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Osaka University
The main purpose of this paper is to improve Theorem 1 in [4]. Let \( G \) be a 4-fold transitive group on \( \Omega = \{1, 2, \ldots, n\} \), \( H = G_{1, 2, 3, 4} \) the subgroup of \( G \) consisting of all the elements fixing the four letters 1, 2, 3 and 4, and let \( \Delta \) be the totality of the letters fixed by \( H \). Then the normalizer \( N \) of \( H \) in \( G \) fixes \( \Delta \). If we denote by \( N^\Delta \) the restriction of \( N \) on \( \Delta \), then by the theorem of Jordan ([2]) and Witt ([5]) \( N^\Delta \) is one of the following groups: \( S_4 \), \( S_5 \), \( A_6 \) or \( M_{11} \).

In the first section, we shall consider the number of fixed letters of an involution. We shall prove especially that if \( N^\Delta \) is \( A_6 \) or \( M_{11} \) then the number \( r \) of the fixed letters of any involution in \( G \) satisfies the relation

\[
n = r^2 + 2
\]

and consequently all involutions have the same number of fixed letters.

Now let \( P \) be a Sylow 2-group of \( H \), \( \Delta' \) the totality of the letters fixed by \( P \) and \( N' \) the normalizer of \( P \) in \( G \). Then, by the theorem of M. Hall ([1], Theorem 5.8.1), \( (N')^\Delta' \) is one of the following groups: \( S_4 \), \( S_5 \), \( A_6 \), \( A_7 \) or \( M_{11} \). In the second section, we shall first consider the case in which \( P \neq 1 \) and \( P \) is transitive on \( \Omega - \Delta' \) and we shall prove that if \( n \geq 35 \) \( (N')^\Delta' \) must be \( S_4 \) or \( S_5 \). As a corollary we have that if \( G \) is not alternating nor symmetric group and if \( P (\neq 1) \) is transitive and regular on \( \Omega - \Delta' \) then \( G \) is \( M_{11} \) or \( M_{23} \). Since a transitive group which is abelian is regular, this gives an improvement of Theorem 1 in [4].

**Notation.** For a set \( X \) let \( |X| \) denote the number of elements of \( X \). For a set \( S \) of permutations on \( \Omega \) the totality of the letters fixed by \( S \) is denoted by \( I(S) \). If a subset \( \Delta \) of \( \Omega \) is a fixed block, i.e. if \( \Delta^S = \Delta \), then the restriction of \( S \) on \( \Delta \) is denoted by \( S^\Delta \). For a permutation group \( G \) the subgroup of \( G \) consisting of all the elements fixing the letters \( i, j, \ldots, k \) is denoted by \( G_{i,j,\ldots,k} \).
1. Number of fixed letters of an involution.

Let $G$ be a 4-fold transitive group on $\Omega = \{1, 2, \ldots, n\}$, $H$ the subgroup of $G$ fixing four letters, $\Delta = I(H)$ and let $N$ be the normalizer of $H$ in $G$. Then $N^\Delta$ must be one of the following groups: $S_4$, $S_5$, $A_6$ or $M_{11}$.

**Proposition 1.** If $N^\Delta = A_6$ or $M_{11}$, then the number $r$ of the fixed letters of any involution in $G$ satisfies the relation

$$n = r^2 + 2.$$ 

Proof. (1) Suppose that $N^\Delta = A_6$. Let $a$ be an arbitrary involution. Since $G$ is 4-fold transitive, taking a conjugate of $a$ if necessary, we may assume that

$$a = (1, 2) \cdots.$$ 

Let $(k, l)$ be a 2-cycle of $a$ different from $(1, 2)$. Then $a$ normalizes $G_{1,2,k,l}$ and by assumption $a$ is an even permutation on $\Delta_1 = I(G_{1,2,k,l})$. Therefore we have

$$a^\Delta_1 = (1, 2)(i)(j)(k, l).$$

Thus at least two letters are fixed by $a$. Now for a subset $\{i, j\}$ of $I(a)$, $a$ normalizes $G_{1,2,i,j}$, therefore for $\Delta_3 = I(G_{1,2,i,j})$ we have

$$a^\Delta_3 = (1, 2)(i)(j)(k, l).$$

Thus $\{i, j\}$ determines uniquely a 2-cycle $(k, l)$ of $a$, and then $G_{1,2,i,j} = G_{1,2,k,l}$ and $\{i, j\} = I(a) \cap I(G_{1,2,k,l})$. We consider the map $\varphi: \{i, j\} \rightarrow (k, l)$ from the family of all subsets of $I(a)$ consisting of two letters into the family of all 2-cycles of $a$ different from $(1, 2)$. From above $\varphi$ is onto. To show that $\varphi$ is one to one, suppose that $\varphi(\{i, j\}) = \varphi(\{i', j'\}) = (k, l)$. Then $I(a) \cap I(G_{1,2,k,l}) = \{i, j\} = \{i', j'\}$. Hence $\varphi$ is one to one and the number of 2-cycles of $a$ different from $(1, 2)$ is $r \cdot C_2$. Thus we have

$$n = 2^2 + 2^2 \cdot r \cdot C_2 = r^2 + 2.$$ 

(2) Suppose that $N^\Delta = M_{11}$ and let $a$ be an arbitrary involution. As in (1), we may assumed that $a = (1, 2) \cdots$, and we can easily see that at least two letters are fixed by $a$. If $\{i_1, i_2\}$ is a subset of $I(a)$, then $a$ normalizes $G_{1,2,i_1,i_2}$ and for $\Delta_1 = I(G_{1,2,i_1,i_2})$ $a^\Delta_1$ is, being an involution of $M_{11}$, of the following form:

$$a^\Delta_1 = (1, 2)(i_1)(i_2)(i_3)(k_1, l_1)(k_2, l_2)(k_3, l_3).$$
Then \( G_{1,2,i_1,i_2} = G_{1,2,i_1} \) and thus \( \{i_1, i_2\} \) determines uniquely a set of three 2-cycles \((k_1, l_1), (k_2, l_2), (k_3, l_3)\). Now consider the map

\[ \varphi: \{i_1, i_2\} \rightarrow \{(k_1, l_1), (k_2, l_2), (k_3, l_3)\} \]

from the family of all subsets of \( I(a) \) consisting of two letters into the family of the sets of three 2-cycles of \( a \) different from \((1, 2)\). If a 2-cycle \((k, l)\) of \( a \) different from \((1, 2)\) is given, then \( a \) normalizes \( G_{1,2,i_1,i_2} \) and for \( \Delta = I(G_{1,2,i_1,i_2}) \) \( a^{\Delta} \) has, being an involution of \( M_{11} \), just three fixed letters \( \{i_1, i_2, i_3\} \). Then \( I(a) \cap I(G_{1,2,i_1,i_2}) = \{i_1, i_2, i_3\} \) and \( \varphi(\{i_1, i_2\}) = \varphi(\{i_1, i_3\}) \supseteq (k, l) \). Now, from the definition of \( \varphi, \varphi(\{i_1, i_2\}) \supseteq (k, l) \) if and only if \( G_{1,2,i_1,i_2} = G_{1,2,i_1,i_3} \), i.e. \( \{i_1, i_2\} \subseteq I(a) \cap I(G_{1,2,i_1,i_2}) \). Hence the set of 2-cycles of \( a \) different from \((1, 2)\) is the disjoint union of the images of \( \varphi \) and each inverse image of \( \varphi \) consists of three subsets. Therefore the number of 2-cycles of \( a \) different from \((1, 2)\) is \( r \) and we have

\[ n = 2 + r + 2rC_2 = r^2 + 2. \]

**Proposition 2.** If \( N^3 = S_6 \), then the number \( r \) of the fixed letters of any involution in \( G \) satisfies the following relation:

\[ r(r-1) \equiv 0 \pmod{3}. \]

**Proof.** We may assume that \( r \geq 2 \) and the given involution is \( a = (1, 2) \cdots \). If \( \{i_1, i_2\} \) is a subset of \( I(a) \), then \( a \) normalizes \( G_{1,2,i_1,i_2} \) and for \( \Delta = I(G_{1,2,i_1,i_2}) \) we have

\[ a^\Delta = (1, 2) (i_1) (i_2) (i_3), \]

and \( G_{1,2,i_1,i_2} = G_{1,2,i_1,i_2} = G_{1,2,i_1,i_2} = G_{1,2,i_1,i_2}. \) Now consider the map

\[ \varphi: \{i_1, i_2\} \rightarrow G_{1,2,i_1,i_2} \]

from the family of all subsets of \( I(a) \) consisting of two letters into the family of the subgroups of \( G \). Then each inverse image of \( \varphi \) consists of three subsets and hence we have

\[ rC_2 = \frac{r(r-1)}{2} \equiv 0 \pmod{3}. \]

**2. Main theorem.**

Let \( G \) be again a 4-fold transitive group on \( \Omega = \{1, 2, \ldots, n\} \). It is known that the only 4-fold transitive (not alternating nor symmetric) groups on less than 35 letters are the four Mathieu groups \( M_{11}, M_{12}, M_{23} \) and \( M_{24} \). Therefore in the following we may assume that \( n \geq 35 \).
Now let \( H = G_{1,2,3,4} \), \( \Delta = I(H) \) and let \( P \) be a Sylow 2-group of \( H \), \( \Delta' = I(P) \). Then \( \Delta' \supseteq \Delta \). We denote the normalizers of \( H \) and \( P \) by \( N \) and \( N' \) respectively. Then \((N')^{\Delta'}\) is one of the following groups: \( S_4, S_5, A_6, A_6 \text{ or } \Lambda_f \). We first prove the following

**Proposition 3.** If \( P \) is transitive on \( \Omega - \Delta' \) and \( n \geq 35 \), then \((N')^{\Delta'}\) must be \( S_4 \) or \( S_5 \).

**Proof.** We first remark that if \( i \in \Delta' - \Delta \) the length of the set of transitivity of \( H \) containing \( i \) is odd since the subgroup of \( H \) fixing \( i \) contains a Sylow 2-group \( P \) of \( H \).

The proof in the following is by contradiction.

1. Suppose that \((N')^{\Delta'} = A_6 \) and \( \Delta' = \{1, 2, 3, 4, 5, 6\} \).
   Then \( N^{A_6} \) must be \( A_6, S_5 \), or \( S_4 \).

   (1.1) Suppose \( N^{A_6} = A_6 \) and let \( a \) be a central involution of \( P \). Then, by Lemma 2 in [3], \( |I(a)| = 6 \) and, by Proposition 1, we have
   \[ n = 6^2 + 2 = 38. \]

   Consider the map \( \phi : i \rightarrow G_{1,2,3,4} \) from the set \( \{4, 5, \ldots, 38\} \) into the family of subgroups of \( G \). If \( I(G_{1,2,3,4}) = \{1, 2, 3, i, j, k\} \) then \( \phi^{-1}(G_{1,2,3,4}) \) consists of the three letters \( i, j, k \). Hence we have
   \[ 38 - 3 = 35 \equiv 0 \pmod{3}, \]
   which is a contradiction.

   (1.2) Suppose that \( N^{A_6} = S_5 \) and \( \Delta = \{1, 2, 3, 4, 5\} \). By the quadruple transitivity \( G \) contains an involution \( a = (1, 2)(3)(4) \cdots \). Since \( H = G_{1,2,3,4} \) is normalized by \( a \), \( \Delta \) is fixed by \( a \) and hence \( a \) fixes the letter 5, i.e.
   \[ a = (1, 2)(3)(4)(5) \cdots. \]

   Now the number of Sylow 2-groups of \( H \) is odd. Therefore there is a Sylow 2-group of \( H \) which is normalized by \( a \). We may assume that it is \( P \). Then \( \Delta' \) is fixed by \( a \) and we have
   \[ a^{\Delta'} = (1, 2)(3)(4)(5)(6). \]
   But this is a contradiction since \( a^{\Delta'} \) must be an even permutation.

   (1.3) Suppose that \( N^{A_6} = S_5 \) and let \( \Gamma = \Omega - \Delta' \). Then the sets of transitivity of \( H \) on \( \Omega - \Delta = \{5, 6\} \cup \Gamma \) can be assumed to be one of the following:

   (i) \( \{5, 6\} \) and \( \Gamma \),
   (ii) \( \{5, 6\} \cup \Gamma \).

   Since \( \Gamma \) is a set of transitivity of the 2-group \( P \) \( |\Gamma| \) is a power of 2.
Hence in both cases the length of the set of transitivity containing the latter $5 (\equiv \Delta' - \Delta)$ is even. This is a contradiction by the first remark.

(2) Suppose that $(N)\Delta' = A$, and $\Delta' = \{1, 2, \ldots, 7\}$. Then $N^\Delta$ must be $A_4$, $S_5$ or $S_3$.

(2.1) Suppose $N^\Delta = A_4$ and let $a$ be a central involution of $P$. Then $|I(a)| = 7$ and we have by Proposition 1

$$n = 7^2 + 2 = 51.$$ 

Since $P$ is transitive on $\Omega - \Delta'$ and $|\Omega - \Delta'| = 51 - 7 = 44$ is not a power of 2 we have a contradiction.

(2.2) Suppose that $N^\Delta = S_5$ and $\Delta = \{1, 2, 3, 4, 5\}$. Then the sets of transitivity of $H$ on $\Omega - \Delta = \{6, 7\} \cup \Gamma$ may be assumed to be one of the following:

(i) $\{6, 7\}$ and $\Gamma$,

(ii) $\{6, 7\} \cup \Gamma$.

But in both cases the length of the set of transitivity containing the letter 6 ($\equiv \Delta' - \Delta$) is even. This is a contradiction by the first remark.

(2.3) Suppose $N^\Delta = S_4$ and let $\Gamma = \Omega - \Delta'$. Then the sets of transitivity of $H$ on $\Omega - \Delta = \{5, 6, 7\} \cup \Gamma$ may be assumed to be one of the following:

(i) $\{5, 6, 7\}$ and $\Gamma$,

(ii) $\{5, 6\}$ and $\{7\} \cup \Gamma$,

(iii) $\{5, 6, 7\} \cup \Gamma$.

The case (ii) does not occur since the length of the set of transitivity containing the letter 5 ($\equiv \Delta' - \Delta$) is even in this case. In the case (iii), $H$ is transitive on $\Omega - \Delta$. Hence $G$ is 5-fold transitive and then the subgroup $G_i$ fixing the letter 1 is 4-fold transitive on $\{2, 3, \ldots, n\}$ and satisfies the assumption in (1). Thus as in (1) we have a contradiction.

We shall now consider the case (i). Let $P'$ be an arbitrary Sylow 2-group of $H$. Then there is an element $x$ of $H$ such that $x^{-1}Px = P'$. Since $\{5, 6, 7\}$ is a set of transitivity of $H$, it is fixed by $x$. Therefore $H_{s,s'} \supset P$ implies $x^{-1}H_{s,s'}x = H_{s,s'} \supset P'$ and hence we have $I(P') = \{1, 2, \ldots, 7\}$. This shows that $I(P')$ is independent of the choice of Sylow 2-group $P'$ of $H$ and is uniquely determined by $H$. Let $a = (1, 2, \ldots)$ be an involution of $G$ which is conjugate to a central involution of $P$. Then $|I(a)| = 7$. If $\{i_1, i_2\}$ is a subset of $I(a)$, then $G_{i_1, i_2}$ is normalized by $a$. Therefore there is a Sylow 2-group $P''$ of $G_{i_1, i_2}$ which is normalized by $a$. Let $I(P'') = \{1, 2, i_1, i_2, i_3, k, l\}$. Since $a$ is an even permutation on $I(P'')$, we may assume

$$a = (1, 2)(i_1)(i_2)(i_3)(k, l) \cdots.$$
Now $I(P'')$ is uniquely determined by $G_{i_1, i_2, i_3, i_4}$, therefore $\{i_1, i_2\}$ determines uniquely a 2-cycle $(k, l)$ of $a$ and we have the map

$$\varphi: \{i_1, i_2\} \rightarrow (k, l)$$

from the family of all the subsets of $I(a)$ consisting of two letters into the family of 2-cycles of $a$ different from $(1, 2)$. By the definition of $\varphi$, it is easy to see that $\varphi(\{i_1, i_2\}) = (k, l)$ if and only if $G_{i_1, i_2, i_3, i_4}$ and $G_{i_1, i_2, k, l}$ have a Sylow 2-group in common, and $\varphi$ is onto. Now suppose that $\varphi(\{i_1, i_2\}) = \varphi(\{j_1, j_2\}) = (k, l)$. Then $G_{i_1, i_2, i_3, i_4}$ and $G_{i_1, i_2, k, l}$ have a Sylow 2-group $P_1$ in common, and $G_{i_1, i_2, j_1, j_2}$ and $G_{i_1, i_2, k, l}$ have a Sylow 2-group $P_2$ in common. Since both $P_1$ and $P_2$ are Sylow 2-groups of $G_{i_1, i_2, k, l}$ we have $I(P_1) = I(P_2)$. Therefore $\{j_1, j_2\} \subset I(a) \cap I(P_1) = \{i_1, i_2, i_3\}$. Thus we have that each inverse image of $\varphi$ consists of three subsets of $I(a)$ and hence the number of 2-cycles of $a$ different from $(1, 2)$ is $\frac{1}{3} \cdot C_2 = 7$. In this way we have

$$n = 2 + 7 + 14 = 23,$$

which contradicts the assumption.

(3) Suppose that $(N')^\prime = M_{11}$ and $\Delta' = \{1, 2, \ldots, 11\}$. Then $N^a$ must be one of the following groups: $M_{11}, A_6, S_5$ or $S_4$.

(3.1) Suppose $N^a = M_{11}$, and let $a$ be a central involution of $P$. Then $|I(a)| = 11$ and by Proposition 1 we have $n = 11^5 + 2 = 123$. Since $P$ is transitive on $\Omega - \Delta'$ and $|\Omega - \Delta'| = 123 - 11 = 112$ is not a power of 2, we have a contradiction.

(3.2) Suppose $N^a = A_6$. In the same way as in (3.1) we have $n = 123$, which is a contradiction.

(3.3) Suppose $N^a = S_5$ and let $a$ be a central involution of $P$. Then $|I(a)| = 11$ and by Proposition 2 we have $11 (11 - 1) = 110 \equiv 0 \pmod{3}$.

This is a contradiction.

(3.4) Suppose $N^a = S_4$. Since the length of a set of transitivity of $H$ containing one of the letters in $\Delta' - \Delta = \{5, 6, \ldots, 11\}$ is odd, the sets of transitivity of $H$ may be assumed to be one of the following:

(i) $\{5, 6, 7\}, \{8, 9, 10\}$ and $\{11\} \cup \Gamma$,
(ii) $\{5, 6, 7, 8, 9, 10, 11\}$ and $\Gamma$,
(iii) $\{5, 6, 7, 8, 9, 10, 11\} \cup \Gamma$.

First consider the case (i). Since $(N')^\prime = M_{11}$, there is an element $x$ in $N'$ such that

$$x^\prime = (1, 2, 3, 4)(i_1, i_2, i_3, i_4)(k)(l)(m).$$
Then \( x \) normalizes \( H = G_{1,2,3,4} \). Hence \( x \) must fix two sets of transitivity \( \{5, 6, 7\} \) and \( \{8, 9, 10\} \) or interchange them. But, from the form of \( x^\lambda \), this is impossible.

In the case (iii), \( G \) is 5-fold transitive. Hence \( X = G_1 \) is 4-fold transitive on \( \{2, 3, \ldots, n\} \) and \( P \) is a Sylow 2-group of \( X_{2,3,4,5} \). Since \(|I(P) - \{1\}| = 10\), we have a contradiction by the theorem of M. Hall ([1], Theorem 5.8.1).

Now consider the case (ii). If \( P' \) is an arbitrary Sylow 2-group of \( H \) there is an element \( x \) of \( H \) such that \( P' = x^{-1}Px \). Since \( \{5, 6, \ldots, 11\} \) is a set of transitivity of \( H \), it is left invariant by \( x \). Therefore \( H_{5,6,\ldots,11} \supset P \) implies \( H_{5,6,\ldots,11} \supset P' \) and we have \( I(P') = \{1, 2, \ldots, 11\} \). This shows that \( I(P') \) is independent of the choice of \( P' \) and is determined uniquely by \( H \). We denote it by \( J(H) \). Let \( a = (1, 2) \cdots \) be an involution which is conjugate to a central involution of \( P \). We consider the map

\[
\varphi : \{i_1, i_2\} \to J(G_{1,2,i_1,i_2})
\]

which assigns \( J(G_{1,2,i_1,i_2}) \) to a subset \( \{i_1, i_2\} \) of \( I(a) \). Since \( a \) normalizes \( G_{1,2,i_1,i_2} \), there is a Sylow 2-group \( P'' \) of \( G_{1,2,i_1,i_2} \) such that \( a^{-1}P''a = P'' \). Let \( \Delta'' = I(P'') = J(G_{1,2,i_1,i_2}) \). Then \( a^{\Delta''} \) is an involution of \( M_{11} \). Hence we have

\[
a^{\Delta''} = (1, 2)(i_1)(i_2)(i_3)(k_1, k_2)(k_3, k_4)(k_5, k_6)
\]

and \( I(a) \cap J(G_{1,2,i_1,i_2}) = \{i_1, i_2, i_3\} \). Now it is easy to see that the inverse image \( \varphi^{-1}(J(G_{1,2,i_1,i_2})) \) consists of three subsets \( \{i_1, i_2\}, \{i_1, i_3\}, \{i_2, i_3\} \). Therefore we have

\[
i_1C_2 = \frac{11 \cdot 10}{2} \equiv 0 \pmod{3},
\]

which is a contradiction.

From Proposition 3 we have easily an improvement of Theorem 1 in [4].

**Theorem.** Let \( G \) be a 4-fold transitive group on \( \Omega = \{1, 2, \ldots, n\} \), excluding \( S_n \) and \( A_n \). If a Sylow 2-group \( P \) of the subgroup fixing four letters is not trivial, and transitive and regular on \( \Omega - I(P) \), then \( G \) must be \( M_{12} \) or \( M_{33} \).

Proof. We use the same notations as before. For \( n < 35 \) the theorem is trivial. Therefore we may assume \( n \geq 35 \), and then, by Proposition 3, we need consider only the case in which \( (N')^\lambda = S_4 \) or \( S_6 \).

1. Suppose \( (N')^\lambda = S_4 \). Then \( G \) is 5-fold transitive. Hence \( X = G_1 \).
is 4-fold transitive on \{2, 3, \ldots, n\}. Since \(X_{2,3,4}=H\) and a Sylow 2-group \(P\) of \(H\) is regular on \{5, 6, \ldots, n\}, \(X_{2,3,4,5}\) is of odd order. Therefore, by the theorem of M. Hall, \(X\) must be one of the following groups: \(S_4, S_5, A_4, A_5\) or \(M_{23}\). But this contradicts the assumption \(n \geq 35\).

(2) Suppose \((N')^*=S_5\) and let \(\Delta' = \{1, 2, 3, 4, 5\}\). Then \(N^*=S_4\) or \(S_5\). If \(N^*=S_4\), then \(H\) is transitive on \{5, 6, \ldots, n\} and hence \(G\) is 5-fold transitive. Then \(X=G_i\) is 4-fold transitive on \{2, 3, \ldots, n\} and satisfies the assumption in (1). Therefore as in (1) we have a contradiction. On the other hand, if \(N^*=S_5\) then by Proposition 2 we have

\[5(5-1) \equiv 0 \pmod{3},\]

since for a central involution \(a\) of \(P\) \(|I(a)|=5\). But this is a contradiction.

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References