<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Degree of symmetric Kählerian submanifolds of a complex projective space</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Takagi, Ryoichi; Takeuchi, Masaru</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 14(3) P.501–P.518</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1977</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/8612">https://doi.org/10.18910/8612</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/8612</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Note</strong></td>
<td></td>
</tr>
</tbody>
</table>
Introduction. Let $P_N(c)$ denote the $N$-dimensional complex projective space $P_N(C)$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature $c > 0$. For an irreducible symmetric Kahlerian manifold $M$ of compact type, Nakagawa-Takagi [5] constructed a series of full equivariant Kahlerian imbeddings

$$f_p : (M, g_p) \to P_{N_p}(c),$$

parametrized by positive integers $p$, and observed that the degree $d(f_p)$ of $f_p$ (See §1 for the definition) is given by

$$d(f_p) = r_p,$$

where $r = \text{rank } M$,

in the case where $p = 1$ or $M$ is a complex quadric or a complex Grassmann manifold.

In this note we shall prove the above equality for general symmetric Kahlerian submanifolds of $P_N(c)$: Let

$$f_i : (M_i, g_i) \to P_{N_i}(c) \quad (1 \leq i \leq s)$$

be the $p_i$-th full Kahlerian imbedding of an irreducible symmetric Kahlerian manifold $M_i$ of rank $r_i$ ($1 \leq i \leq s$). Take the tensor product (See §2 for the definition)

$$f = f_1 \otimes \cdots \otimes f_s : (M_1 \times \cdots \times M_s, g_1 \times \cdots \times g_s) \to P_N(c)$$

of the $f_i$ ($1 \leq i \leq s$). Then (Theorem 2) the degree $d(f)$ is given by

$$d(f) = \sum_{i=1}^{s} r_i p_i.$$

It should be noted that any full Kahlerian immersion $f$ into $P_N(c)$ of a symmetric Kahlerian manifold of compact type is obtained in this way.

*) The second author was supported by Sonderforschungsbereich “Theoretische Mathematik” at Universität Bonn.
1. Degree of Kählerian immersions

Let $V$ be a real vector space of dimension $2n$, equipped with an almost complex structure $J$ and an inner product $g$ satisfying

$$g(Jx, Jy) = g(x, y) \quad \text{for } x, y \in V.$$ 

Such a pair $(J, g)$ will be called a hermitian structure on $V$. Denoting complex linear extensions of $J$ and $g$ to the complexification $V^c$ of $V$ by the same $J$ and $g$ respectively, we define subspaces $V^\pm$ of $V^c$ and a hermitian inner product $\langle \ , \rangle$ on $V^c$ by

$$V^\pm = \{x \in V^c; Jx = \pm \sqrt{-1} x\},$$

$$\langle x, y \rangle = g(x, y) \quad \text{for } x, y \in V^c,$$

where $x \mapsto \bar{x}$ denotes the complex conjugation of $V^c$ with respect to $V$. Then we have $V^\pm = V^c$ and

$$V^c = V^+ \oplus V^- \quad (\text{orthogonal direct sum with respect to } \langle \ , \rangle).$$

A basis $u=(u_1, \cdots, u_n)$ of $V^+$ satisfying $\langle u_i, u_j \rangle = \delta_{ij} \ (1 \leq i, j \leq n)$ is called a unitary frame of $V$.

Let $E$ be a smooth real vector bundle over a smooth manifold $M^*$ with a smooth assignment $(J, g): p \mapsto (J_p, g_p)$ of hermitian structures on fibres $E_p$. $(J, g)$ is called a hermitian structure on $E$. Then, getting together the constructions on fibres $E_p$, we have a hermitian inner product $\langle \ , \rangle$ on the complexification $E^c$ of $E$, and subbundles

$$E^\pm = \bigcup_{p \in M} E^\pm_p,$$

of $E^c$ satisfying

$$E^c = E^+ \oplus E^- \quad (\text{orthogonal Whitney sum}),$$

and the complex conjugation $E^\star \mapsto E^{\star \pm}$. The map on the space of smooth sections induced from the complex conjugation will be also denoted by

$$C^\infty(E^\pm) \mapsto C^\infty(E^{\star \pm}).$$

Let $(M, g)$ be a Kählerian manifold of $\dim \mathcal{C} M = n$. Then the almost complex structure tensor $J$ and the Kählerian metric $g$ give a hermitian structure on the tangent bundle $T(M)$ of $M$. Thus we get a hermitian inner product $\langle \ , \rangle$ on the complexification $T(M)^c$ of $T(M)$ and subbundles $T(M)^\pm$ of $T(M)^c$ such that

\* In this note, a manifold will be always assumed to be connected.
\[ T(M)^c = T(M)^+ \oplus T(M)^- \] (orthogonal Whitney sum).

Denote by \( U_p(M) \) the totality of unitary frames of \( T_p(M) \). Then the union

\[ U(M, g) = \bigcup_{p \in M} U_p(M) \]

has a structure of smooth principal bundle over \( M \) with the structure group \( U(n) \). The Levi-Civita’s connection form \( \omega \) and the canonical form \( \theta \) of \( (M, g) \) will be considered as a \( \mathfrak{u}(n) \)-valued 1-form and \( \mathfrak{c}^n \)-valued 1-form on \( U(M, g) \) respectively. \( \omega_A^B (1 \leq A, B \leq n) \) and \( \theta^A (1 \leq A \leq n) \) denote the components of \( \omega \) and \( \theta \) respectively.

Now let \((M, g)\) and \((M', g')\) be Kahlerian manifolds of complex dimensions \( n \) and \( N \) respectively and be a Kahlerian immersion, i.e., a holomorphic isometric immersion of \((M, g)\) into \((M', g')\). The almost complex structure tensors of \( M \) and \( M' \) will be denoted by \( J \) and \( J' \) respectively. The Levi-Civita connections of \( T(M) \) and \( T(M') \) are denoted by \( \nabla \) and \( \nabla' \) respectively. The induced bundle \( f^*T(M') \) over \( M \) has a hermitian structure \((J', g')\) induced from the one on \( T(M') \). Also it has a connection induced from the Levi-Civita connection on \( T(M') \), which will be also denoted by \( \nabla' \). If we denote the orthogonal complement of \( f^*T_p(M) \) in \( T_{f(p)}(M') \) with respect to \( g'_{f(p)} \) by \( N_p(M) \), the union

\[ N(M) = \bigcup_{p \in M} N_p(M) \]

is a subbundle of \( f^*T(M') \), having a hermitian structure \((J', g')\) induced from the one on \( f^*T(M') \). The hermitian inner products on \( T(M)^c, T(M')^c, f^*T(M')^c \) and \( N(M)^c \) will be denoted by the same \( \langle, \rangle \). We have the following orthogonal Whitney sum decompositions:

\[ f^*T(M') = f^*T(M) \oplus N(M) , \]
\[ f^*T(M')^c = f^*T(M)^c \oplus N(M)^c , \]
\[ f^*T(M')^\perp = f^*T(M)^\perp \oplus N(M)^\perp , \]

where the complex linear extension of the differential \( f_* \) is denoted by the same \( f_* \). The injections \( f_*: T(M) \to f^*T(M'), f_*: T(M)^c \to f^*T(M')^c \) and \( f_*: T(M)^\perp \to f^*T(M')^\perp \) preserve the respective inner products. So we shall often identify \( T(M) \) etc. with a subbundle of \( f^*T(M') \) etc. through the injections \( f_* \). The orthogonal projection \( f^*T(M') \to N(M) \) will be denoted by \( x \mapsto x^\perp \) and the induced projection \( C^\infty(f^*T(M')) \to C^\infty(N(M)) \) will be also denoted by \( x \mapsto x^\perp \). Then the normal connection \( D \) on \( N(M) \) satisfies

\[ D_x \xi = (\nabla_x \xi)^\perp \quad \text{for} \quad X \in C^\infty(T(M)), \xi \in C^\infty(N(M)). \]
Now we shall define the higher fundamental form $H^m$ of $f$ as a smooth section of the complex vector bundle $\text{Hom}(\otimes^m T(M)^+, N(M)^+)$. In the sequel, for a real linear object, its complex linear extension will be denoted by the same notation. For vector spaces $V$ and $W$, the space $\text{Hom}(\otimes^m V, W)$ of linear maps from the $m$-fold tensor product $\otimes^m V$ of $V$ into $W$ will be identified with the space of $m$-multilinear maps on $V$ into $W$. Let $h^2 \in C^\infty(\text{Hom}(\otimes^2 T(M), N(M)))$ be the second fundamental form of $f$, i.e.,

$$h^2(x, y) = (\nabla^*_x Y)^*$$

for $x, y \in T_p(M)$,

where $Y$ is a local smooth vector field on $M$ around $p$ such that $Y_p = y$. It is known (cf. Kobayashi-Nomizu [3]) that

$$h^2(x, y) = h^2(y, x), \quad h^2(Jx, y) = J^* h^2(x, y) \quad \text{for} \quad x, y \in T_p(M),$$

and hence

(1.1) \quad $h^2(T^*_p(M)^-, T^*_p(M)^-) = \{0\}, \quad h^2(T^*_p(M)^+, T^*_p(M)^-) \subset N_p(M)^-.$

We define $h^m \in C^\infty(\text{Hom}(\otimes^m T(M), N(M)))$ ($m \geq 3$) inductively as follows:

(1.2) \quad $h^{m+1}(x_1, \ldots, x_m, x_{m+1}) = D_{x_{m+1}} h^m(x_1, \ldots, x_m)$

$$- \sum_{i=1}^m h^m(x_1, \ldots, \nabla_{x_{m+1}} x_i, \ldots, x_m) \quad \text{for} \quad x_i \in T_p(M),$$

where the $X_i$ are smooth local vector fields on $M$ around $p$ such that $(X_i)_p = x_i$.

Note that (1.1) and (1.2) imply

$$h^m(x_1, \ldots, x_m) \in N_p(M)^+ \quad \text{for} \quad x_1, x_2 \in T^*_p(M)^+ \quad \text{and} \quad x_3, \ldots, x_m \in T^*_p(M)^-.$$

Now $H^m \in C^\infty(\text{Hom}(\otimes^m T(M)^+, N(M)^+))$ ($m \geq 2$) is defined by

$$H^m(x_1, \ldots, x_m) = h^m(x_1, \ldots, x_m) \quad \text{for} \quad x_i \in T^*_p(M)^+.$$

We write

$$\sum_{m \geq 2} h^m \in C^\infty(\text{Hom}(\sum_{m \geq 2} \otimes^m T(M), N(M)))$$

and

$$\sum_{m \geq 2} H^m \in C^\infty(\text{Hom}(\sum_{m \geq 2} \otimes^m (T(M)^+, N(M)^+))$$

by $h$ and $H$ respectively. Note that then we have

(1.3) \quad \left\{ \begin{array}{l}
H(X_1, X_2) = \nabla^*_x X_1 - \nabla^*_x X_2, \\
H(X_1, \ldots, X_m, X_{m+1}) = D_{x_{m+1}} H(X_1, \ldots, X_m)
\end{array} \right.$

$$- \sum_{i=1}^m h(X_1, \ldots, \nabla_{x_{m+1}} X_i, \ldots, X_m) \quad (m \geq 2)$$

for $X_i \in C^\infty(T(M)^+).$
Making use of the higher fundamental form $H$ we shall define the degree $d(f)$ of the Kahlerian immersion $f$. Let $p \in M$. For a positive integer $m$, we define a subspace $\mathcal{H}^m_p(M)$ of $T_{f(p)}(M')^+$ to be the subspace spanned by $T_p(M)^+$ and $H(\sum_{2 \leq k \leq m} \otimes^k T_p(M)^+)$. Then we get a series

$$\mathcal{H}^1_p(M) \subset \mathcal{H}^2_p(M) \subset \cdots \subset \mathcal{H}^m_p(M) \subset \mathcal{H}^{m+1}_p(M) \subset \cdots \subset T_{f(p)}(M')^+$$

of increasing subspaces of $T_{f(p)}(M')^+$. We define $O^m_p(M)$ to be the orthogonal complement of $\mathcal{H}^{m-1}_p(M)$ in $\mathcal{H}^m_p(M)$ with respect to $\langle , \rangle$, where $\mathcal{H}^1_p(M)$ is understood to be $\{0\}$. Thus we have an orthogonal direct sum:

$$\mathcal{H}^m_p(M) = O^1_p(M) \oplus O^2_p(M) \oplus \cdots \oplus O^m_p(M).$$

For each positive integer $m$, we define the set $\mathcal{R}_m$ of $m$-regular points of $M$ inductively as follows. Define $\mathcal{R}_1 = M$. For $m \geq 2$, assume $\mathcal{R}_{m-1}$ is already defined. Then we define

$$\mathcal{R}_m = \{ p \in \mathcal{R}_{m-1}; \dim_C \mathcal{H}^m_p(M) = \max_{p' \in \mathcal{R}_{m-1}} \dim_C \mathcal{H}^m_{p'}(M) \}.$$

We have inclusions: $\mathcal{R}_1 \supset \mathcal{R}_2 \supset \cdots \supset \mathcal{R}_m \supset \mathcal{R}_{m+1} \supset \cdots$. Note that each $\mathcal{R}_m$ is an open non-empty subset of $M$ and that

$$\mathcal{H}^m_p(M) = \bigcup_{p \in \mathcal{R}_m} \mathcal{H}^m_p(M)$$

is a smooth complex vector bundle over $\mathcal{R}_m$ which is a subbundle of $f^*T(M')^+|\mathcal{R}_m$ for each $m$.

**Lemma 1.** Let $p \in \mathcal{R}_m$, $m \geq 1$.

1) For each $x \in T_p(M)^+$ and each local smooth section $Y$ of $\mathcal{H}^m_p(M)$ around $p$ we have

$$\nabla'_x Y \in \mathcal{H}^{m+1}_p(M).$$

2) $O^{m+1}_p(M) = \{0\}$ if and only if for each $x \in T_p(M)^+$ and each local smooth section $Y$ of $\mathcal{H}^m_p(M)$ around $p$ we have

$$\nabla'_x Y \in \mathcal{H}^m_p(M).$$

**Proof.** Induction on $m$. Let $x \in T_p(M)^+$ and $Y$ a local smooth section of $\mathcal{H}^1_p(M) = T_p(M)^+$ around $p$. Then by (1.3)

$$\nabla'_x Y \equiv H(Y, x) \mod \mathcal{H}^1_p(M),$$

which implies the Lemma for $m=1$. Let $m \geq 2$ and $x \in T_p(M)^+$. Each local smooth section $Y$ of $\mathcal{H}^m_p(M)$ around $p$ is written as

$$Y = Z + \sum H(X_1, \cdots, X_m)$$

by a local smooth section $Z$ of $\mathcal{H}^{m-1}_p(M)|\mathcal{R}_m$ and local smooth sections $X_i$ of
From the assumption of the induction, we have $\nabla^i Z \in S^p_T(M)$. Further (1.3) implies

$$\nabla^i H(X_1, \ldots, X_m) \equiv D_x H(X_1, \ldots, X_m) \equiv H((X_1)_p, \ldots, (X_m)_p, x) \mod S^p_T(M),$$

and hence

$$\nabla^i Y \equiv \sum H((X_1)_p, \ldots, (X_m)_p, x) \mod S^p_T(M).$$

This implies the Lemma for $m$. q.e.d.

It follows from Lemma 1, 2) that there exists uniquely a positive integer $d$ such that

$$\begin{cases}
O^d_p(M) \neq \{0\} & \text{for some } p \in R_d, \\
O^{d+1}_p(M) = \{0\} & \text{for each } p \in R_d.
\end{cases}$$

Such integer $d$ is called the degree of the Kählerian immersion $f$ and denoted by $d=d(f)$. We have

$$R_d = R_{d+1} = \cdots.$$

This open subset $R_d$ of $M$ will be denoted by $R$ and called the set of regular points of $M$.

**Lemma 2** (Nakagawa-Takagi [5]). If $(M', g') = P_N(c)$, then:
1) $H^m$ is symmetric multilinear for each $m \geq 2$;
2) For each $u=(u_1, \ldots, u_n) \in U(M, g)$, we have
   (a) $h(u_i, u_j, u_k) = 0$,
   (b) $h(u_1, \ldots, u_m, u_i) = \frac{m-2}{2} \sum_{r=1}^{m} S_{i,r} H(u_1, \ldots, \hat{u}_r, \ldots, u_m)$
   \[= \frac{1}{r!(m-r)!} \sum_{r=1}^{m} \sum_{\sigma} \langle H(u_{i(r+1)}, \ldots, u_{\sigma(m)}), H(u_i, u_j) \rangle \times H(u_i, u_{\sigma(1)}, \ldots, u_{\sigma(r)}) \quad (m \geq 3),\]
where $\sigma$ runs through the permutations of $\{1, 2, \ldots, m\}$.

**Lemma 3** (Nakagawa [4]). Let $M$ be a smooth manifold, $p_0 \in M$ and $f: M \to P_N(C)$
a smooth immersion. Let $\pi: U(P_N(c)) \to P_N(C)$ be the bundle of unitary frames of $P_N(c)$, $\theta^A$ ($1 \leq A \leq N$) and $\omega^B$ ($1 \leq A, B \leq N$) be canonical forms and Levi-Civita's connection forms of $P_N(c)$ respectively. Then, $f(M)$ is contained in an $N'$-dimensional linear subvariety of $P_N(C)$ if and only if we can find $u_0 \in U(P_N(c))$ with $\pi(u_0) = f(p_0)$ such that for each smooth curve $\{p_t\}$ of $M$ through $p_0$ there exists a smooth curve $\{u_t\}$ of $U(P_N(c))$ through $u_0$ with $\pi(u_t) = f(p_t)$ satisfying
Now we prove the following theorem, giving a geometric interpretation of the degree \( d(f) \).

**Theorem 1.** Let \((M', g') = P_N(c)\) and \(f: (M, g) \rightarrow P_N(c)\) be a Kählerian immersion. Then the dimension \( N'(f) \) of the smallest linear subvariety of \( P_N(C) \) containing \( f(M) \) is given by

\[
N'(f) = \text{rank}_C \mathcal{H}^{d(f)}(M).
\]

**Proof.** First we show that for each \( x \in T_p(M) \), \( p \in R_m \) \((m \geq 1)\) and for each local smooth section \( Y \) of \( \mathfrak{h}(m)(M) \) around \( p \), we have

\[

(1.4) \quad \nabla^*_X Y \in \mathcal{H}^{m+1}(M).

\]

By virtue of Lemma 1, it suffices to show (1.4) for \( x \in T_p(M) \). It follows from (1.1) and (1.2) that for each local smooth sections \( X, X_i \) of \( T(M)^+ \) around \( p \) we have

\[

\left\{ \begin{array}{l}
\nabla^*_X X_1 = \nabla^*_X X_1',

D_X H(X_1, \cdots, X_m) = h(X_1, \cdots, X_m, \bar{X})

+ \sum_{i=1}^m H(X_1, \cdots, \nabla^*_X X_i, \cdots, X_m)

\end{array} \right. \quad (m \geq 2).
\]

Here we know that \( h(X_1, \cdots, X_m, \bar{X}) \) is a local smooth section of \( \mathcal{H}^m(M) \) in view of Lemma 2, and hence we can prove (1.4) for \( x \in T_p(M) \) in the same way as Lemma 1.

Take a connected component \( M_0 \) of the set \( R \) of regular points and take \( p_0 \in M_0 \). (1.4) implies

\[

(1.6) \quad \nabla^*_X Y \in \mathcal{H}^{d(f)}(M)
\]

for each \( x \in T_p(M) \), \( p \in M_0 \), and for each local smooth section \( Y \) of \( \mathcal{H}^{d(f)}(M) | M_0 \) around \( p \). Using the notation in Lemma 3, we choose a unitary frame \( u_0 = (u_0(0), \cdots, u_N(0)) \in U(P_N(c)) \) with \( \pi(u_0) = f(p_0) \) such that \( \{u_0(0), \cdots, u_N(0)\} \) spans \( \mathcal{H}^{d(f)}(M) \), where \( N' = \text{rank}_C \mathcal{H}^{d(f)}(M) \). For each smooth curve \( \{p_1\} \) of \( M_0 \) through \( p_0 \), we can choose a smooth curve \( \{u_1(0), \cdots, u_N(0)\} \) of \( U(P_N(c)) \) through \( u_0 \) with \( \pi(u_1) = f(p_1) \) such that \( \{u_1(0), \cdots, u_N(0)\} \) spans \( \mathcal{H}^{d(f)}(M) \). This is possible since \( \mathcal{H}^{d(f)}(M) | M_0 \) is a subbundle of \( \pi^*T(M)^+ | M_0 \). Then (1.6) implies

\[

\theta^{d}(u_t) = \langle f_* \dot{p}_t, u_A(t) \rangle = 0 \quad (N' + 1 \leq A \leq N),
\]

\[

\omega^{d}(u_t) = \langle \nabla_{f_* \dot{p}_t} u_t, u_A(t) \rangle = 0 \quad (N' + 1 \leq A \leq N, 1 \leq B \leq N').
\]
Thus, by Lemma 3, $f(M_0)$ is contained in an $N'$-dimensional linear subvariety $P$ of $P_N(C)$. From the analyticity of the immersion $f$, we conclude $f(M) \subset P$, and hence $N'(f) \leq N'$.

Assume that $f(M)$ is contained in a linear subvariety $P'$ of $P_N(C)$. Since $P'$ is a totally geodesic complex submanifold of $P_N(C)$, we have

$$\mathcal{H}_{p'}(M) \subset T_{f(p)}(P')^+ \quad \text{for } p \in \mathcal{R}.$$ 

This implies $N' \leq N'(f)$ and hence $N' = N'(f)$. q.e.d.

2. **Symmetric Kählerian submanifolds of $P_N(C)$**

A holomorphic immersion $f$ of a complex manifold $M$ into $P_N(C)$ is said to be **full** if $f(M)$ is not contained in any proper linear subvariety of $P_N(C)$. In this section we recall the construction of full Kählerian imbeddings into $P_N(C)$ of a symmetric Kählerian manifold of compact type. (cf. Borel [1], Takeuchi [6], Nakagawa-Takagi [5])

Let $\Pi = \{\alpha_1, \ldots, \alpha_i\}$ be an irreducible Dynkin diagram and $\Sigma$ the root system with the fundamental root system $\Pi$. Take a lexicographic order $>$ on $\Sigma$ such that the set of simple roots in $\Sigma$ with respect to $>$ coincides with $\Pi$. Assume that the highest (with respect to $>$) root $\gamma_1$ of $\Sigma$ has the following expression:

$$\gamma_1 = \alpha_1 + \sum_{i=2}^{l} m_i \alpha_i.$$

Put $\Pi_0 = \{\alpha_2, \ldots, \alpha_i\}$ and fix a positive integer $p$. To the triple $(\Pi, \Pi_0; p)$ we can associate a full Kählerian imbedding of an irreducible symmetric Kählerian manifold into $P_N(C)$ as follows.

Take a compact simple Lie algebra $g$ with the Dynkin diagram $\Pi$. Let $t$ be a maximal abelian subalgebra of $g$ and denote by $g^c$ and $t^c$ the complexifications of $g$ and $t$ respectively. We identify a weight of $g^c$ relative to the Cartan subalgebra $t^c$ with an element of $\mathbb{H}^*$ by means of the duality defined by the Killing form $(\, , \,)$ of $g^c$. Thus the root system $\Sigma$ of $g^c$ relative to $t^c$ is a subset of $\sqrt{-1}t$. Let $\{\Lambda_1, \ldots, \Lambda_l\}$, $\{\epsilon_1, \ldots, \epsilon_l\} \subset \sqrt{-1}t$ be the fundamental weights of $g^c$ and the dual basis for $\Pi$ respectively:

$$2(\Lambda_i, \alpha_j) = \delta_{ij}, \quad (\alpha_i, \epsilon_j) = \delta_{ij}, \quad (1 \leq i, j \leq l).$$

Put $\Sigma^+ = \{\alpha \in \Sigma; \alpha > 0\}$, $\Sigma_0 = \Sigma \cap \{\Pi_0\}^T$ and $\Sigma^+ = \Sigma^+ - \Sigma_0$, where $\{\Pi_0\}$ is denotes the subgroup of $\sqrt{-1}t$ generated by $\Pi_0$. Define subalgebras $t^c$, $m^+$ and $u$ of $g^c$ by
where $g^c$ denotes the root space of $g^c$ for $\alpha \in \Delta$. Let $\mathfrak{k} = \mathfrak{k}^0 \oplus \mathfrak{k}$, which is a real form of $\mathfrak{k}^c$, and $m$ be the orthogonal complement of $\mathfrak{k}$ in $g$ with respect to $(\cdot, \cdot)$. Then the automorphism $\theta = \exp \text{ad} \sqrt{-1} \mathfrak{k}$ of $g$ is involutive and gives the decomposition $g = \mathfrak{k} + m$ with

$$\mathfrak{k} = \{ X \in g; \theta X = X \}, \quad m = \{ X \in g; \theta X = -X \}.$$  

$G$ and $G^c$ denote the adjoint groups of $g$ and $g^c$ respectively, $G$ and $G^c$ are the universal covering groups of $G$ and $G^c$ respectively. We may identify as $G \subset G^c$ and $G \subset G^c$. Let $K$ and $U$ denote the (closed) connected subgroups of $G^c$ generated by $\mathfrak{k}$ and $\mathfrak{k}$ respectively. We define a complex manifold $M$ by

$$M = G^c / U.$$  

Then the natural map $G / K \to G^c / U$ induces the identification $M = G / K$ as smooth manifolds. The tangent space $T_o(M)$ of $M$ at the origin $o = U$ is identified with $m$ and $T_o(M) +$ with $m^+$ in the natural way.

Let $\rho: G \to SU(N+1)$ be an irreducible unitary representation of $G$ with the highest weight $\rho \Lambda$. By virtue of the irreducibility it induces a homomorphism

$$\rho: G \to PU(N+1) = SU(N+1)/\{ e_{1_{N+1}}^{N+1} = 1 \}$$  

such that the diagram

$$\begin{array}{ccc}
G & \xrightarrow{\bar{\rho}} & SU(N+1) \\
\downarrow \pi & & \downarrow \pi \\
G & \xrightarrow{\rho} & PU(N+1)
\end{array}$$

is commutative, where the $\pi$ are respective covering homomorphisms. They are extended holomorphically to $G^c$ and $G^c$ in such a way that the diagram

$$\begin{array}{ccc}
G^c & \xrightarrow{\bar{\rho}} & SL(N+1, C) \\
\downarrow \pi & & \downarrow \pi \\
G^c & \xrightarrow{\rho} & PL(N+1, C) = SL(N+1, C)/\{ e_{1_{N+1}}^{N+1} = 1 \}
\end{array}$$

is commutative, where we have used the same letters for extensions. Let
$P_{\mathcal{N}}(C) = \mathbb{C}^{N+1} - \{0\}/\mathbb{C}^*$ be the complex projective space associated to the representation space $\mathbb{C}^{N+1}$ of $\hat{\rho}$. For $v \in \mathbb{C}^{N+1} - \{0\}$, the equivalence class of $v$ will be denoted by $[v]$. Taking a highest weight vector $v_0 \in \mathbb{C}^{N+1} - \{0\}$, we can define a full holomorphic imbedding $f: M \to P_{\mathcal{N}}(C)$ by

$$f(x_0) = \rho(x)[v_0] \quad \text{for} \ x \in G^C.$$ 

We take the $SU(N+1)$-invariant Fubini-Study metric on $P_{\mathcal{N}}(C)$ of constant holomorphic sectional curvature $c$ and introduce a Kählerian metric $g$ on $M$ in such a way that $f: (M, g) \to P_{\mathcal{N}}(C)$ becomes a Kählerian imbedding. Then $(M, g)$ is an irreducible symmetric Kählerian manifold of compact type. If we denote the group of Kählerian automorphisms of $(M, g)$ and the one of holomorphisms of $M$ by $\text{Aut}(M, g)$ and $\text{Aut}(M)$ respectively, the identity-components $\text{Aut}^0(M, g)$ and $\text{Aut}^0(M)$ are identified with $G$ and $G^C$ respectively. Further $f$ is $G^C$-equivariant by the homomorphism $\rho$:

$$f(xp) = \rho(x)f(p) \quad \text{for} \ x \in G^C, \ p \in M.$$ 

where $\rho(G) \subset PU(N+1) = \text{Aut}(P_{\mathcal{N}}(C))$.

Put

$$\kappa(M) = \# \{ \alpha \in \sum_1^+; \ \alpha - \alpha_i \in \sum \} + 2.$$ 

Then (Nakagawa-Takagi [5]) the scalar curvature $k$ of $(M, g)$ is given by

$$k = \frac{(\dim_c M)\kappa(M)}{p},$$

which gives a geometric characterization of the positive integer $p$. It is also characterized (Nakagawa-Takagi [5]) by

$$g = \frac{\rho(\alpha, \alpha)}{c} g_0,$$

where $g_0$ is a $G$-invariant Kählerian metric on $M$ defined from the inner product $-\langle \cdot, \cdot \rangle$ on $g$. The imbedding $f$ will be called the $p$-th full Kählerian imbedding of $M$.

Now we shall construct a full Kählerian imbedding of a general (not necessarily irreducible) symmetric Kählerian manifold into $P_{\mathcal{N}}(C)$. For complex projective spaces $P_{\mathcal{N}_1}(C)$ and $P_{\mathcal{N}_2}(C)$ associated to $\mathbb{C}^{N_1+1}$ and $\mathbb{C}^{N_2+1}$ respectively, we define a holomorphic imbedding $\iota$ of $P_{\mathcal{N}_1}(C) \times P_{\mathcal{N}_2}(C)$ into the complex projective space $P_{\mathcal{N}_1+N_2+N_1+N_2}(C)$ associated to the tensor product $\mathbb{C}^{N_1+1} \otimes \mathbb{C}^{N_2+1}$ by
\[ \iota: [z_0]_{0 \leq i < N_1} \times [w_j]_{0 \leq j < N_2} \mapsto [z_0 w_j]_{0 \leq i < N_1, 0 < j < N_2}, \]

where \([*]\) denotes the point of the projective space with homogeneous coordinates \(*\). Then it defines a full Kahlerian imbedding

\[ \iota: P_{N_1}(c) \times P_{N_2}(c) \rightarrow P_{N_1 N_2 + N_1 + N_2}(c). \]

Let

\[ f_i: (M_0, g_i) \rightarrow P_{N_1}(c) \quad (i = 1, 2) \]

be two Kahlerian immersions. Then the composite

\[ f_1 \boxtimes f_2 = \iota(f_1 \times f_2): (M_1 \times M_2, g_1 \times g_2) \rightarrow P_{N_1 N_2 + N_1 + N_2}(c) \]

is also a Kahlerian immersion, which will be called the tensor product of \( f_1 \) and \( f_2 \). One can easily check the associativity

\[ (f_1 \boxtimes f_2) \boxtimes f_3 = f_1 \boxtimes (f_2 \boxtimes f_3) \]

of the tensor product, and so the multi-fold tensor product \( f_1 \boxtimes \cdots \boxtimes f_i \) is well-defined.

Now let

\[ f_i: (M_0, g_i) \rightarrow P_{N_1}(c) \quad (1 \leq i \leq s) \]

be full Kahlerian imbeddings of irreducible symmetric Kahlerian manifolds of compact type constructed as before. Then the tensor product

\[ f = f_1 \boxtimes \cdots \boxtimes f_s: (M_1 \times \cdots \times M_s, g_1 \times \cdots \times g_s) \rightarrow P_{N}(c), \]

where \( N = \prod_{i=1}^{s} (N_i + 1) - 1 \), is a full Kahlerian imbedding of the symmetric Kahlerian manifold \((M, g) = (M_1 \times \cdots \times M_s, g_1 \times \cdots \times g_s)\). Note that

\[ G^c = G_1^c \times \cdots \times G_s^c, \quad G = G_1 \times \cdots \times G_s, \]

where \( G^c = \text{Aut}^0(M), G = \text{Aut}^0(M, g), G_1^c = \text{Aut}^0(M_1), G_i = \text{Aut}^0(M_i, g_i) \), and that \( f \) is \( G^c \)-equivariant by the homomorphism \( \rho = \rho_1 \boxtimes \cdots \boxtimes \rho_s \) induced from the external tensor product \( \bar{\rho_1} \boxtimes \cdots \boxtimes \bar{\rho_s} \) of respective representations \( \bar{\rho} \). The tangent space \( T_\phi(M) \) of \( M \) at the origin \( o = o_1 \times \cdots \times o_s \) of \( M \), where \( o \) is the origin of \( M_i \), is identified with the direct sum

\[ m = m_1 \oplus \cdots \oplus m_s \]

of respective complements \( m_i \), and hence \( T_\phi(M)^+ \) with

\[ m^+ = m_1^+ \oplus \cdots \oplus m_s^+. \]

Further the stabilizer \( K \) of the origin \( o \) in \( G \) is the direct product
of respective stabilizers $K_i$.

It is known (Nakagawa-Takagi [5]. See also Takeuchi [8]) that any full Kahlerian immersion into $P_n(c)$ of a symmetric Kahlerian manifold of compact type is obtained in this way.

3. Degree of symmetric Kahlerian submanifolds of $P_n(c)$

Let

$$f: (M, g) \to P_n(c)$$

be the $p$-th full Kahlerian imbedding of an irreducible symmetric Kahlerian manifold $(M, g)$ constructed in §2. We recall first the construction of the Hermann map for $M$ (cf. Takeuchi [7]). Choose root vector $E_{\alpha} \in \mathfrak{g}_\mathbb{C}$ for $\alpha \in \Sigma$ in such a way that

$$[E_{\alpha}, E_{-\alpha}] = -\alpha, \quad (E_{\alpha}, E_{-\alpha}) = -1.$$ 

Then the complex conjugation $X \mapsto \overline{X}$ of $\mathfrak{g}_\mathbb{C}$ with respect to $g$ satisfies $E_{\alpha} = E_{-\alpha}$ for each $\alpha \in \Sigma$. We put

$$X_{\alpha} = \sqrt{\frac{2}{(\alpha, \alpha)}} E_{\alpha}, \quad H_{\alpha} = \frac{2}{(\alpha, \alpha)} \alpha \quad \text{for} \ \alpha \in \Sigma.$$ 

Then we have

$$[X_{\alpha}, X_{-\alpha}] = -H_{\alpha}, \quad (X_{\alpha}, X_{-\alpha}) = -\frac{2}{(\alpha, \alpha)}, \quad X_{\alpha} = X_{-\alpha}.$$ 

Let \{\gamma_1, \ldots, \gamma_r\} $\subset \Sigma^+_{\mathbb{Z}}$ be a maximal system of strongly orthogonal roots containing the highest root $\gamma_1$ such that $r = \text{rank } M$ and $(\gamma_j, \gamma_j) = (\alpha_i, \alpha_i)$ for each $j$ (cf. Helgason [2]). An injective homomorphism $\phi_j: \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}_\mathbb{C}$ is defined by

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_{\gamma_1}, \quad X^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \mapsto X_{-\gamma_1},$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto -H_{\gamma_1}.$$ 

Since $\phi_j(-X) = \overline{\phi_j(X)}$ for $X \in \mathfrak{sl}(2, \mathbb{C})$, we have $\phi_j(\mathfrak{su}(2)) \subset \mathfrak{g}$. If we define a map $\phi$ from the $r$-fold direct sum $\mathfrak{sl}(2, \mathbb{C})^r$ of $\mathfrak{sl}(2, \mathbb{C})$ into $\mathfrak{g}_\mathbb{C}$ by

$$\phi(X_1, \ldots, X_r) = \sum_{j=1}^{r} \phi_j(X_j) \quad \text{for} \ X_j \in \mathfrak{sl}(2, \mathbb{C}),$$

then it is also an injective homomorphism such that $\phi(\mathfrak{su}(2)^r) \subset \mathfrak{g}$. The extension of $\phi$ to the $r$-fold direct product $SL(2, \mathbb{C})^r$ of $SL(2, \mathbb{C})$ is also denoted by
\[ \phi: SL(2, C)^r \rightarrow G^c. \]

It satisfies \( \phi(SU(2))^r \subseteq G. \) Putting

\[ SL(1, 1; C) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, C) ; c = 0 \right\} , \]

we identify the \( r \)-fold direct product \( P_1(C)^r \) of \( P_1(C) \) with \( SL(2, C)^r/SL(1, 1; C)^r. \)

Then the map

\[ xSL(1, 1; C)^r \mapsto \phi(x)p \quad \text{for } x \in SL(2, C)^r \]

defines a holomorphic imbedding

\[ \phi: P_1(C)^r \rightarrow M , \]

which is \( SL(2, C)^r \)-equivariant:

\[ \phi(xp) = \phi(x)\phi(p) \quad \text{for } x \in SL(2, C)^r , \ p \in P_1(C)^r . \]

The imbedding \( \phi \) is called the Hermann map. The Kählerian metric \( h \) on \( P_1(C)^r \) induced from \( (M, g) \) is the direct product \( h_1 \times \cdots \times h_r \) of Kählerian metrics \( h_j \) on \( P_1(C) \) of constant holomorphic sectional curvatures, since \( SU(2)^r \) acts transitively on \( P_1(C)^r \) as Kählerian automorphisms of \( (P_1(C)^r , h). \) The tangent space \( T_x(\phi(P_1(C)^r)) \) will be identified with a subspace \( \mathfrak{p} \) of \( \mathfrak{m}, \) and hence \( T_x(\phi(P_1(C)^r))^+ \) with a subspace \( \mathfrak{p}^+ \) of \( \mathfrak{m}^+. \)

**Lemma 4.** Let

\[ \phi: (P_1(C)^r , h_1 \times \cdots \times h_r) \rightarrow (M, g) \]

be the Hermann map as above. Then:

1) \( \mathfrak{m}^+ = K\mathfrak{p}^+; \)

2) \( \phi \) is totally geodesic;

3) Each \( h_j \) has the holomorphic sectional curvature \( \frac{c}{p}. \)

**Proof.** 1) If we put

\[ U_{\gamma_j} = E_{\gamma_j} + E_{-\gamma_j}, \quad V_{\gamma_j} = \sqrt{-1}(E_{\gamma_j} - E_{-\gamma_j}) \quad (1 \leq j \leq r) , \]

\( \mathfrak{p} \) is spanned over \( R \) by the \( U_{\gamma_j} , V_{\gamma_j} \) \( (1 \leq j \leq r) \). The subspace \( \mathfrak{a} \) of \( \mathfrak{m} \) spanned over \( R \) by the \( U_{\gamma_j} \) \( (1 \leq j \leq r) \) is a maximal abelian subalgebra in \( \mathfrak{m}, \) and hence \( \mathfrak{m} = K\mathfrak{a}. \) Since the projection \( \sigma: \mathfrak{m}^c \rightarrow \mathfrak{m}^+ \) relative to the decomposition \( \mathfrak{m}^c = \mathfrak{m}^+ \oplus \bar{\mathfrak{m}}^+ \) is \( K \)-equivariant, we have \( \mathfrak{m}^+ = K\mathfrak{p}^+ \). But \( \mathfrak{p}^+ \) is spanned over \( R \) by the \( E_{-\gamma_j} \) \( (1 \leq j \leq r) \) and hence is contained in \( \mathfrak{p}^+ = \sigma^+(\mathfrak{p}), \) which is spanned over \( C \) by the \( E_{-\gamma_j} \) \( (1 \leq j \leq r). \) Thus we conclude \( \mathfrak{m}^+ = K\mathfrak{p}^+. \)

2) From the relations
[U_{\gamma}, V_{\gamma}] = 2\sqrt{-1}\gamma, [\sqrt{-1}\gamma, U_{\gamma}] = V_{\gamma}, [\sqrt{-1}\gamma, V_{\gamma}] = -U_{\gamma},

we get [[\hbar, \hbar], \hbar] = \hbar, and hence \phi is totally geodesic (cf. Helgason [2]).

3) Identifying \( X^+ + X^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) with a tangent vector of \( P_t(\mathbb{C}) \) at the origin, we have by (2.2)

\[
h_j(X^+ + X^-, X^+ + X^-) = g(X_{\gamma_j} + X_{-\gamma_j}, X_{\gamma_j} + X_{-\gamma_j})
\]

\[
= -\frac{2p(\gamma_j, \gamma_j)}{c} (X_{\gamma_j} + X_{-\gamma_j}, X_{\gamma_j} + X_{-\gamma_j})
\]

\[
= -\frac{2p(\gamma_j, \gamma_j)}{c} \cdot \frac{2}{(\gamma_j, \gamma_j)} = p \cdot \frac{4}{c}.
\]

It follows that \( h_j \) is \( p \) times the Fubini-Study metric of \( P_t(\mathbb{C}) \), which implies the assertion 3).

q.e.d.

Now we shall prove the following

**Theorem 2.** Let

\[
f_i: (M_i, g_i) \to P_{N_i}(\mathbb{C}) \quad (1 \leq i \leq s)
\]

be the \( p_i \)-th full Kählerian imbedding of an irreducible symmetric Kählerian manifold \( (M, g) \) of compact type, with rank \( M_i = r_i \) \((1 \leq i \leq s)\), and

\[
f: (M, g) \to P_N(\mathbb{C})
\]

be the tensor product of the \( f_i \) \((1 \leq i \leq s)\). Then the degree \( d(f) \) of \( f \) is given by

\[
d(f) = \sum_{i=1}^s r_i p_i.
\]

For the proof of the Theorem we need the following Lemma.

**Lemma 5** (Nakagawa-Takagi [5]). Let

\[
f: (M, g) \to P_N(\mathbb{C})
\]

be a Kählerian immersion of a locally symmetric Kählerian manifold \( (M, g) \). Then:

1) \( \langle H(\otimes^m T_p(M)^\vee), H(\otimes^m T_p(M)^\vee) \rangle = \{0\} \) for \( m \neq m' \), and hence \( O^p(M) = H(\otimes^m T_p(M)^\vee) \) for each \( m \);

2) For each \( u = (u_1, \ldots, u_n) \in U(M, g),
\[
h(u_{i_1}, \ldots, u_{i_m}, u_j) = -\frac{e}{2} \sum_{r=1}^m \delta_{i_r, j} H(u_{i_1}, \ldots, \hat{u}_{i_r}, \ldots, u_{i_m})
\]

\[
+ \sum_{1 \leq k < l \leq m} \langle R(u_{i_k}, u_{i_l})u_{i_k}, u_{i_l} \rangle H(u_{i_k}, u_{i_l}, \ldots, \hat{u}_{i_k}, \ldots, \hat{u}_{i_l}, \ldots, u_{i_m})
\]

\((m \geq 3)\),
where $R$ is the curvature tensor of $(M, g)$.

Proof of Theorem 2. Let $r = r_1 + \cdots + r_s$ be the rank of $M$. We use the notation in the end of §2. Taking the direct product of respective homomorphisms $\phi_i: SL(2, C)^{r_i} \rightarrow G^{r_i}$ for $M_i (1 \leq i \leq s)$ and the one of Hermann maps $\phi_i: P_i \left( \frac{e}{p_i} \right)^{r_i} \rightarrow (M_i, g_i) \ (1 \leq i \leq s)$, we get a homomorphism $\phi: SL(2, C)^r \rightarrow G^r$ such that $\phi(SU(2)^r) \subset G$ and a totally geodesic Kählerian imbedding

$$\phi: P = P_1 \left( \frac{e}{p_1} \right)^{r_1} \times \cdots \times P_s \left( \frac{e}{p_s} \right)^{r_s} \rightarrow (M, g),$$

which is $SL(2, C)^r$-equivariant:

$$\phi(xp) = \phi(x)\phi(p) \quad \text{for } x \in SL(2, C)^r, p \in P.$$

The tangent space $v = T_0(\phi(P))$ of $\phi(P)$ at the origin is the direct sum

$$v = v_1 \oplus \cdots \oplus v_s$$

of respective tangent spaces $v_i$ of $\phi_i(P_i) = \phi_i^{-1}(P_i)$ at the origin, and hence

$$v^+ = v_1^+ \oplus \cdots \oplus v_s^+.\)$$

It follows from Lemma 4 and decompositions (2.3), (2.4) that

$$m^+ = Kp^+.$$

Let us consider a Kählerian imbedding

$$f' = f \circ \phi: P \rightarrow P_N(c).$$

If we put

$$\rho' = \rho \circ \phi: SL(2, C)^r \rightarrow PL(N+1, C),$$

$f'$ is $SL(2, C)^r$-equivariant by the homomorphism $\rho'$:

$$f'(xp) = \rho'(x)f'(p) \quad \text{for } x \in SL(2, C)^r, p \in P,$$

Note that $\rho'(SU(2)^r) \subset PU(N+1) = \text{Aut}(P_N(c))$ and $SU(2)^r$ acts transitively on $P$ as Kählerian automorphisms of $P$. We shall identify as $P \subset (M, g)$ through the imbedding $\phi$. Denote the higher fundamental forms of $f$ and $f'$ by $H$ and $H'$ respectively. We shall prove the following two assertions:

(i) $d(f) = d(f').$

(ii) $d(f') = \sum_{i=1}^s r_i p_i.$

But in view of the $\text{Aut}^0(M, g)$-equivariance of $f$ and Lemma 5, 1), we know that each point of $M$ is regular and $d(f)$ is determined by conditions
\[ H_{\theta}^{(r)} \neq 0 \quad \text{and} \quad H_{\theta}^{(r)+1} = 0. \]

In the same way, \( d(f') \) is determined by conditions
\[ H_{\theta}^{(r)} \neq 0 \quad \text{and} \quad H_{\theta}^{(r)+1} = 0. \]

Here \( H_{\theta}^{1} \) and \( H_{\theta}^{r} \) are understood to be always not 0. Hence the assertion (i) is equivalent to the assertion
\[
(i)' \quad H_{\theta}^{*} = 0 \iff H_{\theta}^{m} = 0 \quad (m \geq 2).
\]

Proof of \((i)\)' . Note first that if we denote by \( X \mapsto kX \) the action of \( k \in K \) on \( N_{c}(M) \) through the differential \( \rho(k) \), we have
\[
H(kX, \ldots, kX) = kH(X, \ldots, X) \quad \text{for each} \quad X \in M^{+}, \quad k \in K,
\]
because of the \( \text{Aut}^{e}(M, g) \)-equivariance of \( f \). Now
\[
H_{\theta}^{m} = 0
\]
\[ \iff H(X, \ldots, X) = 0 \quad \text{for each} \quad X \in M^{+}, \quad \text{by Lemma 2,1),} \]
\[ \iff H(Y, \ldots, Y) = 0 \quad \text{for each} \quad Y \in M^{+} \quad \text{by (3.1), (3.2),} \]
\[ \iff H'(Y, \ldots, Y) = 0 \quad \text{for each} \quad Y \in M^{+} \quad \text{since} \ \phi \ \text{is totally geodesic}, \]
\[ \iff H'(Y, \ldots, Y) = 0 \quad \text{for each} \quad Y \in M^{+} \quad \text{by Lemma 2,1) \}
\[ \iff H_{\theta}^{m} = 0. \]

Proof of \((ii)\). For an index \( j \), \( 1 \leq j \leq r \), we define \( \nu(j) \), \( 1 \leq \nu(j) \leq s \), by
\[ \nu(j) = \nu \quad \text{if} \quad r_{1} + \cdots + r_{s-1} + 1 \leq j \leq r_{1} + \cdots + r_{s-1} + r_{s}. \]

Take a unitary frame \( u = (u_{1}, \ldots, u_{s}) \) of \( P \) at the origin \( o \) such that \( u_{i} \) is tangent to the \( i \)-th factor of \( P \) for each \( i \), and fix it once for all. Then the curvature tensor \( R \) of \( P \) satisfies
\[
\langle R(u_{k}, u_{l})u_{j}, u_{i} \rangle = \frac{c}{p_{s(i)}} \delta_{ij} \delta_{jk} \delta_{kl}. \]

For each \( i, \ldots, i_{m}, j \ (m \geq 2) \), the following equality holds:
\[
k'\left(u_{i_{1}}, \ldots, u_{i_{m}}, u_{j}, u_{j} \right) = \frac{c(a_{j}+1)}{2p_{s(i)}} (a_{j}-p_{s(i)}+1)H'(u_{i_{1}}, \ldots, u_{i_{m}}),
\]
where \( a_{j} \) is an integer given by
Indeed, Lemma 5.2 and (3.3) imply

\[ h'(u_{i_1}, \ldots, u_{i_{m+1}}, u_j) = -\frac{c}{2} \sum_{r=1}^{m+1} \delta_{i_r} \delta_{i_j} H'(u_{i_1}, \ldots, \hat{u}_r, \ldots, u_{i_{m+1}}, u_j) \]

\[ + \frac{c}{p_{(j)}} \sum_{1 \leq s < t \leq m+1} \delta_{i_s} \delta_{i_t} H'(u_{i_1}, \ldots, \hat{u}_s, \ldots, \hat{u}_t, \ldots, u_{i_{m+1}}, u_j) \]

Put \( i_{m+1} = j \). Recalling that \( H' \) is symmetric, we have

\[ h'(u_{i_1}, \ldots, u_{i_m}, u_j, u_j) = -\frac{c}{2} \sum_{r=1}^{m+1} \delta_{i_r} H'(u_{i_1}, \ldots, \hat{u}_r, \ldots, u_{i_m}, u_j) - \frac{c}{2} H'(u_{i_1}, \ldots, u_{i_m}) \]

\[ + \frac{c}{p_{(j)}} \sum_{1 \leq s < t \leq m+1} \delta_{i_s} \delta_{i_t} H'(u_{i_1}, \ldots, \hat{u}_s, \ldots, \hat{u}_t, \ldots, u_{i_m}, u_j) \]

\[ + \frac{c}{p_{(j)}} \sum_{1 \leq s < t \leq m} \delta_{i_s} H'(u_{i_1}, \ldots, \hat{u}_s, \ldots, u_{i_m}, u_j) \]

\[ = \left\{ -\frac{c}{2} a_j - \frac{c}{2} a_j(a_j-1) + \frac{c}{p_{(j)}} a_j \right\} H'(u_{i_1}, \ldots, u_{i_m}) \]

\[ = \frac{c(a_j+1)}{2p_{(j)}} (a_j - p_{(j)}) H'(u_{i_1}, \ldots, u_{i_m}). \]

Now we are in a position to prove (ii). If \( d' = d(f') = 1 \), then \( f' \) is totally geodesic, and hence \( s = 1 \), \( r_i = 1 \), \( p_i = 1 \). So we may assume \( d' \geq 2 \). Then there exist indices \( i_1, \ldots, i_{d'} \) such that \( H'(u_{i_1}, \ldots, u_{i_{d'}}) = 0 \). It follows from (1.5) and \( H''^{d'+1} = 0 \) that

\[ h'(u_{i_1}, \ldots, u_{i_{d'}}, u_j, u_j) = 0 \quad \text{for each } j, 1 \leq j \leq r. \]

Thus (3.4) implies

\[ \# \{ k ; 1 \leq k \leq d', i_k = j \} = p_{(j)} \quad \text{for each } j, 1 \leq j \leq r, \]

and hence

\[ d' = \sum_{j=1}^{r} p_{(j)} = \sum_{i=1}^{s} r_i p_i. \]
References


