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Potential Theory and its Applications, II.

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In the potential theory the Dirichlet Problem is a central one, and this has been discussed by many authors on an abstract Riemann surface. R. Nevanlinna[1] studied it in his paper under the following conditions:

1) *The Riemann surface R is the compact subsurface of another Riemann surface $\underline{R}$.*

2) *The transfinite kernel of $\underline{R}-R$ is non empty.*

On the other hand M. Ohtsuka[2] studied under the following conditions:

1) *The projection of the Riemann surface R on another Riemann surface R is compact.*

2) *When $\underline{R}$ is a closed surface of genus zero or one, $\underline{R}-R$ contains at least three or one point respectively.*

And more precise investigation is done by him under the condition that the connectivity is finite.

But now we shall study also this problem for a non compact surface. This study is incomplete in many points as will be seen in the following. This idea owes to the paper of Brelot[3] and is intimate with those of M. Bader[4] or P. Parreu[5] rather than of R. Nevanlinna or M. Ohtsuka.

1. Let F be an abstract Riemann surface, then it is well known that there exists another non prolongable Riemann surface $\widetilde{F}$ containing the former in it. If F has finite number of genus, then $\widetilde{F}$ is a closed Riemann surface of the same genus, but if $\widetilde{F}$ has an infinite number of genus, then $\widetilde{F}$ will be an open Riemann surface. There occur two cases: either $\widetilde{F}$ is a zero-boundary or a positive-boundary Rieman sur-

1) R. Nevanlinna: Über die Lösbarkeit des Dirichletschen Problemes für eine Riemannsche Fläche, Nachr. Gött. 1 (1938), 181-193.

2) M. Ohtsuka: Dirichlet Problem on Riemann surface and conformal mapping, Nagoya Math. Jour., Vol. 3 (1951), 91-137.

3) M. Brelot: Le Problème de Dirichlet "ramifié", Ann. Grenoble, 22 (1946), 167-200.

4) R. Bader: La théorie du potential sur une surface de Riemann, C. R. Paris, 228 (1939), 2001-2002.

5) M. Parreu: A comportement à la frontière de la tonction de Green d'une surface de Riemann, C. R. Paris, 230 (1950), 709-711

face. If $\widetilde{F}$ is of zero-boundary, then this problem is easy as we have shown in the previous paper.

In the sequel we suppose $\widetilde{F}$ to be also a positive-boundary Riemann surface. On the other hand it is evident that if F is a zero-boundary Riemann surface, then there is no non-constant bounded harmonic function, hence we must first suppose F is of positive-boundary.

2. **Topology** $\mathfrak{A}$. On an abstract Riemann surface the most simple topology is defined by the exhaustion of $\widetilde{F}$: $\bigcup_n \widetilde{F}_n = \widetilde{F}$. In this topology we define ideal boundary elements and we obtain by adding all ideal elements a compact closed space $\overline{\widetilde{F}}$. In our case we introduce topology $\mathfrak{A}$ about $\overline{\widetilde{F}}$, then $\overline{\widetilde{F}}$ is a normal space, therefore $\overline{\widetilde{F}}$ is a metric space. In this topology each neighborhood of an inner point of $\widetilde{F}$ is homeomorphic to the disc of the z-plane.

On an abstract Riemann surface there is defined nothing but Green function from which we begin the theory of harmonic function, accordingly we ought to study it and other harmonic measure of function defined by arbitrary exhaustion $\widetilde{F}_n$ of $\widetilde{F}$.

Let us denote by $G(x, p)$ or simply by $G(x)$ the least harmonic positive function except at an inner point p of $\widetilde{F}$ where $G(x)$ has one logarithmic singularity. By this property $G(x)$ is uniquely determined with respect to p and $\widetilde{F}$. $G(x)$ is continuous at every inner point of $\widetilde{F}$ except for p, but we hardly know the behaviour at the boundary. If $G(x)$ has limit when x converges to an ideal boundary point in topology $\mathfrak{A}$, then this point is called regular in $\mathfrak{A}$.

If $\widetilde{F}$ is of planer character, then $\widetilde{F}$ coincides with the whole z-plane, in which case the singular points for $G(x)$ is a set of capacity zero. In our case this property does not necessarily hold.

3. **Harmonic measure and Green function.** Let us denote by ${}_c\widetilde{F}_n$ the subsurface of $\widetilde{F}$ cut by a finite number of compact analytic curves C from $\widetilde{F}$, and by ${}_c\widetilde{F}_n$ the non decreasing sequence of compact subsurfaces such as $\bigcup_n {}_c\widetilde{F}_n = {}_c\widetilde{F}$.

In denoting by ${}_c\omega_n$ the bounded harmonic function defined in $\widetilde{F}_n \cap {}_c\widetilde{F}$ with the boundary values 0 on C and 1 on the boundary of $\widetilde{F}_n$ contained in ${}_c\widetilde{F}$, and by ω_n the bounded harmonic function with the boundary values 1 on the boundary of $\widetilde{F}_n$ contained in ${}_c\widetilde{F}$ and 0 on the other boundary contained in $\widetilde{F} - {}_c\widetilde{F}$. ${}_c\omega_n$ is decreasing with respect to n, but ω_n is not always so. We define the measure of a boundary contained in ${}_c\widetilde{F}$ with respect to $\widetilde{F}$ by the greatest lower bound of ω_n for every F_n such as $\bigcup_n \widetilde{F}_n = \widetilde{F}$. *Then* $\omega_c = 0$ *is equivalent to* $\omega = 0$. For evidently if $\omega = 0$, then $\omega_c = 0$. Conversely assume that $\omega_c = 0$. Then by maximum principle $\varliminf_{x \in F_c} G(x, p) \geqq \min_{x \in c} G(x) \geqq \delta_0 > 0$, where p is assumed to be $p \overline{\in} {}_c\widetilde{F}$. If $\omega \neq 0$, then there exists a compact subsurface $\widetilde{F}_n$ such that $\omega_n = 1$ on the boundary of $\widetilde{F}_n$ contained in ${}_c\widetilde{F}$, 0 on the other boundary, therefore $G(x) - \delta_0 \omega_n(x) > 0$, for every $\widetilde{F}_n$. Therefore

$$G(x) \geqq G(z) - \delta_0 \omega(x) .$$

But since $G(x) - \delta_0 \omega(x)$ has the same singularity as $G(x)$ at p, this contradicts the uniqueness of $G(x)$.

4. **Transfinite kernel.** For a subsurface $\widetilde{F}_c$ cut by C from $\widetilde{F}$, if the harmonic measure ${}_c\omega$ does not vanish, then the set of the boundary contained in ${}_cF$ is not zero. It is easily seen that the set in which harmonic measure zero is an open set, and that the set for any subsurface ${}_cF$ containing at least a point of this set $\omega_c > 0$, is a closed set. This is called the *transfinite kernel* of the boundary.

By definition we have easily the following proposition.

If and only if $\varliminf_{x \to p \in V_c} G(x) > 0$, *$V_c \cap R$ has no transfinite kernel.*

Take a closed subset A of the sum of boundary elements R, and a non decreasing sequence $\widetilde{F}_n$ of compact subsurfaces such that $\bigcup_n \widetilde{F}_n = \widetilde{F}$. We denote by ${}_c\omega_n^*$ the bounded positive harmonic function defined in $\widetilde{F}_n \cap \widetilde{F}_c$ with the boundary values 1 on the boundary $\gamma_n(A)$ of $\widetilde{F}_n$ such that the non compact subsurface cut by $\gamma_n(A)$ has at least one point of A, and the boundary value 0 on the boundary except for $\gamma_n(A)$ and 0 on C.

Further we denote by ω_n^* the bounded positive harmonic function defined in $\widetilde{F}_n$ with the boundary value 1 on $\gamma_n(A)$, 0 on $\gamma_n(A)$ which are boundaries other than $\gamma_n(A)$.

We define the harmonic measure of $A_c(A)$ by the lower limit of ${}_c\omega_n(\omega_n)$ for every exhaustion of $\widetilde{F}_n$, such that $\bigcup \widetilde{F}_n = \widetilde{F}$.

Proposition. *If and only if* ${}_C\omega_A^* = 0$, *we have* $\omega_A^* = 0$.

For by harmonic majoration we have

$$\omega_A^* \geqq {}_c\omega_A^*$$

Let us suppose ${}_c\omega_A^* = 0$ and $\omega_A^* \neq 0$, then there exists a sequence of F_n such that $\varliminf_n {}_c\omega_n^* = 0$.

We denote by Γ_1 the compact curve near to C in $\widetilde{F}_c$, and by Γ_0 the other curve which covers A (for the closure of the subsurface cut by Γ_0 contains A), and which does not intersect any $\gamma_n + \tilde{\gamma}_n$ for $n \geqq n_0$, and denote by λ_0, λ_1 the maxima of ${}_n\omega_A^*$ on Γ_0 and Γ_1 respectively, then by assumption

$$\omega_c^* (x, \gamma_n(A), \widetilde{F}_n \frown F_c) < \varepsilon : x \in \Gamma_0 + \Gamma_1,$$
$$\omega_n^* (x, \gamma_n(A), \widetilde{F}_n) - {}_c\omega_n^* (x, \gamma_n(A), \widetilde{F}_n \frown \widetilde{F}_c) = 0 : x \in \gamma_n(A) + \tilde{\gamma}_n(A).$$

Since

$${}_c\omega_n^* (x, \gamma_n(A), \widetilde{F}_n \frown {}_c\widetilde{F}) \geqq 0,$$
$$\omega_n^* (x, \gamma_n(A), \widetilde{F}_n) - {}_c\omega_n^* (x, \gamma_n(A), {}_c\widetilde{F} \frown \widetilde{F}_n) \geqq \lambda,$$

where x is contained in the domain bounded by C, Γ_0 and $\gamma_n(A) + \tilde{\gamma}_n(A)$. Especially we have on Γ_1,

$$\lambda_0 - \varepsilon < \lambda_1.$$

On the other hand

$$\omega_n^* (x, \gamma_n(A), \widetilde{F}_n) - \lambda_0 \omega_n^* (x, \Gamma_0 + \gamma_n(A), \widetilde{F}_n) = 0 : x \in \gamma_n(A) + \tilde{\gamma}_n(A)$$
$$\omega_n^* (x, \gamma_n(A), \widetilde{F}_n) - \lambda_0 \omega_n^* (x, \Gamma_0 + \gamma_n(A), \widetilde{F}_n) \geqq 0 : x \in \Gamma_0,$$

therefore the left side term is non positive in the domain bounded by $\gamma_n(A) + \tilde{\gamma}_n(A)$ and Γ_0. Especially on Γ_1, we have

$$\lambda_1 < \lambda_0 \Theta \ : 0 < \Theta < 1.$$

Hence it follows $\lambda_1 = 0$, and we have

$$\underline{\lim_n} \, \omega_n^* (x, \gamma_n(A), \widetilde{F}_n) = 0.$$

Let us denote by ${}_c\omega_n^{**}$ the bounded harmonic positive function such that

$${}_c\omega_n^{**} = 0 : x \in C, \qquad {}_c\omega_n^{**} = 1 : x \in \gamma_n(A),$$
$$0 < \omega_c^{**} < 1 : x \in \tilde{\gamma}_n(A),$$

and

$$\int_{\delta_{n(A)}} \frac{\partial \omega_n^{**}}{\partial n} ds = 0.$$

If there exists a sequence of ${}_c\widetilde{F}_n : \bigcup_n \widetilde{F}_n = \widetilde{F}$, such that $\underline{\lim}\, {}_c\omega_n^{**} = 0$, then we say that A is the set of harmonic capacity zero.

This is the direct generalization of harmonic measure in the plane domain, but ω^* is a not generalization of the former, and it seems as the ordinary linear measure with respect to all the boundaries of $\widetilde{F}$, i.e. the measure of A, when $\widetilde{F}$ is comformally represented in the unit circle. In the sequel we denote this measure by "measure" without the adjective "harmonic". It is evident that the set of harmonic capacity zero is also of measure zero. But in the case when $A \subset R$ is isolated from other boundaries these two measure coincides each other, in the general case the set of harmonic capacity positive may be of measure zero, for instance the inaccessible points set is of measure zero but can be of positive harmonic capacity in the bounded simply connected domain in the z-plane.

5. **Proposition.** *From F we take off disc $F_0 : F_0 = E[G(x) \geqq M]$, and let the compact niveau curve of $G(x) = M$ be denoted by C, then the set $S_\delta = E[G(x) \geqq \delta] \frown R$ is a set of harmonic capacity zero.*

Put $F_\delta = E[G(x) \geqq \delta]$, and $C_a = E[G = a]$.

Let us cover R by a system of neighborhoods v_n, then C_a may converge into R.

$F_{\delta n} = F_\delta - \sum^n v_n(R)$ has the boundary curve $\gamma_{n\delta}$ and $\tilde{\gamma}_{n\delta}$, where $\tilde{\gamma}_{n\delta}$ is the boundary of F_n in $\sum^n v_n(R)$ and irregular for the Dirichlet problem and $G(x) = \delta : x \in \gamma_{n\delta}$.

We denote by $U(x)$ the solution of Dirichlet problem with the boundary value 0 on C, and 1 on C_δ, then $U(x) = \dfrac{M-G}{\delta}$.

Let $N\varepsilon_0 = E[U(x) < 1 - \varepsilon_0] \frown F_\delta$, where $\varepsilon_0 \gneqq 0$, then we have only to show that the set $N\varepsilon_0$ is of capacity zero, where

$$\begin{aligned} U(x) &= 1 &&: x \in \tilde{\gamma}_{n\delta}, \\ U(x) &\leqq 1 - \varepsilon_0 &&: x \in \gamma_{n\delta}. \end{aligned}$$

We denote by $U_n'(x)$ the bounded harmonic function such that

$$\begin{aligned} U_n'(x) &= 1 &&: x \in \gamma_{n\delta}, \\ U_n'(x) &= 1 &&: x \in \tilde{\gamma}_{n\delta}, \\ U_n'(x) &= 0 &&: x \in C, \end{aligned}$$

then

$$U(x) \leqq U_n'(x)$$

$$\frac{\partial U}{\partial n} \geqq \frac{\partial U_n'}{\partial n} \quad \text{on } \gamma_n, \text{ where } n \text{ is inner normal.}$$

But

$$\lim_n U_n'(x) = U(x) : \quad \int_{\tilde{\gamma}_n} \frac{\partial U}{\partial n}\, ds = \lim \int_{\tilde{\gamma}_n} \frac{\partial U_n'}{\partial n}\, ds$$

$$\int_{\tilde{\gamma}_n} \frac{\partial U_n'}{\partial n}(1 + s_n)\, ds = \int_{\tilde{\gamma}_n} \frac{\partial U}{\partial n}\, ds : s_n > 0, \ : \ \lim_n s_n = 0,$$

$$\begin{aligned} V_n(x) &= (1 + s_n) U_n'(x) - U(x) = 0 \ : \ x \in C, \\ V_n(x) &= 1 + s_n - 1 = s_n \ : \ x \in \gamma_n s \ : \ s_n > 0, \\ M > V_n(x) &= 1 + s_n - 1 + \varepsilon_0 \geqq s_n + \varepsilon_0 > \varepsilon_0 \ : \ x \in \gamma_{n\delta}, \end{aligned}$$

where

$$s_n < \varepsilon_0.$$

Therefore $V_n(x) = (1 + s_n) U_n'(x) - U(x)$ is the required function, but

$\lim_n U_n'(x)\ (1+s_n)-U(x)=0.$

It follows that the set of C_δ in which $U<1-\varepsilon_0$ is of capacity zero, but this assertion is valid for every δ and every ε_0, thus the irregular set of Green function is of capacity zero.

Let p be a boundary element of R then if p is of positive measure, we say that p is a boundary component, otherwise boundary point. In the case of the z-plane, if p is a component then p is a continuum and every point of p is regular for Dirichlet problem, but in an abstract Riemann surface this does not hold necessarily.

If we want the theorems obtained in the z-plane should be valid, in our case, we must introduce the topology $\mathfrak{B}$ by means of Green function.

6. **Topology** $\mathfrak{B}$.

Consider the Green function $G(x)$ of $\widetilde{F}$ with a pole at an inner point and a curve L in $\widetilde{F}$.

Definition. 1) *If L converges to an ideal boundary element p in $\mathfrak{A}$, and* 2) *if on L lim $G(x)$ exists, then we say that L determines an accessible boundary point p and define the length of L by the least upper bound of $\sum|a_i, a_{i+1}|$ for all sub-division of L.*

Associated domain. Let p be an accessible boundary point, then there exists a curve L and $\lim G(x)=\gamma$ corresponding to p. We denote by G_ε the domain of $\widetilde{F}$ in which $\gamma-\varepsilon<G<\gamma+\varepsilon$. G_ε is composed of enumerably infinitely disjoint domains. Take a neighbourhood of $V_n(p)$ of p, then $D_{\varepsilon,n}=G_\varepsilon\cap V_n$ is composed of domains. We denote every $D_{\varepsilon,n}$ which has at least an inner point with L except p by ${}_\varepsilon U_n$ and call it the associated domain with the order ε and n.

For two accessible boundary elements p and q, we define $p=q$ if and only if any $U(p)$ is also an associated domain of q and vice versa.

With this definition every inner point is an accessible point and, since $G(x)$ is continuous except at the pole of G, it is evident that $\mathfrak{A}$ topology is the same as $\mathfrak{B}$.

Distance in $\mathfrak{B}$. Let p and q be two accessesible points, then for large ε^N, $G_{\varepsilon^N}(p)$ contains q. If we connect by a curve L in $\widetilde{F}$ two points p and q, and denote by L_{ε^N} the greatest lower bound of lengths of such curves. Evidently L_{ε^N} increases with $\dfrac{1}{\varepsilon^N}$.

We define the distance $p\rightarrow q$ by the lower limit of $\varepsilon+L_\varepsilon$ for any pair of ε and L_ε.

By this definition it is quite easy to see: $p\rightarrow q=q\rightarrow p$, and $(p\leftrightarrows q=0)\equiv(p=q)$ and $|p\ q|+|q\ r|\geqq|p\ r|$.

Since to all accessible points p, L_p and $\lim\limits_{x\in L_p} G$ correspond, we denote by S_γ the accessible boundary point where G has a limit γ. Since by uniqueness of G it is evident that the set of $\sum_{\gamma>0} S_\gamma$ is of measure zero. In $\mathfrak{B}$ S_0 is closed and compact.

In fact let $L_1, L_2, \dots, L_n, \dots$ be a sequence of accessible boundary points in S_0.

Since $G_M \supset R$ for large M, then we can assume without loss of generality

$$|L_i, L_j| < 1.$$

Take a domain associated U_1 to L_1, then there exist infinitely many $L_i^{(1)}$ of L_i of which U_1 is associated domain, and there exists at least one $U_{\frac{1}{2}}$ in U_1 which is an associated domain for infinitely many of $L_i^{(2)}$ of $L_i^{(1)}$. Then we have a sequence of $U_{\frac{1}{i}}$ and a sequence of systems of $L_j^{(i)}$: $i=1, 2, \dots$.

Suppose $a_i \in U_{\frac{1}{i}} - U_{\frac{1}{i+1}}$ and connect $a_1, a_2, \dots$ by a curve L.

Since $\cap U_{\frac{1}{i}}$ determines only one point in topology $\mathfrak{A}$, and $\lim G=0$ on L, therefore in our metric $\lim |L_i^{(i)}, L| = 0$.

Thus we get $\overline{\mathfrak{B}}$ by completion from $\mathfrak{B}$.

7. **Topology $\mathfrak{C}$.** If a curve L in F converges to a point in topology $\mathfrak{A}$ and on it $G(x)$ of $\tilde{F}$ has a limit, then we say that L determines an accessible boundary point of F, and as the associated domain we take the domain $V_n \cap F \cap G_{\gamma\pm\varepsilon}$. The distance $|p_i, p_j|$ is analogously defined as the preceding relative to F, then it is evident that this metric satisfies the axioms of metric. This topology at an inner point of F coincides with $\mathfrak{A}$ and $\mathfrak{B}$ and the relative boundary point of F coincides with the accessible boundary point in the z-plane "ramifié".

8. **Measurability of ideal boundary points in topology $\mathfrak{A}$.**

We have introduced the measure of a closed set $A \subset R$.

Lemma. *If for any finite number of closed sets $A_i \subset R : A_i \cap A_j = 0$, then*

$$\sum_i m(A_i) \leq 1.$$

Since $A_i \cap A_j = 0$, we can cover A_i by $V_i : V_i \supset A_i$ such that $V_i \cap V_j = 0$ and that all the boundaries of V_i are compact in $\tilde{F}$.

On the other hand take a subsurface $\tilde{F}_n$ of which all boundary curves $\gamma_n + \tilde{\gamma}_n$ are contained in $\sum V_i$.

If we denote by ${}_n\gamma_i$ the curves contained in $\sum V_i$, then the non compact domain cut by ${}_n\gamma_i$ contains A_i.

As the measure of A is the lower limit of the harmonic function such that

$$\omega_n(A_i) = 1 : x \in {}_n\gamma_i, \quad \omega_n(A_i) = 0 : x \in \sum_{j \neq i} {}_n\gamma_j + {}_n\tilde{\gamma}_j - {}_n\gamma_i,$$

then $\sum_i \omega_{n',i} \leqq 1$, for any $\widetilde{F}_n$ and V_i, therefore $\sum m(A_i) \leqq 1$.

For a sequence of closed sets $A_i \subset R$ having no common point each other

$$\sum_i m(A_i) = m \sum A_i \text{.}^{1)}$$

Proof. For any given positive number ε, there exists a compact subsurface F_m with the boundary $\gamma_m(\sum(A_i))$ and $\tilde{\gamma}_m(\sum(A_i))$, such that

$$\omega_m(\sum A_i) \geqq m(\sum A_i) - \varepsilon\,,$$

where

$$\begin{aligned} \omega_m(\sum A_i) &= 1 \quad : \quad x \in \gamma_m(\sum A_i) \\ \omega_m(\sum A_i) &= 0 \quad : \quad x \in \tilde{\gamma}_m(\sum A_i)\,. \end{aligned}$$

On the other hand, for this F_m, $\gamma_m(\sum A_i) \subseteqq \sum \gamma_m(A_i)$, and $\tilde{\gamma}_m(\sum_i A_i) \supset \bigcap_i \tilde{\gamma}_m(A_i)$, thus

$$\begin{aligned} \sum \omega_m(A_i) &\geqq 1 \quad : \quad x \in \gamma_m(\sum A_i)\,, \\ \sum \omega_m(A_i) &\geqq 0 \quad : \quad x \in \tilde{\gamma}_m(\sum A_i)\,, \end{aligned}$$

therefore

$$\sum \omega_m(A_i) \geqq \omega_m(\sum A_i) - \varepsilon\,,$$

finally

$$\sum m(A_i) \geqq m(\sum A_i)\,.$$

Since

$$1 \geqq \sum m(A_i)\,,$$

from Lemma above, we can find for any given positive number ε, i_0 such that

$$\sum^{i_0} m(A_i) \geqq \sum m(A_i) - \varepsilon \quad : \quad i \geqq i_0\,.$$

For such i_0, we can find a system of V_i such as $V_i \cap V_j = 0 : V_i \supset A_i$ and we denote by $\widetilde{F}_n'$ the set $\widetilde{F}_n' = \widetilde{F}_n \cap \widetilde{F} - \sum^{i_0} V_i$, then there exists a compact subsurface F_m containing the boundary of $V_i : i \leqq i_0$. We denote by ω_m^i the positive bounded harmonic function such that

$$\begin{aligned} \omega_m^i &= 1 \quad : \quad x \in \gamma_m(A_i)\,, \\ \omega_m^i &= 0 \quad : \quad x \in \tilde{\gamma}_m(A_i)\,, \\ \omega_m &= \sum \omega_m^i = 1 \quad : \quad x \in \sum^{i_0} \gamma_m(A_i)\,, \\ \omega_m &= 0 \quad : \quad x \in \bigcap_{i_0} \tilde{\gamma}_m(A_i)\,. \end{aligned}$$

Thus

$$\sum m(A_i) \leqq \sum^{i_0} m(A_i) + \varepsilon \leqq m \sum^{i_0} (A_i) + \varepsilon\,,$$

which holds for any any F_m, then

$$\sum m(A_i) \leqq m \sum (A_i)\,.$$

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1) In this manner we can prove $\omega(A_i)$ is harmonic.

From this fact any Borel set in $\widetilde{F}$ is measurable in $\mathfrak{A}$, but in this topology one point can have positive measure, such points being at most enumerable, by Lemma.

Measurability in $\mathfrak{B}$ or $\mathfrak{C}$.

We can define closed subsets A of R in $\mathfrak{B}$ or $\mathfrak{C}$, and we can cover A by the sequence of associated domains corresponding to any point of A. Thus defines measure as above, which satisfy the condition of additivity or directly we see that every Borel set in $\mathfrak{B}$ or $\mathfrak{C}$ is also measurable, because the transformation from $\mathfrak{B}$ or $\mathfrak{C}$ to $\mathfrak{A}$ is continuous.

Remark. If F is mapped conformally onto $|z|<1$, and if the image of $A\subset R$ has outer measure zero on $|z|=1$, then it is clear that the measure of A with respect to F is zero.

Measure of inaccessible boundary point set.

We map F onto $|z|<1$ conformally by using the universal covering surface $\widetilde{F}^{\infty}$ of $\widetilde{F}$. We denote by $h(x)$ the conjugate harmonic function of $G(x)$ in $\widetilde{F}$, then $w=e^{-G-ih}=w(x)$ maps F on $|w|<1: w(m(x))=w(z)$: $m(x)=z: |z|<1$.

Taking any one branch of $w(x)$, it is a uniform function defined in the circle, then by Fatow's theorem, the limit of $w(z)$ by radial approaching to $|z|=1$ exists except at most linear measure zero on $|z|=1$, which will be denoted by W_0. We have only to show that the image L in $\widetilde{F}$ of l in $|z|<1$ converges in $\mathfrak{A}$ to a point and $\lim G(x)$ exists on L.

Suppose L does not converge to a point in $\widetilde{F}$, then there exists two sequences p_i and $q_i: \lim\limits_i p_i=p\neq q=\lim q_i$, and sequence of arcs L_i between p_i and q_i, on which $w(x)$ converges to W_0 uniformly.

If p is an ideal boundary point in $\mathfrak{A}$, then there is a neighborhood V_n with only relative boundary curve γ_n such as $V_n\ni q: i>i_0$, in which another boundary curve γ' near to γ_n is taken.

From the compact domain bounded by γ_n and γ', we can find a ring domain bounded by two compact curves, denoted by R. In this ring domain there are infinitely many curves of L_i, having a limit curve L in it. We cut this domain into simply connected domain R', then L_i converges to L and $w(x)$ converges uniformly to W_0 on L. As $w(x)$ is an infinitely many valued function, we take a branch of this function corresponding to L_i, denoted by $w_i(x)$. From this family of functions we can extract a normal family $w_i(x): \lim\limits_i w_i(x)=w(x)$ but $w(x)$ converges to W_0 on L, from which follows that $w(x)\equiv W$, but for any i, $|w_i(x)| =|w(x)|=e^{-G}$. This is a contradiction.

On L, $G(z)$ clearly converges, because $|w|$ is one valued and equal to e^{-G}. Then by definition L determines an accessible point, and we have:

Proposition. *The mesure of inaccessible boundary point is zero.*

9. **Lemma.** *Let $U(x)$ be subharmonic in F bounded from above and for all accessible boundary points $\overline{\lim}\, U(x) \leqq 0$. Then $U(x) \leqq 0$ in F.*

Otherwise if $U(x) > \delta > 0$, we denote by D_δ the domain in which $U(x) > \delta$, then on the boundary of D_δ, $U(x) = \delta$, but this boundary does not converge to any assessibly boundary point, but this boundary is situated in F or converges to inaccessible boundary point, on the other hand the measure of inaccessible boundary point is zero, for any $\varepsilon < \frac{\delta}{2M}$, then there exists a subsurface F_n' bounded by γ', and bounded harmonic function ω_n with the boundary value 1 and 0 on the other boundary.

$$U(x) \leqq M\omega_n : \lim \omega_n = 0\text{, then } \delta < \varepsilon .$$

M. Brelot proved the following lemma.

Lemma. *The upper cover (defined by the supremum at each point) of a class of positive harmonic functions in F is continuous in F and subharmonic or equal to the constant.*

In topology $\mathfrak{C}$, we can define a real Borel measurable function φ on the boundary B of F, the lower class I_φ^F is defined by all the bounded above and continuous subharmonic functions such that $\overset{c}{\underset{p' \to p}{\lim}}\, U(p') \leqq \varphi(p)$ inclusive except at most set of measure zero of B, then it follows by lemma the upper cover $\underline{H}_\varphi^F$ of I_φ^F, which will be called hypo-function, is harmonic or equal to the constant ∞, or $-\infty$ in each component of F. Similarly the upper class S_φ^F and its lower cover $\overline{H}_\varphi^F$ which will be called hyper-function, are defined for superharmonic function and $\overline{H}_\varphi^F$ has the similar character as $\underline{H}_\varphi^F$. $\underline{H}_\varphi^F \leqq \overline{H}_\varphi^F$ on p, and if they coincide at a point, then they areuidentical in the component containing the point. When $\overline{H} = \underline{H}$, we shall denote it by H, and call it the general solution, and moreover if H is finite, φ will be called a resolutive boundary function.

10. **Barrier function.** In topology $\mathfrak{C}$, let us denote all boundary point sets by B. If $U(x)$ has the following property, then it will be called a barrier function at $x \in B$.

1. $U(x) = 0$, where x is an accessible boundary point of B.
2. $U(x)$ is continuous in $F+B$ and $U(x) \geqq 0$.
3. $U(x)$ is super-harmonic in F.
4. For every associated domain $V(x_0)$, there exists a constant a such that $U(x) \geqq a > 0 : x \in F - V(x_0)$.

This definition is the same as in the z-plane when x is an inner point of $\widehat{F}$.

For any function φ defined on accessible boundary points B, if

$$\underline{\lim_{\substack{c \\ p'\to p}}}\varphi(p') \leqq \underline{H} < \overline{H} \leqq \overline{\lim_{\substack{c \\ p'\to p}}}\varphi(p')$$

then p is called regular for Dirichlet problem.

Proposition. *B has barrier function except at a set of measure zero.*

Let $p \in S_0 \cap B$, then p has a barrier function. In fact, we take a sequence of associated domains of $p: V_i' \cap G'_{\varepsilon_i} \cap F$: one component of $V_i \cap G_{\varepsilon_i} \cap F$ where V_i has a *compact boundary* γ *relative* $\widetilde{F}$, then min $G(x)$ on γ is $\delta_i > 0$. Define

$$\omega_i(x) = \frac{1}{\operatorname{Min}(\delta_i,\ \varepsilon_i)} \operatorname{Min}(G,\ \varepsilon_i,\ \delta_i) : x \in \overline{V_i' \cap G'_{\varepsilon_i} \cap F},$$
$$\omega_i(x) = 1 \qquad : x \in \overline{F} - \overline{V_i' \cap G'_{\varepsilon_i}}$$

and

$$p = \overline{\bigcap_i V_i' \bigcap_i G'_{\varepsilon_i} \cap F},$$

then the function $U(x) = \sum_{r=1}^{\infty} \frac{\omega_i}{2^i}$ is a barrier function.

If at p barrier exists, then p is regular for Dirichlet problem.

We denote by $\overline{f}(x)$ or $\underline{f}(x)$ the upper envelope of $f(x)$ in B or lower envelope on B, then $\overline{f}(x)$ $(\underline{f}(x))$ is upper semi-continuous.

Let D_0 be a small associated domain, then

$$\underline{f}(x) - \varepsilon \leqq \underline{f}(x_0) < \overline{f}(x_0) < \overline{f}(x) + \varepsilon$$
$$\varphi(x) = \underline{f}(x_0) - \varepsilon - CU(x) : C > 0,$$
$$\varphi(x) \leqq \underline{f}(x_0) : x \in \overline{F} - D_0, \quad \varphi(x) \leqq \underline{f}(x) - \varepsilon \leqq f(x).$$

Thus $\varphi(x) \in I^F_\varphi$, and analogously

$$\psi(x) = \overline{f}(x_0) + \varepsilon + C(U(x)) : \psi(x) \in S^F_\varphi.$$

Since $U(x_0) = 0$,

$$\varphi(x_0) = \underline{f}(x_0) - \varepsilon, \quad \psi(x_0) \geqq \underline{f}(x_0) - 2\varepsilon,$$
$$\lim_{x \to x_0} \overline{H}(x) \geqq \underline{f}(x_0).$$

Since $\psi(x) = \overline{f}(x_0) + \varepsilon$,

$$\psi(x) \leqq \overline{f}(x_0) + 2\varepsilon : x \in D_0,$$

finally

$$\overline{H}(x_0) \leqq \overline{f}(x_0).$$

As $\overline{H}$ is the lower limit of ψ, then $\overline{H}(x_0) \geqq f(x_0)$, similarly for φ, then

$$\underline{f}(x_0) \leqq \underline{H}(x_0) \leqq \overline{H}(x_0) \leqq \overline{f}(x_0).$$

The boundary of F is accessible except at a set of measure zero, and composed of two different kinds, that is, a boundary point which is an inner point B_1 of F or a boundary B_2 of $\widetilde{F}$ simultaneously. We take an exhaustion $\widetilde{F}_n$ of $\widetilde{F}$, then the boundary of F_n contained in $\widetilde{F}_n$ is all regular except at a set of harmonic capacity zero relative to $\widetilde{F}_n$, by the theorem of Bouligand for every n, and the part B_2 is contained in S_0 and also accessible except at a set of measure zero relative to F, and S_0 is regular except a set of measure zero relative to $\widetilde{F}$, accordingly except a set of measure zero relative to F. Thus every boundary of B has barrier function except the set of measure zero relative to F.

11. **Uniqueness of topology.** The property of existence of barrier at a boundary point is local. If we take a compact surface F_0 and bounded harmonic function $U(x)$ with the boundary value 0 on the boundary of F_0 and 1 on R, then $U(x)$ is called a conducter potential.

Let $G'(z)$ be the Green function with a pole at $p' \neq p$, then G' induces another topology $\mathfrak{B}'$ or $\mathfrak{C}'$.

Proposition. *Topology $\mathfrak{C}'$ is the same as $\mathfrak{C}$ expect a set of measure zero.*

If $p_1=p_2$ in ($\mathfrak{C}$), then for any given positive number $\varepsilon>0$, there exists δ such as $G'(x)\leqq\varepsilon/2: x\in G_\delta: G_\delta=E[G<\delta]$ but $p_1=p_2$ in ($\mathfrak{C}$), then p_1 and p_2 are connected in $G_\delta\cap F$ with a curve L having the length $<\varepsilon/2$, therefore $|p_1\ p_2|<\varepsilon$ in ($\mathfrak{C}'$), whence it follows that $|p_1\ p_2|=0$ and vice versa.

In $F\cap\widetilde{F}$ the accessibility or topology does not depend on G or on G' but on F, and further at every inner point of F we have $\mathfrak{C}=\mathfrak{B}=\mathfrak{A}$.

Dirichlet Problem. *Let φ be a bounded Borel function on B, then φ is resolutive.*

Since every Borel function on B is measurable, there exists by N. Lusin's theorem for any number n such an open set G_n in B, $B\supset G_n$, mes $G_n<\dfrac{1}{n}$. In $B-G_n$, φ is continuous, therefore the solution $\overline{H}_n$, $\underline{H}_n$ determined on the value on $B-G_n$,

$$|\overline{H}_n-\underline{H}_n|<2M\frac{1}{n},$$

accordingly

$$\overline{H}=\underline{H}.$$

Analogously we have the following proposition.

$\underline{H}^F_\varphi$ is the upper cover of $\underline{H}_\psi$, where $\psi\leqq\varphi$ and ψ is bounded above and upper semi-continous. Similarly for $\underline{H}_\varphi$, and

$$\overline{H}(p)=\underline{H}(p)=\int_B \varphi\ dm(p)$$

where $\underset{x(B)}{m}(p)=\underset{x(B)}{\overline{m}}(p)=m(p)$: $x(B)$ is a characteristic function of B.

The fundamental theorem of the cluster set theory holds also.

If p is a point of measure zero and further p_0 is regular for Dirichlet problem for topology $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, then

$$\overline{\lim_{p'\to p_0}}\ (\overline{\lim_{\substack{p\to p' \\ p'\neq p}}}\ U(p)) = \overline{\lim_{p\to p_0}}\ U(p),$$

and if $f(x)$ is one valued analytic function, the we can omit the regularity of p.

If $\widetilde{F}$ has null-boundary, then there exists topology $\mathfrak{A}$ only, consequently we cannot consider Dirichlet problem on $\widetilde{F}$. But for F in this case we have seen that all theorems in z-plane holds except a set of capacity zero. If F has two disjoint positive measure sets in B of $\mathfrak{B}$, $\mathfrak{C}$, then we can define a non constant real boundary function, therefore on F there is a bounded non constant harmonic function. The above condition is only sufficient and not a necessary one. In this meaning topology $\mathfrak{C}$ is also very simple and incomplete but we do not know a more precise topology having an adequate notion of regular point. If we restrict ourselves to the case when all boundary points are pointwise, then our topology is precise.

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