the last part of the proof of Theorem 1, (s, u) satisfies the heat balance (11.2) which implies that (s, u) is the solution of the Stefan problem (S) for the data $\{0, \cdot\}$. Further it can be shown that $s(t)\in C^\infty(]0, T])$ and $u\in C^\infty(D^s)$ in view of Schaeffer [16]. q.e.d.

REMARK 15.1. We have proved the existence of a solution indirectly. But we can also prove it by the finite difference method directly. The difference scheme is as follows.

$0°$ $t_0=0$, $J_0=0$,

$1°$ $t_1=k_1=h$, $J_1=1$, $u_0^1=(I+h\gamma)^{-1}(0)$, $u_1^1=0$.

As for $n=2, 3, \cdots$, we determine k_n and $\{u_j^n\}_{1\leqq j\leqq J_n}$ by using the same rule stated in §2. The proof is essentially similar to that of Theorem 3, so we omit it.

16. Appendix

We shall state the useful three lemmas which are the modifications of the results of Petrovskii [14, p. 357–360]. To get the estimates for the divided differences of the solutions of the difference equations, Petrovskii used a device employed by Bernstein [22] for estimating the derivatives of the solutions of a parabolic equation.

We use the notations introduced in §2 and 3. We need not assume that the moving boundary Γ_h^3: $x_j=s_n$ is non-decreasing. Let $\{u_j^n\}$ be a family of net functions satisfying the difference equation

$$L_h u_j^n \equiv u_{jx\bar{x}}^n - u_{j\bar{t}}^n = 0 \qquad \text{in } D_h\,.$$

In what follows we assume that the uniform space width h and variable time steps

$\{k_n\}$ are sufficiently small. We first state the lemmas.

Lemma 16.1. *Suppose that*

$$(16.1)\qquad 0<s_n\leqq M_1 \qquad (0\leqq t_n\leqq T),$$

$$(16.2)\qquad |u^n_j|\leqq M_2 \qquad \text{on } \bar{D}_h,$$

$$(16.3)\qquad |u^n_{J_n\bar{x}}|\leqq M_3 \quad (0<t_n\leqq T), \qquad |u^0_{jx}|\leqq M_3 \quad (0\leqq l'\leqq x_j<J_n).$$

Then there exists a positive constant $L_{1,d}$ depending on M_1, M_2, M_3 and d such that

$$|u^n_{j\bar{x}}|\leqq L_{1,d} \qquad (l'<d\leqq x_j\leqq s_n,\ 0\leqq t_n\leqq T).$$

Lemma 16.2. *Suppose that* (16.1), (16.2), (16.3) *and*

$$(16.4)\qquad |u^n_{J_n-1,x\bar{x}}|\leqq M_4 \quad (0<t_n\leqq T), \qquad |u^0_{jx\bar{x}}|\leqq M_4 \quad (0\leqq l'\leqq x_j<J_n).$$

Then there exists a positive constant $L_{2,d}$ depending on M_1, M_2, M_3, M_4 and d such that

$$|u^n_{jx\bar{x}}|\leqq L_{2,d} \qquad (l'<d\leqq x_j<s_n,\ 0\leqq t_n\leqq T).$$

Lemma 16.3. *Suppose that*

$$|u^n_{j\bar{x}\bar{x}}|\leqq K_1 \qquad (l_1<x_j<l_2,\ 0\leqq t_n\leqq T),$$
$$|u^0_{jx\bar{x}\bar{x}}|\leqq K_2 \qquad (l_1<x_j<l_2).$$

Then there exists a positive constant K_{d_1,d_2} depending on K_1, K_2, d_1 and d_2 such that

$$|u^n_{jx\bar{x}\bar{x}}|\leqq K_{d_1,d_2} \qquad (l_1<d_1\leqq x_j\leqq d_2<l_2,\ 0\leqq t_n\leqq T).$$

We shall prove the above lemmas.

Proof of Lemma 16.1. First we assume $l'=0$. Let us consider the function

$$(16.5)\qquad z^n_j=(u^n_{jx})^2F(x_j)+Cv^n_j \qquad \text{on } \bar{E}_h,$$

where

$$(16.6)\qquad \begin{aligned}\bar{E}_h &= \{(x_j,t_n);\ h\leqq x_j\leqq s_n-h, \quad 0\leqq t_n\leqq T\},\\ F(x) &= (x-h)^2\{2a-(x-h)\}^2\\ &= \{a^2-(x-h-a)^2\}^2, \quad a=(M_1+1)/2,\end{aligned}$$

$$(16.7)\qquad v^n_j=(u^n_{j+1})^2+(u^n_{j-1})^2,$$

and C is some positive constant. We shall show that if C is sufficiently large,

$$(16.8)\qquad L_hz^n_j\equiv z^n_{jx\bar{x}}-z^n_{j\bar{t}}\geqq 0$$

on $E_h=\{(x_j, t_n); h<x_j<s_n-h, 0<t_n\leqq T\}$. Noting $L_h(u^n_{jx})=0$, after some calculations, we get the following equality (see [14, p. 359]).

$$
\begin{aligned}
(16.9)\qquad L_n z^n_j = & C[(u^n_{j+1,x})^2+(u^n_{jx})^2+k_n(u^n_{j+1,\bar{t}})^2+(u^n_{j-1,x})^2 \\
& +(u^n_{j-1,\bar{x}})^2+k_n(u^n_{j-1,\bar{t}})^2]+(u^n_{jx})^2L_h(F) \\
& +F[(u^n_{jxx})^2+(u^n_{jx\bar{x}})^2+k_n(u^n_{jx\bar{t}})^2] \\
& +F_x[u^n_{jx}+u^n_{j+1,x}]u^n_{jxx}+F_{\bar{x}}[u^n_{jx}+u^n_{j-1,x}]u^n_{jx\bar{x}}\,.
\end{aligned}
$$

Now using the argument of [14, p. 359–360], we observe that

$$
(16.10)\qquad F(u^n_{jxx})^2+F_x[u^n_{jx}+u^n_{j+1,x}]u^n_{jxx}\geqq-(2x'_j+h)^2[3(u^n_{jx})^2+(u^n_{j+1,x})^2]\,,
$$

and

$$
(16.11)\qquad F(u^n_{jx\bar{x}})^2+F_{\bar{x}}[u^n_{jx}+u^n_{j-1,x}]u^n_{jx\bar{x}}\geqq-(2x'_j-h)^2[3(u^n_{jx})^2+(u^n_{j-1,x})^2]\,,
$$

where $x'_j=(x_j-h)-a$. In view of (16.10), (16.11) and the fact that $F>0$, we obtain

$$
\begin{aligned}
(16.12)\qquad L_h z^n_j\geqq & [C+L_h(F)-3(2x'_j+h)^2-3(2x'_j-h)^2](u^n_{jx})^2 \\
& +[C-(2x'_j+h)^2](u^n_{j+1,x})^2+[C-(2x'_j-h)^2](u^n_{j-1,x})^2
\end{aligned}
$$

from (16.9). Obviously, if $C=C_{M_1}$ is sufficiently large, all the terms on the right side of the inequality (16.12) will be nonnegative. Hence we have shown (16.8).

By Lemma 3.1 and (16.8), we see that z^n_j has its largest value on the boundary of $\bar{E}_h$. Therefore we have

$$
z^n_j\leqq M_3^2(M_1+1)^4+2C_{M_1}M_2^2\equiv M \qquad \text{on } \bar{E}_h
$$

by (16.1), (16.2) and (16.3). Consequently we have

$$
(u^n_{jx})^2\leqq M/[(x_j-h)^2\{2a-(x_j-h)\}^2]\,,
$$

which implies

$$
(16.13)\qquad (u^n_{jx})^2\leqq L_1^2/(d^2) \qquad (d\leqq x_j<s_n,\ 0\leqq t_n\leqq T)\,,
$$

where $L_1^2=8M$. We can treat the case $l'>0$ similarly. q.e.d.

Proof of Lemma 16.2. First assume $l'=0$. By (16.1), (16.2), (16.3), we have

$$
(16.14)\qquad (u^n_{jx})^2\leqq L_1^2/(d^2) \qquad (d\leqq x_j<s_n\,,\ 0\leqq t_n\leqq T)
$$

noting (6.13) in the proof of Lemma 16.1. Now using the proof of Lemma 16.1 by taking u^n_{jx} defined on $\{(x_j, t_n); d\leqq x_j<s_n, 0\leqq t_n\leqq T\}$ instead of u^n_j defined on $\bar{D}_h$, we obtain

$$((u^n_{jx})_{\bar{x}})^2 \leqq 8\{M_4^2(M_1+1)^4+2C_{M_1}L_1^2/(d^2)\}/\{d^2\} \qquad (2d \leqq x_j < s_n\,,\ 0 \leqq t_n \leqq T)$$

from (16.1), (16.2) and (16.4) by noting (16.13). We can treat the case $l'>0$ similarly. q.e.d.

Proof of Lemma 16.3. We can show the conclusion by the argument similar to the proof of Lemma 16.1. .q.e.d.

We shall prove Lemma 8.1 (i). We prepare a useful lemma due to Ishii [25, Lemma 3.1]. For the sake of completeness we give the proof of it here.

Lemma 16.4. *Let $v(x, t)$ be a Lipshitz continuous function on $Q=[a_1, a_2]\times[b_1, b_2]$. Then we have*

$$\begin{aligned}(16.15)\qquad & |v(\tilde{x}, \tilde{t})-v(x, t)| \\ & \leqq L\{(\sup_{b_1\leqq t\leqq b_2}\int_{a_1}^{a_2} v_x(x, t)^2 dx)^{1/2}+(\int_{b_1}^{b_2}\int_{a_1}^{a_2} v_t(x, t)^2 dx\,dt)^{1/2}\} \\ & \cdot\{|\tilde{x}-x|^{1/2}+|\tilde{t}-t|^{1/4}\}\end{aligned}$$

for any $(\tilde{x}, \tilde{t}), (x, t)\in Q$, where

$$L = \max\{2A^{1/2}B^{-1/4}, 2A^{-1/2}B^{1/4}, 1\}\,,$$
$$A = a_2-a_1\,,\quad B = b_2-b_1\,.$$

Proof. We put $\eta=2^{-1}AB^{-1/2}|t-\tilde{t}|^{1/2}$, $a=(a_1+a_2)/2$,

$$\varepsilon = \varepsilon(x) = \begin{cases} \eta & (\text{if } x<a)\,, \\ -\eta & (\text{if } x\geqq a)\,, \end{cases}$$

$$C_1 = \sup_{b_1\leqq t\leqq b_2}(\int_{a_1}^{a_2} v_x(x, t)^2 dx)^{1/2}\,,$$

$$C_2 = (\int_{b_1}^{b_2}\int_{a_1}^{a_2} v_t(x, t)^2 dx\,dt)^{1/2}\,.$$

We observe that $x+\varepsilon\in[a_1, a_2]$ for any $(x, \tilde{t}), (x,t)\in Q$. It is easily shown that

$$\begin{aligned}&\varepsilon(v(x, \tilde{t})-v(x, t)) \\ &= \int_x^{x+\varepsilon}\{\int_\xi^x v_\zeta(\zeta, \tilde{t})d\zeta-\int_\xi^x v_\zeta(\zeta, t)d\zeta\}d\xi \\ &\quad +\int_x^{x+\varepsilon}\{\int_t^{\tilde{t}} v_\tau(\xi, \tau)d\tau\}d\xi\,.\end{aligned}$$

Hence we have

$$\begin{aligned}&\eta|v(x, \tilde{t})-v(x, t)| = |\varepsilon|\,|v(x, \tilde{t})-v(x, t)| \\ &\leqq |\int_x^{x+\varepsilon} 2|x-\xi|^{1/2}C_1 d\xi| + |\int_x^{x+\varepsilon}|\tilde{t}-t|^{1/2}(\int_{b_1}^{b_2} v_\tau(\xi, \tau)^2 d\tau)^{1/2}d\xi| \\ &\leqq 2\eta^{3/2}C_1+|\tilde{t}-t|^{1/2}\eta^{1/2}C_2\,.\end{aligned}$$

Thus we obtain

(16.16)
$$\begin{aligned} &|(v(x, \tilde{t})-v(x, t)| \\ &\leqq 2C_1(A\tilde{B}/2)^{1/2}|\tilde{t}-t|^{1/4}+C_2(A\tilde{B}/2)^{-1/2}|\tilde{t}-t|^{1/4}, \ \tilde{B}=B^{-1/2}, \\ &\leqq L(C_1+C_2)|\tilde{t}-t|^{1/4}. \end{aligned}$$

It is easy to show (16.15) by using (16.16). q.e.d.

REMARK 16.1 It is obvious that (16.15) holds for any function v such that $v_x \in L^\infty(b_1, b_2; L^2(a_1, a_2))$ and $v_t \in L^2(Q)$.

Proof of Lemma 8.1. By the definition of $u_h(x, t)$, we have

$$\frac{\partial}{\partial x}u_h(x, t) = \begin{cases} u_{jx}^{n+1}(x_j<x<(1-\theta)x_j+\theta x_{j+1}), \\ u_{jx}^n((1-\theta)x_j+\theta x_{j+1}<x<x_{j+1}), \end{cases}$$

where $t=(1-\theta)t_n+\theta t_{n+1}$, $0 \leqq \theta \leqq 1$, and

$$\frac{\partial}{\partial t}u_h(x, t) = u_{j+1,\bar{t}}^{n+1},$$

where

$$\begin{cases} (1-\theta)x_j+\theta x_{j+1} \leqq x \leqq (1-\theta)x_{j+1}+\theta x_{j+2}, \\ t=(1-\theta)t_n+\theta t_{n+1}, \\ 0 \leqq \theta \leqq 1. \end{cases}$$

Hence we obtain

$$\begin{aligned} &\int_0^\infty \left\{\frac{\partial}{\partial x}u_h(x, t)\right\}^2 dx \\ &= \sum_{j=0}^\infty \int_{x_j}^{x_{j+1}} \left\{\frac{\partial}{\partial x}u_h(x, t)\right\}^2 dx \\ &= \sum_{j=0}^\infty \{(u_{jx}^{n+1})^2\theta h+(u_{jx}^n)^2(1-\theta)h\} \\ &= \theta\sum_{j=0}^\infty (u_{jx}^{n+1})^2 h+(1-\theta)\sum_{j=0}^\infty (u_{jx}^n)^2 h \\ &\leqq 2\tilde{C}_l/t_{n-1}, \qquad t=(1-\theta)t_n+\theta t_{n+1}, \end{aligned}$$

and

$$\begin{aligned} &\int_{t_{m-1}}^{t_n}\int_h^\infty \left\{\frac{\partial}{\partial t}u_h(x, t)\right\}^2 dx\, dt \\ &\leqq \sum_{p=m}^n \sum_{j=1}^\infty (u_{j\bar{t}}^p)^2 hk_p \\ &\leqq \sum_{p=m}^n \sum_{j=1}^{J_p-1} (u_{jx\bar{x}}^p)^2 hk_p \leqq \tilde{C}_l/t_{m-1}, \end{aligned}$$

from Lemma 7.5. Therefore we have

$$\sup_{t_m \leqq t \leqq T} \int_0^\infty \left\{ \frac{\partial}{\partial x} u_h(x, t) \right\}^2 dx \leqq 2\tilde{C}_l / t_{m-1}, \tag{16.17}$$

$$\int_{t_{m-1}}^{t_n} \int_h^\infty \left\{ \frac{\partial}{\partial t} u_h(x, t) \right\}^2 dx\, dt \leqq \tilde{C}_l / t_{m-1}. \tag{16.18}$$

Thus we obtain the conclusion by (16.17), (16.18) and Lemma 16.4. q.e.d.

Proof of Lemma 8.2 (i). It is easily shown that

$$(u^n_{jx\bar{x}} - u^n_{j\bar{t}}) w^n_j = u^n_j w^n_{jx\bar{x}} + u^{n-1}_j w^n_{j\bar{t}} + (u^n_{jx} w^n_j - u^n_j w^n_{jx})_{\bar{x}} - (u^n_j w^n_j)_{\bar{t}}. \tag{16.19}$$

Let $w(x, t) \in C_0^\infty(D)$, that is, w is an infinitely differentiable function which has a compact support in D. By (2.6) and (16,19), we obtain

$$\sum_n \sum_j (u_h(x_j, t_n) w^n_{jx\bar{x}} + u_h(x_j, t_{n-1}) w^n_{j\bar{t}}) h k_n = 0.$$

Letting $h \to 0$ through the subsequence of $\{h\}$,

$$\int_0^T \int_0^{s(t)} u(w_{xx} + w_t) dx\, dt = 0 \tag{16.20}$$

for any $w \in C_0^\infty(D)$. Since $u \in C(D)$ and it satisfies (16.20), it follows that

$$u(x, t) \in C^\infty(D),$$
$$u_{xx} - u_t = 0 \quad \text{in } D,$$

in view of the well-known result concerning the heat equation (see [26, p. 248]).
q.e.d.

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Department of Applied Science
Faculty of Engineering
Miyazaki University
Miyazaki 880
Japan