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A NOTE ON THE DIFFERENTIABLE STRUCTURES OF TOTAL SPACES OF SPHERE BUNDLES OVER SPHERES

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1. Introduction

B. Mazur [3] proved that a homotopy equivalence

$$f: M_1 \rightarrow M_2$$

between two compact differentiable n-manifolds without boundary satisfies the relation

$$f^!\{T(M_2)\} = \{T(M_1)\}\ ,$$

if and only if there exists a diffeomorphism F such that the diagram

$$\begin{array}{ccc} M_1 \times R^k & \xrightarrow{F} & M_2 \times R^k \\ \downarrow p_1 & & \downarrow p_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

is homotopy commutative, where $T(M)$ is the tangent vector bundle over M, $\{T(M)\}$ its stable class, p_i the projection mapping to the first factor $(i=1,2)$ and R^k the k-dimensional Euclidean space for $k \geqq n+2$.

In this paper we shall give a sufficient condition to be able to use the Theorem of Mazur, in the case of total spaces of sphere bundles over spheres.

Let $B^{(i,j)}$ be the total space of a S^j-bundle over S^i.

Let θ be the image of the generator of the homotopy group $\pi_i(S^i)$ under the boundary homomorphism in the homotopy exact sequence for this bundle, and λ be the suspension image of θ. We denote by π the projection mapping of this bundle.

By §3. [2] we have the cellular decomposition

$$B^{(i,j)} = S^j \bigcup_{\theta} e^i \bigcup e^{i+j} \tag{1.0}$$

and the commutative diagram

$$\begin{array}{ccc} S^j \bigcup_\theta e^i & \xrightarrow{I} & B^{(i,j)} \\ & \searrow^{J_1} & \downarrow \\ & & S^i \end{array}$$

where I is the inclusion mapping, and J_1 is the restriction of π to the subcomplex $S^j \bigcup_\theta e^i$.

Note that J_1 is just the smashing mapping of the subcomplex S^j to a point.

For the Puppe sequence

$$S^{i-1} \xrightarrow{\theta} S^j \xrightarrow{I_1} S^j \bigcup_\theta e^i \xrightarrow{J_1} S^i \xrightarrow{\lambda} S^{j+1} \cdots,$$

we have the following exact sequence of the stable KO-groups, say $\widetilde{KO}$,

$$\widetilde{KO}(S^{j+1}) \xrightarrow{\lambda^!} \widetilde{KO}(S^i) \xrightarrow{J_1^!} \widetilde{KO}(S^j \bigcup_\theta e^i) \xrightarrow{I_1^!} \widetilde{KO}(S^j) \to \cdots,$$

and further for the Puppe sequence

$$S^i \bigcup_\theta e^i \xrightarrow{I} B^{(i,j)} \xrightarrow{J} S^{i+j} \to \cdots,$$

we have

$$\widetilde{KO}(S^{i+j}) \xrightarrow{J^!} \widetilde{KO}(B^{(i,j)}) \xrightarrow{I^!} \widetilde{KO}(S^j \bigcup_\theta e^i).$$

Then the commutative diagram

$$\begin{array}{ccccccc} \widetilde{KO}(S^{j+1}) & \xrightarrow{\lambda^!} & \widetilde{KO}(S^i) & \xrightarrow{J_1^!} & \widetilde{KO}(S^j \bigcup_\theta e^i) & \xrightarrow{I_1^!} & \widetilde{KO}(S^j) \cdots \\ & & & \searrow^{\pi^!} & \uparrow I^! & & \\ & & & & \widetilde{KO}(B^{(i,j)}) & & \\ & & & & \uparrow J^! & & \\ & & & & \widetilde{KO}(S^{i+j}) & & \end{array} \tag{1.1}$$

is obtained.

Since every j-sphere S^j is stably parallelizable, by (2.1), (2.2) in [4], we have

$$\{T(B^{(i,j)})\} = \pi^!(\xi^{(i,j)}) \tag{1.2}$$

where $\xi^{(i,j)}$ is the stable vector bundle associated with the sphere bundle over the sphere.

2. S^3-bundles over S^4 and S^7-bundles over S^8

Let $B^{(4,3)}_{m,n}$ be the total space of the S^3-bundle over S^4 which has the characteristic map $m\alpha_3+n\beta_3$, where α_3 and β_3 are the generators of $\pi_3(SO(3))$ and $\pi_3(S^3)$ (see 22.6 [5]). Then we have

$$\theta = n\iota_3 \qquad (2.1),$$

and

$$i_*(m\alpha_3+n\beta_3)=(2m+n)\beta \qquad (2.2),$$

where β is the generator of the stable homotopy group $\pi_3(SO(N))$ for sufficiently large N, and i_* is the homomorphism induced by the inclusion mapping.

By (1.0) and (2.1), we have the following cellular decomposition

$$B^{(4,3)}_{m,n}=S^3 \bigcup_{n\iota_3} e^4 \bigcup e^7 \qquad (2.3),$$

where ι_k is the generator of $\pi_k(S^k)$.

By (2.2)

$$\xi^{(4,3)}_{m,n}=(2m+n)g \qquad (2.4)$$

where g is the generator of $\widetilde{KO}(S^4)$.

By (1.1) we have the commutative diagram

$$\begin{array}{ccccccc}
\widetilde{KO}(S^4) & \xrightarrow{(n\iota_4)^!} & \widetilde{KO}(S^4) & \xrightarrow{J_1^!} & \widetilde{KO}(S^3 \bigcup_{n\iota_3} e^4) & \xrightarrow{I_1^!} & \widetilde{KO}(S^3)=0 \\
 & & & \searrow^{\pi^!} & \uparrow{\scriptstyle I^!} & & \\
 & & & & \widetilde{KO}(B^{(4,3)}_{m,n}) & & \\
 & & & & \uparrow{\scriptstyle J^!} & & \\
 & & & & \widetilde{KO}(S^7)=0\,. & &
\end{array}$$

Then we have

Theorem 1. $\widetilde{KO}(B^{(4,3)}_{m,n})\approx Z_n$ (mod n *integer group*).

Using this theorem and (1.2), (2, 4), we have

Corollary. $\{T(B^{(4,3)}_{m,n})\}\equiv 2m\tilde{g}$ mod $n\tilde{g}$,

where we denote by $\tilde{g}$ the generator of $\widetilde{KO}(B^{(4,3)}_{m,n})$.

By Th. 2.2 [6], we have that

if $m\equiv m'$ mod 12, then the total space $B^{(4,3)}_{m,n}$ and $B^{(4,3)}_{m',n}$ have the same fiber homotopy type.

Denote by f the fiber homotopy equivalence.

Then we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{KO}(B^{(4,3)}_{m,n}) & \overset{\pi^!}{\longleftarrow} & \\ \uparrow f^! & & \widetilde{KO}(S^4)\,, \\ \widetilde{KO}(B^{(4,3)}_{m',n}) & \underset{\bar{\pi}^!}{\longleftarrow} & \end{array}$$

where $\bar{\pi}$ is the projection of $B^{(4,3)}_{m',n}$ onto S^4.

Then by the Th. of Mazur and Th. 1, we have

Theorem 2. *If $m \equiv m'$ mod 12 and $2m \equiv 2m'$ mod n, then there exists a diffeomorphism F such that the diagram*

$$\begin{array}{ccc} B^{(4,3)}_{m,n} \times R^k & \overset{F}{\longrightarrow} & B^{(4,3)}_{m',n} \times R^k \\ \downarrow p_1 & & \downarrow p_2 \\ B^{(4,3)}_{m,n} & \overset{f}{\longrightarrow} & B^{(4,3)}_{m',n} \end{array}$$

is homotopy commutative for same $k \geqq 9$.

REMARK. By Th. 3.1 [7], *if $m \equiv m'$* mod n, *then $B^{(4,3)}_{m,n}$ and $B^{(4,3)}_{m',n}$ are homeomorphic.*

By Th. 6.2 [8], *$B^{(4,3)}_{m,1}$ and $B^{(4,3)}_{m',1}$ are diffeomorphic if and only if* $m(m+1) = m'(m'+1)$ mod 56.

It is easily seen that for S^7-bundles over S^8, we have quite similar results.

In the next section we consider S^{4s-1}-bundles over S^{4s} for $s \geqq 3$.

3. S^{4s-1}-bundles over S^{4s} for $s \geqq 3$.

For the canonical fiber bundle

$$SO(4s-1) \to SO(4s) \to S^{4s-1}\,,$$

we have the homotopy exact sequence

$$\pi_{4s}(S^{4s-1}) \overset{\partial}{\to} \pi_{4s-1}(SO(4s-1)) \overset{i_*}{\to} \pi_{4s-1}(SO(4s))$$
$$\overset{p_*}{\to} \pi_{4s-1}(S^{4s-1}) \overset{\partial}{\to} \pi_{4s-2}(SO(4s-1)) \overset{i_*}{\to} \pi_{4s-2}(SO(4s))\,.$$

By [2]

$$\pi_{4s-1}(SO(4s-1)) \approx Z, \quad \pi_{4s-2}(SO(4s-1)) \approx Z_2\,,$$

and it is wellknown that

$$\pi_{4s-1}(S^{4s-1}) \approx Z, \quad \pi_{4s}(S^{4s-1}) \approx Z_2, \quad \pi_{4s-2}(SO(4s)) = 0\,,$$

then we have the isomorphism

$$\pi_{4s-1}(SO(4s)) \approx \pi_{4s-1}(SO(4s-1)) + Z \approx Z + Z .$$

Denote by g the generator of $\pi_{4s-1}(SO(4s-1))$, then $i_*(g)$ can be chosen as one of the generators of $\pi_{4s-1}(SO(4s))$.

Choose another generator, say h, of $\pi_{4s-1}(SO(4s))$, which satisfies the relation

$$p_*(h) = 2\iota_{4s-1} \qquad (3.1),$$

where ι_{4s-1} is the generator of $\pi_{4s-1}(S^{4s-1})$.

Consider the fiber bundle

$$SO(4s-1) \xrightarrow{i_1} SO(4s+1) \xrightarrow{p_1} V_{4s+1,2},$$

and the homotopy exact sequence

$$\begin{aligned} \to \pi_{4s-1}(SO(4s-1)) &\xrightarrow{i_{1*}} \pi_{4s-1}(SO(4s+1)) \\ \xrightarrow{p_{1*}} \pi_{4s-1}(V_{4s+1,2}) &\xrightarrow{\partial_1} \pi_{4s-2}(SO(4s-1)) \\ &\xrightarrow{i_{1*}} \pi_{4s-2}(SO(4s+1)) \to . \end{aligned}$$

By 25.6 [5], we have

$$\pi_{4s-1}(V_{4s+1,2}) \approx Z_2 ,$$

and since $\pi_{4s-2}(SO(4s+1))=0$, then we have the isomorphism

$$\partial_1 : \pi_{4s-1}(V_{4s+1,2}) \xrightarrow{\approx} \pi_{4s-2}(SO(4s-1)) \approx Z_2$$

and

$$i_{1*} : \pi_{4s-1}(SO(4s-1)) \xrightarrow{\approx} \pi_{4s-1}(SO(4s+1)) \approx Z \qquad (3.2).$$

Now consider the S^{4s-1}-bundle over S^{4s} with the characteristic map

$$\chi = mi_*(g) + nh ,$$

By 3.1, we have the relations

$$\theta = 2n\,\iota_{4s-1}, \quad \lambda = 2n\,\iota_{4s} \qquad (3.3).$$

By the diagram

$$
\begin{array}{ccccc}
 & & \pi_{4s}(S^{4s}) & = & \pi_{4s}(S^{4s}) \\
 & & \downarrow \partial & & \downarrow \bar{\partial} \\
\pi_{4s-1}(SO(4s-1)) & \xrightarrow{i_*} & \pi_{4s-1}(SO(4s)) & \xrightarrow{p_*} & \pi_{4s-1}(S^{4s-1}) \\
 & \searrow & \downarrow i'_* & & \downarrow \bar{\iota}_* \\
 & i_{1*} & \pi_{4s-1}(SO(4s+1)) & & \pi_{4s-1}(V_{4s+1,2}) \\
 & & & & \downarrow \bar{p}_* \\
 & & & & \pi_{4s-1}(S^{4s}),
\end{array}
$$

and by 3.2 we have

$$\xi_{m,n}^{(4s,4s-1)} = (m+xn)\tilde{g}_s \tag{3.4}$$

for some integer x, where $\xi_{m,n}^{(4s,4s-1)}$ is the stable vector bundle associated with the sphere bundle, and $\tilde{g}_s$ is the generator of the group $\widetilde{KO}(S^{4s})$.

By (3.3) and § 3. [5], we have the cellular decomposition of the total space of this sphere bundle

$$B_{m,n}^{(4s,4s-1)} = S^{4s-1}{}_{2n\iota_{4s-1}} \cup e^{4s} \cup e^{8s-1} \tag{3.5}$$

As in § 2, by (1.1) we have easily

Theorem 3. $\widetilde{KO}(B_{m,n}^{(4s,4s-1)}) \approx Z_{2n}$.

By (1.2) (3.4) and this theorem, we have

Corollary. $\{T(B_{m,n}^{(4s,4s-1)})\} \equiv (m+xn)\tilde{g}$ mod $2n\tilde{g}$, *where* $\tilde{g}$ *is the generator of* $\widetilde{KO}(B_{m,n}^{(4s,4s-1)})$).

Denote by l the order of $\pi_{8s-2}(S^{4s-1})$, then by the fiber homotopy classification theorem due to Dold (see e.g. Th. 2.1 [6]}, we have

Theorem 4. *If* $m \equiv m'$ *mod* $2n$ *and mod* l, *then there exists a diffeomorphism* F *such that the diagram*

$$
\begin{array}{ccc}
(B_{m,n}^{(4s,4s-1)}) \times R^k & \xrightarrow{F} & (B_{m',n}^{(4s,4s-1)}) \times R^k \\
\downarrow & & \downarrow \\
B_{m,n}^{(4s,4s-1)} & \xrightarrow{f} & B_{m',n}^{(4s,4s-1)}
\end{array}
$$

is homotopy commutative.

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