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<thead>
<tr>
<th><strong>Title</strong></th>
<th>Degree of symmetric Kählerian submanifolds of a complex projective space</th>
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</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Takagi, Ryoichi; Takeuchi, Masaru</td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 14(3) P.501–P.518</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1977</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/8612">https://doi.org/10.18910/8612</a></td>
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<td><strong>DOI</strong></td>
<td>10.18910/8612</td>
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<td><strong>Note</strong></td>
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Introduction. Let \( P_N(c) \) denote the \( N \)-dimensional complex projective space \( P_N(C) \) endowed with the Fubini-Study metric of constant holomorphic sectional curvature \( c>0 \). For an irreducible symmetric Kahlerian manifold \( M \) of compact type, Nakagawa-Takagi [5] constructed a series of full equivariant Kahlerian imbeddings

\[
f_p : (M, g_p) \to P_{N_p}(c),
\]
parametrized by positive integers \( p \), and observed that the degree \( d(f_p) \) of \( f_p \) (See §1 for the definition) is given by

\[
d(f_p) = rp,
\]
where \( r = \text{rank } M \), in the case where \( p=1 \) or \( M \) is a complex quadric or a complex Grassmann manifold.

In this note we shall prove the above equality for general symmetric Kahlerian submanifolds of \( P_N(c) \): Let

\[
f_i : (M_i, g_i) \to P_{N_i}(c) \quad (1 \leq i \leq s)
\]
be the \( p_i \)-th full Kahlerian imbedding of an irreducible symmetric Kahlerian manifold \( M_i \) of rank \( r_i \) (\( 1 \leq i \leq s \)). Take the tensor product (See §2 for the definition)

\[
f = f_1 \otimes \cdots \otimes f_s : (M_1 \times \cdots \times M_s, g_1 \times \cdots \times g_s) \to P_N(c)
\]
of the \( f_i \) (\( 1 \leq i \leq s \)). Then (Theorem 2) the degree \( d(f) \) is given by

\[
d(f) = \sum_{i=1}^{s} r_i p_i.
\]

It should be noted that any full Kahlerian immersion \( f \) into \( P_N(c) \) of a symmetric Kahlerian manifold of compact type is obtained in this way.

*) The second author was supported by Sonderforschungsbereich “Theoretische Mathematik” at Universität Bonn.
1. Degree of Kählerian immersions

Let $V$ be a real vector space of dimension $2n$, equipped with an almost complex structure $J$ and an inner product $g$ satisfying

$$g(Jx, Jy) = g(x, y) \quad \text{for } x, y \in V.$$  

Such a pair $(J, g)$ will be called a hermitian structure on $V$. Denoting complex linear extensions of $J$ and $g$ to the complexification $V^c$ of $V$ by the same $J$ and $g$ respectively, we define subspaces $V^\pm$ of $V^c$ and a hermitian inner product $\langle , \rangle$ on $V^c$ by

$$V^\pm = \{ x \in V^c; Jx = \pm \sqrt{-1}x \},$$

$$\langle x, y \rangle = g(x, y) \quad \text{for } x, y \in V^c,$$

where $x \mapsto \overline{x}$ denotes the complex conjugation of $V^c$ with respect to $V$. Then we have $V^\pm = V^\mp$ and

$$V^c = V^+ \oplus V^- \quad (\text{orthogonal direct sum with respect to } \langle , \rangle).$$

A basis $u=(u_1, \cdots, u_n)$ of $V^+$ satisfying $\langle u_i, u_j \rangle = \delta_{ij}$ $(1 \leq i, j \leq n)$ is called a unitary frame of $V$.

Let $E$ be a smooth real vector bundle over a smooth manifold $M^*$ with a smooth assignment $(J, g): p \mapsto (J_p, g_p)$ of hermitian structures on fibres $E_p$. $(J, g)$ is called a hermitian structure on $E$. Then, getting together the constructions on fibres $E_p$, we have a hermitian inner product $\langle , \rangle$ on the complexification $E^c$ of $E$, and subbundles

$$E^\pm = \bigcup_{p \in M} E^\pm_p$$

of $E^c$ satisfying

$$E^c = E^+ \oplus E^- \quad (\text{orthogonal Whitney sum}),$$

and the complex conjugation $E^\mp \xrightarrow{\sim} E^\mp$. The map on the space of smooth sections induced from the complex conjugation will be also denoted by

$$C^\infty(E^\pm) \xrightarrow{\sim} C^\infty(E^\mp).$$

Let $(M, g)$ be a Kählerian manifold of dim$_c M = n$. Then the almost complex structure tensor $J$ and the Kählerian metric $g$ give a hermitian structure on the tangent bundle $T(M)$ of $M$. Thus we get a hermitian inner product $\langle , \rangle$ on the complexification $T(M)^c$ of $T(M)$ and subbundles $T(M)^\pm$ of $T(M)^c$ such that

*) In this note, a manifold will be always assumed to be connected.
$T(M)\mathbb{C} = T(M)^+ \oplus T(M)^-$ (orthogonal Whitney sum).

Denote by $U_p(M)$ the totality of unitary frames of $T_p(M)$. Then the union

$$U(M, g) = \bigcup_{p \in M} U_p(M)$$

has a structure of smooth principal bundle over $M$ with the structure group $U(n)$. The Levi-Civita’s connection form $\omega$ and the canonical form $\theta$ of $(M, g)$ will be considered as a $\mathfrak{u}(n)$-valued 1-form and $C^n$-valued 1-form on $U(M, g)$ respectively. $\omega^A_B (1 \leq A, B \leq n)$ and $\theta^A (1 \leq A \leq n)$ denote the components of $\omega$ and $\theta$ respectively.

Now let $(M, g)$ and $(M', g')$ be Kählerian manifolds of complex dimensions $n$ and $N$ respectively and

$$f: (M, g) \to (M', g')$$

be a Kählerian immersion, i.e., a holomorphic isometric immersion of $(M, g)$ into $(M', g')$. The almost complex structure tensors of $M$ and $M'$ will be denoted by $J$ and $J'$ respectively. The Levi-Civita connections of $T(M)$ and $T(M')$ are denoted by $\nabla$ and $\nabla'$ respectively. The induced bundle $f^*T(M')$ over $M$ has a hermitian structure $(J', g')$ induced from the one on $T(M')$. Also it has a connection induced from the Levi-Civita connection on $T(M')$, which will also be denoted by $\nabla'$. If we denote the orthogonal complement of $f^*T_p(M)$ in $T_{f(p)}(M')$ with respect to $g'_{f(p)}$ by $N_p(M)$, the union

$$N(M) = \bigcup_{p \in M} N_p(M)$$

is a subbundle of $f^*T(M')$, having a hermitian structure $(J', g')$ induced from the one on $f^*T(M')$. The hermitian inner products on $T(M)^c$, $T(M')^c$, $f^*T(M')^c$ and $N(M)^c$ will be denoted by the same $\langle, \rangle$. We have the following orthogonal Whitney sum decompositions:

$$f^*T(M') = f^*T(M) \oplus N(M),$$

$$f^*T(M')^c = f^*T(M)^c \oplus N(M)^c,$$

$$f^*T(M')^\perp = f^*T(M)^\perp \oplus N(M)^\perp,$$

where the complex linear extension of the differential $f_*$ is denoted by the same $f_*$. The injections $f_*: T(M) \to f^*T(M')$, $f_*: T(M)^c \to f^*T(M')^c$ and $f_*: T(M)^\perp \to f^*T(M')^\perp$ preserve the respective inner products. So we shall often identify $T(M)$ etc. with a subbundle of $f^*T(M')$ etc. through the injections $f_*$. The orthogonal projection $f^*T(M') \to N(M)$ will be denoted by $x \mapsto x^\perp$ and the induced projection $C^\infty(f^*T(M')) \to C^\infty(N(M))$ will be also denoted by $x \mapsto x^\perp$. Then the normal connection $D$ on $N(M)$ satisfies

$$D_x\xi = (\nabla_x\xi)^\perp \quad \text{for} \quad X \in C^\infty(T(M)), \xi \in C^\infty(N(M)).$$
Now we shall define the higher fundamental form $H^m$ of $f$ as a smooth section of the complex vector bundle $\text{Hom}(\otimes^m T(M)^+, N(M)^+)$.

In the sequel, for a real linear object, its complex linear extension will be denoted by the same notation. For vector spaces $V$ and $W$, the space $\text{Hom}(\otimes^n V, W)$ of linear maps from the $m$-fold tensor product $\otimes^m V$ of $V$ into $W$ will be identified with the space of $m$-multilinear maps on $V$ into $W$. Let $h^2 \in C^\infty(\text{Hom}(\otimes^2 T(M), N(M)))$ be the second fundamental form of $f$, i.e.,

$$h^2(x, y) = (\nabla_x Y)^\perp$$

for $x, y \in T_p(M)$,

where $Y$ is a local smooth vector field on $M$ around $p$ such that $Y_p = y$. It is known (cf. Kobayashi-Nomizu [3]) that

$$h^2(x, y) = h^2(y, x), \quad h^2(Jx, y) = J^\prime h^2(x, y)$$

for $x, y \in T_p(M)$, and hence

(1.1) \hspace{1cm} h^2(T_p(M)^+, T_p(M)^-) = \{0\}, \quad h^2(T_p(M)^+, T_p(M)^-) \subset N_p(M)^\pm.

We define $h^m \in C^\infty(\text{Hom}(\otimes^m T(M), N(M)))$ ($m \geq 3$) inductively as follows:

(1.2) \hspace{1cm} h^{m+1}(x_1, \ldots, x_m, x_{m+1}) = D_{x_{m+1}} h^m(X_1, \ldots, X_m)

$$\quad - \sum_{i=1}^{m} h^m(x_1, \ldots, \nabla_{x_{m+1}} X_i, \ldots, x_m)$$

for $x_i \in T_p(M)$,

where the $X_i$ are smooth local vector fields on $M$ around $p$ such that $(X_i)_p = x_i$.

Note that (1.1) and (1.2) imply

$$h^m(x_1, \ldots, x_m) \in N_p(M)^+$$

for $x_1, x_2 \in T_p(M)^+$ and $x_3, \ldots, x_m \in T_p(M)^-$. Now $H^m \in C^\infty(\text{Hom}(\otimes^m T(M)^+, N(M)^+))$ ($m \geq 2$) is defined by

$$H^m(x_1, \ldots, x_m) = h^m(x_1, \ldots, x_m)$$

for $x_i \in T_p(M)^+$.

We write

$$\sum_{m \geq 2} h^m \in C^\infty(\text{Hom}(\otimes^m T(M), N(M)))$$

and

$$\sum_{m \geq 2} H^m \in C^\infty(\text{Hom}(\otimes^m T(M)^+, N(M)^+))$$

by $h$ and $H$ respectively. Note that then we have

(1.3) \hspace{1cm} \{ H(X_1, X_2) = \nabla_{X_2} X_1 - \nabla_{X_1} X_2, \quad H(X_1, \ldots, X_m, X_{m+1}) = D_{X_{m+1}} H(X_1, \ldots, X_m)

$$\quad - \sum_{i=1}^{m} H(X_1, \ldots, \nabla_{X_{m+1}} X_i, \ldots, X_m)$$

for $X_i \in C^\infty(T(M)^+)$.
Making use of the higher fundamental form \( H \) we shall define the degree \( d(f) \) of the Kahlerian immersion \( f \). Let \( p \in M \). For a positive integer \( m \), we define a subspace \( \mathcal{H}_p^m(M) \) of \( T_{f(p)}(M')^+ \) to be the subspace spanned by \( T_p(M)^+ \) and \( H(\sum_{2 \leq k \leq m} \otimes^k T_p(M)^+) \). Then we get a series
\[
\mathcal{H}_p^0(M) \subset \mathcal{H}_p^1(M) \subset \cdots \subset \mathcal{H}_p^m(M) \subset \mathcal{H}_p^{m+1}(M) \subset \cdots \subset T_{f(p)}(M')^+
\]
of increasing subspaces of \( T_{f(p)}(M')^+ \). We define \( O_p^m(M) \) to be the orthogonal complement of \( \mathcal{H}_p^{m-1}(M) \) in \( \mathcal{H}_p^m(M) \) with respect to \( \langle \ , \rangle \), where \( \mathcal{H}_p^0(M) \) is understood to be \( \{0\} \). Thus we have an orthogonal direct sum:
\[
\mathcal{H}_p^m(M) = O_p^1(M) \oplus O_p^2(M) \oplus \cdots \oplus O_p^m(M).
\]
For each positive integer \( m \), we define the set \( R_m \) of \( m \)-regular points of \( M \) inductively as follows. Define \( R_1 = M \). For \( m \geq 2 \), assume \( R_{m-1} \) is already defined. Then we define
\[
R_m = \{ p \in R_{m-1}; \dim_C \mathcal{H}_p^m(M) = \max_{p' \in R_{m-1}} \dim_C \mathcal{H}_{p'}^m(M) \}.
\]
We have inclusions: \( R_1 \supset R_2 \supset \cdots \supset R_m \supset R_{m+1} \supset \cdots \). Note that each \( R_m \) is an open non-empty subset of \( M \) and that
\[
\mathcal{H}^m(M) = \bigcup_{p \in R_m} \mathcal{H}_p^m(M)
\]
is a smooth complex vector bundle over \( R_m \) which is a subbundle of \( f^* T(M')^+ | R_m \) for each \( m \).

**Lemma 1.** Let \( p \in R_m, m \geq 1 \).

1) For each \( x \in T_p(M)^+ \) and each local smooth section \( Y \) of \( \mathcal{H}^m(M) \) around \( p \) we have
\[
\nabla_x' Y \in \mathcal{H}_p^{m+1}(M).
\]

2) \( O_p^{m+1}(M) = \{0\} \) if and only if for each \( x \in T_p(M)^+ \) and each local smooth section \( Y \) of \( \mathcal{H}^m(M) \) around \( p \) we have
\[
\nabla_x' Y \in \mathcal{H}_p^m(M).
\]

**Proof.** Induction on \( m \). Let \( x \in T_p(M)^+ \) and \( Y \) a local smooth section of \( \mathcal{H}(M) = T(M)^+ \) around \( p \). Then by (1.3)
\[
\nabla_x' Y \equiv H(Y_p, x) \mod \mathcal{H}_p^1(M),
\]
which implies the Lemma for \( m=1 \). Let \( m \geq 2 \) and \( x \in T_p(M)^+ \). Each local smooth section \( Y \) of \( \mathcal{H}^m(M) \) around \( p \) is written as
\[
Y = Z + \sum H(X_1, \ldots, X_m)
\]
by a local smooth section \( Z \) of \( \mathcal{H}^{m-1}(M) | R_m \) and local smooth sections \( X_i \) of
From the assumption of the induction, we have \( \nabla^i Z \in \mathcal{A}^*_p(M) \). Further (1.3) implies
\[
\nabla^i H(X_1, \ldots, X_m) \equiv D_x H(X_1, \ldots, X_m) \equiv H((X_1)_p, \ldots, (X_m)_p, x) \mod \mathcal{A}^*_p(M),
\]
and hence
\[
\nabla^i Y \equiv \sum H((X_1)_p, \ldots, (X_m)_p, x) \mod \mathcal{A}^*_p(M).
\]
This implies the Lemma for \( m \). \( \quad \) q.e.d.

It follows from Lemma 1, 2) that there exists uniquely a positive integer \( d \) such that
\[
\begin{cases}
O^d_p(M) \neq \{0\} & \text{for some } p \in \mathcal{R}_d, \\
O^{d+1}_p(M) = \{0\} & \text{for each } p \in \mathcal{R}_d.
\end{cases}
\]
Such integer \( d \) is called the degree of the Kahlerian immersion \( f \) and denoted by \( d = d(f) \). We have
\[
\mathcal{R}_d = \mathcal{R}_{d+1} = \cdots.
\]
This open subset \( \mathcal{R}_d \) of \( M \) will be denoted by \( \mathcal{R} \) and called the set of regular points of \( M \).

**Lemma 2** (Nakagawa-Takagi [5]). If \((M', g') = P_N(c)\), then:
1) \( H^m \) is symmetric multilinear for each \( m \geq 2 \);
2) For each \( u = (u_1, \ldots, u_n) \in U(M, g) \), we have
   (a) \( h(u_1, u_2, u_3, \ldots, u_m) = 0 \),
   (b) \[
   h(u_1, u_2, \ldots, u_m, u_j) = \frac{m-2}{2} c \sum_{i=1}^{m} s_{i,j} H(u_1, \ldots, \hat{u}_i, \ldots, u_m) \]
   
   \[
   - \sum_{r=1}^{m-2} \frac{1}{r!(m-r)!} \sum_{i=1}^{m} \sum_{\sigma} \langle H(u_{\sigma(r+1)}, \ldots, u_{\sigma(m)}), H(u_1, u_j) \rangle \times H(u_i, u_{\sigma(1)}, \ldots, u_{\sigma(r)}) \quad (m \geq 3),
   \]
   where \( \sigma \) runs through the permutations of \( \{1, 2, \ldots, m\} \).

**Lemma 3** (Nakagawa [4]). Let \( M \) be a smooth manifold, \( p_0 \in M \) and
\[
f: M \to P_N(C)
\]
a smooth immersion. Let \( \pi: U(P_N(c)) \to P_N(C) \) be the bundle of unitary frames of \( P_N(c) \), \( \theta^A \) (\( 1 \leq A \leq N \)) and \( \omega^B_{AB} \) (\( 1 \leq A, B \leq N \)) be canonical forms and Levi-Civita's connection forms of \( P_N(c) \) respectively. Then, \( f(M) \) is contained in an \( N' \)-dimensional linear subvariety of \( P_N(C) \) if and only if we can find \( u_0 \in U(P_N(c)) \) with \( \pi(u_0) = f(p_0) \) such that for each smooth curve \( \{p_t\} \) of \( M \) through \( p_0 \) there exists a smooth curve \( \{u_t\} \) of \( U(P_N(c)) \) through \( u_0 \) with \( \pi(u_t) = f(p_t) \) satisfying
Now we prove the following theorem, giving a geometric interpretation of the degree \( d(f) \).

**Theorem 1.** Let \((M', g')=P_N(c)\) and

\[
f: (M, g) \to P_N(c)
\]

be a Kählerian immersion. Then the dimension \( N'(f) \) of the smallest linear subvariety of \( P_N(c) \) containing \( f(M) \) is given by

\[
N'(f) = \text{rank}_c \mathcal{H}^{d(f)}(M).
\]

**Proof.** First we show that for each \( x \in T_p(M) \), \( p \in R_m \) \((m \geq 1)\) and for each local smooth section \( Y \) of \( \mathcal{H}^{m}(M) \) around \( p \), we have

\[
\nabla' Y \in \mathcal{H}^{m+1}_p(M).
\]

By virtue of Lemma 1, it suffices to show (1.4) for \( x \in T_p(M)^- \). It follows from (1.1) and (1.2) that for each local smooth sections \( X, X_i \) of \( T(M)^+ \) around \( p \) we have

\[
\begin{align*}
\nabla' Y & = \nabla_X Y = \nabla_X X' = 0 \\
D_X H(X_1, \ldots, X_m) & = h(X_1, \ldots, X_m, X) \\
& + \sum_{i=1}^{m} H(X_i, \ldots, \nabla_X X_i, \ldots, X_m)
\end{align*}
\]

\((m \geq 2)\).

Here we know that \( h(X_1, \ldots, X_m, X) \) is a local smooth section of \( \mathcal{H}^{m}(M) \) in view of Lemma 2,2), and hence we can prove (1.4) for \( x \in T_p(M)^- \) in the same way as Lemma 1.

Take a connected component \( M_0 \) of the set \( R \) of regular points and take \( p_0 \in M_0 \). (1.4) implies

\[
\nabla' Y \in \mathcal{H}^{d(f)}(M) \mid M_0
\]

for each \( x \in T_p(M), p \in M_0, \) and for each local smooth section \( Y \) of \( \mathcal{H}^{d(f)}(M) \mid M_0 \) around \( p \). Using the notation in Lemma 3, we choose a unitary frame \( u_0=(u_0, \ldots, u_N(0)) \in U(P_N(c)) \) with \( \pi(u_0)=f(p_0) \) such that \( \{u_0, \ldots, u_{N'}(0)\} \) spans \( \mathcal{H}^{d(f)}(M) \), where \( N' = \text{rank}_c \mathcal{H}^{d(f)}(M) \). For each smooth curve \( \{t \} \) of \( M_0 \) through \( p_0 \), we can choose a smooth curve \( \{u_0=(u_0(t), \ldots, u_N(t)) \} \) of \( U(P_N(c)) \) through \( u_0 \) with \( \pi(u_0)=f(p_t) \) such that \( \{u_0(t), \ldots, u_{N'}(t)\} \) spans \( \mathcal{H}^{d(f)}(M) \). This is possible since \( \mathcal{H}^{d(f)}(M) \mid M_0 \) is a subbundle of \( f^*T(M)^+ \mid M_0 \). Then (1.6) implies

\[
\begin{align*}
\theta^{d}(u_0) &= \langle f_*(\tilde{p}_t), u_A(t) \rangle = 0 \quad (N'+1 \leq A \leq N), \\
\omega^{d}(u_0) &= \langle \nabla_{f_*(\tilde{p}_t)} u_A(t), u_A(t) \rangle = 0 \quad (N'+1 \leq A \leq N, 1 \leq B \leq N').
\end{align*}
\]
Thus, by Lemma 3, \( f(M_0) \) is contained in an \( N' \)-dimensional linear subvariety \( P \) of \( P_N(C) \). From the analyticity of the immersion \( f \), we conclude \( f(M) \subset P \), and hence \( N'(f) \leq N' \).

Assume that \( f(M) \) is contained in a linear subvariety \( P' \) of \( P_N(C) \). Since \( P' \) is a totally geodesic complex submanifold of \( P_N(C) \), we have

\[
\mathcal{H}_p^{\text{fr}}(M) \subset T_{f(p)}(P')^+ \quad \text{for } p \in \mathcal{R}.
\]

This implies \( N' \leq N'(f) \) and hence \( N' = N'(f) \). q.e.d.

2. Symmetric Kählerian submanifolds of \( P_N(c) \)

A holomorphic immersion \( f \) of a complex manifold \( M \) into \( P_N(C) \) is said to be full if \( f(M) \) is not contained in any proper linear subvariety of \( P_N(C) \). In this section we recall the construction of full Kählerian imbeddings into \( P_N(c) \) of a symmetric Kählerian manifold of compact type. (cf. Borel [1], Takeuchi [6], Nakagawa-Takagi [5])

Let \( \Pi^\text{\textsc{I}} = \{\alpha_1, \cdots, \alpha_l\} \) be an irreducible Dynkin diagram and \( \Sigma \) the root system with the fundamental root system \( \Pi \). Take a lexicographic order \( > \) on \( \Sigma \) such that the set of simple roots in \( \Sigma \) with respect to \( > \) coincides with \( \Pi \). Assume that the highest (with respect to \( > \) ) root \( \gamma_1 \) of \( \Sigma \) has the following expression:

\[
\gamma_1 = \alpha_1 + \sum_{i=2}^{l} m_i \alpha_i.
\]

Put \( \Pi_0 = \{\alpha_2, \cdots, \alpha_l\} \) and fix a positive integer \( p \). To the triple \( (\Pi, \Pi_0; p) \) we can associate a full Kählerian imbedding of an irreducible symmetric Kählerian manifold into \( P_N(c) \) as follows.

Take a compact simple Lie algebra \( g \) with the Dynkin diagram \( \Pi \). Let \( \mathfrak{t} \) be a maximal abelian subalgebra of \( g \) and denote by \( g^c \) and \( \mathfrak{t}^c \) the complexifications of \( g \) and \( \mathfrak{t} \) respectively. We identify a weight of \( g^c \) relative to the Cartan subalgebra \( \mathfrak{t}^c \) with an element of \( \mathfrak{t}^* \) by means of the duality defined by the Killing form \((\ , \) \) of \( g^c \). Thus the root system \( \Sigma \) of \( g^c \) relative to \( \mathfrak{t}^c \) is a subset of \( \sqrt{-1}\mathfrak{t} \). Let \( \{\Lambda_1, \cdots, \Lambda_l\} \), \( \{\xi_i, \cdots, \xi_l\} \subset \sqrt{-1}\mathfrak{t} \) be the fundamental weights of \( g^c \) and the dual basis for \( \Pi \) respectively:

\[
2(\Lambda_i, \alpha_j) = \delta_{ij}, \quad (\alpha_i, \xi_j) = \delta_{ij} \quad (1 \leq i, j \leq l).
\]

Put \( \Sigma^+ = \{\alpha \in \Sigma ; \alpha > 0\} \), \( \Sigma_0 = \Sigma \cap \{\Pi_0\}_Z \) and \( \Sigma^*_m = \Sigma^+ - \Sigma_0 \), where \( \{\Pi_0\}_Z \) denotes the subgroup of \( \sqrt{-1}\mathfrak{t} \) generated by \( \Pi_0 \). Define subalgebras \( \mathfrak{t}^c, \mathfrak{m}^c, \) and \( \mathfrak{u} \) of \( g^c \) by
where \( g^c \) denotes the root space of \( g^e \) for \( \alpha \in \Sigma \). Let \( \mathfrak{f} = \mathfrak{f}^c \cap \mathfrak{g} \), which is a real form of \( \mathfrak{f}^c \), and \( m \) be the orthogonal complement of \( \mathfrak{f} \) in \( \mathfrak{g} \) with respect to \( \langle \ , \rangle \). Then the automorphism \( \theta = \exp \text{ad} \sqrt{-1} \epsilon_1 \) of \( \mathfrak{g} \) is involutive and gives the decomposition \( \mathfrak{g} = \mathfrak{f} + m \) with

\[
\mathfrak{f} = \{ X \in \mathfrak{g}; \theta X = X \}, \quad m = \{ X \in \mathfrak{g}; \theta X = -X \}.
\]

\( \mathfrak{f}^c = \mathfrak{f}^c + \sum_{\alpha \in \Sigma_0} g^c_\alpha, \quad \mathfrak{m}^+ = \sum_{\alpha \in \Sigma_+} g^c_\alpha, \)

\[
\mathfrak{u} = \mathfrak{f}^c + \sum_{\alpha \in \Sigma_0 \cup \Sigma_+^0} g^c_\alpha,
\]

where \( g^c_\alpha \) denotes the root space of \( g^e \) for \( \alpha \in \Sigma \). Let \( \mathfrak{f} = \mathfrak{f}^c \cap \mathfrak{g} \), which is a real form of \( \mathfrak{f}^c \), and \( m \) be the orthogonal complement of \( \mathfrak{f} \) in \( \mathfrak{g} \) with respect to \( \langle \ , \rangle \).

Then the automorphism \( \theta = \exp \text{ad} \sqrt{-1} \epsilon_1 \) of \( \mathfrak{g} \) is involutive and gives the decomposition \( \mathfrak{g} = \mathfrak{f} + m \) with

\[
\mathfrak{f} = \{ X \in \mathfrak{g}; \theta X = X \}, \quad m = \{ X \in \mathfrak{g}; \theta X = -X \}.
\]

\( \mathfrak{g} \) and \( \mathfrak{g}^e \) denote the adjoint groups of \( \mathfrak{g} \) and \( \mathfrak{g}^e \) respectively, \( \mathcal{G} \) and \( \mathcal{G}^e \) the universal covering groups of \( \mathcal{G} \) and \( \mathcal{G}^e \) respectively. We may identify as \( \mathcal{G} \subset \mathcal{G}^e \) and \( \mathcal{G}^e \subset \mathcal{G}^e \). Let \( K \) and \( U \) denote the (closed) connected subgroups of \( \mathcal{G}^e \) generated by \( \mathfrak{f} \) and \( \mathfrak{u} \) respectively. We define a complex manifold \( M \) by

\[
M = \mathcal{G}^e / U.
\]

Then the natural map \( \mathcal{G}^e K \rightarrow \mathcal{G}^e U \) induces the identification \( M = \mathcal{G} K \) as smooth manifolds. The tangent space \( T_\mathcal{G}(M) \) of \( M \) at the origin \( o = U \) is identified with \( m \) and \( T_\mathcal{G}(M)^+ \) with \( m^+ \) in the natural way.

Let

\[
\rho: \mathcal{G} \rightarrow SU(N+1)
\]

be an irreducible unitary representation of \( \mathcal{G} \) with the highest weight \( \rho \Lambda_1 \). By virtue of the irreducibility it induces a homomorphism

\[
\rho: \mathcal{G} \rightarrow PU(N+1) = SU(N+1)/\{ \varepsilon_1 N+1; \varepsilon^{N+1} = 1 \}
\]

such that the diagram

\[
\begin{array}{ccc}
\mathcal{G} & \overset{\rho}{\longrightarrow} & SU(N+1) \\
\pi \downarrow & & \pi \\
\mathcal{G} & \overset{\pi}{\longrightarrow} & PU(N+1)
\end{array}
\]

is commutative, where the \( \pi \) are respective covering homomorphisms. They are extended holomorphically to \( \mathcal{G}^e \) and \( \mathcal{G}^e \) in such a way that the diagram

\[
\begin{array}{ccc}
\mathcal{G}^e & \overset{\rho}{\longrightarrow} & SL(N+1, \mathbb{C}) \\
\pi \downarrow & & \pi \\
\mathcal{G}^e & \overset{\rho}{\longrightarrow} & PL(N+1, \mathbb{C}) = SL(N+1, \mathbb{C})/\{ \varepsilon_1 N+1; \varepsilon^{N+1} = 1 \}
\end{array}
\]

is commutative, where we have used the same letters for extensions. Let
$P_n(C) = \mathbb{C}^{n+1}-\{0\}/\mathbb{C}^*$ be the complex projective space associated to the representation space $\mathbb{C}^{n+1}$ of $\tilde{\rho}$. For $v \in \mathbb{C}^{n+1}-\{0\}$, the equivalence class of $v$ will be denoted by $[v]$. Taking a highest weight vector $v_0 \in \mathbb{C}^{n+1}-\{0\}$, we can define a full holomorphic imbedding $f: M \to P_n(C)$ by

$$f(x) = \rho(x)[v_0] \quad \text{for } x \in G^C.$$ 

We take the $SU(N+1)$-invariant Fubini-Study metric on $P_n(C)$ of constant holomorphic sectional curvature $c$ and introduce a Kählerian metric $g$ on $M$ in such a way that $f$ becomes a Kählerian imbedding. Then $(M, g)$ is an irreducible symmetric Kählerian manifold of compact type. If we denote the group of Kählerian automorphisms of $(M, g)$ and the one of holomorphisms of $M$ by $\text{Aut}(M, g)$ and $\text{Aut}(M)$ respectively, the identity-components $\text{Aut}^0(M, g)$ and $\text{Aut}^0(M)$ are identified with $G$ and $G^C$ respectively. Further $f$ is $G^C$-equivariant by the homomorphism $\rho$:

$$f(xp) = \rho(x)f(p) \quad \text{for } x \in G^C, p \in M.$$ 

where $\rho(G) \subset PU(N+1) = \text{Aut}(P_n(C))$.

Put

$$\kappa(M) = \# \{ \alpha \in \Sigma^+; \alpha - \alpha_i \in \Sigma \} + 2.$$ 

Then (Nakagawa-Takagi [5]) the scalar curvature $k$ of $(M, g)$ is given by

$$k = \frac{(\dim_c M) c \kappa(M)}{\rho},$$

which gives a geometric characterization of the positive integer $\rho$. It is also characterized (Nakagawa-Takagi [5]) by

$$g = \frac{\rho(\alpha_1, \alpha_i)}{c} g_0,$$

where $g_0$ is a $G$-invariant Kählerian metric on $M$ defined from the inner product $-(\ ,\ )$ on $g$. The imbedding $f$ will be called the $\rho$-th full Kählerian imbedding of $M$.

Now we shall construct a full Kählerian imbedding of a general (not necessarily irreducible) symmetric Kählerian manifold into $P_n(c)$. For complex projective spaces $P_{n_1}(C)$ and $P_{n_2}(C)$ associated to $\mathbb{C}^{n_1+1}$ and $\mathbb{C}^{n_2+1}$ respectively, we define a holomorphic imbedding $\iota$ of $P_{n_1}(C) \times P_{n_2}(C)$ into the complex projective space $P_{n_1+n_2+n_1+n_2}(C)$ associated to the tensor product $\mathbb{C}^{n_1+1} \otimes \mathbb{C}^{n_2+1}$ by
where \([\ast]\) denotes the point of the projective space with homogeneous coordinates \(\ast\). Then it defines a full Kählerian imbedding

\[ i: P_{N_1}(c) \times P_{N_2}(c) \rightarrow P_{N_1N_2+N_1+N_2}(c). \]

Let

\[ f_i: (M_i, g_i) \rightarrow P_{N_i}(c) \quad (i = 1, 2) \]

be two Kählerian immersions. Then the composite

\[ f_1 \otimes f_2 = i_0(f_1 \times f_2): (M_1 \times M_2, g_1 \times g_2) \rightarrow P_{N_1N_2+N_1+N_2}(c) \]

is also a Kählerian immersion, which will be called the tensor product of \(f_1\) and \(f_2\).

One can easily check the associativity

\[ (f_1 \otimes f_2) \otimes f_3 = f_1 \otimes (f_2 \otimes f_3) \]

of the tensor product, and so the multi-fold tensor product \(f_1 \otimes \cdots \otimes f_s\) is well-defined.

Now let

\[ f_i: (M_i, g_i) \rightarrow P_{N_i}(c) \quad (1 \leq i \leq s) \]

be full Kählerian imbeddings of irreducible symmetric Kählerian manifolds of compact type constructed as before. Then the tensor product

\[ f = f_1 \otimes \cdots \otimes f_s: (M_1 \times \cdots \times M_s, g_1 \times \cdots \times g_s) \rightarrow P_N(c), \]

where \(N = \prod_{i=1}^s (N_i + 1) - 1\), is a full Kählerian imbedding of the symmetric Kählerian manifold \((M, g) = (M_1 \times \cdots \times M_s, g_1 \times \cdots \times g_s)\). Note that

\[ G^c = G^c_1 \times \cdots \times G^c_s, \quad G = G_1 \times \cdots \times G_s, \]

where \(G^c = \text{Aut}^c(M)\), \(G = \text{Aut}^c(M, g)\), \(G^c_i = \text{Aut}^c(M_i)\), \(G_i = \text{Aut}^c(M_i, g_i)\), and that \(f\) is \(G^c\)-equivariant by the homomorphism \(\rho = \rho_1 \otimes \cdots \otimes \rho_s\) induced from the external tensor product \(\hat{\rho_1} \otimes \cdots \otimes \hat{\rho_s}\) of respective representations \(\hat{\rho_i}\). The tangent space \(T_o(M)\) of \(M\) at the origin \(o = o_1 \times \cdots \times o_s\) of \(M\), where \(o_i\) is the origin of \(M_i\), is identified with the direct sum

\[ m = m_1 \oplus \cdots \oplus m_s \]

of respective complements \(m_i\), and hence \(T_o(M)^+\) with

\[ m^+ = m_1^+ \oplus \cdots \oplus m_s^+. \]

Further the stabilizer \(K\) of the origin \(o\) in \(G\) is the direct product
(2.4) \[ K = K_1 \times \cdots \times K_s \]

of respective stabilizers \( K_i \).

It is known (Nakagawa-Takagi [5]. See also Takeuchi [8]) that any full Kahlerian immersion into \( P_n(c) \) of a symmetric Kahlerian manifold of compact type is obtained in this way.

3. Degree of symmetric Kahlerian submanifolds of \( P_n(c) \)

Let \( f: (M, g) \to P_n(c) \) be the \( p \)-th full Kahlerian imbedding of an irreducible symmetric Kahlerian manifold \( (M, g) \) constructed in §2. We recall first the construction of the Hermann map for \( M \) (cf. Takeuchi [7]). Choose root vector \( E_{\alpha} \in \mathfrak{g}_a^c \) for \( \alpha \in \Sigma \) in such a way that

\[ [E_{\alpha}, E_{-\alpha}] = -\alpha, \quad (E_{\alpha}, E_{-\alpha}) = -1. \]

Then the complex conjugation \( X \mapsto \overline{X} \) of \( \mathfrak{g}^c \) with respect to \( \mathfrak{g} \) satisfies \( E_{\alpha} = E_{-\alpha} \) for each \( \alpha \in \Sigma \). We put

\[ X_\alpha = \sqrt{\frac{2}{(\alpha, \alpha)}} E_{\alpha}, \quad H_\alpha = \frac{2}{(\alpha, \alpha)} \alpha \quad \text{for} \quad \alpha \in \Sigma. \]

Then we have

\[ [X_\alpha, X_{-\alpha}] = -H_\alpha, \quad (X_\alpha, X_{-\alpha}) = -\frac{2}{(\alpha, \alpha)}, \quad X_\alpha = X_{-\alpha}. \]

Let \( \{\gamma_1, \ldots, \gamma_r\} \subset \Sigma^+ \) be a maximal system of strongly orthogonal roots containing the highest root \( \gamma_1 \) such that \( r = \text{rank } M \) and \( (\gamma_j, \gamma_j) = (\alpha_i, \alpha_i) \) for each \( j \) (cf. Helgason [2]). An injective homomorphism \( \phi_j: \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}^c \) is defined by

\[ X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_{\gamma_j}, \quad X^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \mapsto X_{-\gamma_j}, \]

\[ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto -H_{\gamma_j}. \]

Since \( \phi_j(-X) = \overline{\phi_j(X)} \) for \( X \in \mathfrak{sl}(2, \mathbb{C}) \), we have \( \phi_j(\mathfrak{su}(2)) \subset \mathfrak{g} \). If we define a map \( \phi \) from the \( r \)-fold direct sum \( \mathfrak{sl}(2, \mathbb{C})^r \) of \( \mathfrak{sl}(2, \mathbb{C}) \) into \( \mathfrak{g}^c \) by

\[ \phi(X_1, \ldots, X_r) = \sum_{j=1}^r \phi_j(X_j) \quad \text{for } X_j \in \mathfrak{sl}(2, \mathbb{C}), \]

then it is also an injective homomorphism such that \( \phi(\mathfrak{su}(2)^r) \subset \mathfrak{g} \). The extension of \( \phi \) to the \( r \)-fold direct product \( SL(2, \mathbb{C})^r \) of \( SL(2, \mathbb{C}) \) is also denoted by
\( \phi: SL(2, \mathcal{C}) \rightarrow G^c. \)

It satisfies \( \phi(SU(2)^r) \subset G. \) Putting

\[
SL(1, 1; \mathcal{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathcal{C}); \ c = 0 \right\},
\]

we identify the \( r \)-fold direct product \( P_1(\mathcal{C})^r \) of \( P_1(\mathcal{C}) \) with \( SL(2, \mathcal{C})/SL(1, 1; \mathcal{C}). \)

Then the map

\[
x SL(1, 1; \mathcal{C})^r \mapsto \phi(x) \rho \quad \text{for} \ x \in SL(2, \mathcal{C})^r
\]
defines a holomorphic imbedding

\[
\phi: P_1(\mathcal{C})^r \rightarrow M,
\]

which is \( SL(2, \mathcal{C})^r \)-equivariant:

\[
\phi(x \rho) = \phi(x) \phi(\rho) \quad \text{for} \ x \in SL(2, \mathcal{C})^r, \ \rho \in P_1(\mathcal{C})^r.
\]

The imbedding \( \phi \) is called the **Hermann map**. The Kählerian metric \( h \) on \( P_1(\mathcal{C})^r \) induced from \( (M, g) \) is the direct product \( h_1 \times \cdots \times h_r \) of Kählerian metrics \( h_j \) on \( P_1(\mathcal{C}) \) of constant holomorphic sectional curvatures, since \( SU(2)^r \) acts transitively on \( P_1(\mathcal{C})^r \) as Kählerian automorphisms of \( (P_1(\mathcal{C}), h) \). The tangent space \( T_\phi( P_1(\mathcal{C})^r)) \) will be identified with a subspace \( \mathfrak{p} \) of \( \mathfrak{m} \), and hence \( T_\phi( P_1(\mathcal{C})^r))^+ \) with a subspace \( \mathfrak{p}^+ \) of \( \mathfrak{m}^+ \).

**Lemma 4.** Let

\[
\phi: (P_1(\mathcal{C})^r, h_1 \times \cdots \times h_r) \rightarrow (M, g)
\]

be the Hermann map as above. Then:

1) \( \mathfrak{m}^+ = K\mathfrak{p}^+; \)

2) \( \phi \) is totally geodesic;

3) Each \( h_j \) has the holomorphic sectional curvature \( \frac{c}{\mathfrak{p}}. \)

**Proof.**

1) If we put

\[
U_{\gamma_j} = E_{\gamma_j} + E_{-\gamma_j}, \quad V_{\gamma_j} = \sqrt{-1}(E_{\gamma_j} - E_{-\gamma_j}) \quad (1 \leq j \leq r),
\]

\( \mathfrak{p} \) is spanned over \( \mathcal{R} \) by the \( U_{\gamma_j}, \ V_{\gamma_j} \ (1 \leq j \leq r) \). The subspace \( \mathfrak{a} \) of \( \mathfrak{m} \) spanned over \( \mathcal{R} \) by the \( U_{\gamma_j} \ (1 \leq j \leq r) \) is a maximal abelian subalgebra in \( \mathfrak{m} \), and hence \( \mathfrak{m} = K\mathfrak{a} \). Since the projection \( \omega: \mathfrak{m}^c \rightarrow \mathfrak{m}^+ \) relative to the decomposition \( \mathfrak{m}^c = \mathfrak{m}^+ \oplus \mathfrak{m}^+ \) is \( K \)-equivariant, we have \( \mathfrak{m}^+ = K\omega^+(\mathfrak{a}) \). But \( \omega^+(\mathfrak{a}) \) is spanned over \( \mathcal{R} \) by the \( E_{-\gamma_j} \ (1 \leq j \leq r) \) and hence is contained in \( \mathfrak{p}^+ = \omega^+(\mathfrak{p}) \), which is spanned over \( \mathcal{C} \) by the \( E_{-\gamma_j} \ (1 \leq j \leq r) \). Thus we conclude \( \mathfrak{m}^+ = K\mathfrak{p}^+ \).

2) From the relations
\[ [U_{\gamma_j}, V_{\gamma_j}] = 2\sqrt{-1} \gamma_j, \quad [\sqrt{-1} \gamma_j, U_{\gamma_j}] = V_{\gamma_j}, \quad [\sqrt{-1} \gamma_j, V_{\gamma_j}] = -U_{\gamma_j}, \]

we get \([[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}] = \mathfrak{p},\] and hence \(\phi\) is totally geodesic (cf. Helgason [2]).

3) Identifying \(X^+X^- = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)\) with a tangent vector of \(P_i(\mathbb{C})\) at the origin, we have by (2.2)

\[
h_j(X^+X^-, X^+X^-) = g(X_{\gamma_j} + X_{-\gamma_j}, X_{\gamma_j} + X_{-\gamma_j})
= -\frac{p(\alpha_1, \alpha_1)}{c}(X_{\gamma_j} + X_{-\gamma_j}, X_{\gamma_j} + X_{-\gamma_j})
= -\frac{2p(\gamma_j, \gamma_j)}{c}(X_{\gamma_j}, X_{-\gamma_j}) = \frac{2p(\gamma_j, \gamma_j)}{c}\frac{2}{(\gamma_j, \gamma_j)}
= p \frac{4}{c}.
\]

It follows that \(h_j\) is \(p\) times the Fubini-Study metric of \(P_i(\mathbb{C})\), which implies the assertion 3). q.e.d.

Now we shall prove the following

**Theorem 2.** Let

\[ f_i: (M_i, g_i) \to P_{N_i}(\mathbb{C}) \quad (1 \leq i \leq s) \]

be the \(p_i\)-th full Kählerian imbedding of an irreducible symmetric Kählerian manifold \((M_i, g_i)\) of compact type, with rank \(M_i = r_i\) \((1 \leq i \leq s)\), and

\[ f: (M, g) \to P_{N}(\mathbb{C}) \]

be the tensor product of the \(f_i\) \((1 \leq i \leq s)\). Then the degree \(d(f)\) of \(f\) is given by

\[ d(f) = \sum_{i=1}^{s} r_i p_i. \]

For the proof of the Theorem we need the following Lemma.

**Lemma 5** (Nakagawa-Takagi [5]). Let

\[ f: (M, g) \to P_{N}(\mathbb{C}) \]

be a Kählerian immersion of a locally symmetric Kählerian manifold \((M, g)\). Then:

1) \(\langle H(\otimes^m T_p(M)^\vee), H(\otimes^m T_p(M)^\vee)\rangle = \{0\}\) for \(m = m'\), and hence \(O^p(M) = H(\otimes^m T_p(M)^\vee)\) for each \(m\);

2) For each \(u = (u_1, \ldots, u_n) \in U(M, g),\)

\[ h(u_{i_1}, \ldots, u_{i_m} \otimes u_i) = -\frac{c}{2} \sum_{2 \leq r \leq m} \delta_{t r} H(u_{i_1}, \ldots, \hat{u}_{i_r}, \ldots, u_{i_m}) \]

\[ + \sum_{1 \leq i < j \leq n} \langle R(u_{i_r}, u_i) u_{i_1}, \ldots, u_{i_m} \rangle H(u_{n_1}, \ldots, \hat{u}_{i_r}, \ldots, u_{i_m}) (m \geq 3), \]
where $R$ is the curvature tensor of $(M, g)$.

Proof of Theorem 2. Let $r = r_1 + \cdots + r_s$ be the rank of $M$. We use the notation in the end of §2. Taking the direct product of respective homomorphisms $\phi_i: SL(2, \mathbb{C})^{r_i} \rightarrow G_i$ for $M_i (1 \leq i \leq s)$ and the one of Hermann maps $\phi_i: P_i \left( \frac{c}{p_i} \right)^{r_i} \rightarrow (M_i, g_i) (1 \leq i \leq s)$, we get a homomorphism $\phi: SL(2, \mathbb{C}) \rightarrow G$ such that $\phi(SU(2)c) \subset G$ and a totally geodesic Kählerian imbedding

$$\phi: P = P_1 \left( \frac{c}{p_1} \right)^{r_1} \times \cdots \times P_s \left( \frac{c}{p_s} \right)^{r_s} \rightarrow (M, g),$$

which is $SL(2, \mathbb{C})^r$-equivariant:

$$\phi(xp) = \phi(x)(p) \quad \text{for } x \in SL(2, \mathbb{C}), p \in P.$$

The tangent space $\mathfrak{p} = T_0(\phi(P))$ of $\phi(P)$ at the origin is the direct sum

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s$$

of respective tangent spaces $\mathfrak{p}_i$ of $\phi_i: P_i \left( \frac{c}{p_i} \right)^{r_i}$ at the origin, and hence

$$\mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \cdots \oplus \mathfrak{p}_s^+.$$

It follows from Lemma 4 and decompositions (2.3), (2.4) that

$$(3.1) \quad m^+ = Kp^+.$$

Let us consider a Kählerian imbedding

$$f' = f \circ \phi: P \rightarrow P_N(c).$$

If we put

$$\rho' = \rho \circ \phi: SL(2, \mathbb{C}) \rightarrow PL(N+1, \mathbb{C}),$$

$f'$ is $SL(2, \mathbb{C})^r$-equivariant by the homomorphism $\rho'$:

$$f'(xp) = \rho'(x)f'(p) \quad \text{for } x \in SL(2, \mathbb{C}), p \in P.$$

Note that $\rho'(SU(2)c) \subset PU(N+1) = \text{Aut}(P_N(c))$ and $SU(2)c$ acts transitively on $P$ as Kählerian automorphisms of $P$. We shall identify as $P \subset (M, g)$ through the imbedding $\phi$. Denote the higher fundamental forms of $f$ and $f'$ by $H$ and $H'$ respectively. We shall prove the following two assertions:

(i) $d(f) = d(f')$.

(ii) $d(f') = \sum r_i p_i$.

But in view of the $\text{Aut}^0(M, g)$-equivariance of $f$ and Lemma 5, 1), we know that each point of $M$ is regular and $d(f)$ is determined by conditions
\[ H_d^{(r)} \neq 0 \quad \text{and} \quad H_n^{(r)+1} = 0. \]

In the same way, \( d(f') \) is determined by conditions
\[ H_d^{(r)} \neq 0 \quad \text{and} \quad H_n^{(r)+1} = 0. \]

Here \( H_d \) and \( H_n \) are understood to be always not 0. Hence the assertion (i) is equivalent to the assertion
\[(i') \quad H^*_n = 0 \iff H^*_n = 0 \quad (m \geq 2).\]

Proof of (i'). Note first that if we denote by \( X \mapsto kX \) the action of \( k \in K \) on \( N_c(M) \) through the differential \( \rho(k)_* \), we have

\[ (3.2) \quad H(kX_1, \ldots, kX_m) = kH(X_1, \ldots, X_m) \quad \text{for} \quad X_i \in \mathfrak{m}^+, \quad k \in K, \]

because of the \( \text{Aut}^e(M, g) \)-equivariance of \( f \). Now
\[ H^*_n = 0 \]
\[ \Leftrightarrow H(X_1, \ldots, X_m) = 0 \quad \text{for each} \quad X_i \in \mathfrak{m}^+, \]
\[ \Leftrightarrow H(X, \ldots, X) = 0 \quad \text{for each} \quad X \in \mathfrak{p}^+ \quad \text{by Lemma 2.1),} \]
\[ \Leftrightarrow H(Y, \ldots, Y) = 0 \quad \text{for each} \quad Y \in \mathfrak{p}^+ \quad \text{by (3.1), (3.2),} \]
\[ \Leftrightarrow H'(Y, \ldots, Y) = 0 \quad \text{for each} \quad Y \in \mathfrak{p}^+ \quad \text{since} \quad \phi \quad \text{is totally geodesic,} \]
\[ \Leftrightarrow H'(Y, \ldots, Y_m) = 0 \quad \text{for each} \quad Y_i \in \mathfrak{p}^+ \quad \text{by Lemma 2.1) \}
\[ \Leftrightarrow H'^*_n = 0. \]

Proof of (ii). For an index \( j, 1 \leq j \leq r \), we define \( \nu(j) \), \( 1 \leq \nu(j) \leq s \), by
\[ \nu(j) = \nu \quad \text{if} \quad r_1 + \cdots + r_{s-1} + 1 \leq j \leq r_1 + \cdots + r_{s-1} + r_s. \]

Take a unitary frame \( u=(u_1, \ldots, u_r) \) of \( P \) at the origin \( o \) such that \( u_i \) is tangent to the \( i \)-th factor of \( P \) for each \( i \), and fix it once for all. Then the curvature tensor \( R \) of \( P \) satisfies
\[ (3.3) \quad \langle R(u_i, u_i)u_j, u_i \rangle = \frac{c}{p_{\nu(i)}} \delta_{ij} \delta_{j \nu} \delta_{kl}. \]

For each \( i, \ldots, i_m, j \) \( (m \geq 2) \), the following equality holds:
\[ (3.4) \quad h'(u_{i_1}, \ldots, u_{i_m}, u_j, u_j) = \frac{c(a_j+1)}{2p_{\nu(i)}} (a_j-p_{\nu(i)})H'(u_{i_1}, \ldots, u_{i_m}), \]
where \( a_j \) is an integer given by
Indeed, Lemma 5.2) and (3.3) imply

\[ h'(u_{i_1}, \ldots, u_{i_{m+1}}, u_j) = -\frac{c}{2} \sum_{i=1}^{m+1} \delta_{i,j} H'(u_{i_1}, \ldots, \hat{u}_i, \ldots, u_{i_{m+1}}) \]

\[ + \frac{c}{p_{\alpha(j)}} \sum_{1 \leq \alpha < \beta \leq m+1} \delta_{\alpha,j} \delta_{\beta,j} H'(u_{i_1}, \ldots, \hat{u}_\alpha, \ldots, \hat{u}_\beta, \ldots, u_{i_{m+1}}, u_j). \]

Put \( i_{m+1} = j \). Recalling that \( H' \) is symmetric, we have

\[ h'(u_{i_1}, \ldots, u_{i_m}, u_j) \]

\[ = -\frac{c}{2} \sum_{i=1}^{m} \delta_{i,j} H'(u_{i_1}, \ldots, \hat{u}_i, \ldots, u_{i_m}) - \frac{c}{2} H'(u_{i_1}, \ldots, u_m) \]

\[ + \frac{c}{p_{\alpha(j)}} \sum_{1 \leq \alpha < \beta \leq m} \delta_{\alpha,j} \delta_{\beta,j} H'(u_{i_1}, \ldots, \hat{u}_\alpha, \ldots, \hat{u}_\beta, \ldots, u_m, u_j) \]

\[ + \frac{c}{p_{\alpha(j)}} \sum_{1 \leq \alpha < \beta \leq m} \delta_{\alpha,j} H'(u_{i_1}, \ldots, \hat{u}_\alpha, \ldots, u_m, u_j) \]

\[ = \left\{ -\frac{c}{2} a_j - \frac{c}{2 p_{\alpha(j)}} + \frac{c}{2} \frac{a_j(a_j-1)}{2} + \frac{c}{p_{\alpha(j)}} \right\} H'(u_{i_1}, \ldots, u_m) \]

\[ = \frac{c(a_j+1)}{2 p_{\alpha(j)}} (a_j - p_{\alpha(j)}) H'(u_{i_1}, \ldots, u_m). \]

Now we are in a position to prove (ii). If \( d' = d(f') = 1 \), then \( f' \) is totally geodesic, and hence \( s = 1, r_1 = 1, p_1 = 1 \). So we may assume \( d' \geq 2 \). Then there exist indices \( i_1, \ldots, i_{d'} \) such that \( H'(u_{i_1}, \ldots, u_{i_{d'}}) \neq 0 \). It follows from (1.5) and \( H^{d'+1} = 0 \) that

\[ h'(u_{i_1}, \ldots, u_{i_{d'}}, u_j, u_j) = 0 \quad \text{for each } j, 1 \leq j \leq r. \]

Thus (3.4) implies

\[ \# \{ k; 1 \leq k \leq d', i_k = j \} = p_{\alpha(j)} \quad \text{for each } j, 1 \leq j \leq r, \]

and hence

\[ d' = \sum_{j=1}^{r} p_{\alpha(j)} = \sum_{i=1}^{s} r_i p_i. \]

q.e.d.
References


