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<th>On a deformation of Riemannian structures on compact manifolds</th>
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1. The purpose of this paper is to prove that every compact $C^\infty$-Riemannian manifold with at least 3 dimensions can be deformed conformally to a $C^\infty$-Riemannian structure of constant scalar curvature.

Let $S$ be a $d$-dimensional $C^\infty$-Riemannian manifold with $d \geq 3$, and denote its fundamental positive definite tensor by $g_{ij}$. Throughout this paper we will use the definitions and notations of the book "Curvature and Betti numbers" by K. Yano and S. Bochner, unless otherwise stated. The volume element is written as $dV$. The total volume is assumed to be 1.

Here we are going to present the outline of the proof. Consider a conformal transformation of a Riemannian structure

\begin{equation}
\bar{g}_{ij} = e^{2\rho} g_{ij}.
\end{equation}

Then the connection coefficients $\bar{\Gamma}_j^i_k$ corresponding to $\bar{g}_{ij}$ are expressed as

\begin{equation}
\bar{\Gamma}_j^i_k = \Gamma_j^i_k + \rho_k \delta_j^i + \rho_j \delta_k^i - \rho^i g_{jk},
\end{equation}

where

\begin{equation}
\rho_i = \frac{\partial \rho}{\partial x^i}.
\end{equation}

From (1.2)

\begin{equation}
\bar{R}_{jkl} = R_{jkl} - \rho_j \delta_k^l + \rho_k \delta_j^l - g_{jk} \rho_l + g_{jl} \rho_k
\end{equation}

where

\begin{equation}
\rho_{jk} = \rho_{j,k} - \rho_j \rho_k + \frac{1}{2} g^{\alpha \beta} \rho_{\alpha \beta} g_{jk}.
\end{equation}

Hence

\begin{equation}
\bar{R}_{jk} = R_{jk} - (d-2) \rho_{jk} - g_{jk} \rho_8
\end{equation}

and

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1) see [5] page 78.
\[ (1.7) \quad \bar{R} = e^{-2p}(R - 2(d-1)\rho^2) . \]

Here \( \bar{R}_{jk}, \bar{R}_{jk} \) and \( \bar{R} \) denote the curvature tensor, the Ricci tensor and the scalar curvature, respectively, of the new structure.

Now let \( \Delta \) denote the Laplace-Beltrami operator corresponding to \( g_{ij} \). Then (1.7) can be written as

\[ (1.8) \quad \bar{R} = e^{-2p}(R - 2(d-1)\rho^2) \]
\[ \quad = e^{-2p}(R - 2(d-1)\left(\Delta \rho + \left(\frac{d}{2} - 1\right)g^{\alpha\beta}\rho\rho_{\alpha\beta}\right)) \]
\[ \quad = e^{-2p}\left(R - \frac{4(d-1)}{d-2} e^{\frac{d}{d-1}p} \Delta (e^{\frac{d}{d-1}p}) \right) . \]

Set

\[ (1.9) \quad \bar{u} = e^{\frac{d}{d-1}p} \]

or

\[ (1.9') \quad (\bar{u})^{\frac{1}{d-2}} = e^{\frac{1}{2}p} , \]

and then

\[ (1.10) \quad -\bar{R}(\bar{u})^{\frac{d+2}{d-2}} = -R\bar{u} + \frac{4(d-1)}{d-2} \Delta \bar{u} . \]

Conversely, we are going to prove

**Theorem A.** There exists a positive \( C^\infty \)-function \( \bar{u} \) satisfying

\[ (1.10') \quad -\bar{u}^{\frac{d+2}{d-2}}C_0 = -R\bar{u} + \frac{4(d-1)}{d-2} \Delta \bar{u} \]

where \( C_0 \) is a constant.

If such a function \( \bar{u} \) be found, we have only to set \( \bar{g}_{ij} = (\bar{u})^{\frac{1}{d-2}}g_{ij} \) to obtain the desired structure.

On the other hand, if there exists a positive extremal \( \bar{v}^{\bar{g}} \) minimizing a variational function \( (q \geq 2) \)

\[ (1.11) \quad F^{(q)}(u) = \frac{\int(4(d-1)\nabla u^2 + Ru^2)\,dV}{\left(\int |u|^q\,dV\right)^{\frac{2}{q}}} \]

to a value \( \mu_{(q)} \), then this function satisfies the corresponding Euler's equation

\[ (1.12) \quad \frac{4(d-1)}{d-2} \Delta \bar{v}^{(q)} - R\bar{v}^{(q)} = -\mu_{(q)}(\bar{v}^{(q)})^{q-1} . \]
In order to prove Theorem A, we shall prove the following two theorems.

**Theorem B.** For any \( q < \frac{2d}{(d-2)} \), there exists a positive function \( v^q \) satisfying (1.12).

**Theorem C.** As \( q \) tends to \( \frac{2d}{(d-2)} \), a uniform limit \( \bar{u} \) of such \( v^q \)'s exists, is positive and satisfies (1.10) with \( C_0 = \mu_0 \frac{c_0}{d} \).

Theorem A is an immediate consequence of Theorem C.

2. Let \( \varepsilon \) be a positive number less than \( \frac{4}{(d-2)} \). Set

\[
(2.1) \quad p_\varepsilon = (2d/(d-2) - \varepsilon),
\]

\[
(2.1') \quad p_\varepsilon' = p_\varepsilon / (p_\varepsilon - 1),
\]

and

\[
(2.2) \quad F_{c,p}^q(u) = F_{c}(u) = \left( \frac{(4d-1)/d - 2}{\int |\nabla u|^2 + Ru^2} \right) dV
\]

\[
(2.3) \quad ||u||_q = \left( \int |u|^q dV \right)^{1/q}.
\]

By \( L_p \) we denote the Banach space of real functions with the norm

\[
|| \cdot ||_p.
\]

**Lemma 1.** Let \( \{u_i\} \) be a sequence of \( C^\infty \)-functions with \( ||u_i||_{p_\varepsilon} = 1 \) such that

\[
(2.4) \quad \lim_i F_{c}(u_i) = \mu_{c,p_\varepsilon} = C_0(\varepsilon) = \min_u F_{c}(u).
\]

Then the sequences \( \{|u_i|\} \) possess a similar property except that \( |u_i| \) might not be differentiable at the zero points of \( u_i \).

Proof is almost evident if one notices that

\[
(2.5) \quad |\nabla u|^2 = |\nabla (|u|)|^2,
\]

except at zero point of \( u \) with non-vanishing \( \Delta u \). The measure of the set of such points is zero. By the measure we understand the measure
Lemma 2. There exists a positive constant $C$, such that for $\frac{p}{q} \leq 2d/(d-2)$

$$\inf_c \|u - c\|_{p_q} \leq C \|\nabla u\|_2,$$

where $c$ is an arbitrary constant and $u$ is assumed to be a smooth function.

This lemma is similar to Sobolev's lemma. The proof is omitted because a minor modification of the proof of Lemma 4 is sufficient for this lemma. However, it should be noted that even when $\varepsilon = 0$, the lemma is valid but this is not necessary for the present paper.

Corollary.

$$F(u) \geq \frac{4(d-1)}{d-2} \frac{1}{C_1} \sup_{P \in S} |R(P)| - 1.$$  

Let $\psi(x)$ be a function over the unit square $E^d = \{x ; -1 \leq x^m \leq 1, m = 1, 2, \ldots, d\}$ in a $d$-dimensional Euclidean space with the property:

$$\|\psi\|_q = \left( \int_{E^d} |\psi|^q dx \right)^{1/q} < \infty$$

where $2 < q < \infty$.

Consider the multiple Fourier (trigonometrical) series of $\psi(x)$:

$$\psi(x) = \sum (a_{i_1} \cdots i_d \cos \pi i_1 x^1 + \cdots + i_d x^d) + b_{i_1} \cdots i_d \sin \pi i_1 x^1 + \cdots + i_d x^d) = \sum (a_I \cos \pi \langle I, x \rangle + b_I \sin \pi \langle I, x \rangle).$$

Here $I$ denote a "vector" $(i_1, \ldots, i_d)$ with integer components and $\langle I, x \rangle$ an inner product of $I$ and another "vector" $(x^1, \ldots, x^d)$. Define $|I|$ by

$$|I| = (i_1^2 + \cdots + i_d^2)^{1/2}.$$  

Lemma 3.

$$\|\psi\|_q \leq \left( |a_0|^{q'} + \sum_{|I| \geq 1} (|a_I|^{q'} + |b_I|^{q'}) \right)^{1/q'},$$

where $q' = q/(q-1)$.

Remark. This is the Hausdorff-Young inequality for multiple Fourier series.

2) see [4]
Proof. Let $E'$ be the discrete space of lattice points in a $d$-dimensional space. Define a one to one operator $T$ onto a functional space over $E$ from another functional space over $E' \times E'$ by

\begin{equation}
(T^{-1}\psi)(I, J) = (a_I, b_J)
\end{equation}

at a point $(I, J)$ in $E' \times E'$. The space $E$ is given the ordinary Lebesgue measure while in $E' \times E'$ a weight 1 is assigned to each point. The norm is defined by

\begin{equation}
\|T^{-1}\psi\|_{q'} = (|a_0|^{q'} + \sum_{|I| \geq 1} (|a_I|^{q'} + |b_I|^{q'}))^{1/q'}.
\end{equation}

Then $T$ becomes simultaneously of the type (2.2) and $(1, \infty)$, to which we can apply the Calderon-Zygmund's generalization of M. Riesz's convexity theorem. Hence for any $q$ between 2 and infinity,

\begin{equation}
\|\psi\|_q \leq \|T^{-1}\psi\|_{q'}
\end{equation}

where $q' = q/(q-1)$. This proves the lemma.

3. It is well known that there are countably many non-positive eigenvalues of the elliptic operator $4(d-1)/(d-2)\Delta$ diverging to $-\infty$. We write them in non-increasing order, $-\lambda_1, -\lambda_2, \ldots, -\lambda_m, \ldots$. To each $\lambda_m$ is attached an eigenfunction $\phi_m$ with $\|\phi_m\|_2 = 1$. These $\phi_m$'s are mutually orthogonal in the sense of $L_2$. The first eigenvalue $\lambda_1 = 0$ and the corresponding function $\phi_1 = 1$. Then every square integrable function $u(P)$ can be expanded into the Fourier series with respect to $\{\phi_m\}$. In particular

\begin{equation}
u_i = \sum_{i=1}^N a_i \phi_i.
\end{equation}

**Lemma 4.** Suppose that $\psi_N$'s are smooth functions on S with $\|\psi_N\|_{p_s} < \infty$, such that for integer $j$ between 1 and $N$,

\begin{equation}
\int_S \psi_N \phi_j dV = 0.
\end{equation}

More generally, if $\psi_N/\|\psi_N\|_{p_s}$ is weakly close to 0, then for any given small $\delta < 0$, there exists an integer $N_0$ such that if $N \geq N_0$, then

\begin{equation}
\inf c \|\psi_N - c\|_{p_s} \leq \delta \|\Delta \psi_N\|_2.
\end{equation}

Proof. Without loss of generality we may assume that $\|\psi_N\|_{p_s}$ is uniformly bounded as $N$ tends to infinity.

3) See Theorem D in page 117 in “On the theorem of Hausdorff and Young” in [2].
Firstly, we consider the case when the carrier of $\psi_N$ is contained in a coordinate neighborhood $E$. In this case we can consider the trigonometrical expansion of $\psi_N$. Take a sufficiently large integer $M$, so that for a preassigned small $\delta' > 0$,

$$(3.3) \quad \sum_{|I| \geq M} |I|^{-2p_\epsilon/(p_\epsilon-2)}(p_\epsilon-2)^{2/p_\epsilon} \leq \delta' .$$

This is possible whenever $\epsilon$ is positive because

$$(3.5) \quad 2p_\epsilon/(p_\epsilon-2) = \frac{d(4/d-2-\epsilon) + (d-1)\epsilon}{d-2-\epsilon} > d .$$

We set $c = a_0$ (and $b_0$ is always assumed to be zero). By virtue of Lemma 3,

$$(3.6) \quad ||\psi_N||_{p_\epsilon} \leq (\sum |a_I|^{p_\epsilon} + |b_I|^{p_\epsilon})^{1/p_\epsilon}$$

$$= \left( \sum_{|I| < M} (|a_I|^{p_\epsilon} + |b_I|^{p_\epsilon}) \right)^{1/p_\epsilon} + \left( \sum_{|I| \geq M} (|a_I|^{p_\epsilon} + |b_I|^{p_\epsilon}) \right)^{1/p_\epsilon}$$

where

$$(3.7) \quad p_\epsilon' = \frac{p_\epsilon}{p_\epsilon-1} < 2 .$$

Set

$$(3.8) \quad q_\epsilon = 2/p_\epsilon'$$

and

$$(3.9) \quad q_\epsilon' = q_\epsilon/(q_\epsilon-1) .$$

By a simple calculation

$$(3.10) \quad p' q'_\epsilon = 2p_\epsilon/(p_\epsilon-2) .$$

The second term of the right hand side of (3.6) will be dominated by

$$(3.11) \quad \left( \sum_{|I| \geq M} (|a_I|^{p_\epsilon} + |b_I|^{p_\epsilon}) \right)^{1/p_\epsilon'}$$

$$= \left( \sum_{|I| \geq M} (|a_I| |I|)^{p_\epsilon} + (|b_I| |I|)^{p_\epsilon} \right)^{1/p_\epsilon'}$$

$$\leq \left( \sum_{|I| \geq M} (|a_I| |I|)^{p_\epsilon} + (|b_I| |I|)^{p_\epsilon} \right)^{1/p_\epsilon'}$$

$$\leq \left( \sum_{|I| \geq M} |I|^{-p_\epsilon q'_\epsilon} \right)^{1/p_\epsilon} \leq \delta' \left( \int |\text{grad} \, \psi_N|^2 \, dx \right)^{1/2} .$$

where

$$(3.13) \quad |\text{grad} \, \psi_N| = \left( \sum_{m=1}^d \left( \frac{\partial \psi_N}{\partial x^m} \right)^2 \right)^{1/2} .$$
On account of the uniform ellipticity of $g^{ij}$, there exists a constant $C_2$ such that

$$C_2^{-1} ||\psi||_{\rho} \leq ||\psi||_{\rho} \leq C_2 ||\psi||_{\rho}$$

and

$$C_2^{-1} ||\nabla\psi|| \leq ||\nabla\psi|| \leq C_2 ||\nabla\psi||.$$

Combining (3.11) and (3.15), we have

$$\Sigma_{|I|\leq M}(|a_I|^r + |b_I|^r) \leq 2^M ||\nabla\psi||.$$  

REMARK. If we set $M=1$, we have Lemma 2 for the case when the carrier of $\psi$ is within a coordinate neighborhood. Namely if $a_0=0$,

$$||\psi||_{\rho} \leq C_2 ||\nabla\psi||.$$  

As for the first term of (3.6), $a_I/||\psi||_{\rho}$'s and $b_I/||\psi||_{\rho}$'s can be taken arbitrarily close to zero if $N_0$ is sufficiently large, because these coefficients are linear functionals over $L^p$. By virtue of (3.14), $a_I/||\psi||_{\rho}$ and $b_I/||\psi||_{\rho}$'s are also small, say less than $1/2 M^{-d}$'. Hence

$$\Sigma_{|I|\leq M}(|a_I|^r + |b_I|^r) \leq 2^M ||\nabla\psi||.$$  

However, by virtue of Lemma 2 (see the remark above),

$$\Sigma_{|I|\leq M}(|a_I|^r + |b_I|^r) \leq 2^M ||\nabla\psi||.$$

Combining (3.6), (3.16), and (3.18),

$$||\psi||_{\rho} \leq \delta(||\nabla\psi||)$$

if $a_0=0$. Here

$$\delta = \delta'(c_1 + c_2).$$

This concludes the proof for the case when the carrier of $\psi$ is in a coordinate neighborhood.

As for the general case, we decompose the manifold $S$ into a union of finitely many, say $l_i$'s closures of coordinate neighborhood $U_1, \ldots, U_{l_0}$ such that $U_i$ is a cubic neighborhood of a point in $U_i$, and any two of these $U_i$'s intersect only at the boundary. The restriction of $\psi$ over $U_i$ is denoted by $\psi_{N,i}$. The mean of $\psi_{N,i}$ over $U_i$ with respect to $dx$ will be denoted by

$$c_{N,i} = \int_{U_i} \psi_{N,i} dx.$$
Then

\[(3.22) \quad \| \psi_N - \sum_{r=0}^{t_0} c_{N,r} \|_{p_e} \leq \sum_{r=1}^{t_0} \| \psi_{N,r} - c_{N,r} \|_{p_e} \cdot \]

However, the trigonometrical Fourier coefficients of a function \( \psi \) in
\( L_p \) over the coordinate systems of \( U_i \)'s are also linear functionals over
\( L_{p_e} \). Hence, if \( N_0 \) is sufficiently large, we can apply the result of the
previous case so that for each \( l \) and a preassigned \( \delta'' = l_0^{-1}\delta \)

\[(3.23) \quad \| \psi_{N,l} - c_{N,l} \| \leq \delta''(\| \nabla \psi_{N,l} \|_2). \]

From this it follows that

\[(3.24) \quad \| \psi_{N,l} - \sum_{r=0}^{t_0} c_{N,l,r} \| \leq l_0\delta''(\sup_l \| \nabla \psi_{N,l} \|_2) \leq \delta || \nabla \psi_N ||_2) \]

This completes the proof of Lemma 4.

**Remark (I).** If \( M=1 \), Lemma 2 follows.

**Remark (II).** This \( S = S(N_0) \) depends upon \( N_0 \) and goes to zero as \( N_0 \)
tends to infinity.

4. Consider the Fourier expansion (3.1) of \( u_i \)'s in (2.4) with \( \| u_i \|_{p_e} = 1 \). It is easily seen that all \( F_s(u_i)'s \) are bounded by a positive constant \( C_s \). From this fact, it follows that the convex closure of the set \( \{u_i\} \) in
\( L_2 \), and more generally in \( L_{p_e} \), is compact strongly. This will be formulated in

**Lemma 5.** The convex closure of \( \{u_i\} \) compact; and a limit \( u^{p_e} \) of
a convergent subsequence is not zero.

**Proof.** The latter half is an immediate consequence of the former
half because for each \( i \), \( \| u_i \|_{p_e} = 1 \).

Now without loss of generality we may assume that \( \{u_i\} \) converges
weakly to \( u^{p_e} \). Consider the Fourier expansion with respect to \( \phi_j \)'s.

\[(4.1) \quad u^{p_e} = \sum_{j=1}^{\infty} b_j \phi_j \cdot \]

Then

\[(4.2) \quad \| u_i - u^{p_e} \|_{p_e} \leq \sum_{j=1}^{\infty} || (a_{ij} - b_j) \phi_j ||_{p_e} \]

By virtue of the previous lemma, there exists a sequence of constants
\( c_i(N) \) such that for \( N \geq N_0 \),

\[(4.3) \quad \| \sum_{j=N}^{\infty} (a_{ij} - b_j) \phi_j - c_i(N) \|_{p_e} \leq \delta(N_0) 2\sum_{\delta} k_j (a_{ij} - b_j)^2 \leq \delta(N_0) 2C_3. \]
Deformation of Riemannian Structures

It is easily seen that these \( c_i(N) \)'s tend uniformly to zero as \( N \) goes to infinity. Now, \( i \) tends to infinity. Then the first term of (4.2) vanishes and

\[
\lim_{i} \| u_i - c_i(N) - v^{p_i} \|_{\rho} \leq 2C_3 \delta(N_0).
\]

However this \( \delta(N_0) \) can be made arbitrarily small. Hence

\[
\lim_{N} \lim_{i} \| u_i - c_i(N) - v^{p_i} \|_{\rho} = 0.
\]

By making use of the fact that \( \lim \| u_i - v^{p_i} \|_{\rho} \) is independent of \( N \), we can easily obtain the relation that

\[
\lim \| u_i - v^{p_i} \|_{\rho} \leq \lim_{N} \lim_{i} c_i(N) = 0.
\]

Thus the lemma has been proved.

**NOTICE.** The function \( v^{p_i} \), being a limit of non-negative \( u_i \)'s, is non-negative.

**Lemma 6.** If \( v^{p_i} \), a non-negative \( C^2 \) function satisfies the equation (1.12) for \( q = p_i \), \( \varepsilon \geq 0 \), then \( v^{p_i} = v^{q_i} \) is positive.

**Proof.** For simplicity, we shall use \( v^{q_i} \) instead of \( v^{p_i} \). Suppose that \( v^{q_i} \) vanishes at a point \( P \). Take the polar coordinates \( r, \theta^m, m = 1, 2, \ldots, d-1 \) of a normal geodesic coordinates around \( P \). The volume element and the Laplace-Beltrami operator with respect to the induced and normalized (total volume 1) Riemannian structure on the concentric sphere \( \Omega(r) \) of the radius \( r \) around \( P \) will be denoted by \( \sqrt{\sigma(r)} \) and \( \Delta_{\alpha} \), respectively. Then

\[
\Delta v^{q_i} = \left( (\partial/\partial r)^2 + (\partial/\partial r)(\log \sqrt{\sigma}) \partial/\partial r + r^{-2} \Delta_{\alpha(r)} v^{q_i} \right).
\]

By integrating (4.6) over \( \Omega(r) \) with the volume element \( \sqrt{\sigma} \, d\theta \)

\[
\int_{\Omega(r)} r^{-d-1} (\partial v^{q_i}/\partial r) \sqrt{\sigma} \, d\theta + r^{-2} \int_{\Omega(r)} \Delta v^{q_i} \sqrt{\sigma} \, d\theta
\]

\[
= \int (R v^{q_i} - \mu_{q_i} v^{q_i}) \sqrt{\sigma} \, d\theta.
\]

When \( r \) ranges over a small interval \( (0, r_0) \), there exists a positive constant \( K_i \) such that

\[
\int (R v^{q_i} - \mu_{q_i} v^{q_i}) \sqrt{\sigma} \, d\theta \leq K_i \int v^{q_i} \sqrt{\sigma} \, d\theta
\]

\(\text{see [1]}\)
because \( q > 2 \). Hence
\[
\int_{Q(r)} (\partial \psi^q / \partial r) \sqrt{\sigma} \, d\theta = \int_{Q(r)} \rho^{d-1} K \int_{Q(r)} \psi^q \sqrt{\sigma} \, d\theta \, d\rho
\]
or
\[
\int_{Q(r)} (\partial \psi^q / \partial r) \sqrt{\sigma} \, d\theta \leq r^{d+1} \int_{Q(r)} K \rho^{d-1} \psi^q \sqrt{\sigma} \, d\theta \, d\rho .
\]
Integrating both sides from 0 to \( s \),

\[
\int_{Q(s)} \psi^q \sqrt{\sigma} (s) \, d\theta - \int_{r} (\psi^q \log \sqrt{\sigma} / \partial r)(r) \sqrt{\sigma} \, d\theta \, d\rho \leq r^{d+1} \int_{Q(r)} K \rho^{d-1} \psi^q \sqrt{\sigma} \, d\theta \, d\rho .
\]

Now set
\[
X(r) = \int_{Q(r)} \psi^q \sqrt{\sigma} (r) \, d\theta ,
\]
and take positive constants \( K_2 \) and \( K_3 \) such that both
\[
|\partial \log \sqrt{\sigma} / \partial r(r)| \leq K_2 r
\]
\[
|\psi^q(r)| \leq K_3
\]
hold for \( 0 \leq r \leq r_0 \).

Then, from (4.7) it follows that

\[
X(s) \leq K_2 \int_{0}^{s} X(r) r \, dr + K_3 \int_{0}^{s} \rho^{d-1} X(r) \, d\rho \, dr
\]
\[
\leq K_2 K_3 s^2 / 2 + K_2 K_3 (s^2 / 2d).
\]

In general, it can be shown that

\[
X(s) \leq K_2 (K_1 + K_2)^N 2^{-n} s^{2n} / n !.
\]

The proof can be given by induction on \( n \). If (4.9) holds up to \( n = N-1 \). Then by substituting \( X(r) \) in the right hand side of (4.8) by (4.9),

\[
X(s) \leq K_3 [(K_1 + K_3)^N 2^{-N+1} (N-1)!] [K_2 s^{2N} / 2N] + K_2 s^{2N} (2(N-1) + d) 2N
\]
\[
\leq K_3 (K_1 + K_2)^N 2^{-N} s^{2N} / N !.
\]

Since \( N \) can be taken arbitrarily large, \( X(s) = 0 \) for all \( s \). From this we can conclude that \( \psi^q = 0 \) around \( P \). This means the zero points of \( \psi^q \) is open. Therefore \( \psi^q \) must be identically zero. This contradiction
proves that \( v^q \) is positive everywhere.

**Lemma 7.** The \( v^q \) is a weak solution of (4.8). Here \( \Delta \) is understood as the extension of Laplace-Beltrami operator over \( L_2 \).

Proof. Take a \( C^\infty \) function \( v \) on \( S \) with

\[
\sup_{P \in S} |v(P)| \leq 1,
\]
and a small positive real \( \eta \). Define a subset \( S_1 \) of \( S \) by

\[
S_1 = \{ P ; v^q(P) > \eta \}
\]
and

\[
S_2 = S' - S_1, \text{ where } S' = \{ P ; v^q(P) > 0 \}.
\]

Set

\[
\eta' = \int_{S_2} dV.
\]

It is easily seen that \( \eta' \) goes to zero as \( \eta \) goes to zero. Take a function

\[
w_\eta = v^q + \eta v.
\]

Then

\[
\|w_\eta\|_q \leq \left( \int_{S_1} |w_\eta|^q dV \right)^{1/q} \leq \frac{1}{q} \left( \int_{S_2} (2\eta)^q dV \right)^{1/q} + \frac{1}{q} \eta^q + O(\eta^2)
\]

\[
= \frac{1}{q} 2^q \eta^q \eta' + \frac{1}{q} \eta^q + O(\eta^2).
\]

However the quantity \( \left( \int_{S_1} |w_\eta|^q dV \right)^{1/q} \) is a \( C^\infty \) function in \( \eta \) and

\[
\left( \int_{S_1} |w_\eta|^q dV \right)^{1/q} - \left( \int_{S_1} (v^q)^q dV \right)^{1/q} - \left( \int_{S_1} (v^q)^q dV \right)^{(1/q)-1} \cdot \int_{S_1} (v^q)^{q-1} v dV \leq C_1 \eta^2,
\]

where \( C_1 \) is a positive constant.

On the other hand

\[
\left( \int_{S} (v^q)^q dV \right)^{1/q} - \left( \int_{S_1} (v^q)^q dV \right)^{1/q} \leq \eta(\eta')^{1/q},
\]

\[
\int_{S_1} (v^q)^q dV - \int_{S_1} (v^q)^q dV = \eta^q \eta'.
\]
These are all obtained in the same manner as (4.15). Combining (4.15), (4.16), (4.17) (4.18) and (4.19) we have

\[
\lim_{\eta \to 0} \frac{1}{\eta} (||w||_q - ||v||_q) = \int_S (v - v')^q - V dV.
\]

Now set

\[
H_\xi(u) = F_{\xi}(u)||u||^2_q
\]

and let \( \eta \) be a real number with small absolute value. Then

\[
F_{\xi}(v') = H_\xi(v') = \lim_i F_{\xi}(u_i) = \lim_i H_\xi(u_i) = \min_u F_{\xi}(u) = \mu_{\xi,P}.
\]

Hence

\[
0 \leq \frac{1}{|\eta|} (F_{\xi}(w_u) - F_{\xi}(v')) = \frac{1}{|\eta|} (F_{\xi}(w_u) - \mu_{\xi,P})
\]

\[
= \frac{1}{|\eta|} \frac{1}{||w_u||_q} (H_\xi(w_u) - \mu_{\xi,P}) ||w_u||^2_q
\]

\[
= \frac{\text{sgn}(\eta)}{||w_u||_q^2} \left( -2 \int_S \frac{4(d-1)}{d-2} \Delta v' + Rv' + \mu_{\xi,P}(v')^{q-1} \right) dV + \Phi_\eta,
\]

where \( \Phi_\eta \) tends to zero as \( \eta \) goes to zero.

In order for this inequality to hold for a positive \( \eta \) as well as a negative \( \eta \),

\[
\int_S \left( \frac{4(d-1)}{d-2} \Delta v' - Rv' + \mu_{\xi,P}(v')^{q-1} \right) dV = 0.
\]

Since \( v \) can range over all \( C^\infty \) functions with \( \sup_P |v(P)| \leq 1 \), \( v' \) must satisfy

\[
\frac{4(d-1)}{d-2} \Delta v' - Rv' - \mu_{\xi,P}(v')^{q-1} = 0.
\]

in the sense of weak solution. This is the same equation as (4.8) and (1.12).

This completes the proof.

**Remark.** \( q \) has only to be larger than 2.
5. In this paragraph we shall prove that a weak solution \( \psi^{q_3} \) of (4.25) which gives the minimal value of \( F_\varepsilon(u) \) for \( q = p_\varepsilon \), is actually a \( C^\infty \) function solution, and thus the gap between Lemma 6 and Lemma 7 will be closed.

Lemma 8. The non-negative function \( \psi^{q_3} \), \( q \geq 2 \), satisfying (4.25) in the sense of a weak solution, is \( C^2 \) everywhere and \( C^\infty \) except at zero points of \( \psi^{q_3} \).

Remark. By virtue of Lemma 6, there is no zero point of \( \psi^{q_3} \).

Proof. Firstly, the boundedness of \( \psi^{q_3} \) will be proved.

By \( G(P, Q) \) we denote the Green's function\(^5\) for \( \frac{4(d-1)}{d-2} \Delta \). The Sobolev's lemma will be formulated in the following form. If

\[
(5.1) \quad u_i(P) = \int G(P, Q)u(Q)dV(Q),
\]

where \( u \) belongs to \( L_{q'} \), then \( u_i \) belongs to \( L_{p'} \) where

\[
(5.2) \quad (p')^{-1} \geq (q')^{-1} - (2/d),
\]

and

\[
(5.3) \quad \|u_i\|_{p'} = C_5\|u\|_{q'}.
\]

Here \( C_5 \) is an absolute constant if \( (p')^{-1} - (q')^{-1} + (2/d) \) is larger than a fixed constant. Applying this to (4.25),

\[
(5.4) \quad \psi^{q_3} = -\int G(P, Q)(-\mu(q)(\psi^{q_3})^{q-1} + R\psi^{q_3})dV(Q) + \int \psi^{q_3}dV(Q)
\]

where the function \( A(Q) \) belongs to \( L_{m_1} \) with

\[
(5.5) \quad m_1 = \left( \frac{2d}{d-2} - \varepsilon \right) \left( \frac{d+2}{d-2} - \varepsilon \right)^{-1}.
\]

Hence \( \psi^{q_3} \) belongs to \( L_{q_1'} \) where

\[
(5.6) \quad (q_1')^{-1} = (m_1)^{-1} - (2/d)
\]

\[
\leq \frac{d-2}{2d} \left( 1 - \frac{d-2}{d} \varepsilon \right) \left( 1 + \frac{d-2}{2d} \varepsilon \right) + O(\varepsilon^3)
\]

\[
= \frac{d-2}{2d} \left( 1 - \frac{d-2}{2d} \varepsilon \right) + O(\varepsilon^3)
\]

\(^5\) see [3]
where \( \varepsilon \) is small. Hence we can find a \( q_1 \) such that

\[
q_1 = \frac{2d}{d-2} + \xi > \frac{2d}{d-2}
\]

where \( \xi \) is a positive real number.

Now, notice that the quantity \( A(Q) \) belongs to \( L_{m_2} \) where

\[
m_2 = \left( \frac{2d}{d-2} + \xi \right) \left( \frac{d+2}{d-2} - \varepsilon \right)^{-1}
\]

Then, again by virtue of Sobolev’s lemma, \( v^{(\rho)} \) belongs to \( L_{q_2} \) with

\[
(q_2)^{-1} = (m_2)^{-1} - (2/d) \leq \frac{d+2}{2d + (d-2)\xi} - \frac{2}{d}
\]

\[
= \frac{d(d+2) - 4d - 2(d-2)\xi}{(2d + (d-2)\xi)d} = \frac{d-2}{2d} \left( 1 - \frac{2}{d} \xi \right) \left( 1 + \frac{d-2}{2d} \xi \right)^{-1}
\]

Therefore \( q_2 \) can be taken as

\[
q_2 \geq \frac{2d}{d-2} \left( 1 - \frac{2}{d} \xi \right)^{-1} \left( 1 + \frac{d-2}{2d} \xi \right)
\]

or in particular

\[
q_2 = \frac{2d}{d-2} + \frac{d+2}{d-2} \xi
\]

By repeating these procedures, we can easily show that \( v^{(\rho)} \) belongs to \( L_{q_n} \), with

\[
q_n = \frac{2d}{d-2} + \left( \frac{d+2}{d-2} \right)^{n-1} \xi
\]

Take an integer \( n \) large enough so that

\[
q_n = \left( \frac{d}{2} + \xi \right) \frac{d+2}{d-2} > \frac{d}{2} \frac{d+2}{d-2}
\]

Here \( \xi_1 \) is a positive real number. Then,

\[
(5.12) \quad \sup_P v^{(\rho)}(P) \leq \sup_P ||(P, Q)||_{q_n, q_{n-1}} ||v^{(\rho)}||_{q_n} + \text{finite number}.
\]

The right hand side is bounded because the part involving the Green’s function is finite.

Once the essential boundedness is established, apply \( G(P, Q) \), and we have the proof immediately\(^6\).

6) See Appendix.
Theorem B follows immediately from Lemmas 6, 7, and 8. We shall proceed to prove Theorem C.

**Lemma 9.** The family of functions \{v^{(q)}\} are uniformly bounded for \(2 < q < 2d/(d-2)\).

Proof. Take a positive fixed \(\xi_2 > 1\). Using the procedure in the proof of Lemma 8, starting at \(q_1 = 2d/(d-2) + \xi_2\), we can see that at each \(i\)

\[
(q_{i+1})^{-1} - (q_i)^{-1} \left(\frac{d+2}{d-2}\right) + \frac{2}{d} = \frac{\xi_2^2}{q_{i+1}, q_1} \left(\frac{d+2}{d-2}\right) > \frac{8}{d^3} \xi_2^2
\]

and if

\[
q_n = \left(\frac{d}{2} + \xi_1\right) \frac{d+2}{d-2},
\]

then

\[
\|v^{(q)}\|_{q_n} \leq C_{5}^{-1}\|v^{(q)}\|_{q_1}.
\]

Here \(C_5\) is defined in (5.3). From this it follows that

\[
\Lambda(q) \leq \sup_{P}\|G(P, Q)\|_{q_n < q < q_{n-1}, C_{5}^{-1}\|v^{(q)}\|_{q_1}} \leq C_4\|v^{(q)}\|_{q_1}
\]

where

\[
\Lambda(q) = \sup_{P} v^{(q)}(P)
\]

and \(C_4\) is an absolute constant. However

\[
\|v^{(q)}\|_{q_1} \leq \Lambda(q)^{\xi_2^{-q}} \Lambda(q)^{q} = \Lambda(q)^{\xi_2^{-q}} = \Lambda(q)^{\xi_2^{-q}}.
\]

Hence for small \(\xi\),

\[
\Lambda(q) \leq C_4\Lambda(q)^{\xi_2}
\]

Notice that \(\Lambda(q)\) may be assumed to be \(\leq 1\). Combining this with (6.3), we can see that

\[
\Lambda(q) \leq C_4\Lambda(q)^{\xi_2},
\]

or

\[
\Lambda(q)^{\xi_2} \leq C_4.
\]
This proves the uniformly boundedness of $v^{(q)}$’s.

**Proof of Theorem C.** Since $q^{(q)}$’s are uniformly bounded,

\begin{align}
\lim_{q \to d/d-2} & \int_S G(P, Q)(Rv^{(q)} - \mu_{C_1}(v^{(q)})^q-1) \, dV(Q) \\
& + \int_S v^{(q)}(Q)dV(Q) = \bar{u}
\end{align}

converges uniformly to a $C^1$ function when we take a suitable sequence of $q$’s. This limit $\bar{u}$ must satisfy

\begin{equation}
\frac{4(d-1)}{(d-2)} \Delta \bar{u} - R\bar{u} = -\mu_{C_1}(d/d-2)(\bar{u})^{d+2}/(d-2)
\end{equation}

weakly. From this we can easily obtain the $C^2$ property for $\bar{u}$ because of the boundedness of $\bar{u}$. Again, Lemma 6 is available and $\bar{u}$ can be proved to be $C^\infty$ because it is bounded $C^2$ function without zero points, satisfying (6.14). This is nothing but the equation (1.10). Thus Theorem C has been proved.

A direct consequence of Therem A is that if $R$ is everywhere non-negative, then $\bar{R}$, the scalar curvature of the new structure, is a non-negative constant and is zero just in case $R$ is everywhere zero. If $R$ is everywhere non-positive and not identically zero, then $\bar{R}$ is negative because it is less than $F^{(d/d-2)}(1) < 0$.

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**Appendix**

Supplement to the proof of Lemma 8.

Once the essential boundedness of $v^{(q)}$ is established, it immediately follows that $v^{(q)}$ is $C^1$. Hence $v^{(q)}$, being a solution of an equation

\[ \Delta v^{(q)} = Rv^{(q)} - \mu_{C_1}(v^{(q)})^q-1 , \]

is a $C^2$ solution except at zero point of $v^{(q)}$. Repeating this kind of procedures, we can see that $v^{(q)}$ is $C^\infty$ except at zero point of $v^{(q)}$. 
REMARK. As is seen very easily, if the original structure is $C^k$, $k \geq 3$, and $\omega$, then $\tilde{u}$ itself is also $C^k$, $k \geq 1$ and $\omega$.

REMARK. Prof. J. Serrin notified the author that Lemma 6 can be proved by using E. Hopf's maximum principle (cf. [5]).

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Reference


