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On an Extension of the Helly's Theorem

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The theorem that we shall prove in this note is an extension of the Helly's theorem¹⁾, which makes it possible to improve the Weierstrass's approximation theorem. The theorem is as follows.

THEOREM. *Let M be a dense and convex set in a normed linear space E , not necessarily complete, and let f_1, f_2, \dots, f_n be n linear functionals over E , Then for any element a in E there exists $b \in M$ in an ε -neighbourhood²⁾ of a so that*

$$f_i(a) = f_i(b), \quad (i = 1, 2, \dots, n).$$

Proof. We may assume that

$$\sum_{i=1}^n \kappa_i f_i(x) \neq \text{const}, \quad (1)$$

unless every κ_i is zero.³⁾

If for $(n+1)$ points x_0, x_1, \dots, x_n arbitrary taken in E , there should exist $(n+1)$ real numbers $\lambda_0, \lambda_1, \dots, \lambda_n$ with

$$\sum_{j=0}^n \lambda_j = 0, \quad \sum_{j=0}^n |\lambda_j| \neq 0 \quad (2)$$

so that

$$\sum_{j=0}^n \lambda_j f_i(x_j) = 0, \quad (i = 1, 2, \dots, n) \quad (3)$$

then

$$\det \begin{vmatrix} f_1(x_0) & \dots & f_1(x_n) \\ \dots & \dots & \dots \\ f_n(x_0) & \dots & f_n(x_n) \\ 1 & \dots & 1 \end{vmatrix} = 0.$$

Therefore there would exist n real numbers κ_i not all of which are zero so that

$$\sum_{i=1}^n \kappa_i f_i(x_j) = 0.$$

This contradicts the hypothesis (1). So there do not exist $(n+1)$ real numbers λ_j ($j = 0, 1, \dots, n$) satisfying the condition (2), (3).

¹⁾ Helly. Monatshefte für Math. und Phys. Bd. 31 (1921).

²⁾ ε -neighbourhood of a means the set $\{x; \|x-a\| < \varepsilon\}$.

³⁾ In this note small greek letters denote real numbers.

Now let us consider the continuous mapping $F(x)$ from E into the n -dimensional euclidean space R^n in such a way that

$$F(x) = (f_1(x), \dots, f_n(x)).$$

Then we may choose $(n+1)$ points a, x_1, \dots, x_n , such that $F(a), F(x_1), F(x_2), \dots, F(x_n)$ are linearly independent in R^n because of that fact above shown. Furthermore we may take $b_j \in M$ ($j = 1, 2, \dots, n$) sufficiently near to x_j so that $F(a), F(b_1), \dots, F(b_n)$ are also linearly independent, and

$$\|b_j - a\| < \varepsilon. \quad (4)$$

Near the point $z = (1+n\delta)a - \delta \sum_{j=1}^n b_j$, where the positive number δ is so small that $\|z-a\| < \varepsilon/2$, we can take b_0 so that

$$\|b_0 - a\| < \varepsilon, \quad (5)$$

and for some μ_j ($1 \leq j \leq n$),

$$F(b_0) = (1+n\delta + \sum_{j=1}^n \mu_j) F(a) - (\sum_{j=1}^n (\delta + \mu_j)) F(b_j). \quad (6)$$

This means

$$f_i(b_0) = (1+n\delta + \sum_{j=1}^n \mu_j) f_i(a) - \sum_{j=1}^n (\delta + \mu_j) f_i(b_j). \quad (i = 1, 2, \dots, n) \quad (7)$$

From (7)

$$f_i(a) = \frac{1}{1+n\delta + \sum_{j=1}^n \mu_j} f_i(b_0) + \sum_{j=1}^n \frac{\delta + \mu_j}{1+n\delta + \sum_{k=1}^n \mu_k} f_i(b_j). \quad (8)$$

Put

$$\left. \begin{aligned} \alpha_0 &= \frac{1}{1+n\delta + \sum_{k=1}^n \mu_k} \\ \alpha_j &= \frac{\delta + \mu_j}{1+n\delta + \sum_{k=1}^n \mu_k} \end{aligned} \right\} \quad (9)$$

and take b_0 so near to z that $|\mu_j| < \delta$, then

$$\alpha_j \geq 0. \quad (j = 0, 1, \dots, n) \quad (10)$$

$$f_i(a) = \sum_{j=0}^n \alpha_j f_i(b_j) = f_i(\sum_{j=0}^n \alpha_j b_j). \quad (11)$$

Put $\bar{b} = \sum_{j=0}^n \alpha_j b_j$ in M . Then

$$f_i(a) = f_i(\bar{b}), \quad (i = 1, 2, \dots, n)$$

and we have from (5) and (11)

$$\|a - \bar{b}\| \leq \sum_{j=0}^m \alpha_j \|a - b_j\| \leq \sum_{j=0}^m \alpha_j \varepsilon = \varepsilon,$$

and thus the theorem is proved.

Now let us take “(C)”, the space of continuous functions defined on $[0, 1]$, in place of E with the norm defined as usual, and P , the aggregate of polynomials, in place of M , then we shall have the theorem:

THEOREM. *For a $g(t) \in (C)$ we can take a polynomial $p(t) \in P$ such that for an arbitrarily small positive ε and for n linear functionals Φ_i ($i = 1, 2, \dots, n$),*

$$\sup_{t \in [0,1]} |g(t) - p(t)| < \varepsilon,$$

and

$$\Phi_i(g(t)) = \Phi_i(p(t)). \quad (i = 1, 2, \dots, n)$$

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