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ÉTALE ENDOMORPHISMS OF 3-FOLDS. II

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Abstract

Up to a finite étale covering, we classify smooth projective 3-folds X with $\kappa(X) = -\infty$ admitting a nonisomorphic étale endomorphism in the case where there exists an FESP $Y_\bullet$ constructed from X by a sequence of blowing-downs of an ESP and an extremal ray $R_\bullet$ of fiber type on $\overline{\mathrm{NE}}(Y_\bullet)$ such that the pair $(Y_\bullet, R_\bullet)$ is of type (C_1) or (C_0).

Contents

1. Introduction

This is the second of a series of articles which provide proofs of results announced in [6]. We shall use the same notation as in [6]. Let us review the background briefly. By an *étale sequence of constant Picard number* ('ESP' for short) $W_\bullet = (\upsilon_n\colon W_n \to W_{n+1})_{n\in\mathbb{Z}}$ of smooth projective k-folds W_n, we mean that for any $n \in \mathbb{Z}$,

- υ_n is a nonisomorphic finite étale covering, and
- the Picard number $\rho(W_n)$ is constant.

In [6], we have applied the minimal model program (called 'MMP', for short) to the constant ESP $X_\bullet = (X, f)$ induced from a nonisomorphic étale endomorphism $f\colon X \to X$ of a smooth projective 3-fold X with $\kappa(X) = -\infty$ and constructed an FESP $Y_\bullet$ from $X_\bullet$ by a sequence of blowing-downs of an ESP (cf. [6, Corollary 1.2]). Hereafter, an endomorphism of a projective variety Z means a surjective morphism (holomorphic map) from Z to itself. Though our étaleness assumption seems to be quite restrictive at first glance, the structure of such varieties is not so simple when compared with the case of $\kappa(X) \geq 0$ (cf. [3], [4]). The main purpose of our series of articles is to study structures of smooth projective 3-folds X with *negative* Kodaira dimension which admit a *nonisomorphic étale endomorphism*. One

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of the difficulties is that there may exist infinitely many extremal rays of $\overline{\mathrm{NE}}(X)$. Here by an *extremal ray R*, we always mean a *K_X-negative* extremal ray of $\overline{\mathrm{NE}}(X)$. Furthermore, it is not so sure that we can find an extremal ray R of $\overline{\mathrm{NE}}(X)$ which is preserved by a suitable power f^k ($k > 0$) of a given étale endomorphism $f\colon X \to X$. Hence the MMP does not necessarily work compatibly with étale endomorphisms. Thus we adapt a method to study the rough structure of the original endomorphism $f\colon X \to X$ through the FESP $Y_\bullet$ constructed from X by a sequence of blowing-downs of an ESP (cf. [6, Definition 3.7]). Here the letter 'F' in the FESP means that there exist extremal rays $R_\bullet$ of fiber type on $\overline{\mathrm{NE}}(Y_\bullet)$. In particular, we shall focus our attention to the finiteness of extremal rays, which is equivalent to the finiteness of extremal rays of *divisorial type*. Once if we know the finiteness of the set of extremal rays, then replacing f by its suitable power f^k ($k > 0$), we can run again the MMP compatibly with étale endomorphisms and obtain another constant FESP (Y, g) induced from a nonisomorphic étale endomorphism $g\colon Y \to Y$. We will consider this problem in several stages. For example, in Part I, we showed the finiteness of extremal rays in the case where there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C_1) or (C_0) constructed from X by a sequence of blowing-downs of an ESP (cf. [6, Theorem 1.4]).

In this Part II article, we shall study the structure of a nonisomorphic étale endomorphism $f\colon X \to X$ in the case where there exist an FESP $Y_\bullet$ constructed from X by a sequence of blowing-downs of an ESP and extremal rays $R_\bullet = (R_n)_n$ ($\subset \overline{\mathrm{NE}}(Y_\bullet)$) of fiber type such that $(Y_\bullet, R_\bullet)$ is of type (C_1) or (C_0), that is, the contraction morphism $\varphi_n := \mathrm{Cont}_{R_n}\colon Y_n \to S_n$ associated to the extremal ray R_n ($\subset \overline{\mathrm{NE}}(Y_n)$) is a conic bundle over a smooth algebraic surface S_n of $\kappa(S_n) = 1$ for any n or of $\kappa(S_n) = 0$ for any n (cf. [6, Definition 3.6]).

Fortunately, in this case, we can run the MMP to the constant FESP $X_\bullet := (X, f)$ compatibly with étale endomorphisms; In fact, the following have been proved in [6].

- There exist at most finitely many extremal rays of $\overline{\mathrm{NE}}(X)$ (cf. [6, Theorem 1.4]).
- If we replace f by its suitable power $f^k (k > 0)$, there exist a birational morphism $\pi\colon X \to Y$ which is a succession of blowing-ups along elliptic curves and a nonisomorphic étale endomorphism $g\colon Y \to Y$ of a smooth projective 3-fold Y such that $\pi \circ f = g \circ \pi$ (cf. [6, Corollary 8.1]).
- Any extremal ray R of $\overline{\mathrm{NE}}(Y)$ is of fiber type and the contraction morphism $\varphi := \mathrm{Cont}_R\colon Y \to S$ associated to R is a conic bundle over a smooth algebraic surface S with $\kappa(S) = 1$ or 0. i.e., The FESP (Y, R) is of type (C_1) or (C_0).

Hereafter, we call $\pi\colon X \to Y$ '*a sequence of equivariant blowing-downs*' of X to the constant FESP Y. The following theorem is our main result.

Theorem 1.1. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Let $Y_\bullet := (Y, g)$ be a constant FESP constructed from $X_\bullet := (X, f)$ by a sequence of equivariant blowing-downs $\pi\colon X \to Y$. Let R be an extremal ray of fiber type on $\overline{\mathrm{NE}}(Y)$ such that the FESP (Y, R) is of type (C_1) or (C_0).*

Then, up to a finite étale covering, X satisfies one of the following conditions. (More precisely, replacing f by its suitable power $f^k (k > 0)$, there exist a finite étale Galois covering $\widetilde{X} \to X$ of X and a nonisomorphic étale endomorphism $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ which is a lift of $f\colon X \to X$. If we replace $f\colon X \to X$ by $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$, then X satisfies one of the following.)

(1) *X is isomorphic to the direct product of an irrational uniruled surface and an elliptic curve.*

(2) *$X \simeq Y$ and X is a $\mathbb{P}^1$-bundle over an abelian surface.*

(3) *$\psi\colon Y \to B$ is a fiber bundle over a smooth curve B of genus $g(B) \geq 1$ whose fiber is the Atiyah surface $\mathbb{S}$ over an elliptic curve E.*

(4) *There exists a smooth morphism $\psi\colon Y \to B$ onto a smooth curve B of genus $g(B) \geq 1$ such that*

- *the general fiber of ψ is isomorphic to the Atiyah surface $\mathbb{S}$ over an elliptic curve E,*
- *the special fiber of ψ is isomorphic to the direct product $E \times \mathbb{P}^1$, and*
- *there exists a fiber bundle structure $\Phi\colon Y \to E$ over an elliptic curve E.*

Furthermore, the π-exceptional divisors $\Theta := \mathrm{Exc}(\pi)$ are simple normal crossings of elliptic ruled surfaces. In the case (3), $\pi(\Theta)$ are canonical sections s_∞ of some fibers of ψ. In the case (4), $\pi(\Theta)$ are canonical sections s_∞ of the general fiber of ψ or fibers of the second projection $E \times \mathbb{P}^1 \to \mathbb{P}^1$ from the special fiber of ψ to $\mathbb{P}^1$.

Here by $\mathcal{F}_r$, we denote a unique indecomposable vector bundle of rank r and degree 0 on an elliptic curve E with $\Gamma(E, \mathcal{F}_r) \neq 0$ (cf. [1]). We are particularly interested in the case of $r = 2$. We call the $\mathbb{P}^1$-bundle $\mathbb{P}_E(\mathcal{F}_2)$ over E associated with $\mathcal{F}_2$ the 'Atiyah surface' and denote it by $\mathbb{S}$. Furthermore, by the canonical section s_∞, we mean the unique section of $\mathbb{S}$ with zero self-intersection number (cf. [6, Definition 5.1]). Actually, we can say more; In the case (3) and (4) in Theorem 1.1, X is a *fiber bundle over an elliptic curve E* (cf. Remark 4.8). This fact will be proved in our subsequent Part III article.

Now, we shall state the idea of the proof of Theorem 1.1. We may assume that all the extremal rays of X are of divisorial type. Since there exist finitely many extremal rays of $\overline{\mathrm{NE}}(X)$, we can take a constant FESP $Y_\bullet = (Y, g)$ induced from a nonisomorphic étale endomorphism $g\colon Y \to Y$. Then the reduction explained in [6, Section 9] can be applied to our situation. For the sake of simplicity, we assume that $\pi\colon X \not\simeq Y$. By replacing X by its suitable finite étale Galois covering $\widetilde{X}$ of X, we may assume the following;

- $\varphi\colon Y \to S$ is a $\mathbb{P}^1$-bundle over the product $S := B \times E$ of an elliptic curve E and a curve B of genus $g(B) \geq 1$.
- If we set $\psi := p \circ \varphi \circ \pi\colon X \to B$ and $\psi' := p \circ \varphi\colon Y \to B$ for the first projection $p\colon S \to B$, then $\psi \circ f = \psi$ and $\psi' \circ g = \psi'$. i.e., Both f and g are nonisomorphic étale endomorphisms over B.

By construction, there is induced a nonisomorphic étale endomorphism $g_t := g|_{Y_t}\colon Y_t \to Y_t$ on each smooth fiber $Y_t := \psi'^{-1}(t)$ $(t \in B)$ which is a $\mathbb{P}^1$-bundle over E. Applying the technique called 'Atiyah reduction' (cf. Proposition 2.2), we may assume that Y_t is isomorphic to either the Atiyah surface $\mathbb{S}$ or the product of $\mathbb{P}^1$ and an elliptic curve for any $t \in B$. Let Λ be a subset of B consisting of points $t \in B$ such that $Y_t \simeq \mathbb{S}$. Then we can show that Λ is a Zariski open subset of B (cf. Lemma 2.3). If $\Lambda = \emptyset$, then a suitable finite étale covering $\widetilde{X}$ of X is isomorphic to the product of an irrational uniruled algebraic surface and an elliptic curve (cf. Theorem 3.2). If $\Lambda \neq \emptyset$, then $\varphi\colon Y \to S$ has a section (cf. Lemma 4.1) and the FESP Y can be described explicitely in terms of vector bundles on the surface $S = B \times E$. In fact, Y is isomorphic to $\mathbb{P}_S(\mathcal{E})$, where $\mathcal{E}$ is a rank two vector bundle on S which is an unsplit extension of $\mathcal{O}_S$ by the line bundle $p^*\ell$ for some $\ell \in \mathrm{Pic} B$ with nonnegative degree (cf. Proposition 4.2). With the aid of this description, we can finish the proof of Theorem 1.1 (cf. Theorems 4.7, 5.1).

2. Basics concerning an FESP of type (C_1)

Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C_1) constructed from X by a sequence of blowing-downs of an ESP. In view of [6, Theorems 1.4 and 9.3], replacing f by its suitable power $f^k (k > 0)$, there exits a nonisomorphic étale endomorphism $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ which is a lift of $f\colon X \to X$. Then if we further replace X by $\widetilde{X}$, we may assume that there exist Cartesian morphisms of constant ESPs

$$X_\bullet = (X, f) \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} S_\bullet = (S, u)$$

such that the following condition (**C**1) is satisfied:

(1) $\pi\colon X \to Y$ is a sequence of equivariant blowing-ups of a smooth projective 3-fold Y along elliptic curves.
(2) $\varphi\colon Y \to S$ is a $\mathbb{P}^1$-bundle over the product $S := C \times E$ of a smooth curve C with $g(C) \geq 2$ and an elliptic curve E.
(3) $u = \mathrm{id}_C \times \alpha$ for some non-zero group homomorphism $\alpha\colon E \to E$.
(4) By φ, the centers of the blowing-up π are mapped onto fibers of the first projection $p\colon S \to C$.

We begin with an easy lemma.

Lemma 2.1. *For any $t \in C$, the surface $Y_t := (p \circ \varphi)^{-1}(t)$ is isomorphic to either $\mathbb{P}_E(\mathcal{E})$, the Atiyah surface $\mathbb{S}$ or $\mathbb{P}_E(\mathcal{O}_E \oplus \ell_t)$, where $\mathcal{E}$ is a stable vector bundle of rank 2 on E and $\ell_t \in \mathrm{Pic}^0(E)$ is of finite order.*

Proof. Since $g\colon Y \to Y$ is a relative endomorphism over C, we infer that $\deg(g|_{Y_t}) = \deg g = \deg f > 1$. Thus the restriction $g|_{Y_t}$ of g to Y_t gives a nonisomorphic étale endomorphism of the elliptic ruled surface Y_t. Hence, the claim follows by [6, Propositions 4.1, 4.8 and 5.10]. □

Let $\mu_n\colon E \to E$ be a multiplication mapping by a positive integer n. We set $\widetilde{X} := X \times_{E,\mu_n} E$ and $\widetilde{Y} := Y \times_{E,\mu_n} E$ respectively. Since $\mu_n \circ \alpha = \alpha \circ \mu_n$, $f\colon X \to X$ (resp. $g\colon Y \to Y$) can be lifted to a nonisomorphic étale endomorphism $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ (resp. $\widetilde{g}\colon \widetilde{Y} \to \widetilde{Y}$) such that $\widetilde{Y}_\bullet := (\widetilde{Y}, \widetilde{g})$ is the constant FESP constructed from of $\widetilde{X}$ by a sequence of equivariant blowing-downs. Furthermore, there also exists a Cartesian morphism of constant ESPs

$$\widetilde{X}_\bullet = (\widetilde{X}, \widetilde{f}) \xrightarrow{\widetilde{\pi}} \widetilde{Y}_\bullet \xrightarrow{\widetilde{\varphi}} S_\bullet = (S, u)$$

which satisfies the conditions (**C**1) as above. The following proposition considerably simplifies the arguments.

Proposition 2.2 (Atiyah reduction)**.** *There esists some positive integer n such that for any $t \in C$, the surface $\widetilde{Y}_t := (p \circ \widetilde{\varphi})^{-1}(t)$ is isomorphic to either the Atiyah surface $\mathbb{S}$ over E or the product $\mathbb{P}^1 \times E$.*

Proof. If $Y_t \simeq \mathbb{P}_E(\mathcal{E})$ for a stable vector bundle $\mathcal{E}$ on E, then [6, Proposition 4.4] implies that $Y_t \times_{E,\mu_2} E \simeq E \times \mathbb{P}^1$ for a multiplication mapping $\mu_2\colon E \to E$ by two. If $Y_t \simeq \mathbb{S}$, then [6, Lemma 4.13] implies that $\mathbb{S} \times_{E,\mu_k} E \simeq \mathbb{S}$ for a multilication mapping $\mu_k\colon E \to E$ by any integer $k > 0$. If $Y_t \simeq \mathbb{P}_E(\mathcal{O}_E \oplus \ell_t)$, then by [6, Corollary 4.9], $\mathrm{ord}(\ell_t)$ divides $\deg(\alpha)$, or

$\deg(\alpha + \mathrm{id}_E)$, or $\deg(\alpha - \mathrm{id}_E)$, where $\alpha\colon E \to E$ is a nonisomorphic group homomorphism of E induced from $f\colon X \to X$. In particular, $\mathrm{ord}(\ell_t)$ is bounded above by a positive constant which is independent of the choice of $t \in C$. Let n be a least common multiple of $2, \deg(\alpha), \deg(\alpha + \mathrm{id}_E)$ and $\deg(\alpha - \mathrm{id}_E)$. Then n satisfies the desired property. □

Hence by replacing $f\colon X \to X$ by its suitable lift $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ as in Proposition 2.2, hereafter we may assume that $Y_t := (p \circ \varphi)^{-1}(t)$ is isomorpic to either $\mathbb{S}$ or $\mathbb{P}^1 \times E$ for any $t \in C$. By $\Lambda(\subset C)$, we denote the set of points $t \in C$ such that $Y_t := (p \circ \varphi)^{-1}(t)$ is isomorphic to $\mathbb{S}$. By construction, if $t \notin \Lambda$, then $Y_t \simeq \mathbb{P}^1 \times E$.

Lemma 2.3. *The set Λ is a Zariski open subset of C.*

Proof. For a relative tangent sheaf $\Theta_{Y/C}$, let us consider the coherent sheaf $\mathcal{F} := (p \circ \varphi)_*\Theta_{Y/C}$ on C. Then there exists a canonical homomorphism $\mathcal{F} \otimes \mathbb{C}(t) \to H^0(Y_t, \Theta_{Y_t})$ for $Y_t := (p \circ \varphi)^{-1}(t)$. We define the $\mathbb{Z}$-valued function, τ on C by $\tau(t) := \dim H^0(Y_t, \Theta_{Y_t})$ $(t \in C)$. Since a holomorphic vector field $H^0(Y_t, \Theta_{Y_t})$ of Y_t is naturally isomorphic to the Lie algebra $\mathrm{Lie}(\mathrm{Aut}^0(Y_t))$ of the automorphism group $\mathrm{Aut}^0(Y_t)$, we see that $\tau(t) = \dim \mathrm{Aut}^0(Y_t)$. Hence by [8, Lemma 10], we see that

$$\tau(t) = \begin{cases} 2, & t \in \Lambda, \\ 4, & t \notin \Lambda. \end{cases}$$

Since τ is upper semi-continuous by the upper semi-continuity of cohomology, Λ is a Zariski open subset of C. □

Hence, under the condition (**C**1) our situation is divided into two cases:

- Case $(\mathrm{C}_{1,a})$: If $\Lambda \neq \emptyset$, then $Y \to C$ is a smooth morphism whose general fiber Y_t is isomorphic to $\mathbb{S}$. Furthermore, $Y \to C$ is a fiber bundle if and only if $\Lambda = C$.
- Case $(\mathrm{C}_{1,b})$: If $\Lambda = \emptyset$, then $Y \to C$ is a fiber bundle whose fiber is isomorphic to $\mathbb{P}^1 \times E$.

Remark 2.4. In the Case $(\mathrm{C}_{1,a})$, if $\Lambda \neq C$, then $Y \to C$ is a smooth morphism but is not a fiber bundle. That is, a jumping phenomenon occurs; i.e.,

$$Y_x \simeq \begin{cases} \mathbb{P}^1 \times E, & x \notin \Lambda, \\ \mathbb{S}, & x \in \Lambda. \end{cases}$$

In fact, we have constructed such an example in our previous particle, Part I (cf. [6, Remark 8.3]).

3. Classifications in the case $(\mathbf{C}_{1,b})$

First, we study the structure of X in the case where there exists a constant FESP $Y_\bullet = (Y, g)$ of type $(\mathrm{C}_{1,b})$ constructed from $X_\bullet = (X, f)$ by a sequence of equivariant blowing-downs.

Proposition 3.1. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists a constant FESP $Y_\bullet = (Y, g)$ of type $(C_{1,b})$ constructed from $X_\bullet = (X, f)$ by a sequence of equivariant blowing-downs.*

Then the following hold:

(1) *There exists an isomorphism $Y \simeq T \times E$ over T, where $q\colon T \to C$ is a $\mathbb{P}^1$-bundle.*
(2) *Let $p\colon Y \to T$ be the first projection. Then there exists a relative automorphism $v \in \mathrm{Aut}(T/C)$ over C such that $p \circ g = v \circ p$.*

Proof. Since all the fibers Y_t of ψ over $t \in C$ are isomorphic, $\psi\colon Y \to C$ is a holomorphic fiber bundle over C by the theorem of Fischer–Grauert [2]. Since the relative anti-canonical bundle $-K_{Y/C}$ is ψ-free, there is induced an elliptic fibration

$$\alpha\colon Y \twoheadrightarrow T \subset \mathbb{P}_C(\psi_*(-K_{Y/C}))$$

over a $\mathbb{P}^1$-bundle $q\colon T \to C$ so that $\psi = q \circ \alpha$. This elliptic fibration α is an elliptic fiber bundle, since for all $t \in C$, $\alpha|_{Y_t}\colon Y_t \to \mathbb{P}^1$ is a trivial elliptic bundle over $\mathbb{P}^1$. Since $-K_{Y/C}$ is ψ-free and $K_{Y/C} \sim g^*K_{Y/C}$, there is induced a relative automorphism $v \in \mathrm{Aut}(T/C)$. In summary, there exists the Cartesian diagram below

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ T & \xrightarrow{v} & T \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ C & = & C \end{array},$$

such that the following hold:

(1) $g\colon Y \to Y$ is a nonisomorphic étale endomorphism of Y.
(2) $\alpha\colon Y \to T$ is an elliptic fiber bundle over a smooth surface T whose fiber is an elliptic curve E.
(3) $v\colon T \simeq T$ is a relative automorphism of T over C.
(4) $q\colon T \to C$ is a $\mathbb{P}^1$-bundle over a smooth curve C of genus $g(C) \geq 2$.
(5) The composite map $\psi := q \circ \alpha\colon Y \to C$ is a fiber bundle whose fiber is isomorphic to $\mathbb{P}^1 \times E$.

For the composite map $\varphi'\colon Y \xrightarrow{\varphi} S = C \times E \xrightarrow{p_2} E$, let us consider the canonical morphism $\Psi := \alpha \times \varphi'\colon Y \to T \times E$. Then by construction, Ψ is an isomorphism, since Ψ is of degree one when restricted to each fiber of ψ. Thus the proof is finished. □

Next, we state the structure theorem in the case where there exists an FESP of type $(\mathrm{C}_{1,b})$ constructed from X by a sequence of equivariant blowing-downs.

Theorem 3.2. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists a constant FESP $Y_\bullet = (Y, g)$ of type $(C_{1,b})$ constructed from X by a sequence of equivariant blowing-downs. Then the following hold*:

(1) *There exists an isomorphism $X \simeq S \times E$ over a surface S, where E is an elliptic curve and S is birational to the product $C \times \mathbb{P}^1$ of a smooth curve C of genus $g(C) \geq 2$ and $\mathbb{P}^1$.*
(2) *There is induced an automorphism $u\colon S \simeq S$ over C such that $p_0 \circ f = u \circ p_0$ for the*

first projection $p_0 \colon X \to S$.

In particular, X satisfies the condition (1) *in* Theorem 1.1.

Proof. Applying Proposition 3.1, there exists a Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{v} & T \end{array}$$

which satisfies the following conditions:

- $\pi \colon X \to Y$ is a sequence of equivariant blowing-ups along elliptic curves.
- $q \colon T \to C$ is a $\mathbb{P}^1$-bundle over a smooth curve C with $g(C) \geq 2$.
- There exists an isomorphism $Y \simeq T \times E$ over T for an elliptic curve E and $p \colon Y \to T$ is the first projection.
- $v \colon T \simeq T$ is an automorphism over C.

Hereafter, we shall use the same notation as in Section 2 (cf. [6, Corollary 1.2]) concerning the construction of an FESP. Let $\pi_i \colon X_i \to X_{i+1}$ for $0 \leq i \leq n-1$ be the blowing-up of X_i along an elliptic curve $C_i (\subset X_i)$ such that $\pi = \pi_{n-1} \circ \cdots \circ \pi_0 \colon X \to Y$. (Here we set $X_0 := X$ and $X_n := Y$.) Furthermore let $f_i \colon X_i \to X_i$ be an induced nonisomorphic étale endomorphism of X_i. Then $f_i^{-1}(C_i) = C_i$ for each i. Now we show that the blowing-up center $C_n \, (\subset X_n := Y)$ of π_{n-1} is some fiber of p. The proof is by contradiction. Suppose that $p(C_n)$ is a curve on T. Since $g^{-1}(C_n) = C_n$ and v is an automorphism of T, we infer that $\deg (C_n / p(C_n)) = \deg g \cdot \deg (C_n / p(C_n))$. This contradicts the assumption that $\deg g = \deg f > 1$. Let S_{n-1} be the blown-up of $S_n := T$ at the point $p(C_n)$. Then $X_{n-1} \simeq S_{n-1} \times E$ and there is induced an automorphism $v_{n-1} \colon S_{n-1} \simeq S_{n-1}$ over C such that $p_{n-1} \circ f_{n-1} = v_{n-1} \circ p_{n-1}$ for the first projection $p_{n-1} \colon X_{n-1} \to S_{n-1}$. By applying the same argument inductively, we infer that:

- There exists an isomorphism $X \simeq S \times E$ over S, where S is birational to $T := S_n$.
- There is induced an automorphism $u \colon S \simeq S$ over C such that $p_0 \circ f = u \circ p_0$ for the first projection $p_0 \colon X \to S$.

Thus the proof is finished. □

4. Classifications in the case $(\mathrm{C}_{1,a})$

In this section, we shall study the structure of a nonisomorphic étale endomorphism $f \colon X \to X$ in the case where there exists a constant FESP $Y_\bullet = (Y, g)$ of type $(\mathrm{C}_{1,a})$ constructed from X by a sequence of equivariant blowing-downs. First, we shall describe the structure of Y in terms of vector bundles on an elliptic ruled surface. Hereafter, we use the same notation as in Section 2. The following lemma is crucial.

Lemma 4.1. *Let* $f \colon X \to X$ *be a nonisomorphic étale endomorphism of a smooth projective* 3*-fold* X *with* $\kappa(X) = -\infty$. *Let* $Y_\bullet := (Y, g)$ *be a constant FESP of type* $(\mathrm{C}_{1,a})$ *constructed* *from X by a sequence of equivariant blowing-downs. Then there exists a smooth irreducible divisor D of Y which satisfies the following three conditions.*

(1) *D is a section of the* $\mathbb{P}^1$*-bundle* $\varphi\colon Y \to S := C \times E$.

(2) *The intersection* $D \cap Y_t$ *equals the canonical section* s_∞ *of* $\mathbb{S}$ *(resp. a fiber of the first projection* $\mathbb{P}^1 \times E \to \mathbb{P}^1$*) if* $t \in \Lambda$ *(resp. if* $t \notin \Lambda$*).*

(3) $g^{-1}(D) = D$.

Proof. Let $p\colon S = C \times E \to C$ be the first projection and set $\psi := p \circ \varphi\colon Y \to C$. Since $-K_{\mathbb{S}} \sim 2s_\infty$ by [6, Proposition 5.2], we infer that $h^0(\mathbb{S}, \mathcal{O}(-K_{\mathbb{S}})) \simeq \mathbb{C}$. Hence by the base change theorem, there exists an isomorphism

$$\psi_*\mathcal{O}_Y(-K_Y) \otimes \mathbb{C}(t) \simeq H^0(\mathbb{S}, \mathcal{O}(2s_\infty)) \simeq \mathbb{C}$$

for any $t \in \Lambda$. Since $\psi_*\mathcal{O}_Y(-K_Y)$ is a torsion free sheaf on a curve C, it is an invertible sheaf on C. Let

$$\Psi\colon \psi^*\psi_*\mathcal{O}_Y(-K_Y) \to \mathcal{O}_Y(-K_Y)$$

be the canonical homomorphism of $\mathcal{O}_Y$-modules. Then

$$\mathcal{O}_Y(-K_Y) \simeq \psi^*\psi_*\mathcal{O}_Y(-K_Y) \otimes \mathcal{O}_Y(G)$$

for some effective divisor G on Y. For $t \in C$, there exists an exact sequence of sheaves;

$$0 \longrightarrow \mathcal{O}_Y(-K_Y) \otimes_{\mathcal{O}_Y} \mathcal{O}(-Y_t) \longrightarrow \mathcal{O}_Y(-K_Y) \longrightarrow \mathcal{O}_Y(-K_Y)|_{Y_t} \longrightarrow 0.$$

Taking direct images, we obtain the following exact sequence;

$$0 \longrightarrow \psi_*\mathcal{O}_Y(-K_Y) \otimes_{\mathcal{O}_C} \mathcal{O}_C(-t) \longrightarrow \psi_*\mathcal{O}_Y(-K_Y) \longrightarrow H^0(Y_t, \mathcal{O}(-K_{Y_t})).$$

Thus there exists an injective homomorphism

$$\psi_*\mathcal{O}_Y(-K_Y) \otimes \mathbb{C}(t) \hookrightarrow H^0(Y_t, \mathcal{O}(-K_{Y_t})).$$

Now there exists the following commutative diagram:

$$\begin{array}{ccc} \psi^*\psi_*\mathcal{O}_Y(-K_Y)|_{Y_t} & \xrightarrow{\Psi_t} & \mathcal{O}_Y(-K_Y)|_{Y_t} \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y_t} \otimes_{\mathbb{C}} (\psi_*\mathcal{O}_Y(-K_Y) \otimes \mathbb{C}(t)) & \longrightarrow & \mathcal{O}_{Y_t}(-K_{Y_t}), \end{array}$$

where Ψ_t is the restriction of Ψ to Y_t ($t \in C$) and two vertical arrows are isomorphisms. Thus Ψ_t is a non-zero homomorphism. Hence for any $t \in C$, $G|_{Y_t}$ is effective and Y_t is not contained in $\operatorname{Supp} G$. By construction, $G|_{Y_t} = 2s_\infty$ for any $t \in \Lambda$. Hence $G = 2D + F$ for a prime divisor D on Y such that $D|_{Y_t} = s_\infty$ for any $t \in \Lambda$ and an effective divisor F supported on the fibers of $p \circ \varphi\colon Y \to C$ over $C \setminus \Lambda$. On the other hand, since F is not contained in any Y_t, we see that $F = 0$ and $G = 2D$. Furthermore, since $g^*K_Y \simeq K_Y$ and $\psi \circ g = \psi$, we have $g^{-1}(D) = D$ and thus $g^{-1}(D|_{Y_t}) = D|_{Y_t}$. The composite map $\varphi|_D = \varphi \circ i\colon D \xrightarrow{i} Y \xrightarrow{\varphi} S = C \times E$ is a birational morphism. We have the following commutative diagram:

$$\begin{array}{ccc} D & \xrightarrow{g|_D} & D \\ {\scriptstyle \varphi|_D}\downarrow & & \downarrow{\scriptstyle \varphi|_D} \\ S & \xrightarrow{u} & S \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ C & = & C. \end{array}$$

We show that $\varphi|_D \colon D \to S$ is an isomorphism. Suppose that there exists an irreducible curve $\Delta(\subset D)$ which is contracted to a point on S by $\varphi|_D$. Then by the above Cartesian diagram, we see that $(g^k)^{-1}(\Delta)$ is also $\varphi|_D$-exceptional for a positive integer k. Note that $u \colon S \to S$ is a nonisomorphic étale endomorphism. If we let $k \to \infty$, then D contains infinitely many $\varphi|_D$-exceptional curves which are disjoint. Thus a contradiction is derived. Hence $\varphi|_D \colon D \to S$ is a finite morphism, and hence an isomorphism by Zariski's main theorem. Hence D is a section of φ and $G = 2D$. Since $(D|_{Y_t}, g)$ is a constant ESP of an elliptic curve such that the inclusion $(D|_{Y_t}, g) \hookrightarrow (D, g)$ is Cartesian, [6, Proposition 6.9] shows that for $t \in C \setminus \Lambda$, $D|_{Y_t}$ is a fiber of the first projection $Y_t \simeq \mathbb{P}^1 \times E \to \mathbb{P}^1$. Thus we are done. □

The following proposition describes the structure of the FESP Y in terms of vector bundles on S.

Proposition 4.2. *Let $p \colon S := C \times E \to C$ be the first projection. Then there exists an unsplit exact sequence of vector bundles on S*

$$0 \longrightarrow p^*\ell \longrightarrow \mathcal{E} \xrightarrow{q} \mathcal{O}_S \longrightarrow 0 \tag{1}$$

which satisfies the properties below:

(i) *ℓ is a line bundle of nonnegative degree on C with $h^0(C, \ell) > 0$.*
(ii) *If $\Lambda \neq C$, then* $\deg \ell > 0$.
(iii) *The surjection q corresponds to the section D of φ in* Lemma 4.1.
(iv) *There exists an isomorphism $Y \simeq \mathbb{P}_S(\mathcal{E})$ over $S = C \times E$.*
(v) *$u^*\mathcal{E} \simeq \mathcal{E}$ and the extension class $\eta \in \mathrm{Ext}^1(\mathcal{O}_S, p^*\ell)$ of* (1) *is preserved by $u \colon S \to S$ up to scalar.*

Proof. There exists the following exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0. \tag{2}$$

We take the direct image of (2) and set $\mathcal{F} := \varphi_*\mathcal{O}_Y(D)$. Since $R^1\varphi_*\mathcal{O}_Y = 0$, we obtain the following exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{F} \longrightarrow \varphi_*\mathcal{O}_D(D) \longrightarrow 0. \tag{3}$$

By construction, $Y \simeq \mathbb{P}_S(\mathcal{F})$ and $\varphi_*\mathcal{O}_D(D)|_{S_t}$ is isomorphic to the normal bundle of $D \cap Y_t$ in Y_t for each $t \in C$, which is trivial. Hence $\varphi_*\mathcal{O}_D(-D) \simeq p^*(\ell)$ for some line bundle ℓ on C. We set $\mathcal{E} := \mathcal{F} \otimes_{\mathcal{O}_S} p^*\ell$. Then after tensoring $p^*\ell$ with (3) , we obtain the above exact sequence (1). We set $S_t := p^{-1}(t)$ for $t \in C$. Then the restriction of (1) to $S_t \simeq E$ gives the following exact sequence:

(4) $$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{E}|_{S_t} \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

Since $Y_t \simeq \mathbb{P}_E(\mathcal{E}|_{S_t})$, we see that

$$\mathcal{E}|_{S_t} \simeq \begin{cases} \mathcal{F}_2, & t \in \Lambda, \\ \mathcal{O}_E \oplus \mathcal{O}_E, & t \notin \Lambda. \end{cases}$$

Since (4) unsplits for $t \in \Lambda$, (1) also unsplits. The assertions (iii) and (iv) are obvious by construction. The former assertion in (v) follows immediately from the fact that $g^{-1}(D) = D$. By the Künneth formula, there exist isomorphisms

(5) $$\mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, p^*\ell) \simeq H^1(S, p^*\ell) \simeq \{H^0(C,\ell) \otimes H^1(E,\mathcal{O}_E)\} \oplus \{H^1(C,\ell) \otimes H^0(E,\mathcal{O}_E)\}.$$

Let $\eta \in \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, p^*\ell)$ be the extension class of (1). Then $\eta = \eta_1 + \eta_2$ for some unique $\eta_1 \in H^0(C,\ell) \otimes H^1(E,\mathcal{O}_E)$ and $\eta_2 \in H^1(C,\ell) \otimes H^0(E,\mathcal{O}_E)$. Note that the group homomorphism $\alpha\colon E \to E$ acts on both vector spaces $H^i(E,\mathcal{O}_E)$ $(i = 0, 1)$ as a multiplication by a common non-zero constant $\mu \in \mathbb{C}^\times$. Hence $u^*\eta = \mu\eta$ and the latter assertion in (v) has been proved. By Leray's spectral sequence, there also exists the following exact sequence of sheaves on C;

$$0 \longrightarrow H^1(C, p_*p^*\ell) \longrightarrow H^1(S, p^*\ell) \longrightarrow H^0(C, R^1p_*p^*\ell) \simeq H^0(C,\ell) \otimes H^1(E,\mathcal{O}_E) \longrightarrow 0.$$

Since $\Lambda \neq \emptyset$ by assumption, we see that $\eta_1 \neq 0$. Hence $h^0(C,\ell) > 0$ and $\deg \ell \geq 0$. Thus the assertion (i) is proved. Moreover, η_1 vanishes exactly at the points of $C \setminus \Lambda$. Hence, if $\Lambda \neq C$, then we have $\deg \ell > 0$. Thus the assertion (ii) has been proved and we are done. □

Corollary 4.3. *We assume the same condition as in* Proposition 4.2. *Then the following hold.*

1. *By the composite map $\alpha\colon Y \xrightarrow{\varphi} S := C \times E \xrightarrow{q} E$, Y is a fiber bundle over the elliptic curve E.*
2. *For a point $a \in E$, let $T_a\colon E \simeq E$ be an automorphism of E defined by $\zeta \mapsto \zeta + a$. Then for any $a \in E$, there exists $g_a \in$* Aut(Y) *such that $\alpha \circ g_a = T_a \circ \alpha$.*

Proof. Hereafter, we use the same notation as in the proof of Proposition 4.2. Set $W_t := \alpha^{-1}(t)$ for $t \in E$. Then the restriction of (1) gives the exact sequence of sheaves on C;

$$0 \longrightarrow \ell \longrightarrow \mathcal{E}|_{W_t} \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

whose extension class is given by $\eta_2|_C$ for $\eta_2 \in H^1(C,\ell) \otimes H^0(E,\mathcal{O}_E)$. Since the non-zero section of $H^0(E,\mathcal{O}_E)$ is nowhere vanishing, $\eta_2|_C$ can be taken to be constant. Hence for any $t \in E$, $W_t \simeq \mathbb{P}_C(\eta_2|_C)$ is isomorphic to each other. Thus by [2], Y is a fiber bundle over E. We set $\widetilde{T}_a := \mathrm{id}_C \times T_a \in \mathrm{Aut}(S)$. Pulling back the exact sequence (1) in Proposition 4.2 by $\widetilde{T}_a$, we get the following exact sequence of vector bundles on S

(6) $$0 \longrightarrow p^*\ell \longrightarrow \widetilde{T}_a^*\mathcal{E} \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

whose extension class is given by $\eta_a := \widetilde{T}_a^*\eta \in \mathrm{Ext}^1(\mathcal{O}_S, p^*\ell)$. Then $\eta_a = \eta$, since in the isomorphism (5), T_a acts on $H^i(E,\mathcal{O}_E)$ $(i = 0, 1)$ trivially. Hence $\widetilde{T}_a^*\mathcal{E} \simeq \mathcal{E}$ and there exists an isomorphism $Y_a := Y \times_{E,T_a} E \simeq \mathbb{P}_S(\widetilde{T}_a^*\mathcal{E}) \simeq \mathbb{P}_S(\mathcal{E}) =: Y$. Then the canonical projection

$Y_a \to Y$ gives an automorphism g_a of Y with $\alpha \circ g_a = T_a \circ \alpha$. Thus we are done. □

The converse to Proposition 4.2 also holds;

Lemma 4.4. *Suppose that there exists an unsplit exact sequence* (1) *of vector bundles on* $S := C \times E$ *which satisfies the condition* (i) *in* Proposition 4.2. *Then* $Y := \mathbb{P}_S(\mathcal{E})$ *admits a nonisomorphic étale endomorphism.*

Proof. Let $\eta \in \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, p^*\ell)$ be the non-zero extension class of (1), which can be decomposed as $\eta = \eta_1 + \eta_2$ for some unique $\eta_1 \in H^0(C, \ell) \otimes H^1(E, \mathcal{O}_E)$ and $\eta_2 \in H^1(C, \ell) \otimes H^0(E, \mathcal{O}_E)$. For an integer $n \geq 2$, let $\mu_n \colon E \to E$ be a multiplication mapping by n and set $u := \mathrm{id}_C \times \mu_n \colon S \to S$. Pulling back the exact sequence (1) by u, we obtain the following exact sequence of vector bundles on S

$$0 \longrightarrow p^*\ell \longrightarrow u^*\mathcal{E} \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

whose extension class is given by $u^*\eta \in \mathrm{Ext}^1(\mathcal{O}_S, p^*\ell)$. Then $u^*\eta = n\eta$, since μ_n acts on both vector spaces $H^i(E, \mathcal{O}_E)$ $(i = 0, 1)$ as a scalar multiplication by n. Hence there exists an isomorphism $\widetilde{Y} := Y \times_{S,u} S \simeq \mathbb{P}_S(u^*\mathcal{E}) \simeq \mathbb{P}_S(\mathcal{E}) =: Y$. Then the canonical projection $\widetilde{Y} \to Y$ induces a nonisomorphic étale endomorphism of Y. □

Remark 4.5. The assertions (i) and (ii) in Proposition 4.2 are related to the *jumping phenomenon* that 'the Atiyah surface $\mathbb{S}$ degenerates to the product of $\mathbb{P}^1$ and an elliptic curve' exactly at the points of C where $\eta_1 \in H^0(C, \ell) \otimes H^1(E, \mathcal{O}_E)$ (as in the proof of Proposition 4.2) vanishes (cf. Remark 2.4).

The following lemma describes the centers of the equivariant blowing-up $\pi \colon X \to Y$ of an FESP Y.

Lemma 4.6. *Let* $f \colon X \to X$ *be a nonisomorphic étale endomorphism of a smooth projective* 3*-fold* X *with* $\kappa(X) = -\infty$. *Suppose that there exists a constant FESP* (Y, g) *of type* (C_1) *or of type* (C_0). *Let*

$$\pi_\bullet \colon (X, f) \xrightarrow{\pi^{(0)}} \cdots \longrightarrow (X^{(i-1)}, f^{(i-1)}) \xrightarrow{\pi^{(i-1)}} (X^{(i)}, f^{(i)}) \longrightarrow \cdots \xrightarrow{\pi^{(k-1)}} (Y, g)$$

be a sequence of equivariant blowing-downs, where we set

$$(X^{(0)}, f^{(0)}) := (X, f), (X^{(k)}, f^{(k)}) := (Y, g)$$

and $\pi^{(i-1)}$ *is (the inverse of) the blowing-up of* $X^{(i)}$ *along an elliptic curve* $C_i (\subset X^{(i)})$ *for each* $1 \leq i \leq k$. *Then for any i, the* $\pi^{(i-1)}$*-exceptional divisor* Δ_i *is isomorphic to either* $\mathbb{S}$ *or an ellptic ruled surface* $\mathbb{P}_{C_i}(\mathcal{O} \oplus \ell_i)$ *for some torsion line bundle* ℓ_i *on* C_i.

Proof. By construction, we have $(f^{(i)})_* R_i = R_i$ for each extremal ray R_i of $\overline{\mathrm{NE}}(X^{(i)})$ such that $\pi^{(i)} = \mathrm{Cont}_{R_i}$. In Particular, $(f^{(i)})^{-1}(\Delta_i) = \Delta_i$ for each i. Thus there is induced a nonisomorphic étalle endomorphism $f^{(i)}|_{\Delta_i} \colon \Delta_i \to \Delta_i$ of Δ_i. Then by [6, Propositions 1,1, 4.8 and Remark 3.2], we see that $\Delta_i \simeq \mathbb{S}$ or $\Delta_i \simeq \mathbb{P}_{C_i}(\mathcal{O} \oplus \ell_i)$ for some torsion line bundle ℓ_i on C_i. □

Now we state our main theorem in this section.

Theorem 4.7. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists a constant FESP $Y_\bullet = (Y, g)$ of type $(C_{1,a})$ constructed from $X_\bullet := (X, f)$ by a sequence of equivariant blowing-downs. Then X satisfies the condition* (3) *or* (4) *in* Theorem 1.1.

Proof. There exist Cartesian morphisms of constant ESPs

$$X_\bullet = (X, f) \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} S_\bullet = (S, u)$$

which satisfy the condition (**C**1) in Section 2. By assumption, $\Lambda \neq \emptyset$. Let

$$\pi\colon X =: X^{(0)} \xrightarrow{\pi^{(0)}} X^{(1)} \longrightarrow \cdots \longrightarrow X^{(i)} \xrightarrow{\pi^{(i)}} X^{(i+1)} \longrightarrow \cdots \xrightarrow{\pi^{(k-1)}} X^{(k)} := Y$$

be a sequence of equivariant blowing-downs to a constant FESP Y, where $\pi^{(i-1)}$ is (the inverse of) the blowing-up along an elliptic curve $C^{(i)}$ on $X^{(i)}$. The structure of Y is described in Lemma 4.1 and Proposition 4.2. By construction, there exists a nonisomorphic étale endomorphism $f^{(i)}\colon X^{(i)} \to X^{(i)}$ such that ${f^{(i)}}^{-1}(C^{(i)}) = C^{(i)}$ for all i. Applying the same method as in the proof of [6, Proposition 7.8], we shall seek for the candidate of $C^{(i)}$. First we consider the case of $i = k$. Since $g^{-1}(C^{(k)}) = C^{(k)}$ for $g := f^{(k)}$, with the aid of [6, Lemma 2.6], we see that $u^{-1}(D_k) = D_k$ for $D_k := \varphi(C^{(k)})$. Since $u = \mathrm{id}_C \times \alpha\colon S \to S$ is a nonisomorphic étale endomorphism, D_k is an elliptic curve which is some fiber of the first projection $p\colon S \to C$. Hence if we set $x := p(D_k) \in C$, then $C^{(k)}$ is contained in the elliptic ruled surface $Y_x := \psi^{-1}(x)$ for $\psi := p \circ \varphi\colon Y \to C$. Since $g^{-1}(Y_x) = Y_x$, there is induced a nonisomorphic étale endomorphism $g|_{Y_x}\colon Y_x \to Y_x$ and $(g|_{Y_x})^{-1}(C^{(k)}) = C^{(k)}$. By [6, Proposition 6.9], we see that if $x \notin \Lambda$ (resp. $x \in \Lambda$), then $C^{(k)}$ equals the canonical section s_∞ of $\mathbb{S}$ over E (resp. some fiber of the second projection $C \times \mathbb{P}^1 \to \mathbb{P}^1$). Applying [6, Lemmas 2.6, 6.10, Corollary 7.9 and Proposition 7.10] successively, we infer the following:

- The π-exceptional locus $\mathrm{Exc}(\pi)$ are simple normal crossing divisors.
- Any irreducible component of $\mathrm{Exc}(\pi)$ is isomorphic to either $\mathbb{S}$ or the $\mathbb{P}^1$-bundle $\mathbb{P}_{C^{(i)}}(\mathcal{O} \oplus \ell_i)$ for a torsion line bundle $\ell_i \in \mathrm{Pic}(C^{(i)})$.
- The image of $C^{(i)}$ in Y equals the canonical section s_∞ of $\mathbb{S}$ (resp. the fiber of the second projection $C \times \mathbb{P}^1 \to \mathbb{P}^1$) if $x \in \Lambda$ (resp. $x \notin \Lambda$).

Hence each $C^{(i)}$ is contained in the proper transform Δ_i of Y_x or $\mathrm{Exc}(\pi_j)$ for some j ($j > i$). If $\Delta_i \simeq \mathbb{S}$ (resp. $\Delta_i \simeq \mathbb{P}_{C_j}(\mathcal{O} \oplus \ell_j)$), then $C^{(i)} = s_\infty$ (resp. $C^{(i)}$ is a multisection of $\Delta_i \to C_j$). In any case, each $C^{(i)}$ dominates C. Then applying Proposition 4.2, Corollary 4.3 and Lemma 4.6, we see that X is of type (3) or of type (4) in Theorem 1.1. □

Remark 4.8. If X is of type (3) or of type (4) in Theorem 1.1, we can show that the composite map $\rho\colon X \xrightarrow{\pi} Y \xrightarrow{\varphi} S \simeq C \times E \xrightarrow{q} E$ is a fiber bundle over E. Applying Corollary 4.3 (2), we shall prove this fact in our subsequent article; Part III.

5. Classifications in the case $(\mathbf{C}_0)$

In this section, we shall study the structure of a nonisomorphic, étale endomorphism $f\colon X \to X$ which admits an FESP $(Y_\bullet, R_\bullet)$ of type (C_0) constructed from $X_\bullet = (X, f)$ by a sequence of equivariant blowing-downs. Since essentially the same arguments as in the case of type (C_1) can be applied to this case, we shall only sketch the outline.

Theorem 5.1. *Let $f: X \to X$ be a nonisomorphic étale endomorphism of a smooth projective* 3*-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C_0) constructed from X by a sequence of equivariant blowing-downs. Then X is of type* (1), (2), (3) *or* (4) *in* Theorem 1.1.

Proof. By [6, Corollary 8.1], replacing X by a suitable finite étale covering $\widetilde{X}$ of X, we may assume that there exist Cartesian morphisms of constant ESPs

$$X_\bullet = (X, f) \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} A_\bullet = (A, u)$$

such that the following condition (**C**0) is satisfied:

(1) $\pi: X \to Y$ is a sequence of equivariant blowing-ups of a smooth projective 3-fold Y along elliptic curves.

(2) $\varphi: Y \to A$ is a $\mathbb{P}^1$-bundle over an Abelian surface A.

If π is an isomorphism, then X is a $\mathbb{P}^1$-bundle over A and is of type (2) in Theorem 1.1. Hence, hereafter we may assume that π is not an isomorphism. Then, applying [6, Theorem 9.5], if we replace f by its suitable power f^k, we may further assume the following:

(3) A is isomorphic to the direct product $E' \times E$ of elliptic curves.

(4) $u = \mathrm{id}_{E'} \times \mu$ for a nonisomorphic group homomorphism $\mu: E \to E$.

(5) The centers of the blowing-up π are mapped by φ onto fibers of the first projection $p: A \to E'$.

If we set $\psi := p \circ \varphi: Y \to E'$, then ψ is a smooth morphism. Applying the same argument as in the proof of Lemma 2.1, any fiber $Y_t := \psi^{-1}(t)$ of ψ is isomorphic to one of the following: $\mathbb{P}_E(\mathcal{E})$, $\mathbb{S}$ or $\mathbb{P}_E(\mathcal{O}_E \oplus \ell_t)$, where $\mathcal{E}$ is a stable vector bundle of rank 2 on E and $\ell_t \in \mathrm{Pic}^0(E)$ is of finite order. Then it follows from Proposition 2.2 that if we replace X by its suitable finite étale covering, we may assume the following: any Y_t is isomorphic to either $\mathbb{S}$ or $\mathbb{P}^1 \times E$. By $\Lambda (\subset E')$, we denote the set of points $t \in E'$ such that $Y_t \simeq \mathbb{S}$. Then by the same argument as in the proof of Lemma 2.3, Theorems 3.2 and 4.7, we infer the following:

- Λ is a Zariski open subset of E'.
- If $\Lambda = \emptyset$, then X is of type (1) in Theorem 1.1.
- If $\Lambda \neq \emptyset$, then X is of type (2), (3) or (4) in Theorem 1.1.

Thus we are done. □

Proof of Theorem 1.1. The proof follows immediately from Theorems 3.2, 4.7 and 5.1. □

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