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The main purpose of this paper is to improve Theorem 1 in [4].

Let $G$ be a 4-fold transitive group on $\Omega = \{1, 2, \ldots, n\}$, $H = G_{1,2,3,4}$ the subgroup of $G$ consisting of all the elements fixing the four letters 1, 2, 3 and 4, and let $\Delta$ be the totality of the letters fixed by $H$. Then the normalizer $N$ of $H$ in $G$ fixes $\Delta$. If we denote by $N^\Delta$ the restriction of $N$ on $\Delta$, then by the theorem of Jordan ([2]) and Witt ([5]) $N^\Delta$ is one of the following groups: $S_4$, $S_5$, $A_6$ or $M_{11}$.

In the first section, we shall consider the number of fixed letters of an involution. We shall prove especially that if $N^\Delta$ is $A_6$ or $M_{11}$ then the number $r$ of the fixed letters of any involution in $G$ satisfies the relation

$$n = r^2 + 2$$

and consequently all involutions have the same number of fixed letters.

Now let $P$ be a Sylow 2-group of $H$, $\Delta'$ the totality of the letters fixed by $P$ and $N'$ the normalizer of $P$ in $G$. Then, by the theorem of M. Hall ([1], Theorem 5.8.1), $(N')^\Delta'$ is one of the following groups: $S_4$, $S_5$, $A_6$, $A_7$ or $M_{11}$. In the second section, we shall first consider the case in which $P \neq 1$ and $P$ is transitive on $\Omega - \Delta'$ and we shall prove that if $n \geq 35$ $(N')^\Delta'$ must be $S_4$ or $S_5$. As a corollary we have that if $G$ is not alternating nor symmetric group and if $P (\neq 1)$ is transitive and regular on $\Omega - \Delta'$ then $G$ is $M_{11}$ or $M_{23}$. Since a transitive group which is abelian is regular, this gives an improvement of Theorem 1 in [4].

**Notation.** For a set $X$ let $|X|$ denote the number of elements of $X$. For a set $S$ of permutations on $\Omega$ the totality of the letters fixed by $S$ is denoted by $I(S)$. If a subset $\Delta$ of $\Omega$ is a fixed block, i.e. if $\Delta^S = \Delta$, then the restriction of $S$ on $\Delta$ is denoted by $S^\Delta$. For a permutation group $G$ the subgroup of $G$ consisting of all the elements fixing the letters $i, j, \ldots, k$ is denoted by $G_{i,j,\ldots,k}$.
1. Number of fixed letters of an involution.

Let \( G \) be a 4-fold transitive group on \( \Omega = \{1, 2, \ldots, n\} \), \( H \) the subgroup of \( G \) fixing four letters, \( \Delta = I(H) \) and let \( N \) be the normalizer of \( H \) in \( G \). Then \( N^\Delta \) must be one of the following groups: \( S_4, S_5, A_6 \) or \( M_{11} \).

**Proposition 1.** If \( N^\Delta = A_6 \) or \( M_{11} \), then the number \( r \) of the fixed letters of any involution in \( G \) satisfies the relation

\[
n = r^2 + 2.
\]

Proof. (1) Suppose that \( N^\Delta = A_6 \). Let \( a \) be an arbitrary involution. Since \( G \) is 4-fold transitive, taking a conjugate of \( a \) if necessary, we may assume that

\[
a = (1, 2) \cdots.
\]

Let \((k, l)\) be a 2-cycle of \( a \) different from \((1, 2)\). Then \( a \) normalizes \( G_{(1,2)} \) and by assumption \( a \) is an even permutation on \( \Delta = I(G_{(1,2)}) \). Therefore we have

\[
a^{a_1} = (1, 2) (i) (j) (k, l).
\]

Thus at least two letters are fixed by \( a \). Now for a subset \( \{i, j\} \) of \( I(a) \), \( a \) normalizes \( G_{(i,j)} \), therefore for \( \Delta_e = I(G_{(i,j)}) \) we have

\[
a^{a_2} = (1, 2) (i) (j) (k, l).
\]

Thus \( \{i, j\} \) determines uniquely a 2-cycle \((k, l)\) of \( a \), and then \( G_{(i,j)} \) and \( \{i, j\} = I(a) \cap I(G_{(i,j)}) \). We consider the map \( \varphi : \{i, j\} \rightarrow (k, l) \) from the family of all subsets of \( I(a) \) consisting of two letters into the family of all 2-cycles of \( a \) different from \((1, 2)\). From above \( \varphi \) is onto. To show that \( \varphi \) is one to one, suppose that \( \varphi(\{i, j\}) = \varphi(\{i', j'\}) = (k, l) \). Then \( I(a) \cap I(G_{(i,j)}) = \{i, j\} = \{i', j'\} \). Hence \( \varphi \) is one to one and the number of 2-cycles of \( a \) different from \((1, 2)\) is \( \mathcal{C}_2 \). Thus we have

\[
n = 2 + r + 2 \cdot \mathcal{C}_2 = r^2 + 2.
\]

(2) Suppose that \( N^\Delta = M_{11} \) and let \( a \) be an arbitrary involution. As in (1), we may assumed that \( a = (1, 2) \cdots \), and we can easily see that at least two letters are fixed by \( a \). If \( \{i_1, i_2\} \) is a subset of \( I(a) \), then \( a \) normalizes \( G_{(i_1,i_2)} \) and for \( \Delta_e = I(G_{(i_1,i_2)}) \) \( a^{a_1} \) is, being an involution of \( M_{11} \), of the following form:

\[
a^{a_1} = (1, 2) (i_1) (i_2) (i_3) (k_1, l_1) (k_2, l_2) (k_3, l_3).
\]
Then $G_{1,2,i_1,i_2} = G_{1,2,b_1,b_2}$ and thus $\{i_1, i_2\}$ determines uniquely a set of three 2-cycles $(k_1, l_1), (k_2, l_2), (k_3, l_3)$. Now consider the map

$$\varphi: \{i_1, i_2\} \to \{(k_1, l_1), (k_2, l_2), (k_3, l_3)\}$$

from the family of all subsets of $I(a)$ consisting of two letters into the family of the sets of three 2-cycles of $a$ different from $(1, 2)$. If a 2-cycle $(k, l)$ of $a$ different from $(1, 2)$ is given, then $a$ normalizes $G_{1,2,i_1,i_2}$ and for $\Delta_2 = I(G_{1,2,k,l})$ has, being an involution of $M_n$, just three fixed letters $\{i_1, i_2, i_3\}$. Then $I(a) \cap I(G_{1,2,k,l}) = \{i_1, i_2, i_3\}$ and $\varphi(\{i_1, i_2\}) = \varphi(\{i_2, i_3\}) \supseteq (k, l)$. Now, from the definition of $\varphi$, $\varphi(\{i_1, i_2\}) \supseteq (k, l)$ if and only if $G_{1,2,i_1,i_2} \subseteq I(a)$, i.e. $\{i_1, i_2\} \subseteq I(a) \cap I(G_{1,2,k,l})$. Hence the set of 2-cycles of $a$ different from $(1, 2)$ is the disjoint union of the images of $\varphi$ and each inverse image of $\varphi$ consists of three subsets. Therefore the number of 2-cycles of $a$ different from $(1, 2)$ is $rC_2$ and we have

$$n = 2 + r + 2 \cdot rC_2 = r^2 + 2.$$  

**Proposition 2.** If $N^a = S_n$, then the number $r$ of the fixed letters of any involution in $G$ satisfies the following relation:

$$r(r-1) \equiv 0 \pmod{3}.$$  

Proof. We may assume that $r \geq 2$ and the given involution is $a = (1, 2) \cdots$. If $\{i_1, i_2\}$ is a subset of $I(a)$, then $a$ normalizes $G_{1,2,i_1,i_2}$ and for $\Delta_1 = I(G_{1,2,i_1,i_2})$ we have

$$a^{\Delta_1} = (1, 2)(i_1)(i_2)(i_3),$$

and $G_{1,2,i_1,i_2} = G_{1,2,i_1,i_3} = G_{1,2,i_2,i_3}$. Now consider the map

$$\varphi: \{i_1, i_2\} \to G_{1,2,i_1,i_2}$$

from the family of all subsets of $I(a)$ consisting of two letters into the family of the subgroups of $G$. Then each inverse image of $\varphi$ consists of three subsets and hence we have

$$rC_2 = \frac{r(r-1)}{2} \equiv 0 \pmod{3}.$$  

**2. Main theorem.**

Let $G$ be again a 4-fold transitive group on $\Omega = \{1, 2, \cdots, n\}$. It is known that the only 4-fold transitive (not alternating nor symmetric) groups on less than 35 letters are the four Mathieu groups $M_{11}, M_{12}, M_{23}$ and $M_{24}$. Therefore in the following we may assume that $n \geq 35$. 
Now let $H = G_{1,2,3,4}$, $\Delta = I(H)$ and let $P$ be a Sylow 2-group of $H$, $\Delta' = I(P)$. Then $\Delta' \supset \Delta$. We denote the normalizers of $H$ and $P$ by $N$ and $N'$ respectively. Then $(N')^{a'}$ is one of the following groups: $S_4$, $S_5$, $A_4$, $A_5$, or $M_{11}$. We first prove the following

**Proposition 3.** If $P$ is transitive on $\Omega - \Delta'$ and $n \geq 35$, then $(N')^{a'}$ must be $S_4$ or $S_5$.

**Proof.** We first remark that if $i \in \Delta' - \Delta$ the length of the set of transitivity of $H$ containing $i$ is odd since the subgroup of $H$ fixing $i$ contains a Sylow 2-group $P$ of $H$.

The proof in the following is by contradiction.

(1) Suppose that $(N')^{a'} = A_4$ and $\Delta' = \{1, 2, 3, 4, 5, 6\}$. Then $N^a$ must be $A_4$, $S_5$ or $S_4$.

(1.1) Suppose $N^a = A_4$ and let $a$ be a central involution of $P$. Then, by Lemma 2 in [3], $|I(a)| = 6$ and, by Proposition 1, we have

$$n = 6^2 + 2 = 38.$$ 

Consider the map $\varphi : i \mapsto G_{1,2,3,i}$ from the set $\{4, 5, \ldots, 38\}$ into the family of subgroups of $G$. If $I(G_{1,2,3,i}) = \{1, 2, 3, i, j, k\}$ then $\varphi^{-1}(G_{1,2,3,i})$ consists of the three letters $i, j, k$. Hence we have

$$38 - 3 = 35 \equiv 0 \pmod{3},$$

which is a contradiction.

(1.2) Suppose that $N^a = S_5$ and $\Delta' = \{1, 2, 3, 4, 5\}$. By the quadruple transitivity $G$ contains an involution $a = (1, 2)(3)(4)\cdots$. Since $H = G_{1,2,3,4}$ is normalized by $a$, $\Delta$ is fixed by $a$ and hence $a$ fixes the letter 5, i.e.

$$a = (1, 2)(3)(4)(5)\cdots.$$ 

Now the number of Sylow 2-groups of $H$ is odd. Therefore there is a Sylow 2-group of $H$ which is normalized by $a$. We may assume that it is $P$. Then $\Delta'$ is fixed by $a$ and we have

$$a^{a'} = (1, 2)(3)(4)(5)(6).$$

But this is a contradiction since $a^{a'}$ must be an even permutation.

(1.3) Suppose that $N^a = S_4$ and let $\Gamma = \Omega - \Delta'$. Then the sets of transitivity of $H$ on $\Omega - \Delta = \{5, 6\} \cup \Gamma$ can be assumed to be one of the following:

(i) $\{5, 6\}$ and $\Gamma$,

(ii) $\{5, 6\} \cup \Gamma$.

Since $\Gamma$ is a set of transitivity of the 2-group $P$ $|\Gamma|$ is a power of 2.
Hence in both cases the length of the set of transitivity containing the
latter \(5 \in \Delta' - \Delta\) is even. This is a contradiction by the first remark.

(2) Suppose that \(\langle N \rangle^\Delta = A_7\) and \(\Delta' = \{1, 2, \ldots, 7\}\). Then \(N^\Delta\) must
be \(A_7, S_8\) or \(S_4\).

(2.1) Suppose \(N^\Delta = A_7\) and let \(a\) be a central involution of \(P\).
Then \(|I(a)| = 7\) and we have by Proposition 1
\[
n = 7^2 + 2 = 51.
\]
Since \(P\) is transitive on \(\Omega - \Delta'\) and \(|\Omega - \Delta'| = 51 - 7 = 44\) is not a power
of 2 we have a contradiction.

(2.2) Suppose that \(N^\Delta = S_8\) and \(\Delta = \{1, 2, 3, 4, 5\}\). Then the sets of
transitivity of \(H\) on \(\Omega - \Delta = \{6, 7\} \cup \Gamma\) may be assumed to be one of the following:

(i) \(\{6, 7\}\) and \(\Gamma\),
(ii) \(\{6, 7\} \cup \Gamma\).

But in both cases the length of the set of transitivity containing the
letter \(6 \in \Delta' - \Delta\) is even. This is a contradiction by the first remark.

(2.3) Suppose \(N^\Delta = S_4\) and let \(\Gamma = \Omega - \Delta'\). Then the sets of transi-
tivity of \(H\) on \(\Omega - \Delta = \{5, 6, 7\} \cup \Gamma\) may be assumed to be one of the
following:

(i) \(\{5, 6, 7\}\) and \(\Gamma\),
(ii) \(\{5, 6\}\) and \(\{7\} \cup \Gamma\),
(iii) \(\{5, 6, 7\} \cup \Gamma\).

The case (ii) does not occur since the length of the set of transitivity
containing the letter \(5 \in \Delta' - \Delta\) is even in this case. In the case (iii),
\(H\) is transitive on \(\Omega - \Delta\). Hence \(G\) is 5-fold transitive and then the
subgroup \(G_1\) fixing the letter 1 is 4-fold transitive on \(\{2, 3, \ldots, n\}\) and
satisfies the assumption in (1). Thus as in (1) we have a contradiction.

We shall now consider the case (i). Let \(P'\) be an arbitrary Sylow
2-group of \(H\). Then there is an element \(x\) of \(H\) such that \(x^{-1}Px = P'\).
Since \(\{5, 6, 7\}\) is a set of transitivity of \(H\), it is fixed by \(x\). Therefore
\(H_{s_5,s_7} \supset P\) implies \(x^{-1}H_{s_5,s_7}x = H_{s_5,s_7} \supset P'\) and hence we have \(I(P') = \{1, 2, \ldots, 7\}\).
This shows that \(I(P')\) is independent of the choice of Sylow 2-
group \(P'\) of \(H\) and is uniquely determined by \(H\). Let \(a = (1, 2) \cdots\) be
an involution of \(G\) which is conjugate to a central involution of \(P\).
Then \(|I(a)| = 7\). If \(\{i_1, i_2\}\) is a subset of \(I(a)\), then \(G_{i_1,i_2}\) is normalized
by \(a\). Therefore there is a Sylow 2-group \(P''\) of \(G_{i_1,i_2}\) which is
normalized by \(a\). Let \(I(P'') = \{1, 2, i_1, i_2, i_3, k, l\}\). Since \(a\) is an even
permutation on \(I(P'')\), we may assume
\[
a = (1, 2)(i_1)(i_2)(i_3)(k, l) \cdots.
\]
Now $I(P'')$ is uniquely determined by $G_{1,2,i_1,i_2}$, therefore \{i_1, i_2\} determines uniquely a 2-cycle $(k, l)$ of $a$ and we have the map

$$\varphi : \{i_1, i_2\} \to (k, l)$$

from the family of all the subsets of $I(a)$ consisting of two letters into the family of 2-cycles of $a$ different from $(1, 2)$. By the definition of $\varphi$, it is easy to see that $\varphi(\{i_1, i_2\})=(k, l)$ if and only if $G_{1,2,i_1,i_2}$ and $G_{1,2,k,l}$ have a Sylow 2-group in common, and $\varphi$ is onto. Now suppose that $\varphi(\{i_1, i_2\})=\varphi(\{j_1, j_2\})=(k, l)$. Then $G_{1,2,i_1,i_2}$ and $G_{1,2,k,l}$ have a Sylow 2-group $P_1$ in common, and $G_{1,2,j_1,j_2}$ and $G_{1,2,k,l}$ have a Sylow 2-group $P_2$ in common. Since both $P_1$ and $P_2$ are Sylow 2-groups of $G_{1,2,k,l}$ we have $I(P_1)=I(P_2)$. Therefore $\{j_1, j_2\} \subseteq I(a) \cap I(P_1)=\{i_1, i_2, i_3\}$. Thus we have that each inverse image of $\varphi$ consists of three subsets of $I(a)$ and hence the number of 2-cycles of $a$ different from $(1, 2)$ is $\frac{1}{3} \cdot C_2=7$.

In this way we have

$$n = 2+7+14 = 23,$$

which contradicts the assumption.

(3) Suppose that $(N')^3'=M_{11}$ and $\Delta' = \{1, 2, \ldots, 11\}$. Then $N^a$ must be one of the following groups: $M_{11}$, $A_4$, $S_5$ or $S_4$.

(3.1) Suppose $N^a=M_{11}$ and let $a$ be a central involution of $P$. Then $|I(a)|=11$ and by Proposition 1 we have $n=11^2+2=123$. Since $P$ is transitive on $\Omega-\Delta'$ and $|\Omega-\Delta'|=123-11=112$ is not a power of 2, we have a contradiction.

(3.2) Suppose $N^a=A_4$. In the same way as in (3.1) we have $n=123$, which is a contradiction.

(3.3) Suppose $N^a=S_5$ and let $a$ be a central involution of $P$. Then $|I(a)|=11$ and by Proposition 2 we have

$$11(11-1) = 110 \equiv 0 \pmod{3}.$$ 

This is a contradiction.

(3.4) Suppose $N^a=S_4$. Since the length of a set of transitivity of $H$ containing one of the letters in $\Delta' - \Delta = \{5, 6, \ldots, 11\}$ is odd, the sets of transitivity of $H$ may be assumed to be one of the following:

(i) $\{5, 6, 7\}$, $\{8, 9, 10\}$ and $\{11\} \cup \Gamma$,

(ii) $\{5, 6, 7, 8, 9, 10, 11\}$ and $\Gamma$,

(iii) $\{5, 6, 7, 8, 9, 10, 11\} \cup \Gamma$.

First consider the case (i). Since $(N')^3'=M_{11}$, there is an element $x$ in $N'$ such that

$$x^3' = (1, 2, 3, 4)(i_1, i_2, i_3, i_4)(k)(l)(m).$$
Then $x$ normalizes $H=G_{1,2,3,4}$. Hence $x$ must fix two sets of transitivity 
\{5, 6, 7\} and \{8, 9, 10\} or interchange them. But, from the form of $x^a$, this is impossible.

In the case (iii), $G$ is 5-fold transitive. Hence $X=G_{1,2,3,4,5}$ and $P$ is a Sylow 2-group of $X_{1,2,3,4,5}$. Since $|I(P) - \{1\}| = 10$, we have a contradiction by the theorem of M. Hall ([1], Theorem 5.8.1).

Now consider the case (ii). If $P'$ is an arbitrary Sylow 2-group of $H$ there is an element $x$ of $H$ such that $P' = x^{-1}Px$. Since $\{5, 6, \ldots, 11\}$ is a set of transitivity of $H$, it is left invariant by $x$. Therefore $H_{5,6,\ldots,11} \supset P$ implies $H_{5,6,\ldots,11} \supset P'$ and we have $I(P') = \{1, 2, \ldots, 11\}$. This shows that $I(P')$ is independent of the choice of $P'$ and is determined uniquely by $H$. We denote it by $J(H)$. Let $a = (1, 2) \ldots$ be an involution which is conjugate to a central involution of $P$. We consider the map

$$\varphi : \{i_1, i_2\} \rightarrow J(G_{1,2,i_1,i_2})$$

which assigns $J(G_{1,2,i_1,i_2})$ to a subset $\{i_1, i_2\}$ of $I(a)$. Since $a$ normalizes $G_{1,2,i_1,i_2}$, there is a Sylow 2-group $P''$ of $G_{1,2,i_1,i_2}$ such that $a^{-1}P''a = P''$. Let $\Delta'' = I(P'') = J(G_{1,2,i_1,i_2})$. Then $a^{\Delta''}$ is an involution of $M_{11}$. Hence we have

$$a^{\Delta''} = (1, 2)(i_1)(i_2)(i_3)(k_1, l_1)(k_2, l_2)(k_3, l_3)$$

and $I(a) \cap J(G_{1,2,i_1,i_2}) = \{i_1, i_2, i_3\}$. Now it is easy to see that the inverse image $\varphi^{-1}(J(G_{1,2,i_1,i_2}))$ consists of three subsets $\{i_1, i_2\}, \{i_1, i_3\}, \{i_2, i_3\}$. Therefore we have

$$c_2 = \frac{11 \cdot 10}{2} \equiv 0 \pmod{3},$$

which is a contradiction.

From Proposition 3 we have easily an improvement of Theorem 1 in [4].

**Theorem.** Let $G$ be a 4-fold transitive group on $\Omega = \{1, 2, \ldots, n\}$, excluding $S_n$ and $A_n$. If a Sylow 2-group $P$ of the subgroup fixing four letters is not trivial, and transitive and regular on $\Omega - I(P)$, then $G$ must be $M_{12}$ or $M_{23}$.

Proof. We use the same notations as before. For $n < 35$ the theorem is trivial. Therefore we may assume $n \geq 35$, and then, by Proposition 3, we need consider only the case in which $(N')^{\Delta'} = S_4$ or $S_5$.

(1) Suppose $(N')^{\Delta'} = S_4$. Then $G$ is 5-fold transitive. Hence $X = G_1$.
is 4-fold transitive on \( \{2, 3, \ldots, n\} \). Since \( X_{2,3,4} = H \) and a Sylow 2-group \( P \) of \( H \) is regular on \( \{5, 6, \ldots, n\} \), \( X_{2,3,4,5} \) is of odd order. Therefore, by the theorem of M. Hall, \( X \) must be one of the following groups: \( S_4, S_5, A_5, A_6, \) or \( M_{11} \). But this contradicts the assumption \( n \geq 35 \).

(2) Suppose \( (N')^r = S_5 \) and let \( \Delta' = \{1, 2, 3, 4, 5\} \). Then \( N^a = S_4 \) or \( S_5 \). If \( N^a = S_4 \), then \( H \) is transitive on \( \{5, 6, \ldots, n\} \) and hence \( G \) is 5-fold transitive. Then \( X = G_1 \) is 4-fold transitive on \( \{2, 3, \ldots, n\} \) and satisfies the assumption in (1). Therefore as in (1) we have a contradiction. On the other hand, if \( N^a = S_5 \) then by Proposition 2 we have

\[
5(5-1) \equiv 0 \pmod{3},
\]

since for a central involution \( a \) of \( P \), \( |I(a)| = 5 \). But this is a contradiction.

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References