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## ADDENDUM TO: MAXIMAL TORI OF EXTRINSIC SYMMETRIC SPACES AND MERIDIANS

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### Abstract

Improving a theorem in [1] we observe that a maximal torus of an extrinsic symmetric space in a euclidean space  $V$  is itself extrinsic symmetric in some affine subspace of  $V$ .

A compact *extrinsic symmetric space* is a submanifold  $X \subset \mathbb{S}^{p-1} \subset \mathbb{R}^p = V$  such that for any point  $x \in X$  the reflection  $s_x$  along the normal space  $N = N_x X$  keeps  $X$  invariant.

Every compact symmetric space  $X$  contains a maximal torus  $T$  which is unique up to congruence. If  $X = \mathbb{S}^n \subset \mathbb{R}^{n+1}$ , the maximal torus is a great circle  $C = X \cap \mathbb{R}^2$  which is reflective, hence extrinsic symmetric, see [1, Theorem 4]. But for most extrinsic symmetric spaces, the maximal torus is not reflective. However, as we will show, it is an “iterated” reflective subspace, and in particular:

**Theorem 1.** *A maximal torus  $T$  of a compact extrinsic symmetric space  $X \subset V$  is itself extrinsic symmetric in some linear subspace  $W \subset V$ . In fact, there is an affine subspace  $W' \subset W$  such that  $T \subset W'$  is extrinsically isometric to a Clifford torus  $(\mathbb{S}^1)^r \subset \mathbb{C}^r$ .*

Example. Consider the Veronese embedding of the real projective plane  $\mathbb{R}P^2 \subset S(\mathbb{R}^3)$  (= space of symmetric  $3 \times 3$ -matrices) given by  $[x] \mapsto xx^T$  for any  $x \in \mathbb{S}^2$ . Then  $T = \mathbb{R}P^1 = \{xx^T : x \in \mathbb{S}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3\} = \left\{ \begin{pmatrix} A & 0 \\ & 0 \end{pmatrix} : A = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \text{ with } c^2 + s^2 = 1 \right\}$ . This is a fixed component of the reflection  $s_3 = \text{diag}(1, 1, -1)$  acting on  $S(\mathbb{R}^3)$  by conjugation; its fixed space under this linear action is  $W = \left\{ \begin{pmatrix} A & \\ & a \end{pmatrix} : A \in S(\mathbb{R}^2), a \in \mathbb{R} \right\}$ . Thus  $\mathbb{R}P^1$  is contained in the affine plane  $W' \subset W$  which consists of the symmetric matrices  $\begin{pmatrix} A & \\ & a \end{pmatrix}$  with  $A_{11} + A_{22} = 1$  and  $a = 0$ .

Proof of Theorem 1. Using a chain of certain reflective subspaces, called meridians, we have shown in [1] that the maximal torus of  $X$  is contained in a submanifold which is extrinsic symmetric in a subspace of  $V$  and which is intrinsically the Riemannian product of some round spheres  $S_1, \dots, S_r$  with dimensions  $\geq 1$ . Thus we may assume that

$$(1) \quad X = S_1 \times \dots \times S_r \subset V.$$

Now the maximal torus  $T$  of  $X$  is the Riemannian product of great circles  $C_i \subset S_i$  for  $i = 1, \dots, r$ . We have to show that this splitting is extrinsic, more precisely, that each  $S_i$  is a round sphere in a subspace  $V_i \subset V$  with  $V = V_1 \oplus \dots \oplus V_r$  (orthogonal direct sum). It is well known from a theorem of FERUS [2] that every extrinsic symmetric space is a certain  $K$ -orbit in a Lie triple  $\mathfrak{p}$  where  $K$  is the connected component of  $\text{Aut}(\mathfrak{p})$ , and it splits extrinsically if the Lie triple  $\mathfrak{p}$  splits. Thus to prove that the splitting (1) is extrinsic we only have to check the list of extrinsic symmetric spaces in simple Lie triples (e.g. cf. [3, p.311]) for intrinsic

Riemannian products: there are none (although local products occur). Further, there is (up to reduction of codimension and extrinsic isometries) at most one extrinsic symmetric embedding for each compact symmetric space; in particular, there is no extrinsic symmetric embedding of the sphere  $\mathbb{S}^p$  other than the standard sphere  $\mathbb{S}^p \subset \mathbb{R}^{p+1}$ . (This can be checked also directly from the possible choices for the  $SO_p$ -equivariant second fundamental form  $\alpha : S(T_x) \rightarrow N_x$  at any base point  $x \in \mathbb{S}^p$ .) Thus a maximal torus of  $X$  is  $C_1 \times \dots \times C_r$  for great circles  $C_i \subset S_i$ , more precisely,  $C_i = S_i \cap E_i$  for some plane  $E_i \subset V_i$ .  $\square$

The sphere product (1) is an iterated reflective subspace [1], and the maximal torus of a sphere product is also reflective, hence it is an iterated reflective subspace of the given extrinsic symmetric space.

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### References

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