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A SYMBOLIC CALCULUS FOR PSEUDO DIFFERENTIAL OPERATORS GENERATING FELLER SEMIGROUPS

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1. Introduction

It is well-known that the generator A of a time-homogenous Markov process in $\mathbb{R}^n$ is typically given by a Lévy-type operator

$$A\varphi(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2\varphi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial\varphi(x)}{\partial x_i} + c(x)\varphi(x)$$

$$(1.1) \qquad + \int_{\mathbb{R}^n\setminus\{0\}} \left(\varphi(x+y) - \varphi(x) - \frac{\langle y, \nabla\varphi(x)\rangle}{1+|y|^2}\right)\mu(x,dy), \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

This follows immediately from the fact that the generator of a transition semigroup satisfies the positive maximum principle, i.e. for any φ in the domain of the generator and $x_0 \in \mathbb{R}^n$ such that $\varphi(x_0) = \sup_{x\in\mathbb{R}^n}\varphi(x) \geq 0$ we have $A\varphi(x_0) \leq 0$ and by a result of Ph. Courrège [4] which characterizes the operators satisfying the positive maximum principle as operators of type (1.1). But Courrège gave also another equivalent representation of this class of operators as pseudo differential operators

$$(1.2) \quad A\varphi(x) = -p(x,D)\varphi(x) = -\int_{\mathbb{R}^n} e^{i(x,\xi)}p(x,\xi)\cdot\hat{\varphi}(\xi)\,đ\xi, \quad \varphi \in C_0^\infty(\mathbb{R}^n),$$

defined by a symbol $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ having the crucial property that for fixed $x \in \mathbb{R}^n$ the function $p(x,\cdot)$ is a continuous negative definite function (see section 2 for the definition). Such symbols we briefly call negative definite symbols. Here $\hat{\varphi} = \int_{\mathbb{R}^n} e^{-i(x,\xi)}\varphi(x)\,dx$ denotes the Fourier transform and $đ\xi = (2\pi)^{-n}\,d\xi$. Conversely, if the symbol is a continuous negative definite function for every fixed $x \in \mathbb{R}^n$ then the operator $-p(x,D)$ satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$.

The relation between (1.1) and (1.2) is given by the Lévy-Khinchin formula, see [2], which represents the continuous negative definite functions $p(x,\cdot)$ (for fixed x) in terms of the coefficients $a_{ij}(x)$, $b_i(x)$, $c(x)$ and the Lévy-measures $\mu(x,dy)$ of (1.1). In this paper we focus on the representation (1.2) as a pseudo differential operator and look for conditions purely in terms of the symbol $p(x,\xi)$ implying that the operator $-p(x,D)$ actually generates a Markov process. We are interested in particular in the case that the second order diffusion part might vanish completely and the non-local integro-differential part in (1.1) becomes dominating. As a final result we will determine a class of operators generating Feller semigroups (see Theorem 5.7).

In the particular case of a symbol $p(x,\xi) = p(\xi)$ which is independent of x the operator $-p(D)$ generates a convolution semigroup and the corresponding process is a Lévy process. Moreover the negative definite symbol function $p(\xi)$ is nothing but the characteristic exponent of the Lévy process and in this way a complete ono-to-one correspondence between negative definite functions and Lévy processes is given (see [14] for a probabilistic interpretation of the symbol in the general case). Even in this simple x-independent case the most standard example of symmetric α-stable processes show that the corresponding symbol $p(\xi) = |\xi|^\alpha$, $0 < \alpha \leq 2$, is not differentiable unless $\alpha = 2$, i.e. in the case of Brownian motion. From this we see that it is an intrinsic property of the regarded symbol class that they are in general not differentiable with respect to ξ. Hence these symbols do not fit into any known symbol class of pseudo differential operators and we cannot apply pseudo differential calculus without further considerations.

For that reason many approaches to Lévy-type operators besides those which study the case of a dominating diffusion term either concentrate on the representation (1.1) with certain integrability conditions on the Lévy-kernel $\mu(x,dy)$ (see [24], [27], [21]) or they make some homogenity assumptions on the symbol with respect to ξ and often consider perturbations of α-stable and so-called stable-like processes (see [17], [18], [1], [23], [16], [19], [15]). For symmetric stable processes perturbed by singular drifts see also [26] and [25].

In [11], [12], [13] N. Jacob took a general continuous negative definite function $a^2(\xi)$ as the starting point and considered symbols $p(x,\xi)$ defined in terms of this function. Thus these symbols are typically not differentiable with respect to ξ. In this situation the Lévy process associated to a^2 deals as a kind of model process for the jump process generated by the operator $-p(x,D)$ with "variable coefficients". Besides the existence of a corresponding transition semigroup Jacob's approach also yields L^2-estimates for the generator which also have some probabilistic consequences for the process. However the perturbation argument used there allows only small perturbations, in particular the oscillations of the symbol must vanish asymptotically as $|x| \to \infty$. For the same type of generators in [7], [8], [9] the process and the semigroup are constructed via the martingale problem. By this method the strict oscillation bounds of [12] and [13] can be avoided and the result is applicable to a much larger class of symbols, but the useful L^2-estimates can not be obtained in this way.

The starting point in this paper is different since we want to construct a calculus of pseudo differntial operators similar to the classical case of Hörmander classes $S^m_{\varrho,\delta}$, i.e. symbols satisfying the estimate

$$(1.3) \qquad \left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leq c_{\alpha\beta}(1+|\xi|^2)^{(1/2)(m-\varrho|\alpha|+\delta|\beta|)}, \quad \alpha,\beta \in \mathbb{N}_0^n,$$

where $0 \leq \delta \leq \varrho \leq 1$ and $m \in \mathbb{R}$ gives the order of the symbol. Hörmander type symbols are no good choice in general for the operators we have in mind, but anyhow such calculus typically yields estimates in L^2-context and on the other hand needs no oscillation bounds for the symbols. In this way it combines in a sense the advantages of both upper approaches.

Again we look at a general Lévy-process as a model case leading to symbols which are not differentiable. But differentiability of the symbol in particular with respect to ξ is indispensable for a symbolic calculus. Therefore we first decompose the symbol into a differentiable part and a remainder part which is considered as a perturbation. For that purpose the Lévy-measures of the kernel $\mu(x, dy)$ in (1.1) are split into a part supported in a bounded neighbourhood of the origin and a part supported on the complement. Due to the fact that most of the mass of a Lévy-measure typically is concentrated around the origin, it turns out that the latter part defines a low order perturbation of the operator. In particular under quite natural assumptions this perturbation is a bounded operator on the space of continuous functions as well as on L^2. For example the property that an operator generates a Feller semigroup is stable under such perturbations and also L^2-estimates are preserved. These aspects are discussed in more detail in the paper [10].

We therefore focus on symbols with Lévy-measures supported in a bounded neighbourhood of the origin. Recall that in probability theory this assumption is often made from the very beginning and corresponds to the fact that the jumps of the associated process are bounded. It turns out that these symbols are differentiable with respect to ξ, see Prop. 2.1. But in order to get a symbolic calculus with good asymptotic expansions, it is important that moreover the derivatives satisfy certain growth estimates at infinity. In the case (1.3) of Hörmander type symbols in $S^m_{\varrho,0}$, $\varrho > 0$, $\partial_\xi^\alpha \partial_x^\beta p(x,\xi)$ is estimated by powers of $(1 + |\xi|^2)^{1/2}$ and the power decreases when $|\alpha|$ is growing. This lead to asymptotic expansion series of symbols of decreasing order. In the situation of symbols considered here it now turns out, and this is the crucial point, that the derivatives satisfy estimates similar to (1.3): When we define the class of symbols in terms of the fixed continuous negative definite function a^2 and let $\lambda(\xi) = (1 + a^2(\xi))^{1/2}$ then we have

$$\left|\partial_\xi^\alpha p(x,\xi)\right| \leq c_\alpha \lambda^{2-\varrho(|\alpha|)}, \quad \alpha \in \mathbb{N}_0^n, \tag{1.4}$$

where the weight function $(1 + |\xi|^2)^{1/2}$ is replaced by $\lambda(\xi)$ and $\varrho(k) = k \wedge 2$. This behaviour follows only by the fact that $p(x, \xi)$ is a negative definite symbol. Therefore it is natural to define for every continuous negative definite function a^2 symbol classes $\mathrm{S}^{m,\lambda}_\varrho$, $m \in \mathbb{R}$, given by

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leq c_{\alpha\beta} \lambda^{m-\varrho(|\alpha|)}, \quad \alpha, \beta \in \mathbb{N}_0^n, \tag{1.5}$$

which in the case $m = 2$ typically contain negative definite symbols.

Symbol classes defined by general so-called basic weight functions $\lambda(\xi)$ had been considered before by H. Kumano-go (see [20]), but his assumptions on λ are not satisfied by continuous negative definite functions in general. Therefore the major part of the work that has to be done is to show that arguments similar to those in [20] can be applied in the situation here. For that purpose we have to exploit certain estimates for continuous negative definite functions that replace estimates for the basic weight functions used in [20]. This will be done in section 3 where a symbolic calculus for $S_\varrho^{m,\lambda}$ is established and expansion formulas are proven. Since the estimate in (1.5) do not improve for $|\alpha| > 2$ these expansion will not be asymptotic, i.e. the series contains only a finite number of terms of decreasing order, whereas the subsequent terms do not improve the expansion. See also the paper [22] of M. Nagase where he also considers basic weight functions and the case of a general behaviour of the derivatives of the symbol in terms of a general function ϱ. In his paper Nagase also lines out how the technique of Friedrichs symmetrization applies to his class of symbols. In section 4 we adapt this procedure to our situation proving also in our case a Friedrichs symmetrization and a sharp Gårding inequality.

In the final section we prove as an application that elliptic elements in the class $S_\varrho^{2,\lambda}$ give examples of generators of Feller semigroups. For that purpose as in the papers [12], [13] of Jacob we use the Hille–Yosida theorem and prove in particular the existence of the resolvent of the operator. This means we look for solutions of the equation $(p(x,D) + \tau)u = f$ for some $\tau > 0$ in appropriate function spaces. The calculus developed so far is then applied to this problem and gives the solution via an approach by modified Hilbert space methods.

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2. Some properties of negative definite functions

A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called a negative definite function if for all $m \in \mathbb{N}$ and all m-tuples $(\xi^1, \ldots, \xi^m)$, $\xi^j \in \mathbb{R}^n$, $1 \le j \le m$, the matrix

$$(\psi(\xi^i) + \overline{\psi(\xi^j)} - \psi(\xi^i - \xi^j))_{i,j=1,\ldots,m}$$

is positive Hermitian, i.e. for all $c_1, \ldots, c_m \in \mathbb{C}$

$$\sum_{i,j=1}^{m} (\psi(\xi^i) + \overline{\psi(\xi^j)} - \psi(\xi^i - \xi^j))\ c_i\overline{c_j} \ge 0.$$

In the following we will restrict to real-valued negative definite functions for simplicity and the term negative definite function always implies real values. For more details and examples concerning negative definite functions and the following results we refer to the monograph [2].

The set of negative definite functions forms a convex cone and we have

$$\psi(\xi) \geq \psi(0) \geq 0.$$

If ψ is moreover continuous, then there is a constant c_ψ such that

$$\psi(\xi) \leq c_\psi(1 + |\xi|^2). \tag{2.1}$$

The following inequality is a very useful substitute for the triangle inequality. Let ψ be a negative definite function, then

$$\psi^{1/2}(\xi + \eta) \leq \psi^{1/2}(\xi) + \psi^{1/2}(\eta), \quad \xi, \eta \in \mathbb{R}^n. \tag{2.2}$$

As a consequence we have the following analogue of Peetre's inequality (see [6])

$$\frac{1 + \psi(\xi)}{1 + \psi(\eta)} \leq c(1 + \psi(\xi - \eta)). \tag{2.3}$$

Finally recall the important Lévy-Khinchin formula: Every (real-valued) continuous negative definite function ψ has a representation

$$\psi(\xi) = c + q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(y, \xi))\mu(dy), \tag{2.4}$$

where $c \geq 0$ is a constant, $q \geq 0$ is a quadratic form and μ is a symmetric Borel measure on $\mathbb{R}^n \setminus \{0\}$ called the Lévy-measure having the property that

$$\int_{\mathbb{R}^n \setminus \{0\}} \frac{|y|^2}{1 + |y|^2} \mu(dy) < \infty.$$

This correspondence is one-to-one.

In general a continuous negative definite function is not differentiable. In order to define reasonable symbol classes we therefore have to restrict to a subclass of continuous negative definite functions. We have the following result.

Proposition 2.1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function with Lévy-Khinchin representation* (2.4). *Suppose that for $l \geq 2$ all absolute moments*

$$M_l = \int_{\mathbb{R}^n \setminus \{0\}} |y|^l \mu(dy)$$

of the Lévy-measure exist. In particular this holds when $\operatorname{supp}\mu$ *is bounded. Then ψ is infinitely often differentiable and we have the estimate*

$$
\left|\partial_\xi^\alpha \psi(\xi)\right| \le c_{|\alpha|} \begin{cases} \psi(\xi) & \alpha = 0 \\ \psi^{1/2}(\xi) \quad \text{if} & |\alpha| = 1 \ , \quad \alpha \in \mathbb{N}_0^n, \\ 1 & |\alpha| \ge 2 \end{cases} \tag{2.5}
$$

where $c_0 = 1$, $c_1 = (2M_2)^{1/2} + 2\Lambda^{1/2}$, $c_2 = M_2 + 2\Lambda$ *and* $c_l = M_l$ *for* $l > 2$ *and* Λ *is the maximal eigenvalue of the quadratic form of* ψ *in* (2.4).

Proof. For $\alpha = 0$ there is nothing to prove. Let $|\alpha| \ge 1$. We may consider all terms in the representation (2.4) of ψ separately. The constant term is trivial and the estimate (2.5) is well-known for the quadratic form with constants $c_1 = 2\Lambda^{1/2}$, $c_2 = 2\Lambda$ and $c_l = 0$ for $l > 2$. So we may restrict to the integral part in (2.4) and assume that

$$
\psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(y,\xi))\mu(dy).
$$

Since the moments M_l, $l \ge 2$ are bounded, we may exchange differentiation and integration and find

$$
\partial_\xi^\alpha \psi(\xi) = -\int_{\mathbb{R}^n\setminus\{0\}} y^\alpha \cos^{(|\alpha|)}(y,\xi)\mu(dy),
$$

which gives for $|\alpha| = 1$ by Cauchy-Schwarz inequality

$$
\begin{aligned}
|\partial_{\xi_i}\psi(\xi)| &\le \left(\int_{\mathbb{R}^n\setminus\{0\}} |y_i|^2 \mu(dy)\right)^{1/2} \cdot \left(\int_{\mathbb{R}^n\setminus\{0\}} \sin^2(y,\xi)\mu(dy)\right)^{1/2} \\
&\le \left(\int_{\mathbb{R}^n\setminus\{0\}} |y|^2 \mu(dy)\right)^{1/2} \cdot \left(2\int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\mu(dy)\right)^{1/2} \\
&= (2M_2)^{1/2} \cdot \psi^{1/2}(\xi)
\end{aligned}
$$

and for $|\alpha| \ge 2$

$$
\begin{aligned}
\left|\partial_\xi^\alpha \psi(\xi)\right| &\le \int_{\mathbb{R}^n\setminus\{0\}} |y^\alpha| \left|\cos^{(|\alpha|)}(\xi,y)\right| \mu(dy) \\
&\le \int_{\mathbb{R}^n} |y|^{|\alpha|} \mu(dy) = M_{|\alpha|}. \qquad \square
\end{aligned}
$$

Remark 2.2. By [8], Lemma 2.2, there is a bounded measure ν_n on $\mathbb{R}^n \setminus \{0\}$ such that the continuous negative definite function $y \mapsto |y|^2/(1+|y|^2)$ has the Lévy-Khinchin representation

$$\frac{|y|^2}{1+|y|^2} = \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\nu_n(d\xi)$$

and

$$\int_{\mathbb{R}^n\setminus\{0\}} (1+|\xi|^2)\nu_n(d\xi) < \infty.$$

Consider in the situation of Proposition 2.1 a family of continuous negative definite functions $(\psi_i)_{i\in I}$ with Lévy-measures μ_i supported in a fixed ball $B_R(0)$. Then for $l \geq 2$

$$\begin{aligned} M_l &= \int_{B_R(0)\setminus\{0\}} |y|^l \,\mu_i(dy) \leq c_{R,l} \int_{B_R(0)\setminus\{0\}} \frac{|y|^2}{(1+|y|^2)}\mu_i(dy) \\ &\leq c_{R,l} \int_{\mathbb{R}^n\setminus\{0\}} \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\mu_i(dy)\,\nu_n(d\xi) \leq c_{R,l} \int_{\mathbb{R}^n\setminus\{0\}} \psi_i(\xi)\,\nu_n(d\xi). \end{aligned}$$

Moreover for the largest eigenvalue Λ_i of the quadratic form q_i of ψ_i we have

$$\Lambda_i = \sup_{|\xi|\leq 1} q_i(\xi) \leq \sup_{|\xi|\leq 1} \psi_i(\xi).$$

Therefore, if all ψ_i, $i \in I$, are uniformly bounded by a fixed continuous negative definite function ψ, i.e.

$$\psi_i(\xi) \leq \psi(\xi) \qquad \text{for all } \xi \in \mathbb{R}^n,$$

the same constant $c_{|\alpha|}$ may be chosen in (2.5) for all ψ_i.

3. The symbol classes $S_\varrho^{m,\lambda}$ and $S_0^{m,\lambda}$

We consider the case of negative definite symbols which are real-valued. The idea to get good estimates for operators of type (1.2) is as in [8] and [13] to require that the symbol satisfies upper and lower estimates in terms of a fixed continuous negative definite function. For that purpose fix for the following a continuous negative definite function

$$a^2 : \mathbb{R}^n \to \mathbb{R}$$

with Lévy-measure supported in some bounded set. This support condition for the Lévy-measure is no restriction since we will consider only symbols with this property and the same cut-off procedure that we will apply to symbols can be applied to a^2.

Our symbol class will be defined in terms of the function a^2 or equivalently, but in a more convenient way, by the square root

$$\lambda(\xi) = (1+a^2(\xi))^{1/2}. \tag{3.1}$$

Furthermore to simplify the notation we introduce

$$\varrho(k) = k \wedge 2, \quad k \in \mathbb{N}_0. \tag{3.2}$$

Consider a symbol $\tilde{p}(x,\xi)$ as in the introduction, i.e. $\tilde{p} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous and $\xi \mapsto \tilde{p}(x,\xi)$ is negative definite for all $x \in \mathbb{R}^n$. Then the Lévy-Khinchin formula yields

$$\tilde{p}(x,\xi) = c(x) + q(x,\xi) + \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(y,\xi))\mu(x,dy),$$

where c, q, and μ satisfy for each $x \in \mathbb{R}^n$ the same conditions as the corresponding terms in (2.4). Let $\theta \in C_0^\infty(\mathbb{R}^n)$, $0 \le \theta \le 1$, be a some even cut-off function such that $\theta(x) = 1$ in a neighbourhood of the origin. Having in mind Proposition 2.1 we decompose

$$\tilde{p}(x,\xi) = p(x,\xi) + p_r(x,\xi)$$

by splitting its Lévy-measures into a leading term

$$p(x,\xi) = c(x) + q(x,\xi) + \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(y,\xi))\,\theta(y)\,\mu(x,dy)$$

and a remainder term

$$p_r(x,\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(y,\xi))\,(1 - \theta(y))\,\mu(x,dy)$$

Then p and $p_r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions and for fixed $x \in \mathbb{R}^n$ negative definite with respect to ξ (see (3.5) and (3.6) in the proof of Lemma 3.6 in [8] for the continuity of p and p_r, the particular choice of θ does not affect that proof). Furthermore Proposition 2.1 applies to the symbol $p(x,\xi)$.

Suppose that $\tilde{p}(x,\xi)$ is comparable with $a^2(\xi)$ in the sense that

$$\tilde{p}(x,\xi) \le c(1 + a^2(\xi)). \tag{3.3}$$

Then with the notation of Remark 2.2 we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\{0\}} (1-\theta(y))\,\mu(x,dy) &\le c\int_{\mathbb{R}^n\setminus\{0\}} \frac{|y|^2}{1+|y|^2}\,\mu(x,dy) \le c\int_{\mathbb{R}^n\setminus\{0\}} \tilde{p}(x,\xi)\,\nu_n(d\xi) \\ &\le c\int_{\mathbb{R}^n\setminus\{0\}} (1 + a^2(\xi))\,\nu_n(d\xi).\end{aligned}$$

Therefore the Lévy-measures $(1 - \theta(y))\mu(x,dy)$ of $p_r(x,\xi)$ have uniformly bounded mass and the representation (1.1)

$$p_r(x,D)\varphi(x) = -\int_{\mathbb{R}^n\setminus\{0\}} (\varphi(x+y)-\varphi(x))(1-\theta(y))\,\mu(x,dy)$$

shows that $p_r(x,D)$ is bounded as an operator on the spaces of bounded Borel measurable functions and bounded continuous functions. Moreover, under a mild additional conditions $C_\infty(\mathbb{R}^n)$, the set of continuous functions vanishing at infinity, is invariant and typically the operator is bounded on $L^2(\mathbb{R}^n)$, see also [10]. Therefore as mentioned in the introduction we regard $p_r(x,D)$ as a perturbation of $p(x,D)$, which doesn't change the major results we have in mind and we will look in following to the part $p(x,\xi)$ which contains the typically dominating part of the Lévy-measure concentrated around the origin.

By Proposition 2.1 we see

$$\begin{aligned}(3.4)\qquad \left|\partial_\xi^\alpha p(x,\xi)\right| &\le c_\alpha p(x,\xi)^{(1/2)(2-\varrho(|\alpha|))} \le c_\alpha(1+a^2(\xi))^{(1/2)(2-\varrho(|\alpha|))}\\ &= c_\alpha\lambda(\xi)^{(2-\varrho(|\alpha|))}\end{aligned}$$

with a constant c_α not depending on x by Remark 2.2. The estimate (3.4) reflects the typical behaviour of negative definite symbols and in order to define a proper symbol class it is quite natural to assume the same estimate for the derivates $\partial_x^\beta p(x,\xi)$ of the symbol. Therefore for $m\in\mathbb{R}$ we define $\mathrm{S}_\varrho^{m,\lambda}$ to be the class of symbols of order m consisting of all C^∞-functions $p(x,\xi):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ such that

$$(3.5)\qquad \left|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)\right| \le c_{\alpha,\beta}\lambda(\xi)^{m-\varrho(|\alpha|)},\quad x\in\mathbb{R}^n,\ \xi\in\mathbb{R}^n,\ \alpha,\beta\in\mathbb{N}_0^n.$$

For example the condition with $m=2$ is fulfilled if the Lévy-kernel of $p(x,\xi)$ has densities with respect to a certain Lévy-measure $\tilde{\mu}$, i.e. $\mu(x,dy)=f(x,y)\,\tilde{\mu}(dy)$, such that $f(\cdot,y)$, $y\in\mathbb{R}^n\setminus\{0\}$, is uniformly bounded in $C_b^\infty(\mathbb{R}^n)$ and also the coefficients of $q(x,\xi)$ and $c(x)$ are in $C_b^\infty(\mathbb{R}^n)$.

Let us also define the larger symbol class $\mathrm{S}_0^{m,\lambda}$, which is an analogue of the Hörmander class $S_{0,0}^m$, consisting of symbols such that

$$(3.6)\qquad \left|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)\right| \le c_{\alpha,\beta}\lambda(\xi)^{m},\quad x\in\mathbb{R}^n,\ \xi\in\mathbb{R}^n,\ \alpha,\beta\in\mathbb{N}_0^n.$$

First we remark that $\lambda^m(\xi)$ gives a generic example of symbols in $\mathrm{S}_\varrho^{m,\lambda}$.

Lemma 3.1. *For $m\in\mathbb{R}$ and $\alpha\in\mathbb{N}_0^n$ we have*

$$(3.7)\qquad \left|\partial_\xi^\alpha\lambda^m(\xi)\right| \le c_\alpha\lambda(\xi)^{m-\varrho(|\alpha|)}.$$

In particular $\lambda^m\in S_\varrho^{m,\lambda}$.

Proof. By Proposition 2.1 we know

$$\left|\partial_\xi^\alpha(1+a^2(\xi))\right| \le c_\alpha(1+a^2(\xi))^{1/2(2-\varrho(|\alpha|))}$$

and therefore

$$\left|\frac{\partial_\xi^\alpha(1+a^2(\xi))}{1+a^2(\xi)}\right| \le c_\alpha(1+a^2(\xi))^{-(1/2)\varrho(|\alpha|)} = c_\alpha\lambda(\xi)^{-\varrho(|\alpha|)} \tag{3.8}$$

holds. Next note that by induction on $|\alpha|$ using Leibniz rule we have

$$\begin{aligned}\partial_\xi^\alpha\lambda^m(\xi) &= \partial_\xi^\alpha[(1+a^2(\xi))^{m/2}] \\ &= (1+a^2(\xi))^{m/2}\sum_{\alpha_1+\ldots+\alpha_{|\alpha|}=\alpha} c(\alpha_1,\ldots,\alpha_{|\alpha|},m)\prod_{i=1}^{|\alpha|}\frac{\partial_\xi^{\alpha_i}(1+a^2(\xi))}{1+a^2(\xi)},\end{aligned}$$

where $\alpha_1,\ldots,\alpha_{|\alpha|}\in\mathbb{N}_0^n$, and therefore

$$\begin{aligned}\left|\partial_\xi^\alpha\lambda^m(\xi)\right| &\le c_\alpha\lambda^m(\xi)\sum_{\alpha_1+\ldots+\alpha_{|\alpha|}=\alpha}\prod_{i=1}^{|\alpha|}\lambda^{-\varrho(|\alpha_i|)} \\ &\le c_\alpha\lambda^m(\xi)\cdot\sum_{\alpha_1+\ldots+\alpha_{|\alpha|}=\alpha}\lambda(\xi)^{-\sum_{i=1}^{|\alpha|}\varrho(|\alpha_i|)} \\ &\le c_\alpha\lambda^{m-\varrho(|\alpha|)}\end{aligned}$$

by subadditivity of ϱ. □

Clearly for two symbols $p_i\in S_0^{m_i,\lambda}$, $i=1,2$, by Leibniz rule we have

$$\begin{aligned}\left|\partial_\xi^\alpha\partial_x^\beta(p_1\cdot p_2)(x,\xi)\right| &\le c\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}}\left|\partial_\xi^{\alpha'}\partial_x^{\beta'}p_1(x,\xi)\right|\cdot\left|\partial_\xi^{\alpha''}\partial_x^{\beta''}p_2(x,\xi)\right| \\ &\le c\lambda^{m_1+m_2}(\xi),\end{aligned} \tag{3.9}$$

i.e. $p_1\cdot p_2\in S_0^{m_1+m_2,\lambda}$ and $(S_0^{m,\lambda})_{m\in\mathbb{R}}$ forms an algebra of symbols in the usual sense. For symbols in $S_\varrho^{m,\lambda}$ and $S_0^{m,\lambda}$ we denote the corresponding classes of operators defined by

$$p(x,D)\varphi(x) = \int_{\mathbb{R}^n} e^{i(x,\xi)}p(x,\xi)\cdot\hat{\varphi}(\xi)\,d\!\!\!{}^-\xi$$

by $\Psi_\varrho^{m,\lambda}$ and $\Psi_0^{m,\lambda}$. As usual we write $D=(D_{x_1},\ldots,D_{x_n})=(-i\partial_{x_1},\ldots,-i\partial_{x_n})$. By (2.1) the operators are well defined on $\mathcal{S}(\mathbb{R}^n)$ and moreover for $u\in\mathcal{S}(\mathbb{R}^n)$, $\alpha,\beta\in$

$\mathbb{N}_0^n$ and $N > |\beta| + m + n$

$$\left|\partial_x^\beta\left(x^\alpha \int_{\mathbb{R}^n} e^{i(x,\xi)} p(x,\xi)\hat{u}(\xi) đ\xi\right)\right| = \left|\partial_x^\beta\left(\int_{\mathbb{R}^n} e^{i(x,\xi)} D_\xi^\alpha(p(x,\xi)\hat{u}(\xi)) đ\xi\right)\right|$$

$$= \left|\int_{\mathbb{R}^n} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} \binom{\alpha}{\alpha_1}\binom{\beta}{\beta_1} (i\xi)^{\beta_1} \cdot e^{i(x,\xi)} \partial_x^{\beta_2} D_\xi^{\alpha_1} p(x,\xi) D_\xi^{\alpha_2}\hat{u}(\xi) đ\xi\right|$$

$$\leq c \int_{\mathbb{R}^n} \langle\xi\rangle^{|\beta|} \cdot \lambda^m(\xi) \cdot \sum_{|\gamma|\leq|\alpha|} \left|\partial_\xi^\gamma \hat{u}(\xi)\right| đ\xi$$

$$\leq c \int_{\mathbb{R}^n} \langle\xi\rangle^{|\beta|+m-N} đ\xi \cdot \sup_{\xi\in\mathbb{R}^n} \left[\langle\xi\rangle^N \sum_{|\gamma|\leq|\alpha|} \left|\partial_\xi^\gamma \hat{u}(\xi)\right|\right].$$

Here we use the usual notation

$$\langle\xi\rangle = (1 + |\xi|^2)^{1/2}.$$

Since the Fourier transform is continuous on $\mathcal{S}(\mathbb{R}^n)$ this gives

Proposition 3.2. *An operator $p(x, D) \in \Psi_0^{m,\lambda}$ maps $\mathcal{S}(\mathbb{R}^n)$ continuously into itself.*

Let us recall the definition of oscillatory integrals (see [20, Chapt.1.6]). A C^∞-function g on $\mathbb{R}^n \times \mathbb{R}^n$ is called of class $\mathcal{A}$ if the estimates

$$\left|\partial_\eta^\alpha \partial_y^\beta g(\eta, y)\right| \leq c_{\alpha\beta} \langle\eta\rangle^{m+\delta|\beta|} \langle y\rangle^\tau, \quad \alpha, \beta \in \mathbb{N}_0^n \tag{3.10}$$

hold for suitable $m \in \mathbb{R}$, $0 \leq \delta < 1$ and $\tau \geq 0$. In this case the oscillatory integral is defined by

$$Os - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y,\eta)} g(\eta, y)\, dyđ\eta = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y,\eta)} \chi(\varepsilon\eta, \varepsilon y) g(\eta, y)\, dyđ\eta. \tag{3.11}$$

where $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ having the property $\chi(0) = 1$. The oscillatory integral is well-defined for any g of class $\mathcal{A}$ and independent of the particular choice of the function χ.

If we choose $l, l' \in \mathbb{N}_0$ sufficiently large (depending on m, δ and τ) the oscillatory integral coincides with the ordinary integral

$$\begin{aligned} &Os - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y,\eta)} g(\eta, y)\, dyđ\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y,\eta)} \langle y\rangle^{-2l'} \langle D_\eta\rangle^{2l'} \left\{\langle\eta\rangle^{-2l} \langle D_y\rangle^{2l} g(\eta, y)\right\} dyđ\eta. \end{aligned} \tag{3.12}$$

Moreover the following partial integration rule holds

$$(3.13)\qquad \begin{aligned} &Os - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y,\eta)}\eta^\alpha g(\eta,y)\,dy\bar{d}\eta \\ &= Os - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y,\eta)} D_y^\alpha g(\eta,y)\,dy\bar{d}\eta, \quad \alpha\in\mathbb{N}_0^n. \end{aligned}$$

We introduce the class of double symbols $S_0^{m_1,m_2,\lambda}$, $m_1,m_2\in\mathbb{R}$, in terms of the weight function λ consisting of all $p\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$ satisfying

$$(3.14)\qquad \left|\partial_\xi^\alpha\partial_x^\beta\partial_{\xi'}^{\alpha'}\partial_{x'}^{\beta'}p(x,\xi,x',\xi')\right| \le c_{\alpha,\beta,\alpha',\beta'}\lambda(\xi)^{m_1}\cdot\lambda(\xi')^{m_2}, \quad \alpha,\beta,\alpha',\beta'\in\mathbb{N}_0^n.$$

For $p\in S_0^{m,m',\lambda}$ we define the corresponding operator

$$(3.15)\qquad \begin{aligned} &p(x,D_x,x',D_{x'})u(x) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i((x-x'),\xi)+i(x',\xi')}p(x,\xi,x',\xi')\hat{u}(\xi')\bar{d}\xi'dx'\bar{d}\xi. \end{aligned}$$

for all $u\in\mathcal{S}(\mathbb{R}^n)$. As in the classical situation it turns out that double symbols determine the same classes of operators $\Psi_0^{m,\lambda}$ as simple symbols, but they are a very useful tool for their investigation. More precisely we have

Theorem 3.3. *Let $p\in S_0^{m,m',\lambda}$ and $u\in\mathcal{S}(\mathbb{R}^n)$. Then the iterated integral in (3.15) exists and defines a pseudo differential operator in the class $\Psi_0^{m+m',\lambda}$. Moreover*

$$(3.16)\qquad p_L(x,\xi) = Os - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y,\eta)}p(x,\xi+\eta,x+y,\xi)dy\bar{d}\eta$$

is a symbol in $S_0^{m+m',\lambda}$ and defines the same operator, i.e.

$$p(x,D_x,x',D_{x'})u = p_L(x,D)u$$

for all $u\in\mathcal{S}(\mathbb{R}^n)$.

Note that by (2.3) and (2.1)

$$\left|\partial_\eta^\alpha\partial_y^\beta p(x,\xi+\eta,x+y,\xi)\right| \le c\lambda^m(\xi+\eta)\lambda^{m'}(\xi) \le c\lambda^{m+m'}(\xi)\lambda^{|m|}(\eta) \le c_\xi\langle\eta\rangle^{|m|}.$$

Therefore the integrand in (3.16) is of class $\mathcal{A}$ and the integral is well defined. $p_L(x,\xi)$ is called the simplified symbol of $p(x,\xi,x',\xi')$.

REMARK 3.4. The oscillatory integral in (3.16) actually defines a symbol p_L in $S_0^{m+m',\lambda}$. To see this we use the representation (3.12) for the oscillatory integral. For

l, l' sufficiently large we get by exchanging differentiation and integration, (3.14) and (2.3)

$$\left|\partial_\xi^\alpha \partial_x^\beta p_L(x,\xi)\right| \le c_{\alpha\beta} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle\eta\rangle^{-2l}\langle y\rangle^{-2l'}\lambda^m(\xi+\eta)\lambda^{m'}(\xi)\, dy đ\eta \le c_{\alpha\beta}\lambda^{m+m'}(\xi). \tag{3.17}$$

Moreover note that the constants $c_{\alpha\beta}$ are expressed in terms of the constants $c_{\alpha\beta\alpha'\beta}$ for the double symbol in (3.14). In particular, if a family of double symbols satisfies (3.14) uniformly for each $\alpha, \beta, \alpha', \beta'$, then also the simplified symbols satisfy an estimate (3.17) with uniform constants $c_{\alpha\beta}$.

Proof of Theorem 3.3. We adapt the consideration in [20], Chapter 2, to our situation. Choose $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\chi(0) = 1$ and note that (see [20, Lemma 1.6.3])

$$\partial_\eta^\alpha \partial_y^\beta [\chi(\varepsilon\eta, \varepsilon y)] \le c_{\alpha\beta}\langle\eta\rangle^{-|\alpha|}\langle y\rangle^{-|\beta|} \quad \text{uniformly for} \quad 0 \le \varepsilon \le 1. \tag{3.18}$$

For $0 \le \varepsilon \le 1$ let $p_\varepsilon(x,\xi,x',\xi') = \chi(\varepsilon(\xi-\xi'), \varepsilon(x'-x))p(x,\xi,x',\xi')$. Then by Leibniz rule and (3.18) have

$$\left|\partial_\xi^\alpha \partial_x^\beta \partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'} p_\varepsilon(x,\xi,x',\xi')\right| \le c_{\alpha,\beta,\alpha',\beta'}\lambda^m(\xi)\lambda^{m'}(\xi') \tag{3.19}$$

with constants $c_{\alpha,\beta,\alpha',\beta'}$ independent of ε. Define

$$\begin{aligned} p_{u,\varepsilon}(x,\xi,x',\xi') &= p_\varepsilon(x,\xi,x',\xi')\hat{u}(\xi') \\ q_{u,\varepsilon}(x,\xi,x') &= \int_{\mathbb{R}^n} e^{i(x',\xi')} p_{u,\varepsilon}(x,\xi,x',\xi') đ\xi' \\ r_{u,\varepsilon}(x,\xi) &= \int_{\mathbb{R}^n} e^{-i(x',\xi)} q_{u,\varepsilon}(x,\xi,x')dx' \end{aligned}$$

and fix $l, n_0 \in \mathbb{N}$ such that $2l > n + m_+$ and $2n_0 > n$. Note that $e^{i(x',\xi')} = \langle x'\rangle^{-2n_0}\langle D_{\xi'}\rangle^{2n_0} e^{i(x',\xi')}$. Thus for all $|\beta'| \le 2l$ by partial integration and Leibniz rule

$$\begin{aligned} \left|\partial_{x'}^{\beta'} q_{u,\varepsilon}(x,\xi,x')\right| &\le \left|\partial_{x'}^{\beta'} \int_{\mathbb{R}^n} \langle x'\rangle^{-2n_0} e^{i(x',\xi')}\langle D_{\xi'}\rangle^{2n_0} p_{u,\varepsilon}(x,\xi,x',\xi') đ\xi'\right| \\ &\le c_{p,u,l,n_0}\lambda^m(\xi)\langle x'\rangle^{-2n_0}, \end{aligned} \tag{3.20}$$

where the estimate is again uniform in ε. Therefore $r_{u,\varepsilon}$ is well defined and as above

$$\begin{aligned} |r_{u,\varepsilon}(x,\xi)| &\le \left|\langle\xi\rangle^{-2l} \int_{\mathbb{R}^n} e^{-i(x',\xi)}\langle D_{x'}\rangle^{2l} q_{u,\varepsilon}(x,\xi,x')dx'\right| \\ &\le c_{p,u,l,n_0}\lambda^m(\xi)\cdot\langle\xi\rangle^{-2l} \le c_{p,u,l,n_0,\lambda}\langle\xi\rangle^{-2l+m_+} \end{aligned} \tag{3.21}$$

uniformly in ε, where the last inequality follows from (2.1). Thus the integral

$$p_\varepsilon(x, D_x, x', D_{x'})u(x) = \int_{\mathbb{R}^n} e^{i(x,\xi)} r_{u,\varepsilon}(x,\xi) đ\xi$$

exists. In particular for $\varepsilon = 0$ we see that the iterated integral in (3.15) is well defined. Moreover, since the estimates (3.20) and (3.21) are uniform w.r.t. $0 \le \varepsilon \le 1$, we find by a successive application of Lebesgue's theorem

$$\begin{aligned} p(x, D_x, x', D_{x'})u(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i((x-x'),\xi)+i(x',\xi')} \lim_{\varepsilon\to 0} p_{u,\varepsilon}(x,\xi,x',\xi') đ\xi' dx đ\xi \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i((x-x'),\xi)+i(x',\xi')} p_{u,\varepsilon}(x,\xi,x',\xi') đ\xi' dx đ\xi \\ &= \lim_{\varepsilon\to 0} p_\varepsilon(x, D_x, x', D_{x'})u(x). \end{aligned} \tag{3.22}$$

For $\varepsilon > 0$ define

$$p_{L,\varepsilon}(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y,\eta)} \chi(\varepsilon\eta, \varepsilon y)) p(x, \xi+\eta, x+y, \xi) dy đ\eta, \tag{3.23}$$

Then by definition of the oscillatory integral

$$\lim_{\varepsilon\to 0} p_{L,\varepsilon}(x,\xi) = p_L(x,\xi) \tag{3.24}$$

and moreover by partial integration for $l_1, l_1' \in \mathbb{N}_0$ such that $2l_1 > |m| + n$, $2l_1' > n$

$$\begin{aligned} |p_{L,\varepsilon}(x,\xi)| &= \Big| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y,\eta)} \langle\eta\rangle^{-2l_1} \langle D_y\rangle^{2l_1} \\ &\qquad \times \Big\{ \langle y\rangle^{-2l_1'} \langle D_\eta\rangle^{2l_1'} \chi(\varepsilon\eta, \varepsilon y) p(x, \xi+\eta, x+y, \xi) \Big\} dy đ\eta \Big| \\ &\le c \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle\eta\rangle^{-2l_1} \langle y\rangle^{-2l_1'} \lambda^m(\xi+\eta) \lambda^{m'}(\xi) dy đ\eta \\ &\le c \int_{\mathbb{R}^n} \langle\eta\rangle^{-2l_1+|m|} \lambda^{m+m'}(\xi) đ\eta \\ &= c\lambda^{m+m'}(\xi) \end{aligned}$$

uniformly in $0 < \varepsilon \le 1$. Therefore by (3.24)

$$\lim_{\varepsilon\to 0} p_{L,\varepsilon}(x,D)u(x) = p_L(x,D)u(x), \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{3.25}$$

On the other hand substituting $x' = x + y$ and $\xi = \xi' + \eta$ shows

$$p_\varepsilon(x, D_x, x', D_{x'})u(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i((x-x'),\xi)+i(x',\xi')} p_\varepsilon(x,\xi,x',\xi') \hat{u}(\xi') đ\xi' dx' đ\xi$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x,\xi')}e^{-i(y,\eta)}\chi(\varepsilon\eta,\varepsilon y)p(x,\xi'+\eta,x+y,\xi')\hat{u}(\xi')đ\xi' dyđ\eta$$

$$= \int_{\mathbb{R}^n} e^{i(x,\xi')}p_{L,\varepsilon}(x,\xi')\hat{u}(\xi')đ\xi'$$

$$(3.26) \quad = p_{L,\varepsilon}(x,D)u(x).$$

Thus combining (3.22), (3.25) and (3.26) gives

$$p(x,D_x,x',D_{x'})u(x) = p_L(x,D)u(x). \qquad \square$$

Theorem 3.3 has a series of useful corollaries. First we consider the composition of two operators.

Corollary 3.5. *Let* $p_i \in S_0^{m_i,\lambda}$, $m_i \in \mathbb{R}$, $i = 1,2$. *Then* $p_1(x,D)\circ p_2(x,D) \in \Psi_0^{m_1+m_2,\lambda}$.

Proof. Put $p(x,\xi,x',\xi') = p_1(x,\xi)\cdot p_2(x',\xi')$. Then $p \in S_0^{m_1,m_2,\lambda}$. Therefore $p_L(x,D) \in \Psi_0^{m_1+m_2,\lambda}$ and for $u \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned} &p_1(x,D)\circ p_2(x,D)u(x) \\ &= \int_{\mathbb{R}^n} e^{i(x,\xi)}p_1(x,\xi)\int_{\mathbb{R}^n} e^{-i(x',\xi)}\int_{\mathbb{R}^n} e^{i(x',\xi')}p_2(x',\xi')\hat{u}(\xi')đ\xi' dx'đ\xi \\ &= p(x,D_x,x',D_{x'})u(x) = p_L(x,D)u(x). \qquad \square \end{aligned}$$

Let $(\cdot,\cdot)_0$ be the inner product in $L^2(\mathbb{R}^n)$. Then we have for the formally adjoint operator

Corollary 3.6. *Let* $p \in S_0^{m,\lambda}$. *Then there is a* $p^* \in S_0^{m,\lambda}$ *such that*

$$(p(x,D)u,v)_0 = (u,p^*(x,D)v)_0$$

for all $u,v \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Define $\tilde{p}(x,\xi,x',\xi') = \overline{p(x',\xi)}$. Then $\tilde{p} \in S_0^{m,0,\lambda}$ and as in the proof of Corollary 2.2.5 in [20]

$$\begin{aligned} (p(x,D)u,v)_0 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x',\xi)}p(x',\xi)\hat{u}(\xi)đ\xi\cdot\overline{v(x')}dx' \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(x,\xi)}u(x)\left\{\int_{\mathbb{R}^n} e^{i(x',\xi)}p(x',\xi)\overline{v(x')}dx'\right\}dxđ\xi \end{aligned}$$

$$= \int_{\mathbb{R}^n} u(x) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(x-x',\xi)} p(x',\xi)\overline{v(x')} dx' đ\xi dx$$
$$= \int_{\mathbb{R}^n} u(x) \overline{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-x',\xi)+i(x',\xi')} \overline{p(x',\xi)} \hat{v}(\xi') đ\xi' dx' đ\xi dx}$$
$$= (u, \tilde{p}(x, D_x, x', D_{x'})v)_0,$$

which proves the corollary with $p^*(x,D) = \tilde{p}_L(x,D)$. Here we applied Fubini's theorem several times. This is possible in particular since

$$\left| \int_{\mathbb{R}^n} e^{i(x',\xi)} p(x',\xi)\overline{v(x')} dx' \right| = \langle\xi\rangle^{-2n_0} \left| \int_{\mathbb{R}^n} e^{i(x',\xi)} \langle D_{x'}\rangle^{2n_0} (p(x',\xi)\overline{v(x')}) dx' \right|$$
$$\leq c\langle\xi\rangle^{-2n_0} \lambda^m(\xi)$$

is integrable w.r.t. ξ for $n_0 \in \mathbb{N}$ sufficiently large. □

Summarizing we find that $\bigcup_{m\in\mathbb{R}} \Psi_0^{m,\lambda}$ is an algebra of pseudo differential operators with multiplication $\circ$ and involution $*$ that respects the graded structure given by $(S_0^{m,\lambda})_{m\in\mathbb{R}}$, i.e.

$$\Psi_0^{m,\lambda} + \Psi_0^{m,\lambda} \subset \Psi_0^{m,\lambda}$$
$$(\Psi_0^{m,\lambda})^* \subset \Psi_0^{m,\lambda}$$
$$\Psi_0^{m,\lambda} \circ \Psi_0^{m',\lambda} \subset \Psi_0^{m+m',\lambda}$$

Next we extend the domain of the operators. Corollary 3.6 immediately implies by duality that $p(x,D) \in \Psi_0^{m,\lambda}$ has a continuous extension $p(x,D) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ defined by

$$\langle p(x,D)u, v\rangle = \langle u, p^*(x,D)v\rangle, \quad u \in \mathcal{S}'(\mathbb{R}^n), v \in \mathcal{S}(\mathbb{R}^n).$$

The order m of an operator $p(x,D) \in \Psi_0^{m,\lambda}$ has a natural interpretation in terms of mapping properties between Sobolev spaces.

For that purpose we introduce a scale of anisotropic Sobolev spaces which are defined in terms of the function λ:

$$H^{s,\lambda}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{s,\lambda} < \infty\}, \quad s \in \mathbb{R},$$

where

$$\|u\|_{s,\lambda} = \left(\int_{\mathbb{R}^n} \lambda^{2s}(\xi) \, |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}$$

Note that $H^{s,\lambda}(\mathbb{R}^n)$ coincides with the space $H^{s/2,a^2}(\mathbb{R}^n)$ defined in [11], in particular $H^{0,\lambda}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{s,\lambda}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

Theorem 3.7. *A pseudo differential operator with symbol $p \in S_0^{m,\lambda}$ is a continuous operator*

$$p(x,D) : H^{s+m,\lambda}(\mathbb{R}^n) \to H^{s,\lambda}(\mathbb{R}^n)$$

for all $s \in \mathbb{R}$ and we have

$$\|p(x,D)u\|_{s,\lambda} \leq c\,\|u\|_{s+m,\lambda} \quad \text{for all } u \in H^{s+m,\lambda}(\mathbb{R}^n). \tag{3.27}$$

Proof. It is sufficient to prove (3.27) for $u \in \mathcal{S}(\mathbb{R}^n)$. First suppose $s = m = 0$. Then $p \in S_0^{0,\lambda}$ has bounded derivatives and by the well-known L^2-continuity result of Calderón and Vaillancourt [3] we find

$$\|p(x,D)u\|_0 \leq c\,\|u\|_0$$

with a constant c depending only on the constants $c_{\alpha\beta}$ in (3.6) for $|\alpha|$, $|\beta| \leq 3$. Next suppose $s = 0$ and m arbitrary. Then

$$p(x,D)u(x) = \int_{\mathbb{R}^n} e^{i(x,\xi)} p(x,\xi)\lambda^{-m}(\xi)\lambda^m(\xi)\hat{u}(\xi)d\!\!\bar{}\,\xi$$

and $p(x,\xi)\lambda^{-m}(\xi)$ is a symbol in $S_0^{0,\lambda}$. Therefore

$$\|p(x,D)u\|_0 \leq c\,\|\lambda^m(D)u\|_0 = c\,\|u\|_{m,\lambda}\,.$$

Finally for the general case observe that $\lambda^s(D) \circ p(x,D) \in \Psi_0^{s+m,\lambda}$ by Corollary 3.5 and thus

$$\|p(x,D)u\|_{s,\lambda} = \|\lambda^s(D)p(x,D)u\|_0 \leq c\,\|u\|_{s+m,\lambda}\,. \qquad \square$$

REMARK 3.8. Observe that from the above proof, Corollary 3.5 and Remark 3.4 it is clear that the same constant c in (3.27) may be chosen for a family of pseudo differential operators which satisfy (3.6) uniformly.

The symbol classes $S_0^{m,\lambda}$ lead to a reasonable algebra of pseudo differential operators, but are bad symbol classes in the sense that all derivatives of the symbols are estimated by the same power m of $\lambda(\xi)$ as in the case of Hörmander class $S_{0,0}^m$ and not by a smaller power. Therefore we cannot expect asymptotic expansion formulas for this type of symbols. On the other hand the symbols of class $S_\varrho^{m,\lambda}$ have a somewhat better behaviour of their derivatives with respect to ξ. This will yield expansion formulas including terms up to order 2. We consider the expansion of the simplified symbol.

Theorem 3.9. *Given a double symbol* $p \in S_0^{m,m',\lambda}$ *such that*

$$\partial_\xi^\alpha p(x,\xi,x',\xi') \in S_0^{m-\varrho(|\alpha|),m',\lambda} \tag{3.28}$$

holds for all $\alpha \in \mathbb{N}_0^n$. *Then for all* $N \in \mathbb{N}$ *the simplified symbol* p_L *satisfies*

$$p_L(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} p_\alpha(x,\xi) \in S_0^{m+m'-\varrho(N),\lambda}, \tag{3.29}$$

where

$$p_\alpha(x,\xi) = D_{x'}^\alpha \partial_\xi^\alpha p(x,\xi,x',\xi')\big|_{\substack{x'=x\\ \xi'=\xi}} \in S_0^{m+m'-\varrho(|\alpha|),\lambda}. \tag{3.30}$$

Proof. We modify the argument given in [22]. By Taylor formula we have

$$\begin{aligned} p(x,\xi+\eta,x+z,\xi) &= \sum_{|\alpha|<N} \frac{\eta^\alpha}{\alpha!} \partial_\xi^\alpha\, p(x,\xi,x+z,\xi')|_{\xi'=\xi} \\ &\quad + N \sum_{|\gamma|=N} \frac{\eta^\gamma}{\gamma!} p_\gamma(x,z,\xi,\eta) \end{aligned}$$

with

$$p_\gamma(x,z,\xi,\eta) = \int_0^1 (1-t)^{N-1} \partial_\xi^\gamma\, p(x,\xi+t\eta,x+z,\xi')|_{\xi'=\xi}\, dt$$

and therefore by (3.16)

$$\begin{aligned} p_L(x,\xi) &= \sum_{|\alpha|<N} \frac{1}{\alpha!} O_S - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(z,\eta)} \eta^\alpha \partial_\xi^\alpha\, p(x,\xi,x+z,\xi')|_{\xi'=\xi}\, dz đ\eta \\ &\quad + \sum_{|\gamma|=N} \frac{N}{\gamma!} O_S - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(z,\eta)} \eta^\gamma p_\gamma(x,z,\xi,\eta)\, dz đ\eta \\ &= \sum_{|\alpha|<N} \frac{1}{\alpha!} I_\alpha(x,\xi) + \sum_{|\gamma|=N} \frac{N}{\gamma!} J_\gamma(x,\xi). \end{aligned}$$

We have to show that

$$I_\alpha = p_\alpha \in S_0^{m+m'-\varrho(|\alpha|),\lambda}, \quad |\alpha| < N \tag{3.31}$$

and

$$J_\gamma \in S_0^{m+m'-\varrho(N),\lambda}. \tag{3.32}$$

Let $|\alpha| < N$ and choose $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$ such that χ_1 and χ_2 equal 1 in a neighbourhood of the origin. Then by definition of I_α and (3.13)

$$\begin{aligned}
I_\alpha &= O_S - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(z,\eta)}\eta^\alpha \partial_\xi^\alpha \, p(x,\xi,x+z,\xi')|_{\xi'=\xi}\, dz d\!\!{}^-\eta \\
&= O_S - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(z,\eta)} D_z^\alpha \partial_\xi^\alpha \, p(x,\xi,x+z,\xi')|_{\xi'=\xi}\, dz d\!\!{}^-\eta \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(z,\eta)}\chi_1(\varepsilon\eta)\chi_2(\varepsilon z) D_{x'}^\alpha \partial_\xi^\alpha \, p(x,\xi,x',\xi')|_{\substack{x'=x+z\\ \xi'=\xi}}\, dz d\!\!{}^-\eta \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} \chi_2(\varepsilon z)\varepsilon^{-n}\hat{\chi_1}\left(\frac{z}{\varepsilon}\right) D_{x'}^\alpha \partial_\xi^\alpha \, p(x,\xi,x',\xi')|_{\substack{x'=x+z\\ \xi'=\xi}}\, dz \\
&= D_{x'}^\alpha \partial_\xi^\alpha \, p(x,\xi,x',\xi')|_{\substack{x'=x\\ \xi'=\xi}} = p_\alpha(x,\xi),
\end{aligned}$$

because $\chi_2(\varepsilon z)\varepsilon^{-n}\hat{\chi_1}(z/\varepsilon)$ converges to the unit mass at 0 as $\varepsilon \to 0$, and $p_\alpha \in S_0^{m+m'-\varrho(|\alpha|),\lambda}$ by (3.28).

Moreover for $|\gamma| = N$ we have by (3.28), (2.3) and (2.1)

$$\begin{aligned}
&\left|\partial_x^\alpha \partial_z^{\alpha'} \partial_\xi^\beta \partial_\eta^{\beta'} p_\gamma(x,z,\xi,\eta)\right| \\
&= \left|\int_0^1 (1-t)^{N-1}\partial_x^\alpha \partial_z^{\alpha'} \partial_\xi^\beta \partial_\eta^{\beta'} \left(\partial_\xi^\gamma \, p(x,\xi+t\eta,x+z,\xi')|_{\xi'=\xi}\right) dt\right| \\
&\le c_{\alpha,\alpha',\beta,\beta',\gamma}\int_0^1 \lambda^{m-\varrho(N)}(\xi+t\eta)\cdot\lambda^{m'}(\xi)dt \\
&\le c_{\alpha,\alpha',\beta,\beta',\gamma}\lambda^{m-\varrho(N)}(\xi)\lambda^{m'}(\xi)\int_0^1 \lambda^{|m-\varrho(N)|}(t\eta)dt \\
&\le c_{\alpha,\alpha',\beta,\beta',\gamma,\lambda}\lambda^{m+m'-\varrho(N)}(\xi)\cdot\langle\eta\rangle^{|m-\varrho(N)|}.
\end{aligned}$$

Hence again by (3.12) for $l, n_0 \in \mathbb{N}$, $2l > N + |m-\varrho(N)| + n$, $2n_0 > n$,

$$\begin{aligned}
&|\partial_\xi^\alpha \partial_x^\beta J_\gamma(x,\xi)| \\
&= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(z,\eta)}\langle\eta\rangle^{-2l}\langle D_z\rangle^{2l}\Big\{\langle z\rangle^{-2n_0}\langle D_\eta\rangle^{2n_0}\left[\eta^\gamma \partial_\xi^\alpha \partial_x^\beta p_\gamma(x,z,\xi,\eta)\right]\Big\} dz d\!\!{}^-\eta\right| \\
&\le c_{l,\alpha,\beta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\langle\eta\rangle^{-2l+N+|m-\varrho(N)|}\cdot\langle z\rangle^{-2n_0}\lambda(\xi)^{m+m'-\varrho(N)} dz d\!\!{}^-\eta \\
&\le c_{l,\alpha,\beta}\lambda(\xi)^{m+m'-\varrho(N)}.
\end{aligned}$$

which gives (3.32). □

REMARK 3.10. The proof also shows that p_α and the remainder term $p_L - \sum_{|\alpha|<N}(1/\alpha!)p_\alpha$ are in the class $S_0^{m+m'-\varrho(|\alpha|),\lambda}$ and $S_0^{m+m'-\varrho(N),\lambda}$, respectively, and

satisfy estimates (3.6) with constants $c_{\alpha\beta}$ that depend only on the constants $c_{\alpha,\beta,\alpha',\beta'}$ in (3.14) of the double symbol $p(x,\xi,x',\xi')$ itself.

We apply Theorem 3.9 to the double symbols of the composition of two pseudo differential operators and of the formally adjoint operator, see Corollaries 3.5, 3.6 and their proofs, and obtain

Corollary 3.11. *Let $p_1 \in S_\varrho^{m_1,\lambda}$, $p_2 \in S_\varrho^{m_2,\lambda}$ and $p \in S_\varrho^{m,\lambda}$. Then the symbols p_c and p^* of the composition $p_c(x,D) = p_1(x,D) \circ p_2(x,D)$ and the formally adjoint $p^*(x,D) = p(x,D)^*$ satisfy*

$$p_c(x,\xi) = p_1(x,\xi) \cdot p_2(x,\xi) + \sum_{j=1}^n \partial_{\xi_j} p_1(x,\xi) \cdot D_{x_j} p_2(x,\xi) + p_{r_1}(x,\xi)$$

and

$$p^*(x,\xi) = \overline{p(x,\xi)} + \sum_{j=1}^n \partial_{\xi_j} D_{x_j} \overline{p(x,\xi)} + p_{r_2}(x,\xi),$$

where $p_{r_1} \in S_0^{m_1+m_2-2,\lambda}$ and $p_{r_2} \in S_0^{m-2,\lambda}$.

In particular the highest order terms are given by the product and the conjugate of the symbols.

Remark 3.12. Since $\varrho(k) \leq 2$, (3.29) gives no better results for $N > 2$. In this sense we obtain expansion formulas with terms up to order two. Obviously this result is due to the choice of the function $\varrho(k) = k \wedge 2$, which is determined by the behaviour of negative definite symbols. Of course the statement itself does not depend on the specific choice of ϱ and choosing another increasing subadditive function $\varrho : \mathbb{N}_0 \to \mathbb{R}_+$ will not affect the proof.

4. Friedrichs symmetrization

It is well-known that a pseudo differential operator with real symbol is in general no symmetric operator if the symbol depends on x, but there is a modification that is symmetric and differs from the original operator only by a lower order perturbation. This modification can be constructed explictly by the so-called Friedrichs symmetrization. The purpose of this section is to show by the results obtained in the previous section that also for symbols in $S_\varrho^{m,\lambda}$ a Friedrichs symmetrization is available. For that end fix a function $q \in C_0^\infty(\mathbb{R}^n)$ such that q is even, non-negative, supported in the unit ball $B_1(0)$ and $\int_{\mathbb{R}^n} q^2(\sigma) d\sigma = 1$ and define

$$F(\xi,\zeta) = \lambda(\xi)^{-n/4} \cdot q((\zeta - \xi) \cdot \lambda^{-1/2}(\xi)). \tag{4.1}$$

For a symbol $p \in S_0^{m,\lambda}$ let us define its Friedrichs symmetrization to be the double symbol p_F not depending on x given by

$$p_F(\xi, x', \xi') = \int_{\mathbb{R}^n} F(\xi, \zeta) p(x', \zeta) F(\xi', \zeta) d\zeta.$$

Then we have

Theorem 4.1. *Let $p \in S_0^{m,\lambda}$. Then*

$$\left|\partial_{x'}^{\alpha'} \partial_\xi^\beta \partial_{\xi'}^{\beta'} p_F(\xi, x', \xi')\right| \leq c_{\alpha',\beta,\beta'} \lambda^{m-(1/2)\varrho(|\beta|)}(\xi) \cdot \lambda^{-(1/2)\varrho(|\beta'|)}(\xi'). \tag{4.2}$$

In particular $p_F \in S_0^{m,0,\lambda}$ and the simplified symbol $p_{F,L} \in S_0^{m,\lambda}$. Moreover, if $p \in S_\varrho^{m,\lambda}$ we have

$$p - p_{F,L} \in S_0^{m-1,\lambda}. \tag{4.3}$$

First we prove

Lemma 4.2. *For all $\beta \in \mathbb{N}_0^n$ we have*

$$\partial_\xi^\beta F(\xi, \zeta) = \lambda(\xi)^{-n/4} \sum_{\substack{|\gamma| \leq |\beta| \\ \gamma_1 \leq \gamma}} \varphi_{\beta,\gamma,\gamma_1}(\xi) \cdot \left((\xi - \zeta) \cdot \lambda^{-1/2}(\xi)\right)^{\gamma_1} \cdot \left(\partial^\gamma q\right)((\xi - \zeta) \cdot \lambda^{-1/2}(\xi)), \tag{4.4}$$

where $\varphi_{\beta.\gamma,\gamma_1} \in S_0^{-(1/2)\varrho(|\beta|),\lambda}$.

Proof. Obviously (4.4) holds true for $\beta = 0$ with $\varphi_{0,0,0} = 1$. Note that

$$\partial_{\xi_i} \lambda^m(\xi) = m\lambda^m(\xi)\lambda^{-1}(\xi) \cdot \partial_{\xi_i} \lambda(\xi).$$

Proceeding by induction we differentiate (4.4)

$$\begin{aligned}
&\partial_{\xi_j} \partial_\xi^\beta F(\xi, \zeta) \\
&= \lambda^{-n/4}(\xi) \cdot \sum_{\substack{|\gamma| \leq |\beta| \\ \gamma_1 \leq \gamma}} \Big\{ \left[\psi_{\beta,\gamma,\gamma_1}^{(1)}(\xi) + \psi_{\beta,\gamma,\gamma_1}^{(2)}(\xi)\right] ((\xi - \zeta) \cdot \lambda^{-1/2}(\xi))^{\gamma_1} \\
&\quad \times (\partial^\gamma q)((\xi - \zeta) \cdot \lambda^{-1/2}(\xi)) \\
&\quad + \psi_{\beta,\gamma,\gamma_1}^{(3)}(\xi)((\xi - \zeta) \cdot \lambda^{-1/2}(\xi))^{\gamma_1 - \epsilon_j} \cdot (\partial^\gamma q)((\xi - \zeta) \cdot \lambda^{-1/2}(\xi))
\end{aligned}$$

$$+\psi^{(4)}_{\beta,\gamma,\gamma_1}(\xi)((\xi-\zeta)\cdot\lambda^{-1/2}(\xi))^{\gamma_1}\cdot(\partial^{\gamma+\varepsilon_j}q)((\xi-\zeta)\cdot\lambda^{-1/2}(\xi))$$
$$+\psi^{(5)}_{\beta,\gamma,\gamma_1}(\xi)\sum_{k=1}^{n}((\xi-\zeta)\cdot\lambda^{-1/2}(\xi))^{\gamma_1+\varepsilon_k}\cdot(\partial^{\gamma+\varepsilon_k}q)((\xi-\zeta)\cdot\lambda^{-1/2}(\xi))\Big\}$$

with

$$\begin{aligned}
\psi^{(1)}_{\beta,\gamma,\gamma_1}(\xi) &= -\varphi_{\beta,\gamma,\gamma_1}(\xi)\cdot\left(\frac{n}{4}+\frac{|\gamma_1|}{2}\right)\lambda^{-1}(\xi)\partial_{\xi_j}\lambda(\xi),\\
\psi^{(2)}_{\beta,\gamma,\gamma_1}(\xi) &= \partial_{\xi_j}\varphi_{\beta,\gamma,\gamma_1}(\xi),\\
\psi^{(3)}_{\beta,\gamma,\gamma_1}(\xi) &= \gamma_{1,j}\lambda^{-1/2}(\xi)\varphi_{\beta,\gamma,\gamma_1}(\xi),\\
\psi^{(4)}_{\beta,\gamma,\gamma_1}(\xi) &= \lambda^{-1/2}(\xi)\varphi_{\beta,\gamma,\gamma_1}(\xi),\\
\psi^{(5)}_{\beta,\gamma,\gamma_1}(\xi) &= -\frac{1}{2}\lambda^{-1}(\xi)\partial_{\xi_j}\lambda(\xi)\varphi_{\beta,\gamma,\gamma_1}(\xi),
\end{aligned}$$

which is of the form claimed in (4.4) and we have to check that $\psi^{(l)}_{\beta,\gamma,\gamma_1}\in S_0^{-(1/2)(\varrho(|\beta|+1)),\lambda}$, $l=1,\ldots,5$. Note that $\lambda^{-1/2}\in S_0^{-1/2,\lambda}$ and $\lambda^{-1}\partial_{\xi_j}\lambda\in S_0^{-1,\lambda}$, see Lemma 3.1. Since $\varphi_{0,0,0}=1$ we see for $\beta=0$ that

$$\psi^{(l)}_{\beta,\gamma,\gamma_1}\in\operatorname{lin}\{\lambda^{-1/2},\lambda^{-1}\partial_{\xi_j}\lambda\}\subset S_0^{-1/2,\lambda},$$

which also implies $\varphi_{\beta,\gamma,\gamma_1}\in S_0^{-1/2,\lambda}$ for $|\beta|=1$. Next note that $\partial_{\xi_k}\lambda^{-1/2}\in S_0^{-3/2,\lambda}$ and $\partial_{\xi_k}(\lambda^{-1}\partial_{\xi_i}\lambda)\in S_0^{-2,\lambda}$, which yields $\partial_{\xi_k}\varphi_{\beta,\gamma,\gamma_1}\in S_0^{-3/2,\lambda}$ for $|\beta|=1$. Thus by the algebra property (3.9) of the symbols we find for $|\beta|=1$ that $\psi^{(l)}_{\beta,\gamma,\gamma_1}\in S_0^{-1,\lambda}$. But $S_0^{-1,\lambda}$ is stable under taking derivatives and therefore again (3.9) yields $\psi^{(l)}_{\beta,\gamma,\gamma_1}\in S_0^{-1,\lambda}$ for all $|\beta|\geq 2$ by induction. □

Proof of Theorem 4.1. By Lemma 4.2 and the support properties of q we have

$$\begin{aligned}
&\left|\partial_{x'}^{\alpha'}\partial_{\xi}^{\beta}\partial_{\xi'}^{\beta'}p_F(\xi,x',\xi')\right|\\
&=\left|\int_{\mathbb{R}^n}\partial_\xi^\beta F(\xi,\zeta)\partial_{x'}^{\alpha'}p(x',\zeta)\partial_{\xi'}^{\beta'}F(\xi',\zeta)d\zeta\right|\\
&\leq\lambda(\xi)^{-n/4}\lambda(\xi')^{-n/4}\sum_{\substack{|\gamma|\leq|\beta|\\ \gamma_1\leq\gamma}}\sum_{\substack{|\gamma'|\leq|\beta'|\\ \gamma_1'\leq\gamma'}}\left|\varphi_{\beta,\gamma,\gamma_1}(\xi)\cdot\varphi_{\beta',\gamma',\gamma_1'}(\xi')\right|\cdot\\
&\quad\cdot\left|\int_{\substack{|\xi-\zeta|\leq\lambda^{1/2}(\xi)\\ |\xi'-\zeta|\leq\lambda^{1/2}(\xi')}}((\xi-\zeta)\cdot\lambda^{-1/2}(\xi))^{\gamma_1}((\xi'-\zeta)\cdot\lambda^{-1/2}(\xi'))^{\gamma_1'}\right.\\
&\quad\left.\cdot(\partial^\gamma q)((\xi-\zeta)\cdot\lambda^{-1/2}(\xi))(\partial^{\gamma'}q)((\xi'-\zeta)\cdot\lambda^{-1/2}(\xi'))\cdot\partial_{x'}^{\alpha'}p(x',\zeta)d\zeta\right|
\end{aligned}$$

$$(4.5)\qquad \le c_{\alpha',\beta,\beta'}\lambda(\xi)^{-n/4}\lambda(\xi')^{-n/4}\lambda(\xi)^{-(1/2)\varrho(|\beta|)}\lambda(\xi')^{-(1/2)\varrho(|\beta'|)}\cdot I,$$

where

$$I=\int_{\substack{|\xi-\zeta|\le\lambda^{1/2}(\xi)\\ |\xi'-\zeta|\le\lambda^{1/2}(\xi')}}\left|\partial_{x'}^{\alpha'}p(x',\zeta)\right|d\zeta.$$

Observe that by (2.2) and (2.1) for $|\sigma|\le 1$

$$\begin{aligned}\lambda(\xi+\lambda^{1/2}(\xi)\sigma) &\le \lambda(\xi)+\lambda(\lambda^{1/2}(\xi)\sigma)\\ &\le \lambda(\xi)+c(1+\lambda^{1/2}(\xi)\,|\sigma|)\\ &\le c\lambda(\xi).\end{aligned}\tag{4.6}$$

Hence using the substitution $\zeta=\xi+\lambda^{1/2}(\xi)\cdot\sigma$ we find by Cauchy-Schwarz inequality

$$\begin{aligned}|I| &\le \left(\int_{|\xi-\zeta|\le\lambda^{1/2}(\xi)}\left|\partial_{x'}^{\alpha'}p(x',\zeta)\right|^2 d\zeta\right)^{1/2}\left(\int_{|\xi'-\zeta|\le\lambda^{1/2}(\xi')}1d\zeta\right)^{1/2}\\ &= \lambda^{n/4}(\xi)\left(\int_{|\sigma|\le 1}\left|\partial_{x'}^{\alpha'}p(x',\xi+\lambda^{1/2}(\xi)\cdot\sigma)\right|^2 d\sigma\right)^{1/2}\lambda^{n/4}(\xi')\left(\int_{|\sigma|\le 1}d\sigma\right)^{1/2}\\ &\le c\lambda^{n/4}(\xi)\lambda^{n/4}(\xi')\lambda^m(\xi),\end{aligned}$$

which together with (4.5) gives (4.2). □

In order to prove (4.3) we need the following

Lemma 4.3. *Let $p\in S_0^{m,\lambda}$, $t\in\mathbb{R}$ and $\sigma\in\mathbb{R}^n$. Then*

$$\begin{aligned}&\partial_x^\alpha\partial_\xi^\beta(p(x,\xi+t\lambda^{1/2}(\xi)\cdot\sigma))\\ &\quad=\sum_{\substack{|\gamma|\le|\beta|\\ \gamma_1\le\gamma}}\psi_{\beta,\gamma,\gamma_1}(\xi)(\partial_x^\alpha\partial_\xi^\gamma p)(x,\xi+t\lambda^{1/2}(\xi)\cdot\sigma)\cdot(t\sigma)^{\gamma_1},\end{aligned}\tag{4.7}$$

where $\psi_{\beta,\gamma,\gamma_1}\in S_0^{0,\lambda}$.

Proof. Since also $\partial_x^\alpha p(x,\xi)\in S_0^{m,\lambda}$ for all $\alpha\in\mathbb{N}_0^n$ as well, we may replace p by $\partial_x^\alpha p$ and assume $\alpha=0$. With $\psi_{0,0,0}=1$ there is nothing to prove for $\beta=0$. Let $\tilde\xi=\xi+t\lambda^{1/2}(\xi)\cdot\sigma$. Then by induction

$$\partial_{\xi_i}\partial_\xi^\beta p(x,\tilde\xi)=\sum_{\substack{|\gamma|\le|\beta|\\ \gamma_1\le\gamma}}\Big\{\partial_{\xi_i}\psi_{\beta,\gamma,\gamma_1}(\xi)(\partial_\xi^\gamma p)(x,\tilde\xi)\cdot(t\sigma)^{\gamma_1}$$

$$+\psi_{\beta,\gamma,\gamma_1}(\xi)(\partial_\xi^{\gamma+\varepsilon_i}p)(x,\tilde{\xi})\cdot(t\sigma)^{\gamma_1}$$
$$\left.+\sum_{k=1}^{n}\psi_{\beta,\gamma,\gamma_1}(\xi)(\partial_\xi^{\gamma+\varepsilon_k}p)(x,\tilde{\xi})\cdot\partial_{\xi_i}\lambda^{1/2}(\xi)\cdot(t\sigma)^{\gamma_1+\varepsilon_k}\right\},$$

which proves the lemma, since $\partial_{\xi_i}\lambda^{1/2}\in S_0^{0,\lambda}$. □

Proof of (4.3). By the expansion formula (3.29) (replace $\varrho(\cdot)$ by $(1/2)\varrho(\cdot)$, which does not affect the proof) and (4.2) we know that

$$p_{F,L}-p_{F,0}-\sum_{|\alpha|=1}p_{F,\alpha}\in S_0^{m-1,\lambda}.$$

Thus it is enough to prove

$$p_{F,\alpha}\in S_0^{m-1,\lambda}\quad\text{for}\quad|\alpha|=1 \tag{4.8}$$

and

$$p_{F,0}-p\in S_0^{m-1,\lambda}. \tag{4.9}$$

Let $|\alpha|=1$. Then

$$\begin{aligned}&\partial_\xi^\alpha F(\xi,\eta)=\partial_\xi^\alpha\left(\lambda^{-n/4}(\xi)q((\eta-\xi)\lambda^{-1/2}(\xi))\right)\\&=\lambda^{-n/4}(\xi)\Big[-\frac{n}{4}q((\eta-\xi)\lambda^{-1/2}(\xi))\cdot\lambda^{-1}(\xi)\partial_\xi^\alpha\lambda(\xi)\\&\quad+\sum_{k=1}^{n}(\partial_kq)((\eta-\xi)\lambda^{-1/2}(\xi))\cdot(\eta_k-\xi_k)\cdot\lambda^{-1/2}(\xi)\lambda^{1/2}(\xi)\partial_\xi^\alpha\lambda^{-1/2}(\xi)\\&\quad-(\partial^\alpha q)((\eta-\xi)\lambda^{-1/2}(\xi))\cdot\lambda^{-1/2}(\xi)\Big]\end{aligned}$$

and consequently with $\sigma=(\eta-\xi)\lambda^{-1/2}(\xi)$

$$\begin{aligned}p_{F,\alpha}&=D_{x'}^\alpha\partial_\xi^\alpha\,p_F(\xi,x',\xi')|_{\substack{x'=x\\ \xi'=\xi}}=\int_{\mathbb{R}^n}\partial_\xi^\alpha F(\xi,\eta)\cdot D_x^\alpha p(x,\eta)F(\xi,\eta)d\eta\\&=-\frac{n}{4}\lambda^{-1}(\xi)\partial_\xi^\alpha\lambda(\xi)\cdot\int_{\mathbb{R}^n}q^2(\sigma)D_x^\alpha p(x,\xi+\lambda^{1/2}(\xi)\sigma)d\sigma\\&\quad+\sum_{k=1}^{n}\lambda^{1/2}(\xi)\partial_\xi^\alpha\lambda^{-1/2}(\xi)\cdot\int_{\mathbb{R}^n}\sigma_k\partial_kq(\sigma)\cdot q(\sigma)D_x^\alpha p(x,\xi+\lambda^{1/2}(\xi)\sigma)d\sigma\\&\quad-\lambda^{-1/2}(\xi)\cdot\int_{\mathbb{R}^n}(\partial^\alpha q)(\sigma)q(\sigma)D_x^\alpha p(x,\xi+\lambda^{1/2}(\xi)\sigma)d\sigma\\&=I_1+I_2+I_3.\end{aligned}$$

We consider each term separately. Observe that $\int_{\mathbb{R}^n} q^2(\sigma)D_x^\alpha p(x,\xi+\lambda^{1/2}(\xi)\sigma)d\sigma$ is a symbol in $S_0^{m,\lambda}$, since using Lemma 4.3 and (4.6)

$$\begin{aligned}
&\left|\partial_x^\delta\partial_\xi^\beta\int_{\mathbb{R}^n} q^2(\sigma)D_x^\alpha p(x,\xi+\lambda^{1/2}(\xi)\sigma)d\sigma\right| \\
&\leq \left|\int_{\mathbb{R}^n} q^2(\sigma)\partial_x^\delta\partial_\xi^\beta D_x^\alpha p(x,\xi+\lambda^{1/2}(\xi)\sigma)d\sigma\right| \\
&\leq \sum_{\substack{|\gamma|\leq|\beta| \\ \gamma_1\leq\gamma}} |\psi_{\beta,\gamma,\gamma_1}|\int_{\mathbb{R}^n} q^2(\sigma)\left|\left(\partial_x^\delta\partial_\xi^\beta D_x^\alpha p\right)(x,\xi+\lambda^{1/2}(\xi)\sigma)\right||\sigma^{\gamma_1}|\,d\sigma \\
&\leq c\int_{\mathbb{R}^n} q^2(\sigma)\lambda(\xi+\lambda^{1/2}(\xi)\sigma)^m d\sigma \leq c\int_{\mathbb{R}^n} q^2(\sigma)d\sigma\cdot\lambda^m(\xi) = c\lambda^m(\xi)
\end{aligned}$$

and $\lambda^{-1}\partial_\xi^\alpha\lambda \in S_0^{-1,\lambda}$ gives $I_1 \in S_0^{m-1,\lambda}$.
Analogously

$$\int_{\mathbb{R}^n} \sigma_k\partial_k q(\sigma)\cdot q(\sigma)D_x^\alpha p(x,\xi+\lambda^{1/2}(\xi)\sigma)d\sigma \in S_0^{m,\lambda}$$

and thus by $\lambda^{1/2}\partial_\xi^\alpha\lambda^{-1/2} \in S_0^{-1,\lambda}$ we have $I_2 \in S_0^{m-1,\lambda}$.
Moreover concerning I_3 we have by Taylor formula

$$\begin{aligned}
&\int_{\mathbb{R}^n} \partial^\alpha q(\sigma)q(\sigma)D_x^\alpha p(x,\xi+\lambda^{1/2}(\xi)\sigma)d\sigma \\
&= \int_{\mathbb{R}^n} \partial^\alpha q(\sigma)q(\sigma)d\sigma\cdot D_x^\alpha p(x,\xi) \\
&\quad +\lambda^{1/2}(\xi)\int_{\mathbb{R}^n} \partial^\alpha q(\sigma)q(\sigma)\sum_{k=1}^n \sigma_k\cdot\int_0^1 \left(\partial_{\xi_k}D_x^\alpha p\right)(x,\xi+\lambda^{1/2}(\xi)t\sigma)dtd\sigma.
\end{aligned}$$

By the symmetry of q the first term vanishes and we find for the derivatives of the second integral using again Lemma 4.3

$$\begin{aligned}
&\left|\partial_x^\delta\partial_\xi^\beta\int_{\mathbb{R}^n} \partial^\alpha q(\sigma)q(\sigma)\sigma_k\cdot\int_0^1 \left(\partial_{\xi_k}D_x^\alpha p\right)(x,\xi+\lambda^{1/2}(\xi)t\sigma)dtd\sigma\right| \\
&\leq \int_{\mathbb{R}^n} |\partial^\alpha q(\sigma)q(\sigma)\sigma_k|\cdot\int_0^1 \left|\partial_x^\delta\partial_\xi^\beta\left(D_x^\alpha\partial_{\xi_k}p\right)(x,\xi+\lambda^{1/2}(\xi)t\sigma)\right|dtd\sigma \\
&\leq c\int_{\mathbb{R}^n} |\partial^\alpha q(\sigma)q(\sigma)\sigma_k|\cdot\int_0^1 \lambda^{m-1}(\xi+\lambda^{1/2}(\xi)t\sigma)dtd\sigma \leq c\cdot\lambda^{m-1}(\xi).
\end{aligned}$$

Hence $\int_{\mathbb{R}^n} \partial^\alpha q(\sigma)q(\sigma)D_x^\alpha p(x,\xi+\lambda^{1/2}(\xi)\sigma)d\sigma$ is in $S_0^{m-1/2,\lambda}$, which means $I_3 \in$

$S_0^{m-1,\lambda}$ and we have (4.8). Let us turn to (4.9). By Taylor formula we find

$$\begin{aligned}
p_{F,0}(x,\xi) &= \int_{\mathbb{R}^n} q^2(\sigma)p(x,\xi+\lambda^{1/2}(\xi)\sigma)d\sigma \\
&= \int_{\mathbb{R}^n} q^2(\sigma)\Big\{p(x,\xi)+\sum_{k=1}^{n}\lambda^{1/2}(\xi)\cdot\sigma_k\cdot\partial_{\xi_k}p(x,\xi) \\
&\qquad + \int_0^1 (1-t)\sum_{|\gamma|=2}\frac{2}{\gamma!}\lambda(\xi)\sigma^\gamma(\partial_\xi^\gamma p)(x,\xi+t\lambda^{1/2}(\xi)\sigma)dt\Big\}d\sigma.
\end{aligned}$$

By the symmetry of q the integral over the first order term again vanishes and therefore

$$\begin{aligned}
&p_{F,0}(x,\xi)-p(x,\xi) \\
&= \sum_{|\gamma|=2}\frac{2}{\gamma!}\lambda(\xi)\int_{\mathbb{R}^n}\int_0^1 (1-t)q^2(\sigma)\sigma^\gamma(\partial_\xi^\gamma p)(x,\xi+t\lambda^{1/2}(\xi)\sigma)dtd\sigma.
\end{aligned}$$

Using again Lemma 4.3 and (4.6) we see as above that the integral defines a symbol in $S_0^{m-2,\lambda}$, which gives (4.9) □

The next theorem summarizes the important properties of the Friedrichs symmetrization.

Theorem 4.4. *Assume $p \in S_0^{m,\lambda}$ is real-valued. Then $p_F(D_x,x',D_{x'})$ is a symmetric operator on $\mathcal{S}(\mathbb{R}^n)$. If moreover $p(x,\xi)$ is non-negative, then $p_F(D_x,x',D_{x'})$ is non-negative.*

Proof. This is clear, because for u, $v \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
&(p_F(D_x,x',D_{x'})u,v)_0 \\
&= \int_{\mathbb{R}^n}\mathcal{F}^{-1}_{\xi\to x}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(x',\xi)+i(x',\xi')}p_F(\xi,x',\xi')\hat{u}(\xi')d\xi'dx'\right)(x)\overline{v(x)}dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(x',\xi)+i(x',\xi')}\int_{\mathbb{R}^n} F(\xi,\eta)p(x',\eta)F(\xi',\eta)d\eta\,\hat{u}(\xi')d\xi'dx'\,\overline{\hat{v}(\xi)}d\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p(x',\eta)\int_{\mathbb{R}^n} e^{i(x',\xi')}F(\xi',\eta)\hat{u}(\xi')d\xi'\cdot\overline{\int_{\mathbb{R}^n} e^{i(x',\xi)}F(\xi,\eta)\hat{v}(\xi)d\xi}d\eta dx'. \square
\end{aligned}$$

5. Generators of Feller semigroups

In this section we want to apply the results of the previous sections to pseudo differential operators with negative definite symbols. In particular we assume the symbols to be real-valued. As we have seen it is a natural condition to assume that the symbols are of class $S_\varrho^{2,\lambda}$ for some convenient $\lambda(\xi)$. To prove that a pseudo differential operator fulfills the assumptions of the Hille-Yosida theorem and therefore is the generator of an operator semigroup to most extent amounts to solve the equation

$$p(x,D)u + \tau u = f. \tag{5.1}$$

We will solve this problem for elliptic elements in $S_\varrho^{m,\lambda}$. In order to apply modified Hilbert space methods we need some estimates for the operator and the corresponding bilinear form. As an application of the Friedrichs symmetrization we first prove the sharp Gårding inequality which gives a first non-trivial lower bound for the bilinear form.

Theorem 5.1. *Let $p \in S_\varrho^{m,\lambda}$ be nonnegative. There is a $K \geq 0$ such that*

$$\mathrm{Re}(p(x,D)u,u)_0 \geq -K\,\|u\|^2_{(m-1)/2,\lambda}\,.$$

Proof. By Theorem 4.1 we know that $p(x,D) - p_F(D_x, x', D_{x'})$ is of order $m-1$. Since $p(x,\xi) \geq 0$ we have by Theorem 4.4

$$\begin{aligned}\mathrm{Re}(p(x,D)u,u)_0 &= \mathrm{Re}(p_F(D_x,x',D_{x'})u,u)_0 + \mathrm{Re}((p(x,D) - p_F(D_x,x',D_{x'}))u,u)_0\\ &\geq \mathrm{Re}\Big(\lambda^{-(m-1)/2}(D)(p(x,D) - p_F(D_x,x',D_{x'}))u, \lambda^{(m-1)/2}(D)u\Big)_0\\ &\geq -K\,\|u\|^2_{(m-1)/2,\lambda}\,.\end{aligned}$$

□

We are interested in further bounds for the bilinear form, in partcular in the elliptic case.

Theorem 5.2. *Let $p \in S_\varrho^{m,\lambda}$ be real-valued. Then*

$$|(p(x,D)u,v)_0| \leq c\,\|u\|_{m/2,\lambda} \cdot \|v\|_{m/2,\lambda}\,, \quad u,v \in \mathcal{S}(\mathbb{R}^n) \tag{5.2}$$

and the bilinear form extends continuously to $H^{m/2,\lambda}(\mathbb{R}^n)$. If moreover

$$p(x,\xi) \geq \delta\lambda^m(\xi), \quad |\xi| > R, \tag{5.3}$$

for some $\delta > 0$ and some $R > 0$, then for $m \geq 1$ the Gårding inequality

$$\mathrm{Re}(p(x,D)u,u)_0 \geq \delta\,\|u\|^2_{m/2,\lambda} - c\,\|u\|^2_{(m-1)/2,\lambda}\,, \quad u \in H^{m/2,\lambda}(\mathbb{R}^n), \tag{5.4}$$

holds.

Proof. We know that

$$|(p(x,D)u,u)_0| = \left|(\lambda^{-m/2}(D)p(x,D)u,\lambda^{m/2}(D)u)_0\right| \leq c\,\|u\|_{m/2,\lambda}\cdot\|v\|_{m/2,\lambda}\,,$$

since $\lambda^{-m/2}(D)\circ p(x,D)$ is of order $m/2$.
Now assume (5.3). Let $p_\tau(x,\xi) = p(x,\xi)+\tau$. Then for τ sufficiently large

$$p_\tau(x,\xi) \geq \delta\lambda^m(\xi)$$

holds for all $\xi\in\mathbb{R}^n$. We put $q(x,\xi) = p_\tau(x,\xi) - \delta\lambda^m(\xi)\geq 0$. Theorem 5.1 implies

$$\mathrm{Re}(p(x,D)u,u)_0 - \delta\,\|u\|^2_{m/2,\lambda} + \tau\,\|u\|^2_0 = \mathrm{Re}(q(x,D)u,u)_0 \geq -K\,\|u\|^2_{(m-1)/2,\lambda}\,. \ \square$$

Let us turn next to estimates for the operator itself. The operator $p(x,D)\in\Psi_0^{m,\lambda}$ is a continuous operator between the Sobolev spaces $H^{s,\lambda}(\mathbb{R}^n)$, see Theorem 3.7, i.e. $\|p(x,D)u\|_{s,\lambda}\leq c\,\|u\|_{s+m,\lambda}$. If moreover (5.3) holds, we even have a converse inequality.

Theorem 5.3. *Let $p\in S_\varrho^{m,\lambda}$ be real-valued and assume the ellipticity condition* (5.3). *Then for $s\in\mathbb{R}$ such that $m+s\geq 1/2$*

$$\delta^2\,\|u\|^2_{s+m,\lambda} \leq \|p(x,D)u\|^2_{s,\lambda} + c\,\|u\|^2_{s+m-(1/2),\lambda}\,. \tag{5.5}$$

Proof. Let $q_s(x,\xi) = p(x,\xi)^2\lambda^{2s}(\xi)\geq\delta^2\lambda^{2(m+s)}(\xi)$ for $|\xi|$ large. By Corollary 3.11 we know that the highest order term in the expansion of the symbol of $p^*(x,\xi)$ is given by $\overline{p(x,\xi)} = p(x,\xi)$. Thus

$$\begin{aligned}\|p(x,D)u\|^2_{s,\lambda} &= (\lambda^s(D)p(x,D)u,\lambda^s(D)p(x,D)u)_0\\ &= (p^*(x,D)\lambda^{2s}(D)p(x,D)u,u)_0 = \mathrm{Re}(q_s(x,D)u,u)_0 + \mathrm{Re}(q(x,D)u,u)_0,\end{aligned}$$

where $q(x,D)\in S_0^{2(m+s)-1,\lambda}$. Hence Theorem 5.2 implies

$$\|p(x,D)u\|^2_{s,\lambda} \geq \delta^2\,\|u\|^2_{m+s,\lambda} - c\,\|u\|^2_{m+s-(1/2),\lambda} - c'\,\|u\|^2_{m+s-(1/2),\lambda}\,. \qquad \square$$

To prove regularity results for solutions of (5.1) we will have to use certain commutators involving Friedrichs mollifiers. We introduce the Friedrichs mollifier J_ε : $L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$, $\varepsilon>0$, defined by $J_\varepsilon u = j_\varepsilon * u$, where

$$j_\varepsilon(x) = \varepsilon^{-n} j\left(\frac{x}{\varepsilon}\right),\ x\in\mathbb{R}^n, \quad\text{and}\quad j(x) := \begin{cases} c_0\cdot e^{1/(|x|^2-1)} & \text{for } |x|<1\\ 0 & \text{for } |x|\geq 1\end{cases},$$

and c_0 is chosen such that $\int_{\mathbb{R}^n} j(x)\,dx = 1$. Because of $(J_\varepsilon u)\hat{}(\xi) = \hat{\jmath}(\varepsilon\xi)\cdot\hat{u}(\xi)$ and $\hat{\jmath} \in \mathcal{S}(\mathbb{R}^n)$, we have $J_\varepsilon u \in H^{s,\lambda}(\mathbb{R}^n)$ for all $s \geq 0$ and, if moreover $u \in H^{t,\lambda}(\mathbb{R}^n)$, $J_\varepsilon u \to u$ in $H^{t,\lambda}(\mathbb{R}^n)$ as $\varepsilon \to 0$, since $\hat{\jmath}(0) = 1$.
Obviously J_ε is a pseudo differential operator with symbol $\hat{\jmath}(\varepsilon\xi)$ in $S_\varrho^{0,\lambda}$ and the constants $c_{\alpha,\beta}$ in the corresponding estimate (3.5) are uniformly bounded for $0 < \varepsilon \leq 1$, cf. [20], Lemma 1.6.3. Let $p \in S_\varrho^{m,\lambda}$. We consider the commutator

$$[p(x,D), J_\varepsilon] = p(x,D)J_\varepsilon - J_\varepsilon p(x,D).$$

Recall that the commutator is described by the difference of the double symbols $p(x,\xi)\cdot\hat{\jmath}(\varepsilon\xi')$ and $\hat{\jmath}(\varepsilon\xi)\cdot p(x',\xi')$. Since the highest order terms in the expansion series (3.29) cancel, $[p(x,D), J_\varepsilon]$ is an operator of order $m-1$. Moreover the remaining terms of the expansion are controlled uniformly with repect to ε, see Remark 3.8 and Remark 3.10. Therefore we get

Proposition 5.4. *Let $p \in S_\varrho^{m,\lambda}$ and $s \in \mathbb{R}$. There is a constant $c \geq 0$ not depending on $0 < \varepsilon \leq 1$ such that*

$$\|[p(x,D), J_\varepsilon]u\|_{s,\lambda} \leq c\,\|u\|_{m+s-1,\lambda}\,.$$

We summarize the results obtained so far and solve equation (5.1).

Theorem 5.5. *Let $p \in S_\varrho^{m,\lambda}$, $m \geq 2$ be a real-valued symbol, $s \geq 0$ and assume that (5.3) holds. If $\tau > 0$ is sufficiently large, then for $f \in H^{s,\lambda}(\mathbb{R}^n)$ there is a unique solution $u \in H^{s+m,\lambda}(\mathbb{R}^n)$ of the equation*

$$p(x,D)u + \tau u = f.$$

Proof. By Theorem 5.2 we know that

$$(u,v) \mapsto ((p(x,D)+\tau)u, v)_0$$

is a continuous coercive bilinear form on $H^{m/2,\lambda}(\mathbb{R}^n)$ for τ large enough. Thus there is a unique weak solution $u \in H^{m/2,\lambda}(\mathbb{R}^n)$ of

$$((p(x,D)+\tau)u, v)_0 = (f,v)_0 \qquad \text{for all } v \in H^{m/2,\lambda}(\mathbb{R}^n)$$

and the proof is complete, if we show that $u \in H^{s+m,\lambda}(\mathbb{R}^n)$. Let $u_\varepsilon = J_\varepsilon u$. Then $u_\varepsilon \in H^{t+m,\lambda}(\mathbb{R}^n)$ for all $t \leq s$, $0 < \varepsilon \leq 1$ and by Therorem 5.3 and Proposition 5.4 we have

$$\|u_\varepsilon\|_{t+m,\lambda} \leq c\,\|p(x,D)J_\varepsilon u\|_{t,\lambda} + c\,\|J_\varepsilon u\|_{t+m-(1/2),\lambda}\,.$$

$$\leq c\,\|J_\varepsilon(p(x,D)+\tau)u\|_{t,\lambda} + c\,\|J_\varepsilon u\|_{t,\lambda} + c\,\|[p(x,D),J_\varepsilon]u\|_{t,\lambda} + c\,\|J_\varepsilon u\|_{t+m-(1/2),\lambda}$$
$$\leq c\,\|J_\varepsilon f\|_{t,\lambda} + c\,\|u\|_{t,\lambda} + c\,\|u\|_{t+m-1,\lambda} + c\,\|u\|_{t+m-(1/2),\lambda}$$
$$\leq c\,\|f\|_{s,\lambda} + c\,\|u\|_{t+m-(1/2),\lambda}\,.$$

So $u \in H^{t+m-(1/2),\lambda}(\mathbb{R}^n)$ implies that $(u_\varepsilon)_{0<\varepsilon\leq 1}$ is bounded in $H^{t+m,\lambda}(\mathbb{R}^n)$. Since $u_\varepsilon \to u$ in $H^{t+m-(1/2),\lambda}(\mathbb{R}^n)$ as $\varepsilon \to 0$, this implies $u \in H^{t+m,\lambda}(\mathbb{R}^n)$. A recursive application of this conclusion starting with $t = (1-m)/2$ proves the theorem. □

Recall the theorem of Hille–Yosida–Ray [5] for generators of Feller semigroup-s, i.e. strongly continuous, positivity preserving contraction semigroups on the space $C_\infty(\mathbb{R}^n)$ of continuous functions vanishing at infinity:

Theorem 5.6. *Let $A: D(A) \to C_\infty(\mathbb{R}^n)$ be a linear operator in $C_\infty(\mathbb{R}^n)$. Then A is closable and the closure generates a Feller semigroup if and only if*
(i) *$D(A)$ is dense,*
(ii) *A satisfies the positive maximum principle on $D(A)$ and*
(iii) *for some $\tau > 0$ the range of $\tau - A$ is dense.*

We finally state our result about generators of Feller semigroups. For that purpose we have to assume that there is a constant $r > 0$, arbitrarily small, but strictly positive such that

$$\lambda(\xi) \geq c\,|\xi|^r \tag{5.6}$$

for some $c > 0$ and $|\xi|$ large. This is a non-degeneracy condition for operators in $\Psi_\varrho^{m,\lambda}$. Under this condition for $s > n/2r$ the dense and continuous embedding (see [12])

$$H^{s,\lambda}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n)$$

holds. Now we have

Theorem 5.7. *Assume that* (5.6) *holds. If $p(x,\xi)$ is a negative definite symbol of class $S_\varrho^{2,\lambda}$ and moreover*

$$p(x,\xi) \geq \delta\lambda^2(\xi)$$

for some $\delta > 0$ and $|\xi|$ large, then $-p(x,D)$ has an extension that generates a Feller semigroup.

Proof. Choose $s > n/2r$. Then the operator $A = -p(x,D) : H^{s+2,\lambda}(\mathbb{R}^n) \to H^{s,\lambda}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$ is a densely defined operator in $C_\infty(\mathbb{R}^n)$ with domain

$H^{s+2,\lambda}(\mathbb{R}^n)$ und thus A fulfills condition (i) of Theorem 5.6. Moreover by the result of Courrège A satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$ and therefore also on $H^{s+2,\lambda}(\mathbb{R}^n)$, see [12], Theorem 9.3. This is (ii) of Theorem 5.6 and finally (iii) is the claim of Theorem 5.5. □

REMARK 5.8. Note that the estimates for $p(x, D)$ proven in Section 5 imply that the probabilistic consequences for the associated process as they are stated for instance in [12], Section 11, do also hold in this case.

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