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SECOND ORDER WEAKLY HYPERBOLIC OPERATORS WITH COEFFICIENTS SUM OF POWERS OF FUNCTIONS

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Abstract

We consider the Cauchy problem for the operator $D_t^2 - D_x a(t,x) D_x$ in the Gevrey classes. We show that if the coefficient $a(t,x)$ is given by a finite sum of non negative functions then the Cauchy problem is well posed in the wider Gevrey class for the larger powers. We also give an example showing that the order of the Gevrey class obtained here is optimal.

1. Introduction

In this paper we are interested in the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} Pu = D_t^2 u - D_x a(t,x) D_x u = 0 \\ u(0,x) = u_0(x), \ D_t u(0,x) = u_1(x) \end{cases}$$

on $[0,T] \times \mathbf{R}$, where we always assume that $a(t,x) \geq 0$.

For a space X of functions $v(x)$ in $\mathbf{R}$, we say that (CP) is well posed in X if for every $u_0, u_1 \in X$ there is a unique solution $u \in C^2([0,T]; X)$.

In this paper we prove that, if the coefficient a is given by a sum of powers of functions, or even by a suitable series of them, then the Cauchy problem (CP) is well posed in the wider space X for the larger powers. Actually in this note we take X as $\gamma^{(s)}(\mathbf{R})$, the Gevrey classes of order s for some $s > 1$. Since we are interested in studying the influence of the principal part of the symbol and in order to avoid Levi conditions, we do not allow terms of order one, but only a zero order term to be added to the principal part.

This Cauchy problem, for $a(t,x) = a(t)$, in the more general case of n space variables, has been considered in [2], where they proved in particular that, if the coefficient $a(t) \in C^h([0,T])$, then (CP) is $\gamma^{(s)}(\mathbf{R})$ well posed for $s < s_0$, where

$$s_0 = 1 + \frac{h}{2}. \tag{1.1}$$

Moreover they proved by suitable counterexamples that this index s_0 is optimal. In [7] these results have been extended to the case of coefficients depending also on space

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variables, but only for Gevrey index $s_0 \leq 2$.

In Section 2 we shall consider the case $a(t,x) = a(t)$ and we prove the following result:

Theorem 1.1. *Assume that* $a(t,x) = a(t)$ *and*

$$a(t) = \sum_{j=1}^{+\infty} a_j^n(t), \quad a_j^n \geq 0, \quad a_j(t) \in C^h([0,T]), \quad \sum_{j=1}^{+\infty} \|a_j\|_{C^h}^{1/h} < +\infty \tag{1.2}$$

where n *and* h *are positive integers. Then the Cauchy problem* (CP) *is* $\gamma^{(s)}(\mathbf{R})$ *well posed for*

$$1 < s < s^* = 1 + \frac{nh}{2}. \tag{1.3}$$

The Gevrey index $s^* = 1 + (nh/2)$ is optimal, as proved by the following:

Theorem 1.2. *For every positive integer* n *and* h *there exists* $a_1(t) \in C^h([0,T])$, *satisfying* $a_1(t) \geq 0$, *such that the Cauchy problem* (CP) *with* $a(t,x) = a_1^n(t)$ *is not* $\gamma^{(s)}(\mathbf{R})$ *well posed for any* $s > 1 + (nh/2)$.

We give now an easy consequence of Theorem 1.1, related to problem of writing a nonnegative function f as sum, or series, of squares of functions f_j, with f_j of given regularity.

REMARK. In [1] J.-M. Bony proves that any nonnegative function of class C^{2m} defined in an interval is the sum of two squares of functions g_j of class C^m; moreover, he proves that it is not possible, in general, to improve this result and find functions g_j more regular than C^m. We remark now that, thanks to Theorem 1.1, one can give another proof of the sharpness of this result, which, although very indirect, is a little more general. In fact, from [2], it is known that, for every integer m, there exists a function $a(t) \in C^{2m}([0,T])$ such that the corresponding Cauchy problem (CP) is not well posed in $\gamma^{(s)}(\mathbf{R})$ for $s > 1 + (2m/2) = 1 + m$. Then, taking Theorem 1.1 into account, for any $l < \infty$ or also for $l = +\infty$ and for any $p > m$, it is not possible to write this function $a(t)$ as $\sum_{j=1}^{l} a_j^2(t)$, with $a_j \in C^p$ and $\sum_{j=1}^{l} \|a_j\|_{C^p}^{1/p} < +\infty$.

In Sections 3 and 4 we study the case of $a(t,x)$ depending also on x, but we limit to consider $h = 2$:

$$P = D_t^2 - D_x a(t,x) D_x. \tag{1.4}$$

We say that $a(t,x) \in C^2([0,T]; \gamma^{(s)}(\mathbf{R}))$ if

$$\left|\partial_t^j \partial_x^k a(t,x)\right| \leq C_j A_j^k k!^s, \quad (t,x) \in [0,T] \times \mathbf{R}, \quad k = 0, 1, \ldots$$

for $j = 0, 1, 2$ and for some constants C_j and A_j. Then we have

Theorem 1.3. *Assume that*

$$a(t,x) = \sum_{j=1}^{l} a_j(t,x)^n, \quad 0 \le a_j(t,x) \in C^2([-\delta, T+\delta]; \gamma^{(s)}(\mathbf{R}))$$

with a positive integer n and some $\delta > 0$. Then the Cauchy problem for P is $\gamma^{(s)}(\mathbf{R})$ well posed if

$$1 \le s < 1 + n = 1 + \frac{2n}{2}.$$

2. Case of $a(t,x) = a(t)$

We give now the proof of Theorem 1.1; more precisely we prove an energy estimate from which by a standard argument one can obtain the well posedness result.

Proof of Theorem 1.1. Let us consider the operator P in $[0,T] \times \mathbf{R}$

$$P = \partial_t^2 - a(t)\partial_x^2 \tag{2.1}$$

under the assumptions (1.2). Let $u(t,x)$ be a solution of the equation $Pu = 0$. For the Fourier transform $v(t,\xi)$ of u with respect to x, we define the energy

$$E_\varepsilon = |\partial_t v(t,\xi)|^2 + |\xi|^2(a(t)+\varepsilon)|v(t,\xi)|^2,$$

with

$$\varepsilon = |\xi|^{-\sigma},$$

$\sigma > 0$ to be chosen later. From

$$\partial_t^2 v(t,\xi) + |\xi|^2 a(t) v(t,\xi) = 0$$

we have

$$\partial_t E_\varepsilon \le \left(\frac{|a'(t)|}{a(t)+\varepsilon} + \varepsilon^{1/2}|\xi| \right) E_\varepsilon, \tag{2.2}$$

which gives

$$E_\varepsilon(t,\xi) \le E_\varepsilon(0,\xi) \exp\left(t\varepsilon^{1/2}|\xi| + \int_0^t \frac{|a'(\tau)|}{a(\tau)+\varepsilon}\, d\tau \right) \tag{2.3}$$

by Gronwall inequality. Let us consider now the integral in (2.3); thanks to assumptions (1.2), we have:

(2.4)
$$\int_0^T \frac{|a'(t)|}{a(t)+\varepsilon}\,dt \le \int_0^T \frac{\sum_{j=1}^\infty |(a_j^n(t))'|}{\sum_{j=1}^\infty a_j^n(t)+\varepsilon}\,dt \le n\sum_{j=1}^\infty \int_0^T \frac{|a_j'||a_j|^{n-1}}{|a_j|^n+\varepsilon}\,dt$$
$$\le n2^{n-1}\sum_{j=1}^\infty \int_0^T \frac{|a_j'||a_j|^{n-1}}{(|a_j|+\varepsilon^{1/n})^n}\,dt \le n2^{n-1}\sum_{j=1}^\infty \int_0^T \frac{|a_j'|}{|a_j|+\varepsilon^{1/n}}\,dt.$$

From Corollary 2.5 in [3] (see also [9]), we know that

$$\int_0^T \frac{|a_j'|}{|a_j|+\varepsilon^{1/n}}\,dt \le M\|a_j\|_{C^h}^{1/h}\varepsilon^{-1/nh}$$

with $M = M(h,T)$. From this fact and from (2.4) we obtain:

$$\int_0^T \frac{|a'(t)|}{a(t)+\varepsilon}dt \le n2^{n-1}M\sum_{j=1}^\infty \|a_j\|_{C^h}^{1/h}\varepsilon^{-1/nh}. \tag{2.5}$$

From (2.5) we deduce:

$$E_\varepsilon(t,\xi) \le E_\varepsilon(0,\xi)\exp\left\{T\varepsilon^{1/2}|\xi| + \int_0^T \frac{|a'(\tau)|}{a(\tau)+\varepsilon}\,d\tau\right\}. \tag{2.6}$$

Taking (2.3) and (2.5) into account, we obtain

$$\sup_{t\in[0,T]} E_\varepsilon(t,\xi) \le E_\varepsilon(0,\xi)\exp C\{\varepsilon^{1/2}|\xi| + \varepsilon^{-1/nh}\}. \tag{2.7}$$

The best choice of σ,

$$\sigma = \frac{2nh}{nh+2},$$

and (2.7) yield finally

$$\sup_{t\in[0,T]} E_\varepsilon(t,\xi) \le E_\varepsilon(0,\xi)\exp(C|\xi|^{2/(nh+2)}). \tag{2.8}$$

This allows us to solve the Cauchy problem for P in Gevrey classes provided that the Gevrey index s is related to the C^h regularity of a_j and to the exponent n by the assumption (1.3). □

Now we prove by construction of a counterexample that the condition (1.3) is sharp. Our construction is inspired in part by the examples in [4] and in [2].

Proof of Theorem 1.2. Let us take a real, non-negative, 2π-periodic C^∞ function φ such that $\varphi(\tau) = 0$ for τ in a neighborhood of $\tau = 0$ and

$$\int_0^{2\pi} \varphi(\tau) \cos^2 \tau \, d\tau = \pi .$$

Then, for every $\tau \in \mathbf{R}$, we define

$$\alpha(\tau) = 1 + 4\varepsilon\varphi(\tau) \sin 2\tau - 2\varepsilon\varphi'(\tau) \cos^2 \tau - 4\varepsilon^2\varphi^2(\tau) \cos^4 \tau ,$$
$$\tilde{w}(\tau) = \cos \tau \exp\left(-\varepsilon\tau + 2\varepsilon \int_0^\tau \varphi(s) \cos^2 s \, ds \right),$$
$$w(\tau) = \tilde{w}(\tau)e^{\varepsilon\tau} ,$$

where ε is fixed in such a way that $1/2 \le \alpha(\tau) \le 3/2$, and let us denote

$$M = \|\alpha'\|_{L^\infty} .$$

So, $\alpha(\tau)$ and $\tilde{w}(\tau)$ are 2π-periodic C^∞ functions; furthermore w is the solution of the Cauchy problem

$$w''(\tau) + \alpha(\tau)w(\tau) = 0, \quad w(0) = 1, \ w'(0) = 0. \tag{2.9}$$

Let now $\beta(\tau)$ be a non increasing C^∞ function such that $\beta(\tau) = 1$ for $\tau \le 0$, $\beta(\tau) = 0$ for $\tau \ge 1$. We use also four positive monotone sequences $\{\delta_k\}, \{\varrho_k\}, \{\nu_k\}, \{h_k\}$, where ν_k are positive integers, such that

$$\begin{gathered} h_k \to +\infty, \quad \nu_k \to +\infty, \quad \delta_k \to 0, \quad \varrho_k \to 0; \quad \nu_k \in \mathbf{N}, \\ \delta_1 \le 1, \quad 2\sum_{k=1}^{\infty} \varrho_k = T < 1. \end{gathered} \tag{2.10}$$

Finally let us define two families of intervals I_k and J_k, $k \ge 1$, by setting

$$\begin{gathered} I_k = \left[t_k - \frac{\varrho_k}{2}, t_k + \frac{\varrho_k}{2} \right], \quad J_k = \left[t_k + \frac{\varrho_k}{2}, t_k + \frac{3\varrho_k}{2} \right] \\ t_k = \frac{\varrho_k}{2} + 2\sum_{j=1}^{k-1} \varrho_j , \quad \left(t_1 = \frac{\varrho_1}{2} \right). \end{gathered} \tag{2.11}$$

Now we are ready to construct the coefficient $a(t)$ for $t \in [0, 1]$ as follows

$$a(t) = \begin{cases} \delta_k \alpha\left(4\pi\nu_k \dfrac{t - t_k}{\varrho_k} \right) & \text{for } t \in I_k \\ \delta_{k+1} + (\delta_k - \delta_{k+1})\beta\left(\dfrac{t - t_k}{\varrho_k} - \dfrac{1}{2} \right) & \text{for } t \in J_k \\ 0 & \text{for } t \ge T , \end{cases} \tag{2.12}$$

and we define

$$a_1(t) = a^{1/n}(t).$$

It is easy to see that $a_1 \in C^\infty([0,T[)$. To estimate $\|a_1\|_{C^h}$ on J_k and on I_k, we use Faà di Bruno's formula (see [5]), with $F(x) = x^{1/n}$ and $a(t)$ given in (2.12). We obtain:

$$(F \circ a)^{(h)} = \sum_{j=1}^{h} (F^{(j)} \circ a) \sum_{p(h,j)} h! \prod_{i=1}^{h} \frac{(a^{(i)})^{\lambda_i}}{(\lambda_i!)(i!)^{\lambda_i}},$$

where we denote $\varphi^{(m)}(y) = (d/dy)^m \varphi(y)$ and where

$$p(h,j) = \left\{ (\lambda_1, \dots, \lambda_h);\ \lambda_i \ge 0,\ \sum_{i=1}^{h} \lambda_i = j,\ \sum_{i=1}^{h} i\lambda_i = h \right\}. \tag{2.13}$$

Then on J_k we have, taking (2.10) and (2.13) into account,

$$\begin{aligned} \|a_1\|_{C^h(J_k)} &\le C_1(n,h,\|\beta\|_{C^h}) \sum_{j=1}^{h} \delta_{k+1}^{(1/n)-j} \sum_{p(h,j)} \prod_{i=1}^{h} (\delta_k \rho_k^{-i})^{\lambda_i} \\ &\le C_2(n,h,\|\beta\|_{C^h})\, \rho_k^{-h} (\delta_{k+1})^{1/n} \left(\frac{\delta_k}{\delta_{k+1}} \right)^h. \end{aligned} \tag{2.14}$$

On the other hand, on I_k one easily obtains

$$\|a_1\|_{C^h(I_k)} \le C_3(n,h,\|\alpha\|_{C^h}) \delta_k^{1/n} \left(\frac{\nu_k}{\rho_k} \right)^h. \tag{2.15}$$

Now we define a solution $u \in C^\infty([0,T[;\gamma^s(\mathbf{T}))$ for any $s > s_0$ of $Pu = 0$, P as in (2.1), and we take $u_0 = u(0,x)$, $u_1 = \partial_t u(0,x)$ as Cauchy data. Here $\mathbf{T}$ denotes the one dimensional torus $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. Let us set

$$u(t,x) = \sum_{k=1}^{\infty} v_k(t) e^{ih_k x}. \tag{2.16}$$

We have

$$v_k''(t) + h_k^2 a(t) v_k(t) = 0 \tag{2.17}$$

hence, if we impose $v_k(t_k) = 1$, $v_k'(t_k) = 0$, we have, thanks to (2.9),

$$v_k(t) = w\left(4\pi \nu_k \frac{t - t_k}{\varrho_k} \right), \quad t \in I_k, \tag{2.18}$$

provided that

$$h_k^2 = \left(\frac{4\pi\nu_k}{\varrho_k}\right)^2 \delta_k^{-1}. \tag{2.19}$$

In particular

$$v_k\left(t_k - \frac{\varrho_k}{2}\right) = e^{-2\pi\varepsilon\nu_k}, \quad v_k'\left(t_k - \frac{\varrho_k}{2}\right) = 0, \tag{2.20}$$

$$v_k\left(t_k + \frac{\varrho_k}{2}\right) = e^{2\pi\varepsilon\nu_k}, \quad v_k'\left(t_k + \frac{\varrho_k}{2}\right) = 0. \tag{2.21}$$

Now we define the energy:

$$E_k(t) = |v_k'(t)|^2 + h_k^2 a(t)|v_k(t)|^2.$$

Taking (2.17) and (2.20) into account, we obtain then, for $t \le t_k - \varrho_k/2$,

$$\begin{aligned} E_k(t) &\le E_k\left(t_k - \frac{\varrho_k}{2}\right) \exp\left[\int_0^{t_k-\varrho_k/2} \frac{|a'(t)|}{a(t)}\,dt\right] \\ &= \left(\frac{4\pi\nu_k}{\rho_k}\right)^2 \exp\left[-4\pi\varepsilon\nu_k + \sum_{j=1}^{k-1}\int_{I_j}\frac{|a'(t)|}{a(t)}\,dt + \sum_{j=1}^{k-1}\int_{J_j}\frac{|a'(t)|}{a(t)}\,dt\right]. \end{aligned} \tag{2.22}$$

But

$$\int_{I_j}\frac{|a'(t)|}{a(t)}\,dt \le 8\pi M\nu_j,$$

$$\int_{J_j}\frac{|a'(t)|}{a(t)}\,dt = \log\left(\frac{1}{\delta_{j+1}}\right) - \log\left(\frac{1}{\delta_j}\right)$$

so, finally, for $t \le t_k - \varrho_k/2$, taking (2.22) into account, we obtain

$$E_k(t) \le C\exp\left[-4\pi\varepsilon\nu_k + 8\pi M\sum_{j=1}^{k-1}\nu_j + \log\left(\frac{1}{\delta_k}\right) + 2\log\left(\frac{\nu_k}{\varrho_k}\right)\right]. \tag{2.23}$$

Now we choose

$$\varrho_k = (k+k_0)^{-2}$$

in such a way that

$$\sum_{k=1}^{\infty}\varrho_k < \frac{1}{2}$$

and

$$\nu_k = \mu^k$$

with μ a large integer to be chosen later, and finally

$$\delta_k = \left(\frac{\nu_k}{\rho_k}\right)^{-hn} k^{-1}. \tag{2.24}$$

It is easy to see that, thanks to these choices, the right hand members of (2.14) and (2.15) go to 0 as k goes to $+\infty$ and so $a_1 \in C^h([0,1])$. On the other hand, one has

$$\log\left(\frac{1}{\delta_k}\right) \leq (hn+1)\log\left(\frac{\nu_k}{\rho_k}\right). \tag{2.25}$$

Now we choose μ an integer so large that

$$4\pi\varepsilon\nu_k > 8\pi M \sum_{j=1}^{k-1} \nu_j + (hn+3)\log\left(\frac{\nu_k}{\varrho_k}\right) + \varepsilon\nu_k. \tag{2.26}$$

From (2.23), (2.24), (2.25) and (2.26), we obtain

$$E_k(t)\exp\bigl(h_k^{1/s}\bigr) \leq C\exp\left[-\varepsilon\nu_k + C_1 k^{1/2s}\left(\frac{\nu_k}{\varrho_k}\right)^{(hn+2)/2s}\right]$$

and this expression goes to 0 for $k \to \infty$, for any $s > 1 + nh/2$.

So, for u defined by (2.16), $u(0,x)$ and $\partial_t u(0,x)$ are in $\gamma^{(s)}(\mathbf{T})$ for any $s > s_0$.

On the other hand, from (2.21) it follows immediately that $u(t,\cdot)$ is not bounded in $\mathcal{D}'(\mathbf{T})$ as $t \to T$, for any $s > s_0$. $\square$

3. General case

We first study the case $l = 1$ and we shall make a remark for the general case at the end of the last section. Instead of P in (1.4) we may study

$$\tilde{P} = D_t^2 - \mu^2 D_x a(t,x)^n D_x$$

with a small parameter $0 < \mu \leq 1$. Indeed this is achieved by a different scaling of the coordinates t and x. Actually we consider

$$P = D_t^2 - \langle\mu D\rangle a(t,x)^n \langle\mu D\rangle \tag{3.1}$$

which differs from $\tilde{P}$ by a zeroth order term which is irrelevant to our result, where

$$D_t = \frac{1}{i}\frac{\partial}{\partial t}, \quad \langle\mu D\rangle = (1+\mu^2 D_x^2)^{1/2}.$$

To prove the well posedness of the Cauchy problem we derive an a priori estimate for P.

To derive an a priori estimate it is convenient to use a specified class of pseudodifferential operators motivated by [8] which is suited to the operator P. To define the class we introduce the metric

$$g_z(dx, d\xi) = (a(t,x) + \langle \mu\xi\rangle^{-\delta})^{-1}\, dx^2 + \langle \xi\rangle_\mu^{-2}\, d\xi^2 \tag{3.2}$$

where $z = (x, \xi)$, $0 < \delta < 2$ and $\langle \xi\rangle_\mu = \mu^{-1}\langle \mu\xi\rangle$. Note that $\langle \mu\xi\rangle^s \in S(\langle \mu\xi\rangle^s, dx^2 + \langle \xi\rangle_\mu^{-2}\, d\xi^2)$ and $\langle \mu\xi\rangle^s \langle \xi\rangle_\mu^{-t} = \mu^t \langle \mu\xi\rangle^{s-t}$ for $t \geq 0$. Here we recall that $a(t,x) \geq 0$ verifies

$$\left|\partial_t^j \partial_x^k a(t,x)\right| \leq C_j A_j^k k!^s, \quad (t,x) \in [0,T] \times \mathbf{R}$$

for $j = 0, 1, 2$ and $k = 0, 1, \ldots$. We use Weyl-Hörmander calculus of pseudodifferential operators (see [6]). We denote by a^w the Weyl quantization of $a(x,\xi)$ but sometimes the suffix w is omitted if there is no confusion.

Lemma 3.1. *Let* $0 < \delta < 2$. *Then* g *is slowly varying and* σ *temperate.*

Proof. Let us write $z = (x, \xi)$, $w = (y, \eta)$. If $g_w(z - w) < c^2$ and hence $|\xi - \eta| < c\langle \xi\rangle_\mu$ then we see easily

$$\frac{\langle \mu\xi\rangle}{C} \leq \langle \mu\eta\rangle \leq C\langle \mu\xi\rangle, \quad \frac{\langle \xi\rangle_\mu}{C} \leq \langle \eta\rangle_\mu \leq C\langle \xi\rangle_\mu \tag{3.3}$$

with C independent of μ. With $\phi(t, x, \xi) = \sqrt{a(t,x) + \langle \mu\xi\rangle^{-\delta}}$ we have

$$a(t,x) = a(t,y) + \tau\phi(t,w)\partial_x a(t,y) + r,$$

$$|r| \leq \frac{c^2}{2}\big(\sup\left|\partial_x^2 a(t,x)\right|\big)\phi(t,w)^2, \quad |\tau| < c$$

if $|x - y| < c\phi(t,w)$. Since $a(t,x) \geq 0$, the right-hand side is bounded by

$$a(t,y) + cB(a(t,y) + \langle \mu\eta\rangle^{-\delta}).$$

Noting (3.3) it is easy to see that

$$a(t,x) + \langle \mu\xi\rangle^{-\delta} \leq (1 + cB')(a(t,y) + \langle \mu\eta\rangle^{-\delta}). \tag{3.4}$$

Repeating the same arguments we conclude that $a(t,x) + \langle \mu\xi\rangle^{-\delta}$ is g continuous and this together with (3.3) proves that g is slowly varying.

We next show that g is σ temperate. It is enough to show

$$g_w(T) \leq C g_z(T)(1 + g_w^\sigma(z - w))^N, \quad \forall T$$

with some C, N when $g_w(z-w) \ge c^2$. Note that

$$g_w^\sigma(z-w) = \langle\eta\rangle_\mu^2\phi(t,w)^2 g_w(z-w) \ge c^2\langle\eta\rangle_\mu^2\phi(t,w)^2 \ge c^2\langle\mu\eta\rangle^{2-\delta} \tag{3.5}$$

and

$$|x-y|^2 \le \langle\eta\rangle_\mu^{-2} g_w^\sigma(z-w) \le g_w^\sigma(z-w),$$
$$|\xi-\eta|^2 \le \phi(t,w)^{-2} g_w^\sigma(z-w) \le \langle\mu\eta\rangle^\delta g_w^\sigma(z-w).$$

Note now that

$$a(t,x) \le a(t,y) + B(a(t,y) + |x-y|^2)$$

and, by (3.5), $|x-y|^2 \le g_w^\sigma(z-w) \le C\langle\mu\eta\rangle^{-\delta} g_w^\sigma(z-w)^{1+\delta/(2-\delta)}$. One obtains then:

$$a(t,x) \le C(a(t,y) + \langle\mu\eta\rangle^{-\delta})(1 + g_w^\sigma(z-w))^{1+\delta/(2-\delta)}.$$

It is easy to see

$$\langle\mu\xi\rangle^{-\delta} \le C\langle\mu\eta\rangle^{-\delta}(1+|\xi-\eta|)^\delta \le C\langle\mu\eta\rangle^{-\delta}(1+g_w(z-w)^{1/2+\delta/2(2-\delta)})^\delta$$

and hence one has

$$a(t,x) + \langle\mu\xi\rangle^{-\delta} \le C(a(t,y) + \langle\mu\eta\rangle^{-\delta})(1 + g_w^\sigma(z-w))^N$$

with some N. The same reasoning shows that

$$\langle\xi\rangle_\mu^2 \le C\langle\eta\rangle_\mu^2(1+|\xi-\eta|)^2 \le C\langle\eta\rangle_\mu^2(1+g_w^\sigma(z-w))^{N'}$$

with some N'. These prove the assertion. □

Let us recall Theorem 18.5.4 in [6].

Proposition 3.1. *Let $p_i \in S(m_i, g)$, $i = 1, 2$ where $m_i > 0$ are σ, g temperate. Then we have $p_1^w p_2^w = (p_1 \# p_2)^w$ where*

$$p_1 \# p_2 - \sum_{\alpha+\beta<k} \frac{(-1)^\beta}{2^{\alpha+\beta}\alpha!\,\beta!} p_{1(\beta)}^{(\alpha)} p_{2(\alpha)}^{(\beta)} \in S\big(m_1 m_2 \langle\xi\rangle_\mu^{-k}(a + \langle\mu\xi\rangle^{-\delta})^{-k/2}, g\big)$$

with $p_{(\beta)}^{(\alpha)} = \partial_\xi^\alpha(-i\partial_x)^\beta p$.

Assume that p_i are real then it is clear that

$$\sum_{\alpha+\beta=\mathrm{odd}} \frac{(-1)^\beta}{2^{\alpha+\beta}\alpha!\,\beta!} p_{1(\beta)}^{(\alpha)} p_{2(\alpha)}^{(\beta)}$$

are pure imaginary and

$$\begin{aligned}\text{Re}(p_1 \# p_2) - \sum_{\alpha+\beta=\text{even}<k} \frac{(-1)^\beta}{2^{\alpha+\beta}\alpha!\,\beta!} p_{1(\beta)}^{(\alpha)} p_{2(\alpha)}^{(\beta)} \\ \in S\big(m_1 m_2 \langle\xi\rangle_\mu^{-k}(a + \langle\mu\xi\rangle^{-\delta})^{-k/2}, g\big).\end{aligned} \tag{3.6}$$

It is also clear that if $p \in S(m, g)$ is real then

$$\begin{aligned}p \# p - \sum_{\alpha+\beta=\text{even}<k} \frac{(-1)^\beta}{2^{\alpha+\beta}\alpha!\,\beta!} p_{(\beta)}^{(\alpha)} p_{(\alpha)}^{(\beta)} \\ \in S\big(m^2 \langle\xi\rangle_\mu^{-k}(a + \langle\mu\xi\rangle^{-\delta})^{-k/2}, g\big).\end{aligned} \tag{3.7}$$

4. Proof of Theorem 1.3

Let $\beta \le 1/s$ and set

$$P^\sharp = e^{-\gamma t\langle\mu D\rangle^\beta} P e^{\gamma t\langle\mu D\rangle^\beta}.$$

Eventually we take $\beta = 1/(1+n)$. We see that

$$\begin{aligned}P^\sharp &= (D_t - i\gamma\langle\mu D\rangle^\beta)^2 - \langle\mu D\rangle a^n \langle\mu D\rangle \\ &\quad - \langle\mu D\rangle b(t,x,D)\langle\mu D\rangle + R \\ &= A^2 - \langle\mu D\rangle a^n\langle\mu D\rangle - \langle\mu D\rangle b\langle\mu D\rangle + R\end{aligned}$$

where $A = D_t - i\gamma\langle\mu D\rangle^\beta$ and

$$\begin{aligned}b(t,z) &= \sum_{1\le k+j<N} c_{kj} D_x^{j+k} a(t,x)^n \partial_\xi^j e^{-\gamma t\langle\mu\xi\rangle^\beta} \partial_\xi^k e^{\gamma t\langle\mu\xi\rangle^\beta}, \\ R &\in \mu^N S\big(\langle\mu\xi\rangle^{-(1-\beta)N}, dx^2 + \langle\xi\rangle_\mu^{-2+2\beta} d\xi^2\big).\end{aligned}$$

Note that to prove Theorem 1.3 it suffices to derive an apriori estimate for $P^\sharp$ because $\beta \le 1/s$. We introduce the energy:

$$E(u) = \|Au\|^2 + \text{Re}(a^n\langle\mu D\rangle u, \langle\mu D\rangle u) + \|\langle\mu D\rangle^\beta u\|^2. \tag{4.1}$$

Then we see easily that:

$$\begin{aligned}\frac{d}{dt}E &= -2\gamma\ \text{Re}\big[\|\langle\mu D\rangle^{\beta/2}Au\|^2 + \|\langle\mu D\rangle^{3\beta/2}u\|^2 + (a^n\langle\mu D\rangle^{1+\beta}u, \langle\mu D\rangle u)\big] \\ &\quad - 2\,\text{Im}\big(\langle\mu D\rangle^\beta Au, \langle\mu D\rangle^\beta u\big) + \text{Re}\big(na^{n-1}a'\langle\mu D\rangle u, \langle\mu D\rangle u\big) \\ &\quad - 2\,\text{Im}\big[(\langle\mu D\rangle b\langle\mu D\rangle u, Au) - (Ru, Au) + (P^\sharp u, Au)\big].\end{aligned}$$

We now prove that one can bound dE/dt from above by constant times

$$\|\langle\mu D\rangle^{-\beta/2}P^{\sharp}u\|^2.$$

We first remark

Lemma 4.1. *Let $2\beta \geq \delta$ and $K \in S\big((a+\langle\mu\xi\rangle^{-\delta})^{n-1}\langle\mu\xi\rangle^{-\beta}, g\big)$. Then there are C_1, C such that*

$$\begin{aligned} &C_1\,\mathrm{Re}\big([(a+\langle\mu\xi\rangle^{-\delta})^n\langle\mu\xi\rangle^{\beta}]^w u, u\big) - \mathrm{Re}(K^w u, u)\\ &\geq -C\mu^2\|\langle\mu D\rangle^{-1+\beta/2}u\|^2. \end{aligned}$$

Proof. Let us put $T = \mathrm{Re}\,K$ and consider

$$q = (a+\langle\mu\xi\rangle^{-\delta})^{n/2}\langle\mu\xi\rangle^{\beta/2}\big[1 - C_1^{-1}T(a+\langle\mu\xi\rangle^{-\delta})^{-n}\langle\mu\xi\rangle^{-\beta}\big]^{1/2}$$

so that

$$\mathrm{Re}\big(\big[(a+\langle\mu\xi\rangle^{-\delta})^n\langle\mu\xi\rangle^{\beta}\big]^w u, u\big) - C_1^{-1}\,\mathrm{Re}(K^w u, u) = \mathrm{Re}([q^2]^w u, u). \tag{4.2}$$

Noting that $T(a+\langle\mu\xi\rangle^{-\delta})^{-n}\langle\mu\xi\rangle^{-\beta} \in S(1, g)$ by the assumption $2\beta \geq \delta$ we see that $q \in S\big((a+\langle\mu\xi\rangle^{-\delta})^{n/2}\langle\mu\xi\rangle^{\beta/2}, g\big)$ and then (3.7) gives

$$q \# q = q^2 + \mu^2 S\big((a+\langle\mu\xi\rangle^{-\delta})^{n-1}\langle\mu\xi\rangle^{\beta-2}, g\big).$$

Hence the right-hand side of (4.2) is bounded from below by $-C\mu^2\|\langle\mu D\rangle^{-1+\beta/2}u\|^2$ which proves the assertion. □

Lemma 4.2. *Let $\delta n + 2\beta \geq 2$. Then there are C_2, C_2' such that*

$$\begin{aligned} C_2\,\mathrm{Re}\big(a^n\langle\mu D\rangle^{1+\beta}u, \langle\mu D\rangle u\big) \geq{}& \mathrm{Re}\big(\big[(a+\langle\mu\xi\rangle^{-\delta})^n\langle\mu\xi\rangle^{\beta}\big]^w\langle\mu D\rangle u, \langle\mu D\rangle u\big)\\ &- C_2'\|\langle\mu D\rangle^{3\beta/2}u\|^2. \end{aligned}$$

Proof. Note that (3.6) shows

$$\mathrm{Re}\{a^n \# \langle\mu\xi\rangle^{\beta}\} = a^n\langle\mu\xi\rangle^{\beta} + R, \quad R \in \mu^2 S(\langle\mu\xi\rangle^{\beta-2}, g).$$

From the assumption $n\delta + 2\beta \geq 2$ it follows that

$$\mathrm{Re}\big(\langle\mu D\rangle^{-n\delta+\beta}\langle\mu D\rangle u, \langle\mu D\rangle u\big) \leq C\|\langle\mu D\rangle^{3\beta/2}u\|^2 \tag{4.3}$$

with some $C > 0$. This proves that

$$\begin{aligned} C_2\,\mathrm{Re}\big(a^n\langle\mu D\rangle^{1+\beta}u, \langle\mu D\rangle u\big) \geq{}& \mathrm{Re}\big(\big[(a^n+\langle\mu\xi\rangle^{-n\delta})\langle\mu\xi\rangle^{\beta}\big]^w\langle\mu D\rangle u, \langle\mu D\rangle u\big)\\ &- C_2'\|\langle\mu D\rangle^{3\beta/2}u\|^2. \end{aligned}$$

We prove now that one can replace $\big([(a^n+\langle\mu\xi\rangle^{-n\delta})\langle\mu\xi\rangle^{\beta}]^w\langle\mu D\rangle u,\langle\mu D\rangle u\big)$ by constant times $\big([(a+\langle\mu\xi\rangle^{-\delta})^n\langle\mu\xi\rangle^{\beta}]^w\langle\mu D\rangle u,\langle\mu D\rangle u\big)$. This proves the assertion. Let us set

$$q=(a+\langle\mu\xi\rangle^{-\delta})^{n/2}\langle\mu\xi\rangle^{\beta/2}\left[\frac{a^n+\langle\mu\xi\rangle^{-n\delta}}{(a+\langle\mu\xi\rangle^{-\delta})^n}-B^{-1}\right]^{1/2}.$$

Take B large so that

$$(a^n+\langle\mu\xi\rangle^{-n\delta})(a+\langle\mu\xi\rangle^{-\delta})^{-n}-B^{-1}\ge c>0.$$

Since $(a^n+\langle\mu\xi\rangle^{-n\delta})\in S((a+\langle\mu\xi\rangle^{-\delta})^n,g)$ and $(a+\langle\mu\xi\rangle^{-\delta})^{-n}\in S((a+\langle\mu\xi\rangle^{-\delta})^{-n},g)$ one has

$$\big[(a^n+\langle\mu\xi\rangle^{-n\delta})(a+\langle\mu\xi\rangle^{-\delta})^{-n}-B^{-1}\big]^{1/2}\in S(1,g).$$

Then it follows that

$$q\in S\big((a+\langle\mu\xi\rangle^{-\delta})^{n/2}\langle\mu\xi\rangle^{\beta/2},g\big)$$

and it suffices to repeat the proof of Lemma 4.1. □

We now estimate $\mathrm{Re}(a^{n-1}a'\langle\mu D\rangle u,\langle\mu D\rangle u)$. Write

$$a^{n-1}a'=(a+\langle\mu\xi\rangle^{-\delta})^{(n-1)/2}a'\langle\mu\xi\rangle^{\beta/2}\,\#\,K+R$$

where $K=a^{n-1}(a+\langle\mu\xi\rangle^{-\delta})^{-(n-1)/2}\langle\mu\xi\rangle^{-\beta/2}$ and $\mathrm{Re}\,R\in\mu^2S(\langle\mu\xi\rangle^{-2},g)$. Then it is clear that

$$\begin{aligned}2\big|\big(a^{n-1}a'\langle\mu D\rangle u,\langle\mu D\rangle u\big)\big|&\le\big\|\big[(a+\langle\mu\xi\rangle^{-\delta})^{(n-1)/2}a'\langle\mu\xi\rangle^{\beta/2}\big]^w\langle\mu D\rangle u\big\|^2\\&\quad+\|K^w\langle\mu D\rangle u\|^2+C\mu^2\|u\|^2.\end{aligned}\tag{4.4}$$

Noting that $K\in S\big((a+\langle\mu\xi\rangle^{-\delta})^{(n-1)/2}\langle\mu\xi\rangle^{-\beta/2},g\big)$ and hence

$$K\,\#\,K\in S\big((a+\langle\mu\xi\rangle^{-\delta})^{n-1}\langle\mu\xi\rangle^{-\beta},g\big),$$

one can apply Lemma 4.1 to estimate $\|K^w\langle\mu D\rangle u\|^2$; take

$$\delta=\frac{2}{n+1},\quad \beta=\frac{1}{n+1}\tag{4.5}$$

so that $2\beta=\delta$ and $\delta n+2\beta=2$. Then we have:

$$\begin{aligned}&C_2\,\mathrm{Re}\big(\big[(a+\langle\mu\xi\rangle^{-\delta})^n\langle\mu\xi\rangle^{\beta}\big]^w\langle\mu D\rangle u,\langle\mu D\rangle u\big)\\&\ge\|K^w\langle\mu D\rangle u\|^2-C_2'\mu^2\|\langle\mu D\rangle^{3\beta/2}u\|^2.\end{aligned}\tag{4.6}$$

We estimate the first term on the right-hand side of (4.4). Note that

$$(a+\langle\mu\xi\rangle^{-\delta})^{(n-1)/2}a'\langle\mu\xi\rangle^{\beta/2}\,\#\,(a+\langle\mu\xi\rangle^{-\delta})^{(n-1)/2}a'\langle\mu\xi\rangle^{\beta/2}$$
$$=(a+\langle\mu\xi\rangle^{-\delta})^{n-1}a'^2\langle\mu\xi\rangle^{\beta}+R,\qquad R\in\mu^2S(\langle\mu\xi\rangle^{\beta-2},g)$$

by (3.7). Let us consider

$$(a+\langle\mu\xi\rangle^{-\delta})^{n}\langle\mu\xi\rangle^{\beta}-B^{-1}(a+\langle\mu\xi\rangle^{-\delta})^{n-1}a'^2\langle\mu\xi\rangle^{\beta}$$
$$=(a+\langle\mu\xi\rangle^{-\delta})^{n}\langle\mu\xi\rangle^{\beta}\big[1-B^{-1}a'^2(a+\langle\mu\xi\rangle^{-\delta})^{-1}\big]=q^2.$$

Lemma 4.3. *Let $\phi=a'^2(a+\langle\mu\xi\rangle^{-\delta})^{-1}$. Then we have*

$$(a+\langle\mu\xi\rangle^{-\delta})^{k/2}D_x^k\phi\in S(1,g),\quad k=0,1,2.$$

Proof. It is enough to note that $a'^2\in S((a+\langle\mu\xi\rangle^{-\delta}),g)$ which follows from the assumption $0\le a(t,x)\in C^2\big([-\delta,T+\delta];\gamma^{(s)}(\mathbf{R})\big)$ with some $\delta>0$ and Glaeser inequality. □

From Lemma 4.3 and (3.7) it follows that

$$q\,\#\,q=q^2+R,\quad R\in\mu^2S(\langle\mu\xi\rangle^{\beta-2},g)$$

and then

$$\begin{aligned}&C_3\,\mathrm{Re}\Big(\big[(a+\langle\mu\xi\rangle^{-\delta})^{n}\langle\mu\xi\rangle^{\beta}\big]^w\langle\mu D\rangle u,\langle\mu D\rangle u\Big)\\ &\ge B^{-1}\big\|\big[(a+\langle\mu\xi\rangle^{-\delta})^{(n-1)/2}a'\langle\mu\xi\rangle^{\beta/2}\big]^w\langle\mu D\rangle u\big\|^2-C_3'\mu^2\|\langle\mu D\rangle^{\beta/2}u\|^2.\end{aligned}\tag{4.7}$$

From Lemma 4.2 and (4.4), (4.6), (4.7) we have

$$\begin{aligned}&\big|\big(a^{n-1}a'\langle\mu D\rangle u,\langle\mu D\rangle u\big)\big|\\ &\le C_4\,\mathrm{Re}\big(a^n\langle\mu D\rangle^{1+\beta}u,\langle\mu D\rangle u\big)+C_4'\|\langle\mu D\rangle^{3\beta/2}u\|^2.\end{aligned}\tag{4.8}$$

Finally we estimate the remainder terms. Let us study

$$\big(\langle\mu D\rangle b\langle\mu D\rangle u,Au\big)=\big(\langle\mu D\rangle^{1-\beta/2}b\langle\mu D\rangle u,\langle\mu D\rangle^{\beta/2}Au\big).$$

We have

Lemma 4.4. *Taking $\mu>0$ small we have*

$$\mathrm{Re}(a^n\langle\mu D\rangle^{1+\beta}u,\langle\mu D\rangle u)-\|\langle\mu D\rangle^{1-\beta/2}b\langle\mu D\rangle u\|^2$$
$$\ge -C\mu^2\|\langle\mu D\rangle^{3\beta/2}u\|^2.$$

Proof. Thanks to Lemma 4.2 it is enough to show

$$C_0^{-1}\,\mathrm{Re}\Big(\big[(a+\langle\mu\xi\rangle^{-\delta})^n\langle\mu\xi\rangle^{\beta}\big]^w\langle\mu D\rangle u,\langle\mu D\rangle u\Big)-\|\langle\mu D\rangle^{1-\beta/2}b\langle\mu D\rangle u\|^2$$
$$\geq -C\mu^2\|\langle\mu D\rangle^{3\beta/2}u\|^2.$$

Recall that

$$b(x,\xi)=\sum_{1\leq j+k<N}c_{jk}D_x^{j+k}a^n\partial_\xi^j e^{-\gamma t\langle\mu\xi\rangle^{\beta}}\partial_\xi^k e^{\gamma t\langle\mu\xi\rangle^{\beta}}=c_1+c_2,$$

where $c_1=\gamma t\partial_\xi\langle\mu\xi\rangle^{\beta}D_x a^n$ and $c_2\in\mu^2 S(\langle\mu\xi\rangle^{-2+2\beta},g)$. Note that

$$\langle\mu\xi\rangle^{1-\beta/2}\#c_2\#\langle\mu\xi\rangle=b_2$$

with $b_2\in\mu^2 S(\langle\mu\xi\rangle^{3\beta/2},g)$. On the other hand it is clear that

$$\langle\mu\xi\rangle^{1-\beta/2}\#c_1=\gamma t\langle\mu\xi\rangle^{1-\beta/2}\partial_\xi\langle\mu\xi\rangle^{\beta}D_x a^n+\mu S(\langle\mu\xi\rangle^{\beta/2-1},g).$$

With

$$T=\langle\mu\xi\rangle^{1-\beta/2}D_x a^n\partial_\xi\langle\mu\xi\rangle^{\beta}$$

one has

$$\big\|\langle\mu D\rangle^{1-\beta/2}c_1^w\langle\mu D\rangle u\big\|\leq C\|T^w\langle\mu D\rangle u\|+C\mu\|\langle\mu D\rangle^{\beta/2}u\|.$$

Since

$$\overline{T}\#T=\langle\mu\xi\rangle^{2-\beta}(\partial_x a^n\partial_\xi\langle\mu\xi\rangle^{\beta})^2+\mu^4 S(\langle\mu\xi\rangle^{\beta-2},g)$$

by (3.7) it is enough to study

$$(a+\langle\mu\xi\rangle^{-\delta})^n\langle\mu\xi\rangle^{\beta}-C\langle\mu\xi\rangle^{2-\beta}(\partial_x a^n\partial_\xi\langle\mu\xi\rangle^{\beta})^2=(a+\langle\mu\xi\rangle^{-\delta})^n q^2$$

where

$$q=\langle\mu\xi\rangle^{\beta/2}\sqrt{1-C\langle\mu\xi\rangle^{2-2\beta}(\partial_x a^n\partial_\xi\langle\mu\xi\rangle^{\beta})^2(a+\langle\mu\xi\rangle^{-\delta})^{-n}}.$$

Since $a(t,x)\geq 0$ it is easy to see that, with $\phi=\langle\mu\xi\rangle^{2-2\beta}(\partial_x a^n\partial_\xi\langle\mu\xi\rangle^{\beta})^2(a+\langle\mu\xi\rangle^{-\delta})^{-n}$, we have

$$(a+\langle\mu\xi\rangle^{-\delta})^{k/2}D_x^k\phi\in\mu^2 S(1,g),\quad k=0,1,2,$$

and hence from (3.7)

$$(a+\langle\mu\xi\rangle^{-\delta})^{n/2}q\#(a+\langle\mu\xi\rangle^{-\delta})^{n/2}q=(a+\langle\mu\xi\rangle^{-\delta})^n q^2+\mu^2 S(\langle\mu\xi\rangle^{\beta-2},g)$$

which proves the assertion. □

Taking $\mu > 0$ small and γ large, from Lemma 4.4 and (4.8) one gets $dE/dt \le \|\langle \mu D\rangle^{-\beta/2} P^{\sharp} u\|^2$ and hence

$$E(u;t) \le E(u;0) + \int_0^t \|\langle \mu D\rangle^{-\beta/2} P^{\sharp} u\|^2 \, ds.$$

This shows

$$\gamma \|\langle \mu D\rangle^{\beta} u(t)\|^2 + \gamma \|Au(t)\|^2 \le C\left\{\|\langle \mu D\rangle^{\beta} u(0)\|^2 + \|D_t u(0)\|^2\right\} + C\int_0^t \left\|\langle \mu D\rangle^{-\beta/2} e^{-\gamma s\langle \mu D\rangle^{\beta}} P e^{\gamma s\langle \mu D\rangle^{\beta}} u(s)\right\|^2 ds.$$

Replacing u by $e^{-\gamma t\langle \mu D\rangle^{\beta}} u$ we have an apriori estimate for P.

Theorem 4.5. *Let $\mu > 0$ be small and $\gamma \ge \gamma_0$. Then there exists $C > 0$ such that we have*

$$\gamma \left\|\langle \mu D\rangle^{\beta} e^{-\gamma t\langle \mu D\rangle^{\beta}} u(t)\right\|^2 + \gamma \left\|e^{-\gamma t\langle \mu D\rangle^{\beta}} D_t u(t)\right\|^2 \le C\left\{\|\langle \mu D\rangle^{\beta} u(0)\|^2 + \|D_t u(0)\|^2\right\} + C\int_0^t \left\|\langle \mu D\rangle^{-\beta/2} e^{-\gamma s\langle \mu D\rangle^{\beta}} P u(s)\right\|^2 ds$$

for $0 \le t \le T$.

It is clear that this estimate still holds if we add a zeroth order term to P. Since $P^* = P$, we see that Theorem 4.5 holds for P^*. Then the standard duality arguments prove Theorem 1.3.

In order to prove Theorem 1.3 for the general

$$a(t,x) = \sum_{j=1}^{l} a_j(t,x)^n$$

we take the energy

$$E(u) = \|Au\|^2 + \sum_{j=1}^{l} \mathrm{Re}(a_j^n \langle \mu D\rangle u, \langle \mu D\rangle u) + \|\langle \mu D\rangle^{\beta} u\|^2.$$

Then we have

$$\begin{aligned}\frac{d}{dt}E = -2\gamma\, \mathrm{Re}\left[\|\langle \mu D\rangle^{\beta/2} Au\|^2 + \|\langle \mu D\rangle^{3\beta/2} u\|^2 + \sum_{j=1}^{l}(a_j^n \langle \mu D\rangle^{1+\beta} u, \langle \mu D\rangle u)\right] \\ - 2\,\mathrm{Im}\big(\langle \mu D\rangle^{\beta} Au, \langle \mu D\rangle^{\beta} u\big) + \sum_{j=1}^{l} \mathrm{Re}\big(n a_j^{n-1} a_j' \langle \mu D\rangle u, \langle \mu D\rangle u\big) \\ - 2\,\mathrm{Im}\left[\sum_{j=1}^{l}(\langle \mu D\rangle b_j \langle \mu D\rangle u, Au) - (Ru, Au) + (P^{\sharp} u, Au)\right].\end{aligned}$$

To bound dE/dt from above by constant times $\|\langle\mu D\rangle^{-\beta/2}P^\sharp u\|^2$ we employ the same arguments as in Section 4 to estimate each

$$\mathrm{Re}(a_j^n\langle\mu D\rangle^{1+\beta}u, \langle\mu D\rangle u), \quad \mathrm{Re}(na_j^{n-1}a_j'\langle\mu D\rangle u, \langle\mu D\rangle u), \quad (\langle\mu D\rangle b_j\langle\mu D\rangle u, Au)$$

$j = 1, 2, \ldots, l$, applying the calculus in $S(m, g_j)$ with

$$g_j = (a_j + \langle\mu\xi\rangle^{-\delta})^{-1}\,dx^2 + \langle\xi\rangle_\mu^{-2}\,d\xi^2.$$

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