Takagi, R. and Takeuchi, M.
Osaka J. Math.
14 (1977), 501–518

DEGREE OF SYMMETRIC KÄHLERIAN SUBMANIFOLDS OF A COMPLEX PROJECTIVE SPACE

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(Received June 10, 1976)

Introduction. Let $P_N(c)$ denote the N-dimensional complex projective space $P_N(\boldsymbol{C})$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature $c>0$. For an irreducible symmetric Kählerian manifold M of compact type, Nakagawa-Takagi [5] constructed a series of full equivariant Kählerian imbeddings

$$f_p\colon (M, g_p) \to P_{N_p}(c),$$

parametrized by positive integers p, and observed that the degree $d(f_p)$ of f_p (See §1 for the definition) is given by

$$d(f_p) = rp\,, \quad \text{where} \quad r = \operatorname{rank} M\,,$$

in the case where $p=1$ or M is a complex quadric or a complex Grassmann manifold.

In this note we shall prove the above equality for general symmetric Kählerian submanifolds of $P_N(c)$: Let

$$f_i\colon (M_i, g_i) \to P_{N_i}(c) \qquad (1 \leqslant i \leqslant s)$$

be the p_i-th full Kählerian imbedding of an irreducible symmetric Kählerian manifold M_i of rank r_i $(1 \leqslant i \leqslant s)$. Take the tensor product (See §2 for the definition)

$$f = f_1 \boxtimes \cdots \boxtimes f_s\colon (M_1 \times \cdots \times M_s, g_1 \times \cdots \times g_s) \to P_N(c)$$

of the f_i $(1 \leqslant i \leqslant s)$. Then (Theorem 2) the degree $d(f)$ is given by

$$d(f) = \sum_{i=1}^{s} r_i p_i\,.$$

It should be noted that any full Kählerian immersion f into $P_N(c)$ of a symmetric Kählerian manifold of compact type is obtained in this way.

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*) The second author was supported by Sonderforschungsbereich "Theoretische Mathematik" at Universität Bonn.

1. Degree of Kählerian immersions

Let V be a real vector space of dimension $2n$, equipped with an almost complex structure J and an inner product g satisfying

$$g(Jx, Jy) = g(x, y) \qquad \text{for } x, y \in V .$$

Such a pair (J, g) will be called a *hermitian structure* on V. Denoting complex linear extensions of J and g to the complexification V^C of V by the same J and g respectively, we define subspaces $V^{\pm}$ of V^C and a hermitian inner product $\langle\ ,\ \rangle$ on V^C by

$$V^{\pm} = \{x \in V^C;\ Jx = \pm\sqrt{-1}x\} ,$$
$$\langle x, y\rangle = g(x, \bar{y}) \qquad \text{for } x, y \in V^C ,$$

where $x \mapsto \bar{x}$ denotes the complex conjugation of V^C with respect to V. Then we have $\bar{V}^{\pm} = V^{\mp}$ and

$$V^C = V^+ \oplus V^- \text{ (orthogonal direct sum with respect to } \langle\ ,\ \rangle) .$$

A basis $u = (u_1, \cdots, u_n)$ of V^+ satisfying $\langle u_i, u_j\rangle = \delta_{ij}$ $(1 \leqslant i, j \leqslant n)$ is called a *unitary frame* of V.

Let E be a smooth real vector bundle over a smooth manifold M*) with a smooth assignment $(J, g): p \mapsto (J_p, g_p)$ of hermitian structures on fibres E_p. (J, g) is called a *hermitian structure* on E. Then, getting together the constructions on fibres E_p, we have a hermitian inner product $\langle\ ,\ \rangle$ on the complexifiction E^C of E, and subbundles

$$E^{\pm} = \bigcup_{p \in M} E^{\pm}_p$$

of E^C satisfying

$$E^C = E^+ \oplus E^- \text{ (orthogonal Whitney sum),}$$

and the complex conjugation $E^{\pm} \xrightarrow{-} E^{\mp}$. The map on the space of smooth sections induced from the complex conjugation will be also denoted by

$$C^\infty(E^{\pm}) \xrightarrow{-} C^\infty(E^{\mp}) .$$

Let (M, g) be a Kählerian manifold of $\dim_C M = n$. Then the almost complex structure tensor J and the Kählerian metric g give a hermitian structure on the tangent bundle $T(M)$ of M. Thus we get a hermitian inner product $\langle\ ,\ \rangle$ on the complexification $T(M)^C$ of $T(M)$ and subbundles $T(M)^{\pm}$ of $T(M)^C$ such that

*) In this note, a manifold will be always assumed to be connected.

$$T(M)^C = T(M)^+ \oplus T(M)^- \text{ (orthogonal Whitney sum).}$$

Denote by $U_p(M)$ the totality of unitary frames of $T_p(M)$. Then the union

$$U(M, g) = \bigcup_{p \in M} U_p(M)$$

has a structure of smooth principal bundle over M with the structure group $U(n)$. The Levi-Civita's connection form ω and the canonical form θ of (M, g) will be considered as a $\mathfrak{u}(n)$-valued 1-form and $\boldsymbol{C}^n$-valued 1-form on $U(M, g)$ respectively. ω_B^A $(1 \leqslant A, B \leqslant n)$ and θ^A $(1 \leqslant A \leqslant n)$ denote the components of ω and θ respectively.

Now let (M, g) and (M', g') be Kählerian manifolds of complex dimensions n and N respectively and

$$f\colon (M, g) \to (M', g')$$

be a Kählerian immersion, i.e., a holomorphic isometric immersion of (M, g) into (M', g'). The almost complex structure tensors of M and M' will be denoted by J and J' respectively. The Levi-Civita connections of $T(M)$ and $T(M')$ are denoted by ∇ and ∇' respectively. The induced bundle $f^*T(M')$ over M has a hermitian structure (J', g') induced from the one on $T(M')$. Also it has a connection induced from the Levi-Civita connection on $T(M')$, which will be also denoted by ∇'. If we denote the orthogonal complement of $f_*T_p(M)$ in $T_{f(p)}(M')$ with respect to $g'_{f(p)}$ by $N_p(M)$, the union

$$N(M) = \bigcup_{p \in M} N_p(M)$$

is a subbundle of $f^*T(M')$, having a hermitian structure (J', g') induced from the one on $f^*T(M')$. The hermitian inner products on $T(M)^C$, $T(M')^C$, $f^*T(M')^C$ and $N(M)^C$ will be denoted by the same $\langle\ ,\ \rangle$. We have the following orthogonal Whitney sum decompositions:

$$\begin{aligned} f^*T(M') &= f_*T(M) \oplus N(M)\,, \\ f^*T(M')^C &= f_*T(M)^C \oplus N(M)^C\,, \\ f^*T(M')^{\pm} &= f_*T(M)^{\pm} \oplus N(M)^{\pm}\,, \end{aligned}$$

where the complex linear extension of the differential f_* is denoted by the same f_*. The injections $f_*\colon T(M) \to f^*T(M')$, $f_*\colon T(M)^C \to f^*T(M')^C$ and $f^*\colon T(M)^{\mp} \to f^*T(M')^{\pm}$ preserve the respective inner products. So we shall often identify $T(M)$ etc. with a subbundle of $f^*T(M')$ etc. through the injections f_*. The orthogonal projection $f^*T(M') \to N(M)$ will be denoted by $x \mapsto x^{\perp}$ and the induced projection $C^\infty(f^*T(M')) \to C^\infty(N(M))$ will be also denoted by $\xi \mapsto \xi^{\perp}$. Then the normal connection D on $N(M)$ staisfies

$$D_X \xi = (\nabla'_X \xi)^{\perp} \qquad \text{for } X \in C^\infty(T(M)),\ \xi \in C^\infty(N(M))\,.$$

Now we shall define the *higher fundamental form* H^m of f as a smooth section of the complex vector bundle $\mathrm{Hom}(\otimes^m T(M)^+, N(M)^+)$. In the sequel, for a real linear object, its complex linear extension will be denoted by the same notation. For vector spaces V and W, the space $\mathrm{Hom}(\otimes^m V, W)$ of linear maps from the m-fold tensor product $\otimes^m V$ of V into W will be identified with the space of m-multilinear maps on V into W. Let $h^2 \in C^\infty(\mathrm{Hom}(\otimes^2 T(M), N(M)))$ be the second fundamental form of f, i.e.,

$$h^2(x, y) = (\nabla'_x Y)^\perp \qquad \text{for } x, y \in T_p(M),$$

where Y is a local smooth vector field on M around p such that $Y_p = y$. It is known (cf. Kobayashi-Nomizu [3]) that

$$h^2(x, y) = h^2(y, x), \ h^2(Jx, y) = J'h^2(x, y) \qquad \text{for } x, y \in T_p(M),$$

and hence

$$(1.1) \qquad h^2(T_p(M)^+, T_p(M)^-) = \{0\}, \ h^2(T_p(M)^\pm, T_p(M)^\pm) \subset N_p(M)^\pm .$$

We define $h^m \in C^\infty(\mathrm{Hom}(\otimes^m T(M), N(M)))$ $(m \geqslant 3)$ inductively as follows:

$$(1.2) \qquad \begin{aligned} h^{m+1}(x_1, \cdots, x_m, x_{m+1}) &= D_{x_{m+1}} h^m(X_1, \cdots, X_m) \\ &\quad -\sum_{i=1}^m h^m(x_1, \cdots, \nabla_{x_{m+1}} X_i, \cdots, x_m) \quad \text{for } x_i \in T_p(M), \end{aligned}$$

where the X_i are smooth local vector fields on M around p such that $(X_i)_p = x_i$. Note that (1.1) and (1.2) imply

$$h^m(x_1, \cdots, x_m) \in N_p(M)^+ \qquad \text{for } x_1, x_2 \in T_p(M)^+ \text{ and } x_3, \cdots, x_m \in T_p(M)^C .$$

Now $H^m \in C^\infty(\mathrm{Hom}(\otimes^m T(M)^+, N(M)^+))$ $(m \geqslant 2)$ is defined by

$$H^m(x_1, \cdots, x_m) = h^m(x_1, \cdots, x_m) \qquad \text{for } x_i \in T_p(M)^+ .$$

We write

$$\sum_{m \geqslant 2} h^m \in C^\infty(\mathrm{Hom}(\sum_{m \geqslant 2} \otimes^m T(M), N(M)))$$

and

$$\sum_{m \geqslant 2} H^m \in C^\infty(\mathrm{Hom}(\sum_{m \geqslant 2} \otimes^m (T(M)^+, N(M)^+))$$

by h and H respectively. Note that then we have

$$(1.3) \quad \begin{cases} H(X_1, X_2) = \nabla'_{X_2} X_1 - \nabla_{X_2} X_1 , \\ H(X_1, \cdots, X_m, X_{m+1}) = D_{X_{m+1}} H(X_1, \cdots, X_m) \\ \qquad -\sum_{i=1}^m H(X_1, \cdots, \nabla_{X_{m+1}} X_i, \cdots, X_m) \ (m \geqslant 2) \\ \qquad \text{for } X_i \in C^\infty(T(M)^+) . \end{cases}$$

Making use of the higher fundamental form H we shall define the degree $d(f)$ of the Kählerian immersion f. Let $p\in M$. For a positive integer m, we define a subspace $\mathcal{H}_p^m(M)$ of $T_{f(p)}(M')^+$ to be the subspace spanned by $T_p(M)^+$ and $H(\sum_{2\leqslant k\leqslant m}\otimes^k T_p(M)^+)$. Then we get a series

$$\mathcal{H}_p^1(M)\subset\mathcal{H}_p^2(M)\subset\cdots\subset\mathcal{H}_p^m(M)\subset\mathcal{H}_p^{m+1}(M)\subset\cdots\subset T_{f(p)}(M')^+$$

of increasing subspaces of $T_{f(p)}(M')^+$. We define $O_p^m(M)$ to be the orthogonal complement of $\mathcal{H}_p^{m-1}(M)$ in $\mathcal{H}_p^m(M)$ with respect to $\langle\ ,\ \rangle$, where $\mathcal{H}_p^0(M)$ is understood to be $\{0\}$. Thus we have an orthogonal direct sum:

$$\mathcal{H}_p^m(M)=O_m^1(M)\oplus O_p^2(M)\oplus\cdots\oplus O_p^m(M)\,.$$

For each positive integer m, we define the set $\mathcal{R}_m$ of ***m-regular*** points of M inductively as follows. Define $\mathcal{R}_1=M$. For $m\geqslant 2$, assume $\mathcal{R}_{m-1}$ is already defined. Then we define

$$\mathcal{R}_m=\{p\in\mathcal{R}_{m-1};\ \dim_C\mathcal{H}_p^m(M)=\max_{p'\in\mathcal{R}_{m-1}}\dim_C\mathcal{H}_{p'}^m(M)\}\,.$$

We have inclusions: $\mathcal{R}_1\supset\mathcal{R}_2\supset\cdots\supset\mathcal{R}_m\supset\mathcal{R}_{m+1}\supset\cdots$. Note that each $\mathcal{R}_m$ is an open non-empty subset of M and that

$$\mathcal{H}^m(M)=\bigcup_{p\in\mathcal{R}_m}\mathcal{H}_p^m(M)$$

is a smooth complex vector bundle over $\mathcal{R}_m$ which is a subbundle of $f^*T(M')^+\,|\,\mathcal{R}_m$ for each m.

Lemma 1. *Let $p\in\mathcal{R}_m$, $m\geqslant 1$.*

1) *For each $x\in T_p(M)^+$ and each local smooth section Y of $\mathcal{H}^m(M)$ around p we have*

$$\nabla'_x Y\in\mathcal{H}_p^{m+1}(M)\,.$$

2) *$O_p^{m+1}(M)=\{0\}$ if and only if for each $x\in T_p(M)^+$ and each local smooth section Y of $\mathcal{H}^m(M)$ around p we hvae*

$$\nabla'_x Y\in\mathcal{H}_p^m(M)\,.$$

Proof. Induction on m. Let $x\in T_p(M)^+$ and Y a local smooth section of $\mathcal{H}^1(M)=T(M)^+$ around p. Then by (1.3)

$$\nabla'_x Y\equiv H(Y_p,\,x)\qquad\mathrm{mod}\ \mathcal{H}_p^1(M)\,,$$

which implies the Lemma for $m=1$. Let $m\geqslant 2$ and $x\in T_p(M)^+$. Each local smooth section Y of $\mathcal{H}^m(M)$ around p is written as

$$Y=Z+\Sigma H(X_1,\,\cdots,\,X_m)$$

by a local smooth section Z of $\mathcal{H}^{m-1}(M)\,|\,\mathcal{R}_m$ and local smooth sections X_i of

$T(M)^{+}$around p. From the assumption of the induction, we have $\nabla'_x Z \in \mathcal{H}_p^m(M)$. Further (1.3) implies

$$\nabla'_x H(X_1, \cdots, X_m) \equiv D_x H(X_1, \cdots, X_m) \equiv H((X_1)_p, \cdots, (X_m)_p, x) \qquad \text{mod } \mathcal{H}_p^m(M)\,,$$

and hence

$$\nabla'_x Y \equiv \sum H((X_1)_p, \cdots, (X_m)_p, x) \qquad \text{mod } \mathcal{H}_p^m(M)\,.$$

This implies the Lemma for m. q.e.d.

It follows from Lemma 1, 2) that there exists uniquely a positive integer d such that

$$\begin{cases} O_p^d(M) \neq \{0\} & \text{for some } p \in \mathcal{R}_d\,, \\ O_p^{d+1}(M) = \{0\} & \text{for each } p \in \mathcal{R}_d\,. \end{cases}$$

Such integer d is called the *degree* of the Kählerian immersion f and denoted by $d=d(f)$. We have

$$\mathcal{R}_d = \mathcal{R}_{d+1} = \cdots\,.$$

This open subset $\mathcal{R}_d$ of M will be denoted by $\mathcal{R}$ and called the set of *regular points* of M.

Lemma 2 (Nakagawa-Takagi [5]). *If* $(M', g')=P_N(c)$, *then*:

1) H^m *is symmetric multilinear for each* $m \geqslant 2$;

2) *For each* $u=(u_1, \cdots, u_n) \in U(M, g)$, *we have*

(a) $h(u_i, u_j, \bar{u}_k)=0$,

(b) $$h(u_{i_1}, \cdots, u_{i_m}, \bar{u}_j) = \frac{m-2}{2} c \sum_{r=1}^{m} \delta_{i_r j} H(u_{i_1}, \cdots, \hat{u}_{i_r}, \cdots, u_{i_m})$$
$$- \sum_{r=1}^{m-2} \frac{1}{r!(m-r)!} \sum_{l=1}^{n} \sum_{\sigma} \langle H(u_{i_{\sigma(r+1)}}, \cdots, u_{i_{\sigma(m)}}), H(u_l, u_j) \rangle$$
$$\times H(u_l, u_{i_{\sigma(1)}}, \cdots, u_{i_{\sigma(r)}}) \qquad (m \geqslant 3)\,,$$

where σ *runs through the permutations of* $\{1, 2, \cdots, m\}$.

Lemma 3 (Nakagawa [4]). *Let M be a smooth manifold*, $p_0 \in M$ *and*

$$f\colon M \to P_N(\boldsymbol{C})$$

a smooth immersion. Let $\pi\colon U(P_N(c)) \to P_N(\boldsymbol{C})$ *be the bundle of unitary frames of* $P_N(c)$, θ^A $(1 \leqslant A \leqslant N)$ *and* ω_B^A $(1 \leqslant A, B \leqslant N)$ *be canonical forms and Levi-Civita's connection forms of* $P_N(c)$ *respectively. Then*, $f(M)$ *is contained in an* N'-*dimensional linear subvariety of* $P_N(\boldsymbol{C})$ *if and only if we can find* $u_0 \in U(P_N(c))$ *with* $\pi(u_0)=f(p_0)$ *such that for each smooth curve* $\{p_t\}$ *of* M *through* p_0 *there exists a smooth curve* $\{u_t\}$ *of* $U(P_N(c))$ *through* u_0 *with* $\pi(u_t)=f(p_t)$ *satisfying*

$$\begin{cases} \theta^A(\dot{u}_t) = 0 & (N'+1 \leqslant A \leqslant N), \\ \omega_B^A(\dot{u}_t) = 0 & (N'+1 \leqslant A \leqslant N,\ 1 \leqslant B \leqslant N'). \end{cases}$$

Now we prove the following theorem, giving a geometric interpretation of the degree $d(f)$.

Theorem 1. *Let* $(M', g') = P_N(c)$ *and*

$$f\colon (M, g) \to P_N(c)$$

be a Kählerian immersion. Then the dimension $N'(f)$ of the smallest linear subvariety of $P_N(\boldsymbol{C})$ containing $f(M)$ is given by

$$N'(f) = \operatorname{rank}_C \mathcal{H}^{d(f)}(M).$$

Proof. First we show that for each $x \in T_p(M)$, $p \in \mathcal{R}_m$ $(m \geqslant 1)$ and for each local smooth section Y of $\mathcal{H}^m(M)$ around p, we have

$$\nabla'_x Y \in \mathcal{H}_p^{m+1}(M). \tag{1.4}$$

By virtue of Lemma 1, 1), it suffices to show (1.4) for $x \in T_p(M)^-$. It follows from (1.1) and (1.2) that for each local smooth sections X, X_i of $T(M)^+$ around p we have

$$\begin{cases} \nabla'_{\bar{X}} X_1 = \nabla_{\bar{X}} X_1, \\ D_{\bar{X}} H(X_1, \cdots, X_m) = h(X_1, \cdots, X_m, \bar{X}) \\ \qquad + \sum_{i=1}^{m} H(X_1, \cdots, \nabla_{\bar{X}} X_i, \cdots, X_m) \qquad (m \geqslant 2). \end{cases} \tag{1.5}$$

Here we know that $h(X_1, \cdots, X_m, \bar{X})$ is a local smooth section of $\mathcal{H}^m(M)$ in view of Lemma 2,2), and hence we can prove (1.4) for $x \in T_p(M)^-$ in the same way as Lemma 1.

Take a connected component M_0 of the set $\mathcal{R}$ of regular points and take $p_0 \in M_0$. (1.4) implies

$$\nabla'_x Y \in \mathcal{H}_p^{d(f)}(M) \tag{1.6}$$

for each $x \in T_p(M)$, $p \in M_0$, and for each local smooth section Y of $\mathcal{H}^{d(f)}(M) \mid M_0$ around p. Using the notation in Lemma 3, we choose a unitary frame $u_0 = (u_1(0), \cdots, u_N(0)) \in U(P_N(c))$ with $\pi(u_0) = f(p_0)$ such that $\{u_1(0), \cdots, u_{N'}(0)\}$ spans $\mathcal{H}_{p_0}^{d(f)}(M)$, where $N' = \operatorname{rank}_C \mathcal{H}^{d(f)}(M)$. For each smooth curve $\{p_t\}$ of M_0 through p_0, we can choose a smooth curve $\{u_t = (u_1(t), \cdots, u_N(t))\}$ of $U(P_N(c))$ through u_0 with $\pi(u_t) = f(p_t)$ such that $\{u_1(t), \cdots, u_{N'}(t)\}$ spans $\mathcal{H}_{p_t}^{d(f)}(M)$. This is possible since $\mathcal{H}^{d(f)}(M) \mid M_0$ is a subbundle of $f^*T(M')^+ \mid M_0$. Then (1.6) implies

$$\begin{aligned} \theta^A(\dot{u}_t) &= \langle f_*(\dot{p}_t), u_A(t) \rangle = 0 && (N'+1 \leqslant A \leqslant N), \\ \omega_B^A(\dot{u}_t) &= \langle \nabla'_{f_*(\dot{p}_t)} u_B(t), u_A(t) \rangle = 0 && (N'+1 \leqslant A \leqslant N,\ 1 \leqslant B \leqslant N'). \end{aligned}$$

Thus, by Lemma 3, $f(M_0)$ is contained in an N'-dimensional linear subvariety P of $P_N(\boldsymbol{C})$. From the analyticity of the immersion f, we conclude $f(M)\subset P$, and hence $N'(f)\leqslant N'$.

Assume that $f(M)$ is contained in a linear subvariety P' of $P_N(\boldsymbol{C})$. Since P' is a totally geodesic complex submanifold of $P_N(c)$, we have

$$\mathcal{H}_p^{d(f)}(M)\subset T_{f(p)}(P')^{+} \qquad \text{for } p\in\mathcal{R}\ .$$

This implies $N'\leqslant N'(f)$ and hence $N'=N'(f)$. q.e.d.

2. Symmetric Kählerian submanifolds of $P_N(c)$

A holomorphic immersion f of a complex manifold M into $P_N(\boldsymbol{C})$ is said to be *full* if $f(M)$ is not contained in any proper linear subvariety of $P_N(\boldsymbol{C})$. In this section we recall the construction of full Kählerian imbeddings into $P_N(c)$ of a symmetric Kählerian manifold of compact type. (cf. Borel [1], Takeuchi [6], Nakagawa-Takagi [5])

Let $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ be an irreducible Dynkin diagram and Σ the root system with the fundamental root system Π. Take a lexicographic order $>$ on Σ such that the set of simple roots in Σ with respect to $>$ coincides with Π. Assume that the highest (with respect to $>$) root γ_1 of Σ has the following expression:

$$\gamma_1 = \alpha_1+\sum_{i=2}^{l} m_i\alpha_i\ .$$

Put $\Pi_0=\{\alpha_2, \cdots, \alpha_l\}$ and fix a positive integer p. To the triple $(\Pi, \Pi_0; p)$ we can associate a full Kahlerian imbedding of an irreducible symmetric Kählerian manifold into $P_N(c)$ as follows.

Take a compact simple Lie algebra $\mathfrak{g}$ with the Dynkin diagram Π. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ and denote by $\mathfrak{g}^C$ and $\mathfrak{t}^C$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$ respectively. We identify a weight of $\mathfrak{g}^C$ relative to the Cartan subalgebra $\mathfrak{t}^C$ with an element of $\sqrt{-1}\mathfrak{t}$ by means of the duality defined by the Killing form $(\ ,\ )$ of $\mathfrak{g}^C$. Thus the root system Σ of $\mathfrak{g}^C$ relattive to $\mathfrak{t}^C$ is a subset of $\sqrt{-1}\mathfrak{t}$. Let $\{\Lambda_1, \cdots, \Lambda_l\}$, $\{\varepsilon_1, \cdots, \varepsilon_l\}\subset\sqrt{-1}\mathfrak{t}$ be the fundamental weights of $\mathfrak{g}^C$ and the dual basis for Π respectively:

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}\,, \quad (\alpha_i, \varepsilon_j) = \delta_{ij} \qquad (1\leqslant i, j\leqslant l)\,.$$

Put $\Sigma^+=\{\alpha\in\Sigma;\ \alpha>0\}$, $\Sigma_0=\Sigma\cap\{\Pi_0\}_Z$ and $\Sigma_{\mathfrak{m}}^+=\Sigma^+-\Sigma_0$, where $\{\Pi_0\}_Z$ denotes the subgroup of $\sqrt{-1}\mathfrak{t}$ generated by Π_0. Define subalgebras $\mathfrak{k}^C$, $\mathfrak{m}^+$ and $\mathfrak{u}$ of $\mathfrak{g}^C$ by

$$\mathfrak{k}^C = \mathfrak{t}^C + \sum_{\alpha\in\Sigma_0} \mathfrak{g}_\alpha^C\,, \quad \mathfrak{m}^+ = \sum_{\alpha\in\Sigma_{\mathfrak{m}}^+} \mathfrak{g}_{-\alpha}^C\,,$$

$$\mathfrak{u} = \mathfrak{t}^C + \sum_{\alpha\in\Sigma_0\cup\Sigma_{\mathfrak{m}}^+} \mathfrak{g}_\alpha^C\,,$$

where $\mathfrak{g}_\alpha^C$ denotes the root space of $\mathfrak{g}^C$ for $\alpha\in\Sigma$. Let $\mathfrak{k}=\mathfrak{k}^C\cap\mathfrak{g}$, which is a real form of $\mathfrak{k}^C$, and $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $(\ ,\ )$. Then the automorphism $\theta=\exp \operatorname{ad}\sqrt{-1}\,\varepsilon_1$ of $\mathfrak{g}$ is involutive and gives the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ with

$$\mathfrak{k} = \{X\in\mathfrak{g};\, \theta X = X\}\,, \quad \mathfrak{m} = \{X\in\mathfrak{g};\, \theta X = -X\}\,.$$

G and G^C denote the adjoint groups of $\mathfrak{g}$ and $\mathfrak{g}^C$ respectively, $\tilde{G}$ and $\tilde{G}^C$ the universal covering groups of G and G^C repectively. We may identify as $G\subset G^C$ and $\tilde{G}\subset\tilde{G}^C$. Let K and U denote the (closed) connected subgroups of G^C generated by $\mathfrak{k}$ and $\mathfrak{u}$ respectively. We define a complex manifold M by

$$M = G^C/U\,.$$

Then the natural map $G/K\to G^C/U$ induces the identification $M=G/K$ as smooth manifolds. The tangent space $T_o(M)$ of M at the origin $o=U$ is identified with $\mathfrak{m}$ and $T_o(M)^+$ with $\mathfrak{m}^+$ in the natural way.

Let

$$\tilde{\rho}\colon \tilde{G}\to SU(N+1)$$

be an irreducible unitary representation of $\tilde{G}$ with the highest weight $p\Lambda_1$. By virtue of the irreducibility it induces a homomorphism

$$\rho\colon G\to PU(N+1) = SU(N+1)/\{\varepsilon 1_{N+1};\, \varepsilon^{N+1}=1\}$$

such that the diagram

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{\rho}} & SU(N+1) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ G & \xrightarrow[\rho]{} & PU(N+1) \end{array}$$

is commutative, where the π are respective covering homomorphisms. They are extended holomorphically to $\tilde{G}^C$ and G^C in such a way that the diagram

$$\begin{array}{ccc} \tilde{G}^C & \xrightarrow{\tilde{\rho}} & SL(N+1, \boldsymbol{C}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ G^C & \xrightarrow[\rho]{} & PL(N+1, \boldsymbol{C}) = SL(N+1, \boldsymbol{C})/\{\varepsilon 1_{N+1};\, \varepsilon^{N+1}=1\} \end{array}$$

is commutative, where we have used the same letters for extensions. Let

$P_N(\boldsymbol{C})=\boldsymbol{C}^{N+1}-\{0\}/\boldsymbol{C}^*$ be the complex projective space associated to the representation space $\boldsymbol{C}^{N+1}$ of $\tilde{\rho}$. For $v\in\boldsymbol{C}^{N+1}-\{0\}$, the equivalence class of v will be denoted by $[v]$. Taking a highest weight vector $v_0\in\boldsymbol{C}^{N+1}-\{0\}$, we can define a full holomorphic imbedding $f\colon M\to P_N(\boldsymbol{C})$ by

$$f(xo)=\rho(x)[v_0] \qquad \text{for } x\in G^C.$$

We take the $SU(N+1)$-invariant Fubini-Study metric on $P_N(\boldsymbol{C})$ of constant holomorphic sectional curvature c and introduce a Kählerian metric g on M in such a way that

$$f\colon (M, g)\to P_N(c)$$

becomes a Kählerian imbedding. Then (M, g) is an irreducible symmetic Kählerian manifold of compact type. If we denote the group of Kählerian automorphisms of (M, g) and the one of holomorphisms of M by $\mathrm{Aut}(M, g)$ and $\mathrm{Aut}(M)$ respectively, the identity-components $\mathrm{Aut}^0(M, g)$ and $\mathrm{Aut}^0(M)$ are identified with G and G^C respectively. Further f is G^C-equivariant by the homomorphism ρ:

$$f(xp)=\rho(x)f(p) \qquad \text{for } x\in G^C,\ p\in M.$$

where $\rho(G)\subset PU(N+1)=\mathrm{Aut}(P_N(c))$.

Put

$$\kappa(M)=\#\{\alpha\in\Sigma_{\mathfrak{m}}^+;\ \alpha-\alpha_1\in\Sigma\}+2.$$

Then (Nakagawa-Takagi [5]) the scalar curvature k of (M, g) is given by

$$k=\frac{(\dim_C M)c\kappa(M)}{p}, \tag{2.1}$$

which gives a geometric characterization of the positive integer p. It is also characterized (Nakagawa- Takagi [5]) by

$$g=\frac{p(\alpha_1, \alpha_1)}{c}g_0, \tag{2.2}$$

where g_0 is a G-invariant Kählerian metric on M defined from the inner product $-(\ ,\ )$ on $\mathfrak{g}$. The imbedding f will be called the *p-th full Kählerian imbedding* of M.

Now we shall construct a full Kählerian imbedding of a general (not necessarily irreducible) symmetric Kählerian manifold into $P_N(c)$. For complex projective spaces $P_{N_1}(\boldsymbol{C})$ and $P_{N_2}(\boldsymbol{C})$ associated to $\boldsymbol{C}^{N_1+1}$ and $\boldsymbol{C}^{N_2+1}$ respectively, we define a holomorphic imbedding ι of $P_{N_1}(\boldsymbol{C})\times P_{N_2}(\boldsymbol{C})$ into the complex projective space $P_{N_1N_2+N_1+N_2}(\boldsymbol{C})$ associated to the tensor product $\boldsymbol{C}^{N_1+1}\otimes\boldsymbol{C}^{N_2+1}$ by

$$\iota\colon [z_i]_{0\leq i\leq N_1}\times[w_j]_{0\leq j\leq N_2}\mapsto[z_iw_j]_{\substack{0\leq i\leq N_1\\0\leq j\leq N_2}},$$

where [*] denotes the point of the projective space with homogeneous coordinates *. Then it defines a full Kählerian imbedding

$$\iota\colon P_{N_1}(c)\times P_{N_2}(c)\to P_{N_1N_2+N_1+N_2}(c)\,.$$

Let

$$f_i\colon (M_i, g_i)\to P_{N_i}(c)\qquad (i=1,2)$$

be two Kählerian immersions. Then the composite

$$f_1\boxtimes f_2=\iota\circ(f_1\times f_2)\colon (M_1\times M_2, g_1\times g_2)\to P_{N_1N_2+N_1+N_2}(c)$$

is also a Kählerian immersion, which will be called the *tensor product* of f_1 and f_2. One can easily check the associativity

$$(f_1\boxtimes f_2)\boxtimes f_3=f_1\boxtimes(f_2\boxtimes f_3)$$

of the tensor product, and so the multi-fold tensor product $f_1\boxtimes\cdots\boxtimes f_s$ is well-defined.

Now let

$$f_i\colon (M_i, g_i)\to P_{N_i}(c)\qquad (1\leq i\leq s)$$

be full Kählerian imbeddings of irreducible symmetric Kählerian manifolds of compact type constructed as before. Then the tensor product

$$f=f_1\boxtimes\cdots\boxtimes f_s\colon (M_1\times\cdots\times M_s, g_1\times\cdots\times g_s)\to P_N(c)\,,$$

where $N=\prod_{t=1}^{s}(N_t+1)-1$, is a full Kählerian imbedding of the symmetric Kählerian manifold $(M, g)=(M_1\times\cdots\times M_s, g_1\times\cdots\times g_s)$. Note that

$$G^C=G_1^C\times\cdots\times G_s^C\,,\quad G=G_1\times\cdots\times G_s\,,$$

where $G^C=\mathrm{Aut}^0(M)$, $G=\mathrm{Aut}^0(M,g)$, $G_i^C=\mathrm{Aut}^0(M_i)$, $G_i=\mathrm{Aut}^0(M_i,g_i)$, and that f is G^C-equivariant by the homomorphism $\rho=\rho_1\boxtimes\cdots\boxtimes\rho_s$ induced from the external tensor product $\tilde\rho_1\boxtimes\cdots\boxtimes\tilde\rho_s$ of respective representations $\tilde\rho_i$. The tangent space $T_o(M)$ of M at the origin $o=o_1\times\cdots\times o_s$ of M, where o_i is the origin of M_i, is identified with the direct sum

$$\mathfrak{m}=\mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_s$$

of respective complements $\mathfrak{m}_i$, and hence $T_o(M)^+$ with

$$\mathfrak{m}^+=\mathfrak{m}_1^+\oplus\cdots\oplus\mathfrak{m}_s^+\,. \tag{2.3}$$

Further the stabilizer K of the origin o in G is the direct product

$$K = K_1 \times \cdots \times K_s \tag{2.4}$$

of respective stabilizers K_i.

It is known (Nakagawa-Takagi [5]. See also Takeuchi [8]) that any full Kählerian immersion into $P_N(c)$ of a symmetric Kählerian manifold of compact type is obtained in this way.

3. Degree of symmetric Kählerian submanifolds of $P_N(c)$

Let

$$f\colon (M, g) \to P_N(c)$$

be the p-th full Kählerian imbedding of an irreducible symmetric Kählerian manifold (M, g) constructed in §2. We recall first the construction of the Hermann map for M (cf. Takeuchi [7]). Choose root vector $E_\alpha \in \mathfrak{g}^C_\alpha$ for $\alpha \in \Sigma$ in such a way that

$$[E_\alpha, E_{-\alpha}] = -\alpha\,, \quad (E_\alpha, E_{-\alpha}) = -1\,.$$

Then the complex conjugation $X \mapsto \bar{X}$ of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$ satisfies $\bar{E}_\alpha = E_{-\alpha}$ for each $\alpha \in \Sigma$. We put

$$X_\alpha = \sqrt{\frac{2}{(\alpha, \alpha)}} E_\alpha\,, \quad H_\alpha = \frac{2}{(\alpha, \alpha)}\alpha \qquad \text{for } \alpha \in \Sigma\,.$$

Then we have

$$[X_\alpha, X_{-\alpha}] = -H_\alpha\,, \quad (X_\alpha, X_{-\alpha}) = -\frac{2}{(\alpha, \alpha)}\,, \quad \bar{X}_\alpha = X_{-\alpha}\,.$$

Let $\{\gamma_1, \cdots, \gamma_r\} \subset \Sigma^+_{\mathfrak{m}}$ be a maximal system of strongly orthogonal roots containing the highest root γ_1 such that $r = \operatorname{rank} M$ and $(\gamma_j, \gamma_j) = (\alpha_1, \alpha_1)$ for each j (cf. Helgason [2]). An injective homomorphism $\phi_j\colon \mathfrak{sl}(2, \boldsymbol{C}) \to \mathfrak{g}^C$ is defined by

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_{\gamma_j}\,, \quad X^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \mapsto X_{-\gamma_j}\,,$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto -H_{\gamma_j}\,.$$

Since $\phi_j(-{}^t\bar{X}) = \overline{\phi_j(X)}$ for $X \in \mathfrak{sl}(2, \boldsymbol{C})$, we have $\phi_j(\mathfrak{su}(2)) \subset \mathfrak{g}$. If we define a map ϕ from the r-fold direct sum $\mathfrak{sl}(2, \boldsymbol{C})^r$ of $\mathfrak{sl}(2, \boldsymbol{C})$ into $\mathfrak{g}^C$ by

$$\phi(X_1, \cdots, X_r) = \sum_{j=1}^r \phi_j(X_j) \qquad \text{for } X_j \in \mathfrak{sl}(2, \boldsymbol{C})\,,$$

then it is also an injective homomorphism such that $\phi(\mathfrak{su}(2)^r) \subset \mathfrak{g}$. The extension of ϕ to the r-fold direct product $SL(2, \boldsymbol{C})^r$ of $SL(2, \boldsymbol{C})$ is also denoted by

$$\phi\colon SL(2, \boldsymbol{C})^r \to G^C.$$

It satisfies $\phi(SU(2)^r)\subset G$. Putting

$$SL(1, 1; \boldsymbol{C}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{C});\ c = 0\right\},$$

we identify the r-fold direct product $P_1(\boldsymbol{C})^r$ of $P_1(\boldsymbol{C})$ with $SL(2, \boldsymbol{C})^r/SL(1, 1; \boldsymbol{C})^r$. Then the map

$$x\,SL(1, 1; \boldsymbol{C})^r \mapsto \phi(x)o \qquad \text{for } x\in SL(2, \boldsymbol{C})^r$$

defines a holomorphic imbedding

$$\phi\colon P_1(\boldsymbol{C})^r \to M,$$

which is $SL(2, \boldsymbol{C})^r$-equivariant:

$$\phi(xp) = \phi(x)\phi(p) \qquad \text{for } x\in SL(2, \boldsymbol{C})^r,\ p\in P_1(\boldsymbol{C})^r.$$

The imbedding ϕ is called the *Hermann map*. The Kählerian metric h on $P_1(\boldsymbol{C})^r$ induced from (M, g) is the direct product $h_1\times\cdots\times h_r$ of Kählerian metrics h_j on $P_1(\boldsymbol{C})$ of constant holomorphic sectional curvatures, since $SU(2)^r$ acts transitively on $P_1(\boldsymbol{C})^r$ as Kählerian automorphisms of $(P_1(\boldsymbol{C})^r, h)$. The tangent space $T_o(\phi(P_1(\boldsymbol{C})^r))$ will be identified with a subspace $\mathfrak{p}$ of $\mathfrak{m}$, and hence $T_o(\phi(P_1(\boldsymbol{C})^r))^+$ with a subspace $\mathfrak{p}^+$ of $\mathfrak{m}^+$.

Lemma 4. *Let*

$$\phi\colon (P_1(\boldsymbol{C})^r, h_1\times\cdots\times h_r) \to (M, g)$$

be the Hermann map as above. Then:

1) $\mathfrak{m}^+=K\mathfrak{p}^+$;

2) *ϕ is totally geodesic*;

3) *Each h_j has the holomorphic sectional curvature $\dfrac{c}{p}$.*

Proof. 1) If we put

$$U_{\gamma_j} = E_{\gamma_j}+E_{-\gamma_j},\quad V_{\gamma_j} = \sqrt{-1}(E_{\gamma_j}-E_{-\gamma_j}) \qquad (1\leqslant j\leqslant r),$$

$\mathfrak{p}$ is spanned over $\boldsymbol{R}$ by the U_{γ_j}, V_{γ_j} $(1\leqslant j\leqslant r)$. The subspace $\mathfrak{a}$ of $\mathfrak{m}$ spanned over $\boldsymbol{R}$ by the U_{γ_j} $(1\leqslant j\leqslant r)$ is a maximal abelian subalgebra in $\mathfrak{m}$, and hence $\mathfrak{m}=K\mathfrak{a}$. Since the projection $\varpi\colon \mathfrak{m}^C\to\mathfrak{m}^+$ relative to the decomposition $\mathfrak{m}^C=\mathfrak{m}^+\oplus\overline{\mathfrak{m}}^+$ is K-equivariant, we have $\mathfrak{m}^+=K\varpi^+(\mathfrak{a})$. But $\varpi^+(\mathfrak{a})$ is spanned over $\boldsymbol{R}$ by the $E_{-\gamma_j}$ $(1\leqslant j\leqslant r)$ and hence is contained in $\mathfrak{p}^+=\varpi^+(\mathfrak{p})$, which is spanned over $\boldsymbol{C}$ by the $E_{-\gamma_j}$ $(1\leqslant j\leqslant r)$. Thus we conclude $\mathfrak{m}^+=K\mathfrak{p}^+$.

2) From the relations

$$[U_{\gamma_j}, V_{\gamma_j}] = 2\sqrt{-1}\gamma_j,\ [\sqrt{-1}\gamma_j, U_{\gamma_j}] = V_{\gamma_j},\ [\sqrt{-1}\gamma_j, V_{\gamma_j}] = -U_{\gamma_j},$$

we get $[[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}]=\mathfrak{p}$, and hence ϕ is totally geodesic (cf. Helgason [2]).

3) Identifying $X^+ + X^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with a tangent vector of $P_1(\boldsymbol{C})$ at the origin, we have by (2.2)

$$\begin{aligned} h_j(X^+ + X^-, X^+ + X^-) &= g(X_{\gamma_j} + X_{-\gamma_j}, X_{\gamma_j} + X_{-\gamma_j}) \\ &= -\frac{p(\alpha_1, \alpha_1)}{c}(X_{\gamma_j} + X_{-\gamma_j}, X_{\gamma_j} + X_{-\gamma_j}) \\ &= -\frac{2p(\gamma_j, \gamma_j)}{c}(X_{\gamma_j}, X_{-\gamma_j}) = \frac{2p(\gamma_j, \gamma_j)}{c}\frac{2}{(\gamma_j, \gamma_j)} \\ &= p\frac{4}{c}. \end{aligned}$$

It follows that h_j is p times the Fubini-Study metric of $P_1(c)$, which implies the assertion 3). q.e.d.

Now we shall prove the following

Theorem 2. *Let*

$$f_i\colon (M_i, g_i) \to P_{N_i}(c) \qquad (1 \leqslant i \leqslant s)$$

be the p_i-th full Kählerian imbedding of an irreducible symmetric Kählerian manifold (M_i, g_i) of compact type, with rank $M_i = r_i$ $(1 \leqslant i \leqslant s)$, *and*

$$f\colon (M, g) \to P_N(c)$$

be the tensor product of the f_i $(1 \leqslant i \leqslant s)$. Then the degree $d(f)$ of f is given by

$$d(f) = \sum_{i=1}^{s} r_i p_i .$$

For the proof of the Theorem we need the following Lemma.

Lemma 5 (Nakagawa-Takagi [5]). *Let*

$$f\colon (M, g) \to P_N(c)$$

be a Kählerian immersion of a locally symmetric Kählerian manifold (M, g). Then:

1) $\langle H(\otimes^m T_p(M)^+), H(\otimes^{m'} T_p(M)^+)\rangle = \{0\}$ *for $m \neq m'$, and hence $O_p^m(M) = H(\otimes^m T_p(M)^+)$ for each m;*

2) *For each $u=(u_1, \cdots, u_n) \in U(M, g)$,*

$$\begin{aligned} h(u_{i_1}, \cdots, u_{i_m}, \bar{u}_j) = &-\frac{c}{2}\sum_{r=1}^{m} \delta_{i_r j} H(u_{i_1}, \cdots, \hat{u}_{i_r}, \cdots, u_{i_m}) \\ &+ \sum_{\substack{1 \leqslant a < b \leqslant m \\ 1 \leqslant k \leqslant n}} \langle R(u_{i_b}, \bar{u}_j)u_{i_a}, u_k\rangle H(u_k, u_{i_1}, \cdots, \hat{u}_{i_a}, \cdots, \hat{u}_{i_b}, \cdots, u_{i_m}) \\ &\hspace{20em} (m \geqslant 3), \end{aligned}$$

where R is the curvature tensor of (M, g).

Proof of Theorem 2. Let $r=r_1+\cdots+r_s$ be the rank of M. We use the notation in the end of §2. Taking the direct product of respective homomorphisms ϕ_i: $SL(2, \boldsymbol{C})^{r_i}\to G_i^C$ for M_i $(1\leqslant i\leqslant s)$ and the one of Hermann maps ϕ_i: $P_1\left(\frac{c}{p_i}\right)^{r_i}\to(M_i, g_i)$ $(1\leqslant i\leqslant s)$, we get a homomrphism ϕ: $SL(2, \boldsymbol{C})^r\to G^C$ such that $\phi(SU(2)^r)\subset G$ and a totally geodesic Kählerian imbedding

$$\phi\colon P=P_1\left(\frac{c}{p_1}\right)^{r_1}\times\cdots\times P_1\left(\frac{c}{p_s}\right)^{r_s}\to(M, g)\,,$$

which is $SL(2, \boldsymbol{C})^r$-equivarient:

$$\phi(xp)=\phi(x)\phi(p) \qquad \text{for } x\in SL(2, \boldsymbol{C})^r\,,\ p\in P\,.$$

The tangent space $\mathfrak{p}=T_o(\phi(P))$ of $\phi(P)$ at the origin is the direct sum

$$\mathfrak{p}=\mathfrak{p}_1\oplus\cdots\oplus\mathfrak{p}_s$$

of respective tangent spaces $\mathfrak{p}_i$ of $\phi_i\left(P_1\left(\frac{c}{p_i}\right)^{r_i}\right)$ at the origin, and hence

$$\mathfrak{p}^+=\mathfrak{p}_1^+\oplus\cdots\oplus\mathfrak{p}_s^+\,.$$

It follows from Lemma 4 and decompositions (2.3), (2.4) that

$$\mathfrak{m}^+=K\mathfrak{p}^+\,. \tag{3.1}$$

Let us consider a Kählerian imbedding

$$f'=f\circ\phi\colon P\to P_N(c)\,.$$

If we put

$$\rho'=\rho\circ\phi\colon SL(2, \boldsymbol{C})^r\to PL(N+1, \boldsymbol{C})\,,$$

f' is $SL(2, \boldsymbol{C})^r$-equivariant by the homomorphism ρ':

$$f'(xp)=\rho'(x)f'(p) \qquad \text{for } x\in SL(2, \boldsymbol{C})^r,\ p\in P\,,$$

Note that $\rho'(SU(2)^r)\subset PU(N+1)=\mathrm{Aut}(P_N(c))$ and $SU(2)^r$ acts transitively on P as Kählerian automorphisms of P. We shall identify as $P\subset(M, g)$ through the imbedding ϕ. Denote the higher fundamental forms of f and f' by H and H' respectively. We shall prove the following two assertions:

(i) $d(f)=d(f')$.

(ii) $d(f')=\sum_{i=1}^{s} r_i p_i$.

But in view of the $\mathrm{Aut}^0(M, g)$-equivariance of f and Lemma 5, 1), we know that each point of M is regular and $d(f)$ is determined by conditions

$$H_o^{d(f)} \neq 0 \quad \text{and} \quad H_o^{d(f)+1} = 0 .$$

In the same way, $d(f')$ is determined by conditions

$$H'^{d(f')}_o \neq 0 \quad \text{and} \quad H'^{d(f')+1}_o = 0 .$$

Here H_0^1 and H'^1_0 are understood to be always not 0. Hence the assertion (i) is equivalent to the assertion

(i)′ $H_o^m = 0 \Leftrightarrow H'^m_o = 0 \qquad (m \geqslant 2)$.

Proof of (i)′. Note first that if we denote by $X \mapsto kX$ the action of $k \in K$ on $N_o(M)$ through the differential $\rho(k)_*$, we have

$$H(kX_1, \cdots, kX_m) = kH(X_1, \cdots, X_m) \qquad \text{for } X_i \in \mathfrak{m}^+,\ k \in K , \tag{3.2}$$

because of the $\mathrm{Aut}^0(M, g)$-equivarience of f. Now

$$\begin{aligned}
& H_o^m = 0 \\
\Leftrightarrow\ & H(X_1, \cdots, X_m) = 0 && \text{for each } X_i \in \mathfrak{m}^+ , \\
\Leftrightarrow\ & H(\underbrace{X, \cdots, X}_{m}) = 0 && \text{for each } X \in \mathfrak{m}^+ \quad \text{by Lemma 2,1)}, \\
\Leftrightarrow\ & H(\underbrace{Y, \cdots, Y}_{m}) = 0 && \text{for each } Y \in \mathfrak{p}^+ \quad \text{by (3.1), (3.2)}, \\
\Leftrightarrow\ & H'(\underbrace{Y, \cdots, Y}_{m}) = 0 && \text{for each } Y \in \mathfrak{p}^+ \quad \text{since } \phi \text{ is totally geodesic}, \\
\Leftrightarrow\ & H'(Y_1, \cdots, Y_m) = 0 && \text{for each } Y_i \in \mathfrak{p}^+ \quad \text{by Lemma 2,1)} \\
\Leftrightarrow\ & H'^m_o = 0 .
\end{aligned}$$

Proof of (ii). For an index j, $1 \leqslant j \leqslant r$, we define $\nu(j)$, $1 \leqslant \nu(j) \leqslant s$, by

$$\nu(j) = \nu \quad \text{if} \quad r_1 + \cdots + r_{\nu-1} + 1 \leqslant j \leqslant r_1 + \cdots + r_{\nu-1} + r_\nu .$$

Take a unitary frame $u = (u_1, \cdots, u_r)$ of P at the origin o such that u_i is tangent to the i-th factor of P for each i, and fix it once for all. Then the curvature tensor R of P satisfies

$$\langle R(u_k, \bar{u}_l)u_j, u_i \rangle = \frac{c}{p_{\nu(i)}} \delta_{ij} \delta_{jk} \delta_{kl} . \tag{3.3}$$

For each $i_1, \cdots, i_m, j$ $(m \geqslant 2)$, the following equality holds:

$$h'(u_{i_1}, \cdots, u_{i_m}, u_j, \bar{u}_j) = \frac{c(a_j + 1)}{2p_{\nu(j)}} (a_j - p_{\nu(j)}) H'(u_{i_1}, \cdots, u_{i_m}) , \tag{3.4}$$

where a_j is an integer given by

$$a_j = \#\{k;\ 1 \leqslant k \leqslant m,\ i_k = j\}\ .$$

Indeed, Lemma 5,2) and (3.3) imply

$$h'(u_{i_1}, \cdots, u_{i_{m+1}}, \bar{u}_j) = -\frac{c}{2}\sum_{t=1}^{m+1} \delta_{i_t j} H'(u_{i_1}, \cdots, \hat{u}_{i_t}, \cdots, u_{i_{m+1}})$$
$$+\frac{c}{p_{\nu(j)}} \sum_{1 \leqslant a < b \leqslant m+1} \delta_{i_a j}\delta_{i_b j} H'(u_{i_1}, \cdots, \hat{u}_{i_a}, \cdots, \hat{u}_{i_b}, \cdots, u_{i_{m+1}}, u_j)\ .$$

Put $i_{m+1}=j$. Recalling that H' is symmetric, we have

$$\begin{aligned} &h'(u_{i_1}, \cdots, u_{i_m}, u_j, \bar{u}_j) \\ &= -\frac{c}{2}\sum_{t=1}^{m} \delta_{i_t j} H'(u_{i_1}, \cdots, \hat{u}_{i_t}, \cdots, u_{i_m}, u_j) - \frac{c}{2} H'(u_{i_1}, \cdots, u_{i_m}) \\ &\quad + \frac{c}{p_{\nu(j)}} \sum_{1 \leqslant a < b \leqslant m} \delta_{i_a j}\delta_{i_b j} H'(u_{i_1}, \cdots, \hat{u}_{i_a}, \cdots, \hat{u}_{i_b}, \cdots, u_{i_m}, u_j, u_j) \\ &\quad + \frac{c}{p_{\nu(j)}} \sum_{t=1}^{m} \delta_{i_t j} H'(u_{i_1}, \cdots, \hat{u}_{i_t}, \cdots, u_{i_m}, u_j) \\ &= \left\{-\frac{c}{2}a_j - \frac{c}{2} + \frac{c}{p_{\nu(j)}} \cdot \frac{a_j(a_j-1)}{2} + \frac{c}{p_{\nu(j)}} a_j\right\} H'(u_{i_1}, \cdots, u_{i_m}) \\ &= \frac{c(a_j+1)}{2p_{\nu(j)}} (a_j - p_{\nu(j)}) H'(u_{i_1}, \cdots, u_{i_m})\ . \end{aligned}$$

Now we are in a position to prove (ii). If $d'=d(f')=1$, then f' is totally geodesic, and hence $s=1$, $r_1=1$, $p_1=1$. Thus $\sum_{\iota=1}^{s} r_\iota p_\iota = 1$. So we may assume $d' \geqslant 2$. Then there exist indices $i_1, \cdots, i_{d'}$ such that $H'(u_{i_1}, \cdots, u_{i_{d'}}) \neq 0$. It follows from (1.5) and $H'^{d'+1}=0$ that

$$h'(u_{i_1}, \cdots, u_{i_{d'}}, u_j, \bar{u}_j) = 0 \qquad \text{for each } j,\ 1 \leqslant j \leqslant r\ .$$

Thus (3.4) implies

$$\#\{k;\ 1 \leqslant k \leqslant d',\ i_k = j\} = p_{\nu(j)} \qquad \text{for each } j,\ 1 \leqslant j \leqslant r\ ,$$

and hence

$$d' = \sum_{j=1}^{r} p_{\nu(j)} = \sum_{\iota=1}^{s} r_\iota p_\iota\ . \qquad \text{q.e.d.}$$

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