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Supplement to Note on Brauer's Theorem of Simple Groups. II

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The aim of this note is to complete the proof of the following theorem: *Let $\mathfrak{G}$ be a finite group which contains an element P of prime order p which commutes only with its own powers* (*condition* (*)) *and assume that $\mathfrak{G}$ is equal to its commutator-subgroup $\mathfrak{G}'$* (*condition* (**)). *Then the order g of $\mathfrak{G}$ is expressed as $g=p(p-1)(1+np)/t$, where $1+np$ is the number of conjugate subgroups of order p and t is the number of classes of conjugate elements of order p. If $n\leqq p+2$ and $t\not\equiv 0$* (*mod.* 2), *then p is of the form $2^{\mu}-1$ and $\mathfrak{G}\cong LF(2, 2^{\mu})$.*

In [3], the theorem was proved for the case $n<p+2$ and $t\not\equiv 0$ (mod. 2): *If $n<p+2$ and $t\not\equiv 0$* (*mod.* 2), *under* (*) *and* (**), *then p is of the form $2^{\mu}-1$ and $\mathfrak{G}\cong LF(2, 2^{\mu})$.* In [4], the case $n=p+2$ and $t\not\equiv 0$ (mod. 2) are discussed, but the equation in p. 230, line 6 is not correct[1], this value should be $\omega^{j(\mu+\nu)}\cdot(-1)^{jt}$. So the representation of degree $p+1$ may occur. Therefore, in this note, we shall assume that the irreducible representation of degree $p+1$ occurs besides the assumptions (*), (**), $n=p+2$ and $t\not\equiv 0$ (mod. 2). Under these assumptions we shall prove that such a group does not exist.

We shall use the same notations as Brauer [1]. First of all, we shall assume that $n=p+2=F(p, 1, 2)=F(p, u, 1)$ with positive integer u. For, if n does not have the expression $F(p, u, 1)$ with positive integer u, then the character-relations in $B_1(p)$ yields a contradiction easily. Simple computations show that the possible values of the irreducible characters in $B_1(p)$ are 1, $p+1$, $up+1$, $(u-1)p-1$, $(up+1)/t$ and $((u-1)p-1)/t$. In order to consider such characters, we shall prove following lemmas, essentially due to Brauer.

Lemma 1. *Under assumptions* (*), (**), $n=p+2$ $t\not\equiv 0$ (*mod.* 2), *if $\mathfrak{G}$ has an irreducible character A of degree $up+1 (u>1)$, then for the element I of order 2 in the normalizer $\mathfrak{N}(\mathfrak{P})$ of a p-Sylow subgroup $\mathfrak{P}$*

1) W. F. Reynolds kindly pointed out this error and gave the auther many useful suggestions. By this error, Theorem in [5] (p. 107) should be corrected as follows; If $2p-3<n\leqq 2p+3$, $t\not\equiv 0$ (mod. 2) and $t>1$, then $2p+1$ is a prime power and $\mathfrak{G}\cong LF(2, 2p+1)$, *unless the irreducible representation of degree $p+1$ occurs.* But Theorem in [5] (p. 116) is valid.

$$A(I) = \begin{cases} u+1, & \textit{if } u \textit{ is even.} \\ 0, & \textit{if } u \textit{ is odd.} \end{cases}$$

The normalizer $\mathfrak{N}(\mathfrak{P})$ of a p-Sylow subgroup $\mathfrak{P}$ is a metacyclic group $\{P, Q\}$ of order $pq=p(p-1)/t$ and has q linear characters ω_μ and t p-conjugate characters $Y^{(\tau)}$ of degree q. If we consider the character A in $\mathfrak{N}(\mathfrak{P})$, then A is decomposed into two parts $\widetilde{A}$ and A_0, where $\widetilde{A}$ is a sum of $u+1$ linear characters ω_μ and A_0 is a linear homogeneous combination of $Y^{(\tau)}$. Now, as $u>1$, n $(=p+2)$ has an expression $F(p, u, 1)=(up+u^2+u+1)/(u+1)$. This implies $p=u^2-u-1$ and $g=p\ q(up+1)(p+u+1)/(u+1)$. Since $gn(G)^{-1}\cdot(D_gA)^{-1}\cdot A(G)$ is an algebraic integer, where $n(G)$ is the order of the normalizer of G in $\mathfrak{G}$ and D_gA is the degree of A, $gn(Q^j)^{-1}\cdot(up+1)^{-1}\cdot A(Q^j)$ is, and also $A(Q^j)/(u+1)$ is an algebraic integer. But $A(Q^j)=\widetilde{A}(Q^j)$ for $j\not\equiv 0 \pmod q$. Then applying Burnside's method, we have $\widetilde{A}(Q^j)=0$ or $(u+1)\omega^{\mu j}$: that is, $A(Q^j)=0$ or $(u+1)\omega^{\mu j}$ for $j\not\equiv 0$ (mod q). Assume $\widetilde{A}=\alpha_1\omega_{\mu_1}+\alpha_2\omega_{\mu_2}+\cdots+\alpha_r\omega_{\mu_r}$ is a decomposition of $\widetilde{A}$ in $\mathfrak{N}(\mathfrak{P})$. Let m be the least positive integer satisfying $\widetilde{A}(Q^m)\neq 0$. Then m is a divisor of q and any integer x satisfying $\widetilde{A}(Q^x)\neq 0$ is a multiple of m. Now there exist q/m integers satisfying $\widetilde{A}(Q^x)\neq 0$. From the orthogonality-relations, we have

$$\sum_{j=0}^{q-1}\widetilde{A}(Q^j)\bar{\omega}_{\mu_i}(Q^j) = \alpha_i\cdot q\,.$$

On the other hand, $\sum_j\widetilde{A}(Q^j)\bar{\omega}_{\mu_i}(Q^j)=\sum_{\lambda=1}^{q/m}\widetilde{A}(Q^{\lambda m})\bar{\omega}_{\mu_i}(Q^{\lambda m})=(u+1)\ q/m$. Hence $\alpha_i q=(u+1)\ q/m$. This means $\alpha_i=(u+1)/m$ for $i=1, 2, \cdots, r$. Therefore $\widetilde{A}=\dfrac{u+1}{m}(\omega_{\mu_1}+\omega_{\mu_2}+\cdots+\omega_{\mu_m})$. Furthermore $\widetilde{A}(Q^m)\neq 0$ implies $\omega^{\mu_1 m}=\omega^{\mu_2 m}=\cdots=\omega^{\mu_m m}$. This means $\mu_1\equiv\mu_2\equiv\cdots\equiv\mu_m$ (mod q/m). Then we can put $\mu_1=a$, $\mu_2=a+q/m, \cdots, \mu_m=a+(m-1)\ q/m$. Thus

$$\text{(D)} \qquad A = \frac{u+1}{m}(\omega_a+\omega_{a+q/m}+\cdots+\omega_{a+(m-1)q/m})+A_0\,.$$

Next consider its determinant for Q^j for $j\not\equiv 0 \pmod q$. This value must be 1.

$$\mathrm{Det}(A(Q^j)) = \omega^{ja(u+1)}(-1)^{j(1-\frac{u+1}{m})}\,.$$

Suppose $(u+1)/m\equiv 0 \pmod 2$, then $\omega^{ja(u+1)}=(-1)^j$. For $j=1$, we have $a(u+1)\equiv q/2 \pmod q$, $a(u+1)\not\equiv 0 \pmod q$. These yield $u-2\equiv 0$ (mod 2). This contradicts $u+1\equiv 0 \pmod 2$. Now we have $(u+1)/m\not\equiv 0$ (mod 2) and then $a(u+1)\equiv 0 \pmod q$. From (D),

$$A(I)=\frac{u+1}{m}((-1)^a+(-1)^{a+q/m}+\cdots+(-1)^{a+(m-1)q/m}).$$

i) If u is even, then m is odd. And q/m must be even. From $a(u+1)\equiv 0 \pmod{q}$, a is even. Thus $A(I)=u+1$.

ii) If u is odd, then $\frac{q}{m}=\frac{u+1}{m}\frac{u-2}{t}$ is odd. Hence $A(I)=0$. This proves lemma 1.

For other type of irreducible characters, similar results can be proved.

Lemma 2.[2] *Under the same assumptions as in Lemma* 1, *if* $\mathfrak{G}$ *has an irreducible charactor* B *of degree* $(u-1)p-1$, *then for an involution* I (*the element of order* 2) *in* $\mathfrak{N}(\mathfrak{P})$

$$B(I)=\begin{cases} 0, & \textit{if } u \textit{ is even.}\\ u-2, & \textit{if } u \textit{ is odd.}\end{cases}$$

Lemma 3.[3] *Under the same assumptions as in Lemma* 1, *if* $\mathfrak{G}$ *has an irreducible character* C *of degree* $(up+1)/t$, *then for an involution* I *of* $\mathfrak{N}(\mathfrak{P})$

$$C(I)=\begin{cases} (u+1)/t, & \textit{if } u \textit{ is even.}\\ 0, & \textit{if } u \textit{ is odd.}\end{cases}$$

Lemma 4.[4] *Under the same assumptions as in Lemma* 1, *if* $\mathfrak{G}$ *has an irreducible character* C *of degree* $((u-1)p-1)/t$, *then for an involution* I *of* $\mathfrak{N}(\mathfrak{P})$

$$C(I)=\begin{cases} 0, & \textit{if } u \textit{ is even.}\\ (u-2)/t, & \textit{if } u \textit{ is odd.}\end{cases}$$

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2) If $u-2=1$, then the Burnside's method yields nothing. But $u-2=1$ yields $p=5$. For $p=5$, $g=5\cdot 4\cdot(1+7\cdot 5)/t$. Since t is odd, $t=1$. Then $B_1(5)$ consists of the principal character, x characters of degree 6, y characters of degree 16 and z characters of degree 9. And we have $1+x+y+z=5$ and $1+6x+16y=9z$. This is a contradiction. Hence $u-2>1$.

3) Let $u+1=t$. If the irreducible character of degree $up+1$ occurs, then $q\equiv 0 \pmod{(u+1)}$. And $u=2$. This contradicts (**). Therefore $B_1(p)$ may consists of the 1-character, x characters of degree $p+1$, y characters of degree $(u-1)p-1$ and t characters of degree $(up+1)/t$. Then $1+x+y=(p-1)/t=u-2$ and $x+1=(u-1)y$. This is also a contradiction. Hence $(u+1)/t>1$.

4) Let $u-2=t$. If the irreducible character of degree $(u-1)p-1$ occurs, then $q\equiv 0 \pmod{(u-2)}$. And either $u=5$ or $u=3$. If $u=5$, then $p=19$. $B_1(19)$ may consist of the 1-character, x characters of degree 20, y characters of degree 96, z characters of degree 75 and t characters of degree 25. Then $1+x+y+z=6$ and $x+5y=4z+1$. This is a contradiction. If $u=3$, then $p=5$. So $B_1(5)$ may consists of the 1-character, x characters of degree 6, y characters of degree 16 and z characters of degree 9. Then $1+x+y+z=5$ and $1+6x+19y=9z$. This is a contradiction. If the irreducible character of degree $(u-1)p-1$ does not occur, then $B_1(p)$ may consist of the 1-character, x characters of degree $p+1$, y characters of degree $up+1$ and t characters of degree $((u-1)p-1)/(u-2)$. Then $1+x+y=u+1$ and $x+uy=1$. This is a contradiction. Hence $(u-2)/t>1$.

Lemma 5. *Under the same assumptions as Lemma* 1, *let X be an irreducible character of degree $p+1$, then for an involution I of* $\mathfrak{N}(\mathfrak{P})$

$$X(I) = \begin{cases} 0, & \textit{if } q \not\equiv 0 \ (\textit{mod } 4). \\ \textit{either} +2 \textit{ or } -2, & \textit{if } q \equiv 0 \ (\textit{mod } 4). \end{cases}$$

In the latter case, we denote by y_1 and y_2 respectively the numbers of characters whose values for I are $+2$ and -2.

Now, we shall consider two cases.

Case I: $\mathfrak{G}$ contains an irreducible character of degree $(up+1)/t$;

Let $B_1(p)$ contain x characters of degree $up+1$, y characters of degree $p+1$, z characters of degree $(u-1)p-1$. From the character-relations in $B_1(p)$, we have

$$\begin{aligned} 1+x+y+z &= (p-1)/t\,, \\ ux+y+(u+1)/t &= (u-1)z\,, \\ p &= u^2-u-1. \end{aligned}$$

The character-relation which holds for p-regular elements shows for an involution I that

$$\text{(I)} \qquad 1+\sum A(I)+\sum X(I)+C(I)=\sum B(I)\,.$$

Eliminate y and p, then $(u-1)x-(u-1)z+(u^2-1)/t=z+1$. Put $z+1=\alpha(u-1)$. Then $x=-(u+1)/t+\alpha u-1$, $y=(u^2-1)/t-2\alpha u+\alpha+1$ and $z=\alpha u-\alpha-1$. As $x\geqq 0$, $\alpha\geqq 1$. And $\alpha\geqq 2$ for $t=1$.

Now consider (I) for even u and for odd u separately.

Case Ia: Case where u is even; From (I), none of $X(I)$ can be zero. Hence we have

$$1+x(u+1)+2(y_1-y_2)+(u+1)/t = 0\,.$$

But $y_2-y_1\leqq y$. Then $1+x(u+1)+(u+1)/t\leqq 2y$. Substitute above values for x and y, then we have

$$\alpha(u_2+5u-2)-u-2\leqq (3u^2+u-2)/t\,.$$

This inequality yields $\alpha=0$ for $t\neq 1$ and $\alpha\leqq 2$ for $t=1$. Hence we have $t=1$ and $\alpha=2$.

Case Ib: Case where u is odd; From (I), none of $X(I)$ can be zero. Hence we have

$$1+2(y_1-y_2) = (u-2)z\,.$$

But $y_1-y_2\leqq y$. Then $(u-2)z-1\leqq 2y$. Substitute the above values for y and z, then we have

$$\alpha(u^2+u)-u-1 \leqq 2(u^2-1)/t,$$
$$\alpha u-1 \leqq 2(u-1)/t.$$

This inequality yields $\alpha=0$ for $t \neq 1$ and $\alpha<2$ for $t=1$. This is a contradiction.

Case II: $\mathfrak{G}$ contains an irreducible character of degree $((u-1)p-1)/t$; Let $B_1(p)$ contain x characters of degree $up+1$, y characters of degree $p+1$, z characters of degree $(u-1)p-1$. From the character-relations in $B_1(p)$,

$$1+x+y+z = (p-1)/t,$$
$$uz+y = (u-1)z+(u-2)/t,$$
$$p = u^2-u-1,$$
$$1+\sum A(I)+\sum X(I) = \sum B(I)+C(I). \tag{I'}$$

Eliminate y and p, then $x+1=ux-uz+u(u-2)/t$. Put $x+1=\alpha u$. Then $z=(u-2)/t+\alpha u-\alpha-1$, $y=u(u-2)/t-2\alpha u+\alpha+1$ and $x=\alpha u-1$. Of course α is a positive integer.

Case IIa: Case where u is even; As Case Ia, from (I'), we have $1+x(u+1) \leqq 2y$. Substitute the above values for x and y, then we have

$$\alpha(u^2+5u-2)-u-2 \leqq 2u(u-2)/t.$$

This yields $t=1$ and $\alpha=1$.

Case IIb: Case where u is odd; We have from (I'),

$$1+2(y_1-y_2) = (u-2)z+(u-2)/t.$$

As Case Ib, we have

$$(u-2)z+(u-2)/t-1 \leqq 2y,$$
$$\alpha(u^2+u)-u-1 \leqq (u-1)(u-2)/t,$$
$$\alpha u-1 \leqq (u-2)/t.$$

This inequality yields $\alpha<1$. This is a contradiction.

Combining the above cases, the only posible case occurs when $t=1$ for even u. In this case $B_1(p)$ consists of the 1-character, $u-1$ characters of degree $up+1$, u^2-4u+2 characters of degree $p+1$ and $2u-3$ characters of degree $(u-1)p-1$.

Denote the sum of the elements in the conjugate class containing G by $\langle G\rangle$. Now we consider the coefficient of $\langle I\rangle^2$ in the group ring of its center. From the orthogonality relations the coefficient a_p of $\langle P\rangle$ is

$$a_p = gn(I)^{-2}\sum (D_g X)^{-1}X(I)^2\cdot\bar{X}(P),$$

where the summation ranges over all the irreducible characters of $\mathfrak{G}$. (cf. [2], § 5). On the other hand this coefficient is equal to the number of pairs of conjugate elements T and S of $\langle I \rangle$ such that $TS=P$. If $TS=P$, then $TPT^{-1}=P^{-1}$. By condition (*), this number of pairs is p. Hence we get

$$p = gn(I)^{-2}\sum (D_g X)^{-1} X(I)^2 \overline{X}(P),$$

where sum ranges over all irreducible characters of $\mathfrak{G}$.

Applying this, we have

$$n(I)^2 p = g\{1+(up+1)^{-1}(u+1)^2(u-1)+(p+1)^{-1}4(u^2-4u+2)\} .$$
$$n(I)^2 = 2u(u-2)^2(u-1)(3u-2)(u+1) .$$

Since $n(I)$ is a multiple of $p-1=(u+1)(u-2)$ and u is even, we have $u+1=5$. Hence $n(I)^2=5^2 2^6 3$. This number is not a square.

Thus for $n=p+2$ and $t \not\equiv 0 \pmod 2$, such a group $\mathfrak{G}$ does not exist. This completes the proof of the theorem.

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