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# HOMOGENEITY OF MAXIMAL ANTIPODAL SETS

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**Abstract**

We introduce a concept of connectedness of antipodal sets of compact Riemannian symmetric spaces and construct a method to make a bigger antipodal set from a given antipodal set. Moreover, using the connectedness we give a sufficient condition that a given maximal antipodal set is homogeneous.

## 1. Introduction

Let $M$ be a compact Riemannian symmetric space and denote the geodesic symmetry at $x \in M$ by $s_x$. In this paper, we assume that $M$ is connected. If $s_x(y) = y$ for two points $x, y \in M$, we say that $x, y$ are antipodal. A subset $S$ of $M$ is an antipodal set, if any two points of $S$ are antipodal. The 2-number $\#_2M$ of $M$ is the maximum of the cardinalities of antipodal sets of $M$. We call an antipodal set $S$ in $M$ great if $\#S = \#_2M$. An antipodal set $S$ is called maximal if there are no anitipodal sets including $S$ properly. These notions were introduced by Chen-Nagano [1]. In general, any antipodal set of any Riemannian symmetric space of noncompact type is a one-point set, so we consider only compact symmetric spaces in this paper. We say that an antipodal set $A \subset M$ is homogeneous if there is a subgroup of the isometry group of $M$ acting on $A$ transitively.

It is known that any compact Lie group $G$ is a Riemannian symmetric space with respect to a biinvariant metric and any maximal antipodal set including the unit element of $G$ becomes a subgroup of $G$. Therefore, any maximal antipodal set of $G$ is homogeneous. Moreover, Tanaka and Tasaki proved that any great antipodal set of any symmetric $R$-space is homogeneous [7]. Thus, we consider the following problem:

**Problem 1.1.** *Is any maximal antipodal set of any compact Riemannian symmetric space homogeneous ?*

We consider this problem in the present paper introducing a concept of connectedness of antipodal sets. Moreover, we construct a method to make a bigger antipodal set from a given antipodal set using this connectedness.

The present paper is organized as follows. In Section 2, we consider shortest closed geodesics on a compact Riemannian symmetric space and prove that any two shortest closed geodesics through two antipodal points $p, q$ are congruent under the action of some subgroup of the isometry group. In Section 3, we construct a totally geodesic sphere from two antipodal points through which there is a shortest closed geodesic. The Section 4 is the main content in this paper. We introduce a concept of connectedness of antipodal sets. Using

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this connectedness, we construct a subgroup $G_W$ of the isometry group from a given antipodal set $A$ satisfying some condition and prove that $G_W(A)$ is an antipodal set. This is the method to make a bigger antipodal set. We study this expanded antipodal sets in this section and we give a sufficient condition that maximal antipodal sets become homogeneous. In Section 5, we observe an example of the above method in the oriented real Grassmannians $SO(10)/SO(5) \times SO(5)$. In Section 6, we decide the homogeneity of maximal antipodal sets of some compact symmetric spaces. The author would like to thank Professor H.Tasaki for his encouragement.

## 2. Shortest closed geodsics and meridians

We introduce some notations used in this paper.

Notation 2.1. Let $(M, g)$ be a compact Riemannain symmetric space.

- $s_x$ : the geodesic symmety at $x \in M$.
- $G$ : the subgroup of the isometry group of $M$ generated by all geodesic symmetries.
- $K_p := \{h \in G; h(p) = p\}$ $(p \in M)$. Then, $(G, K_p)$ is a compact Riemannian symmetric pair.
- $\mathfrak{g}$ : the Lie algebra of $G$.
- $\sigma_x$ : the involutive inner automorphism of $G$ with respect to $s_x$ $(x \in M)$. The involutive automorphism of $\mathfrak{g}$ indued by $\sigma_x$ is denoted by the same notation $\sigma_x$.
- We fix $o \in M$.
  $\mathfrak{k}$ : the Lie algebra of $K_o$.
  $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ : the eigenspace decomposition of $\mathfrak{g}$ with respect to $\sigma_o$ and $\mathfrak{k}, \mathfrak{m}$ are eigenspaces corresponding to the eigenvalues $+1, -1$ respectively. Then, $T_oM \cong \mathfrak{m}$.
- $\langle X, Y\rangle$ $(X, Y \in \mathfrak{m})$ : the $K_o$-invariant inner product on $\mathfrak{m}$ induced by $g$.

Let $A$ be a maximal flat torus of $M$ through $o \in M$ and $\mathfrak{a} = T_oA$. Then, $\mathfrak{a}$ becomes a maximal abelian subspace of $\mathfrak{m}$ under the identification of $T_oM$ and $\mathfrak{m}$. We set the unit lattice $\Gamma = \{H \in \mathfrak{a}; \exp H \cdot o = o\} = \{H \in \mathfrak{a}; \exp H \in K_o\}$. In the following, for any geodesic $\gamma(t)$ in $M$ we set $\gamma = \{\gamma(t) \in M; t \in \mathbb{R}\}$. Moreover, for any closed geodesic $\gamma(t)$ $(0 \leq t \leq c, \gamma(0) = \gamma(c))$ considered in the following, we assume that $\gamma(t) \neq \gamma(0)$ for any $0 < t < c$.

**Proposition 2.2.** *Let $A$ be a maximal flat torus through $o \in M$ and $\gamma(t)$ be a shortest closed geodesic in $A$ such that $\gamma(0) = \gamma(1) = o$ and $p = \gamma(1/2) \in A$. Then there are no shortest closed geodesics of $A$ containing $o, p$ except for $\gamma(t)$ and $\gamma(-t)$.*

Proof. We remark that $A = \mathfrak{a}/\Gamma$ and for any closed geodesic $\delta(t)$ $(0 \leq t \leq 1)$ of $A$ such that $\delta(0) = \delta(1) = o$ there is $H \in \Gamma$ such that $\delta(t) = \exp tH \cdot o$ and the length of $\delta$ is $\|H\| = \langle H, H\rangle^{\frac{1}{2}}$. Let $c = \min_{H\in\Gamma}\|H\|$ and $\Gamma_0 = \{H \in \Gamma; \|H\| = c\}$. The set of all shortest closed geodesics of A thorough $o \in M$ is $\{\exp tH \cdot o (t \in \mathbb{R}); H \in \Gamma_0\}$.

Let $H_p \in \Gamma_0$ satisfy $\gamma(t) = \exp tH_p \cdot o$. We see $\exp H_p \cdot o = o$ and $\exp\frac{1}{2}H_p \cdot o = p$. It is sufficient to prove $\exp\frac{1}{2}H \cdot o \neq \exp\frac{1}{2}H_p \cdot o$ for any $H \in \Gamma_0, H \neq \pm H_p$. It follows that

$$\exp\frac{1}{2}H \cdot o \neq \exp\frac{1}{2}H_p \cdot o \Leftrightarrow \frac{1}{2}(H + H_p) \notin \Gamma.$$

Hence, we show $\frac{1}{2}(H + H_p) \notin \Gamma$ for any $H \in \Gamma_0, H \neq \pm H_p$.

$\|H\| = \|H_p\| = c$ from $H, H_p \in \Gamma_0$, so

$$\begin{aligned}\|\frac{1}{2}(H + H_p)\|^2 =& \frac{1}{4}(c^2 + c^2 + 2\|H\|\|H_p\|\cos\theta)\\ \leq& \frac{1}{4}(4c^2) = c^2,\end{aligned}$$

where $0 \leq \theta \leq \pi$ is the angle made by $H, H_p$ and the equality is valid if and only if $H = H_p$. Hence, for any $H \in \Gamma_0, H \neq \pm H_p$ we see $\|\frac{1}{2}(H + H_p)\| < c$. By the definition of $c$, we obtain $\frac{1}{2}(H + H_p) \notin \Gamma$. □

We recall fundamental results of polars and meridians introduced by Chen-Nagano[2].

Definition 2.3. For an isometry $h$ of $M$, we set $F(h, M) := \{x \in M; h(x) = x\}$.

(1) A connected component of $F(s_o, M)$ is called a *polar* of $o$. The polar containing $p$ ($p \in F(s_o, M)$) is denoted by $M_o^+(p)$. If a polar is a one-point set, then we call this polar a *pole*. We call $\{o\}$ the trivial pole.
(2) For every $p \in F(s_o, M)$ we denote the connected componet of $F(s_o s_p, M)$ containing $p$ by $M_o^-(p)$. We call $M_o^-(p)$ the *meridian* of $o$ through $p$.

Each of a polar and a meridian is a totally geodesic submanifold of $M$. In $T_pM$ ($p \in F(s_o, M)$), $T_pM = T_pM_o^+(p) + T_pM_o^-(p)$ is an orthgonal direct sum decomposition with respect to the metric $g$. In the following, we recall some properties of polars and meridians from [2].

**Lemma 2.4** ([2])**.** *The following three conditions are equivalent for $o, p \in M$.*

(1) *$p$ is a pole of $o$.*
(2) *$K_o = K_p$.*
(3) *$s_o = s_p$.*

**Lemma 2.5** ([2])**.** *Let $p$ be an antipodal point of $o$. The followings are true.*

(1) *If $A$ is a maximal flat torus containing $o, p$, then $A \subset M_o^-(p)$.*
(2) *$p$ is a pole of $o$ in $M_o^-(p)$.*
(3) *Any closed geodesic of $M$ through $o, p$ is included in $M_o^-(p)$.*

Let $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$ be the eigenspace decomposition with respect to the involutive automorphism $\sigma_p$ of $\mathfrak{g}$ and $\mathfrak{g}^+, \mathfrak{g}^-$ be the eigenspaces corresponding to the eigenvalues $+1, -1$ respectively. Because $o$ and $p$ are antipodal, $s_o$ and $s_p$ are commutative. Hence, $\sigma_o$ and $\sigma_p$ are commutative and $\sigma_p(\mathfrak{k}) \subset \mathfrak{k}, \sigma_p(\mathfrak{m}) \subset \mathfrak{m}$. We set $\mathfrak{k} = \mathfrak{k}^+ + \mathfrak{k}^-$ and $\mathfrak{m} = \mathfrak{m}^+ + \mathfrak{m}^-$ as eigenspace decompositions of $\mathfrak{k}, \mathfrak{m}$ with respect to $\sigma_p$. We see $\mathfrak{g}^+ = \mathfrak{k}^+ + \mathfrak{m}^+$ and $\mathfrak{g}^- = \mathfrak{k}^- + \mathfrak{m}^-$.

**Lemma 2.6** ([2])**.** *$M_o^-(p) = \exp\mathfrak{m}^- \cdot o$.*

Let $G^-$ be the identity component of the fixed point set $F(\sigma_o\sigma_p, G)$. The Lie algebra of $G^-$ is $\mathfrak{k}^+ + \mathfrak{m}^-$ and the Lie algebra of $G^- \cap K_o$ is $\mathfrak{k}^+$.

**Lemma 2.7** ([2])**.** *$M_o^-(p) = G^- \cdot o \cong G^-/G^- \cap K_o$.*

The pair $(G^-, G^- \cap K_o)$ becomes a compact Riemannian symmetric pair by the involutive automorphism $\sigma_o$ of $G^-$ and $M_o^-(p) = G^- \cdot o \cong G^-/G^- \cap K_o$. We define $K(o, p)$ as the identity component of $G^- \cap K_o$. The Lie algebra of $K(o, p)$ is $\mathfrak{k}^+$. The Lie algebra of $K_o$ is $\mathfrak{k} = \mathfrak{k}^+ + \mathfrak{k}^-$ and that of $K_p$ is $\mathfrak{k}_\mathfrak{p} = \mathfrak{k}^+ + \mathfrak{m}^+$, so that of $K_o \cap K_p$ is $\mathfrak{k}^+$. Hence, the identity component of $K_o \cap K_p$ is $K(o, p)$. We remark that for any two maximal flat tori $A_1, A_2$ of $M_o^-(p)$ through $o$ there is $k \in K(o, p)$ such that $A_1 = k(A_2)$.

**Proposition 2.8.** *Let $\gamma(t)$ be a shortest closed geodesic of $M$ and $\gamma(0) = \gamma(1) = o$. Set $p = \gamma(\frac{1}{2})$. If $\delta(t)$ $(\delta \neq \gamma)$ is a shortest closed geodesic such that $\delta(0) = \delta(1) = o$ and $\delta(\frac{1}{2}) = p$, then there is $k \in K(o, p)$ such that $k\delta = \gamma$.*

Proof. Let $A$ and $B$ be maximal flat tori such that $\gamma \subset A$ and $\delta \subset B$. We see $A, B \subset M_o^-(p)$ from Lemma 2.5. There is $k \in K(o, p)$ such that $kB = A$. $k\delta(t)$ is a shortest closed geodesic on $A$ and satisfies $k\delta(0) = k\delta(1) = o$ and $k\delta(\frac{1}{2}) = p$ since $K(o, p) \subset K_o \cap K_p$. Thus, we obtain $k\delta = \gamma$ from Proposition 2.2. □

From Propsotion 2.8, we obtain for any shortest closed geodesic $\gamma(t)$ such that $\gamma(0) = \gamma(1) = o$ and $\gamma(\frac{1}{2}) = p$,

$$\left\{\delta(t)\ ;\ t \in \mathbb{R},\ \begin{array}{c}\delta(s)\ (s \in \mathbb{R}) \text{ is a shortest closed geodesic of } M \\ \text{such that } \delta(0) = \delta(1) = o, \delta(\frac{1}{2}) = p.\end{array}\right\} = K(o, p)\gamma = (K_o \cap K_p)\gamma.$$

In the next section, we study $K(o, p)\gamma$.

## 3. Totally geodesic spheres and shortest closed geodsics

Firstly, we prepare the restricted root system. In the following, we denote $\sigma_o$ by $\sigma$ simply.

By the definition, $\mathfrak{g}$ is a compact Lie algebra, so it is known $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}(\mathfrak{g})$, where $[\mathfrak{g}, \mathfrak{g}]$ becomes a compact semisimple subalgebra of $\mathfrak{g}$ and $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. We denote $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}(\mathfrak{g})$ by $\mathfrak{g}_s$ and $\mathfrak{g}_c$. Since $\sigma : \mathfrak{g} \to \mathfrak{g}$ is an involutive automorphism, it follows that $\sigma(\mathfrak{g}_s) \subset \mathfrak{g}_s$ and $\sigma(\mathfrak{g}_c) \subset \mathfrak{g}_c$. We obtain eigenspace decompositions with respect to $\sigma$

$$\mathfrak{g}_s = \mathfrak{k}_s + \mathfrak{m}_s, \quad \mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{m}_c,$$

where $\mathfrak{k}_s, \mathfrak{k}_c$ are corresponding to the eigenvalue $+1$ and $\mathfrak{m}_s, \mathfrak{m}_c$ are corresponding to the eigenvalue $-1$. Moreover, it is true that $\mathfrak{k}, \mathfrak{m}$ have following direct sum decompositions:

$$\mathfrak{k} = \mathfrak{k}_s + \mathfrak{k}_c, \quad \mathfrak{m} = \mathfrak{m}_s + \mathfrak{m}_c.$$

We denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}^{\mathbb{C}}$ . Then, we obtain a direct sum decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_s^{\mathbb{C}} + \mathfrak{g}_c^{\mathbb{C}}$. Reamrk that $\mathfrak{g}_s^{\mathbb{C}}$ is complex semisimple and $\mathfrak{g}_s$ is the compact real form of $\mathfrak{g}_s^{\mathbb{C}}$. Let $\mathfrak{n} = \mathfrak{k}_s + (\mathfrak{m}_s)_*$, where $(\mathfrak{m}_s)_* = \sqrt{-1}\mathfrak{m}_s$. Then, $\mathfrak{n}$ is a non-compact real form of $\mathfrak{g}_s^{\mathbb{C}}$. Let $(\mathfrak{a}_s)_*$ be a maximal abelian subspace of $(\mathfrak{m}_s)_*$, put $\mathfrak{a}_s = \sqrt{-1}(\mathfrak{a}_s)_*$ and extend $\mathfrak{a}_s$ to a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}_s$. Then, $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_s^{\mathbb{C}}$. Let $\Delta$ be the corresponding root system and $\Sigma$ be the corresponding restricted root system. We denote the multiplicity of each restricted root $\lambda \in \Sigma$ by $n(\lambda)$. Since it is known that each restricted root takes real values on $(\mathfrak{a}_s)_*$, we select some linear order of $(\mathfrak{a}_s)_*$ and denote the set of all positive restricted roots by $\Sigma^+$.

For each linear form $\lambda$ on $(\mathfrak{a}_s)^{\mathbb{C}}$ set

$$\mathfrak{k}_\lambda = \{T \in \mathfrak{k}_s; (\mathrm{ad}H)^2 T = \lambda(H)^2 T \text{ for } H \in \mathfrak{a}_s\},$$
$$\mathfrak{m}_\lambda = \{X \in \mathfrak{m}_s; (\mathrm{ad}H)^2 X = \lambda(H)^2 X \text{ for } H \in \mathfrak{a}_s\}.$$

Then it is true that $\mathfrak{k}_\lambda = \mathfrak{k}_{-\lambda}$ and $\mathfrak{m}_\lambda = \mathfrak{m}_{-\lambda}$. $\mathfrak{k}_0$ is the centrailzer of $\mathfrak{a}_s$ in $\mathfrak{k}_s$. In this setting, it is known that the following direct sum decompositions are true.

$$\mathfrak{k}_s = \mathfrak{k}_0 + \sum_{\lambda\in\Sigma^+} \mathfrak{k}_\lambda, \quad \mathfrak{m}_s = \mathfrak{a}_s + \sum_{\lambda\in\Sigma^+} \mathfrak{m}_\lambda.$$

We set $\mathfrak{a} = \mathfrak{a}_s + \mathfrak{m}_c$ and $\mathfrak{s} = \mathfrak{k}_0 + \mathfrak{k}_c$. Then, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{m}$. We extend every root $\lambda \in \Sigma$ to $\mathfrak{a}^{\mathbb{C}}$ to be 0 on $\mathfrak{m}_c^{\mathbb{C}}$ and denote the extended root and the set of all extended roots by the same symbol $\lambda$ and $\Sigma$. In these setting, we obtain direct sum decompositions of $\mathfrak{k}$ and $\mathfrak{m}$ as follows:

$$\mathfrak{k} = \mathfrak{s} + \sum_{\lambda\in\Sigma^+} \mathfrak{k}_\lambda, \quad \mathfrak{m} = \mathfrak{a} + \sum_{\lambda\in\Sigma^+} \mathfrak{m}_\lambda.$$

Let $\langle\ ,\ \rangle$ be the $K$-invariant inner product on $\mathfrak{m}$ induced by the $G$-inavriant metric $g$ on $M$. The restriction of $\langle\ ,\ \rangle$ to $\mathfrak{m}_s$ is the restriction of a negative constant multiple of the Killing form on the semisimple algebra $\mathfrak{g}_s$. It is known that $\mathfrak{m} = \mathfrak{m}_s + \mathfrak{m}_c$ is an orthogonal direct sum decomposition of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$. We see that $\mathfrak{a} = \mathfrak{a}_s + \mathfrak{m}_c$ is an orthogonal direct sum decomposition. Then we obtain the inner product on $\sqrt{-1}\mathfrak{a}$ by $\langle\ ,\ \rangle$ and denote it by the same letter, that is $\langle\sqrt{-1}H_1, \sqrt{-1}H_2\rangle = \langle H_1, H_2\rangle$ for $H_1, H_2 \in \mathfrak{a}$. Every restricted root takes real values on $\sqrt{-1}\mathfrak{a}$, so there is some $(A_\lambda)_* \in \sqrt{-1}\mathfrak{a}$ such that

$$\langle (A_\lambda)_*, H\rangle = \lambda(H) \text{ for } H \in \sqrt{-1}\mathfrak{a}.$$

Set $A_\lambda = \sqrt{-1}(A_\lambda)_*$. We see $A_\lambda \in \mathfrak{a}_s$ easily. Denote $\mathbb{R}A_\lambda$ by $\mathfrak{a}_\lambda$.

**Lemma 3.1** ([10, Ch.VII, Lemma 11.4, Lemma 11.5])**.** *Let* $\lambda, \mu \in \Sigma^+ \cup \{0\}$ $(\lambda \neq \mu)$ *and* $H \in \mathfrak{a}$. *Then it follows that*

$$[\mathfrak{k}_\lambda, \mathfrak{k}_\mu] \subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}, \quad [\mathfrak{m}_\lambda, \mathfrak{m}_\mu] \subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu},$$
$$[\mathfrak{k}_\lambda, \mathfrak{m}_\mu] \subset \mathfrak{m}_{\lambda+\mu} + \mathfrak{m}_{\lambda-\mu}, \quad [\mathfrak{k}_\lambda, \mathfrak{m}_\lambda] \subset \mathfrak{m}_{2\lambda} + \mathfrak{a}_\lambda,$$
$$\mathrm{ad}(H)\mathfrak{k}_\lambda \subset \mathfrak{m}_\lambda, \quad \mathrm{ad}(H)\mathfrak{m}_\lambda \subset \mathfrak{k}_\lambda.$$

Set $\langle\lambda, \mu\rangle = \langle A_\lambda, A_\mu\rangle$ for $\lambda, \mu \in \Sigma$ and $\hat{A_\lambda} = \frac{2\pi}{\langle\lambda,\lambda\rangle} A_\lambda$ for any $\lambda \in \Sigma$. We recall the unit lattice $\Gamma$ of $\mathfrak{a}$.

**Lemma 3.2** ([10, Ch.VII, Proposition 11.9 Proof])**.** $\hat{A_\lambda} \in \Gamma$ *for any* $\lambda \in \Sigma$.

**Lemma 3.3** ([10, Ch.VII, Section 8])**.** $\lambda(H) \in \pi\sqrt{-1}\mathbb{Z}$ *for any* $H \in \Gamma, \lambda \in \Sigma$.

Suppose that $A$ is the maximal flat torus corresponding to $\mathfrak{a}$. Let $\gamma(t) = \exp tH_p \cdot o$ $(H_p \in \mathfrak{a})$ be a shortest closed geodesic of $A$ such that $\gamma(0) = \gamma(2) = o$ and put $p = \gamma(1)$. In this setting we see $2H_p \in \Gamma$. Let $\Gamma_p = \{H \in \mathfrak{a}\,; \exp H \cdot o = p\} = \{H_p + J\,; J \in \Gamma\}$. We define a subset $\Sigma_p$ of $\Sigma$ as follows:

$$\Sigma_p = \{\lambda \in \Sigma\,; \lambda(X) \in \pi\sqrt{-1}\mathbb{Z} \text{ for any } X \in \Gamma_p\} = \{\lambda \in \Sigma\,; \lambda(H_p) \in \pi\sqrt{-1}\mathbb{Z}\}.$$

We introduce an order of $\Sigma$ satisfying $\lambda \in \Sigma^+ \Rightarrow \lambda(-\sqrt{-1}H_p) \geq 0$. Set $\Sigma_p^+ = \Sigma_p \cap \Sigma^+$. We recall the identity component of $K_o \cap K_p$ is $K(o,p)$. The Lie algebra of $K(o,p)$ is $\mathfrak{k}^+$.

**Lemma 3.4.** $\mathfrak{k}^+ = \mathfrak{s} + \sum_{\lambda\in\Sigma_p^+}\mathfrak{k}_\lambda$.

Proof. Let $X \in \mathfrak{k}$. Then,

$$\begin{aligned} X \in \mathfrak{k}_p &\Leftrightarrow \exp tX \cdot p = p \ (t \in \mathbb{R}) \\ &\Leftrightarrow \exp t\mathrm{Ad}(\exp(-H_p))X \cdot o = o \ (t \in \mathbb{R}) \\ &\Leftrightarrow \mathrm{Ad}(\exp(-H_p))X \in \mathfrak{k}. \end{aligned}$$

Suppose that $X = X_0 + \sum_{\lambda\in\Sigma^+} X_\lambda$ is the decompostion of $X$ corresponding to the direct sum decomposition $\mathfrak{k} = \mathfrak{s} + \sum_{\lambda\in\Sigma^+}\mathfrak{k}_\lambda$. Then,

$$\begin{aligned} e^{\mathrm{ad}(-H_p)}(X_0 + \sum_{\lambda\in\Sigma^+} X_\lambda) &= \sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{ad}(-H_p)^n(X_0 + \sum_{\lambda\in\Sigma^+} X_\lambda) \\ &= X_0 + \sum_{\lambda\in\Sigma^+,\lambda(H_p)\neq 0}\Big(\cos(-\sqrt{-1}\lambda(H_p))X_\lambda + \frac{\sin(-\sqrt{-1}\lambda(H_p))}{\sqrt{-1}\lambda(H_p)}[H_p, X_\lambda]\Big) \\ &\quad + \sum_{\lambda\in\Sigma^+,\lambda(H_p)=0} X_\lambda. \end{aligned}$$

We remark $[H_p, X_\lambda] \in \mathfrak{m}$. Hence, $\mathrm{Ad}(\exp(-H_p))X \in \mathfrak{k} \Leftrightarrow X_\lambda = 0$ for $\lambda(H_p) \notin \pi\sqrt{-1}\mathbb{Z}$. Thus, we showed that $X \in \mathfrak{k}^+$ holds if and only if $X \in \mathfrak{s} + \sum_{\lambda\in\Sigma_p^+}\mathfrak{k}_\lambda$. □

Since $\hat{A}_\lambda \in \Gamma$ for any $\lambda \in \Sigma$ by Lemma 3.2, we see that $\exp t\hat{A}_\lambda \cdot o$ $(0 \leq t \leq 1)$ is a closed geodesic of $M$. Therefore, $\|2H_p\| \leq \|\hat{A}_\lambda\|$ because of the minimality of the length of $\gamma(t)$. We consider the following three cases (A-1),(A-2) and (B).

(A-1) $\|2H_p\| < \|\hat{A}_\lambda\|$ for any $\lambda \in \Sigma$.
(A-2) $\|2H_p\| = \|\hat{A}_\lambda\|$ for some $\lambda \in \Sigma$ and $2H_p \neq \hat{A}_\mu$ for any $\mu \in \Sigma$.
(B) $2H_p = \hat{A}_\lambda$ for some $\lambda \in \Sigma$.

**Lemma 3.5.** *For three cases* (A-1),(A-2) *and* (B), *followings are true:*
(1) (A-1),(A-2) $\Rightarrow \mu(2H_p) = 0$ *or* $\mu(2H_p) = \pi\sqrt{-1}$ *for any* $\mu \in \Sigma^+$.
(2) (B) $\Rightarrow \lambda(2H_p) = 2\pi\sqrt{-1}$ *and* $\mu(2H_p) = 0$ *or* $\mu(2H_p) = \pi\sqrt{-1}$ *for any* $\mu \in \Sigma^+, \mu \neq \lambda$.

Proof. Let $m \in \mathbb{Z}$ and set $L_\mu(m\pi) = \{H \in \mathfrak{a}\,; \langle H, A_\mu\rangle = m\pi\} = \{H \in \mathfrak{a}\,;\ \mu(H) = m\sqrt{-1}\mathbb{Z}\}$ for any $\mu \in \Sigma^+$ which is a hyper plane of $\mathfrak{a}$. The point of $L_\mu(m\pi)$ which has the shortest length from 0 is $\frac{m\pi}{\langle\mu,\mu\rangle}A_\mu$, so $\|\frac{m\pi}{\langle\mu,\mu\rangle}A_\mu\| \leq \|H\|$ for any $H \in L_\mu(m\pi)$. For any $\mu \in \Sigma^+$, it follows that $\mu(2H_p) \in \pi\sqrt{-1}\mathbb{Z}$ by Lemma 3.3. We see $\mu(2H_p) = 0, \pi\sqrt{-1}, 2\pi\sqrt{-1}$. In fact, if $\mu(2H_p) = m\pi\sqrt{-1} (m \geq 3)$, then

$$\|\hat{A}_\mu\| = \|\frac{2\pi}{\langle\mu,\mu\rangle}\hat{A}_\mu\| < \|\frac{m\pi}{\langle\mu,\mu\rangle}\hat{A}_\mu\| \leq \|2H_p\|$$

from the above remark. However, this contradicts to the minimality of $\|2H_p\|$ by Lemma 3.2.

- the case (A-1),(A-2)

  We assume $\lambda(2H_p) = 2\pi\sqrt{-1}$ for some $\lambda \in \Sigma^+$. Then it follows that $\|\hat{A}_\lambda\| \leq \|2H_p\|$ from $H \in L_\lambda(2\pi)$. From the minimality of $\|2H_p\|$, $\hat{A}_\lambda = 2H_p$. However this contradicts to the assumption of (A-1),(A-2). Thus $\lambda(2H_p) = 0, \pi\sqrt{-1}$.

- the case (B)

  Suppose $2H_p = \hat{A}_\lambda$. It is obvious that $\lambda(2H_p) = 2\pi\sqrt{-1}$. We assume $\mu(2H_p) = 2\pi\sqrt{-1}$ for some $\mu \in \Sigma^+, \mu \neq \lambda$. Then $\|A(\mu)\| \leq \|2H_p\|$ from $H \in L_\lambda(2\pi)$. Moreover, it follows that $\hat{A}_\lambda = 2H_p$ from the minimality of $\|2H_p\|$. This implies $\lambda = \mu$. However, this is a contradiction. Thus $\mu(2H_p) = 0, \pi\sqrt{-1}$. □

We consider three subsets $\Sigma^+(0), \Sigma^+(\frac{\pi}{2}), \Sigma^+(\pi)$ of $\Sigma^+$:

$$\Sigma^+(0) = \{\lambda \in \Sigma^+ \,;\, \lambda(2H_p) = 0\} = \{\lambda \in \Sigma^+ \,;\, \lambda(H_p) = 0\},$$
$$\Sigma^+(\frac{\pi}{2}) = \{\lambda \in \Sigma^+ \,;\, \lambda(2H_p) = \pi\sqrt{-1}\} = \{\lambda \in \Sigma^+ \,;\, \lambda(H_p) = \frac{\pi}{2}\sqrt{-1}\},$$
$$\Sigma^+(\pi) = \{\lambda \in \Sigma^+ \,;\, \lambda(2H_p) = 2\pi\sqrt{-1}\} = \{\lambda \in \Sigma^+ \,;\, \lambda(H_p) = \pi\sqrt{-1}\}.$$

By the proof of Lemma 3.5, it is true that $\Sigma^+ = \Sigma^+(0) \sqcup \Sigma^+(\frac{\pi}{2}) \sqcup \Sigma^+(\pi)$. Moreover, we see $\Sigma_p^+ = \Sigma^+(0) \sqcup \Sigma^+(\pi)$. The following lemma is obvious.

**Lemma 3.6.** (A-1),(A-2)$\Rightarrow \Sigma^+(\pi) = \phi$, (B)$\Rightarrow \Sigma^+(\pi) = \{\lambda\}$.

Set $\mathfrak{a}_p = \mathbb{R}H_p$. We define a subspace $\mathfrak{m}_p$ of $\mathfrak{m}$ as follows:

$$\mathfrak{m}_p = \mathfrak{a}_p + \sum_{\lambda \in \Sigma^+(\pi)} \mathfrak{m}_\lambda.$$

**Proposition 3.7.** *$\mathfrak{m}_p$ is a Lie triple system of $\mathfrak{m}$.*

Proof. In (A-1),(A-2), we see $\Sigma^+(\pi) = \phi$ from Lemma 3.6, so $\mathfrak{m}_p = \mathfrak{a}_p$. Hence the statement is obvious.

In (B), suppose $2H_p = \hat{A}_\lambda$. Then $\mathfrak{a}_p = \mathfrak{a}_\lambda$ and $\mathfrak{m}_p = \mathfrak{a}_\lambda + \mathfrak{m}_\lambda$. In this case, we see $2\lambda \notin \Sigma^+$. In fact, if $2\lambda \in \Sigma^+$, then $\exp t\hat{A}_{2\lambda} \cdot o (t \in \mathbb{R})$ is a closed geodesic of $M$ and its length is $\|\hat{A}_{2\lambda}\|$. Then,

$$\|\hat{A}_{2\lambda}\| = \|\frac{2\pi}{\langle 2\lambda, 2\lambda\rangle}\hat{A}_{2\lambda}\| = \frac{2\pi}{\langle 2\lambda, 2\lambda\rangle}\|2\hat{A}_\lambda\| = \frac{1}{2}\|\hat{A}_\lambda\| < \|\hat{A}_\lambda\| = \|2H_p\|.$$

This contradicts to the minimality of $\|2H_p\|$. Hence, $2\lambda \notin \Sigma^+$. By Lemma 3.1,

$$[\mathfrak{a}_\lambda + \mathfrak{m}_\lambda, [\mathfrak{a}_\lambda + \mathfrak{m}_\lambda, \mathfrak{a}_\lambda + \mathfrak{m}_\lambda]] \subset [\mathfrak{a}_\lambda + \mathfrak{m}_\lambda, \mathfrak{k}_\lambda + \mathfrak{s}] \subset \mathfrak{a}_\lambda + \mathfrak{m}_\lambda.$$

Therefore, we showed that $\mathfrak{m}_p$ is a Lie triple system of $\mathfrak{m}$. □

From Proposition 3.7, we see that $\exp\mathfrak{m}_p \cdot o$ is a totally geodesic submanifold of $M$. In the following we denote $\exp\mathfrak{m}_p \cdot o$ as $M_p$. In particular, $M_p$ is a compact Riemannian symmetric space of rank one since $\mathfrak{a}_p$ is a maximal abelian subspace of $\mathfrak{m}_p$ and $\dim \mathfrak{a}_p = 1$.

**Lemma 3.8** ([10, Ch.VII, Theorem 10.3])**.** *Let $N$ be a compact Riemannian symmetric space of rank one and $q \in N$. Let $2L$ denote the common length of the geodesics in $N$. Then the exponential map* Exp : $T_qN \longrightarrow N$ *is a diffeomorphism of the open ball* $B(0, L) = \{X \in T_qN \,;\, \|X\| < L\}$ *in $T_qN$ onto $N - F(s_q, N)$.*

**Theorem 3.9.** *$K(o, p)\gamma = M_p$. Moreover, $M_p$ is a totally geodesic sphere of $M$. Moreover,* (A-1),(A-2)$\Rightarrow \dim M_p = 1$ *and* (B)$\Rightarrow \dim M_p = \dim \mathfrak{m}_p = n(\lambda) + 1$.

Proof. Since $K(o, p)\gamma(t) = \exp t\mathrm{Ad}(K(o, p))H_p \cdot o$, we consider $\mathrm{Ad}(K(o, p))H_p$ in every cases (A-1),(A-2),(B).

- the case (A-1),(A-2)
  In this case, we see $M_p = \exp\mathfrak{m}_p \cdot o = \exp\mathfrak{a}_p \cdot o = \gamma$. For the Lie algebra $\mathfrak{k}^+$ of $K(o, p)$, it follows that $\mathfrak{k}^+ = \mathfrak{s} + \sum_{\lambda\in\Sigma^+(0)} \mathfrak{k}_\lambda$ from Lemma 3.6. Hence, $\mathrm{Ad}(K(o, p))H_p = H_p$ because $[\mathfrak{k}^+, H_p] = \{0\}$. Thus,
  $$K(o, p)\gamma = \{\exp tkH_p \cdot o\ ;\ k \in \mathrm{Ad}(K(o, p)), 0 \leq t \leq 2\} = \exp\mathfrak{a}_p \cdot o = \gamma = M_p.$$
  In paticular, $K(o, p)\gamma$ is a totally geodesic sphere of $M$ since $\gamma$ is a closed geodesic.
- the case (B)
  Let $2H_p = \hat{A}_\lambda$. Then, $\mathfrak{m}_p = \mathfrak{a}_\lambda + \mathfrak{m}_\lambda$ and $\mathfrak{k}^+ = \mathfrak{s} + \mathfrak{k}_\lambda + \sum_{\mu\in\Sigma^+(0)} \mathfrak{k}_\mu$ from Lemma 3.6. It follows that
  $$[\mathfrak{k}^+, \mathfrak{a}_\lambda] = [\mathfrak{k}_\lambda, \mathfrak{a}_\lambda] = \mathfrak{m}_\lambda,$$
  $$[\mathfrak{k}^+, \mathfrak{m}_\lambda] = [\mathfrak{s} + \mathfrak{k}_\lambda + \sum_{\mu\in\Sigma^+(0)} \mathfrak{k}_\mu, \mathfrak{m}_\lambda] \subset \mathfrak{m}_\lambda + \sum_{\mu\in\Sigma^+(0)} \left(\mathfrak{m}_{\lambda+\mu} + \mathfrak{m}_{\lambda-\mu}\right) + \mathfrak{a}_\lambda,$$
  from Lemma 3.1 and the proof of Proposition 3.7. We see $\mathfrak{m}_{\lambda\pm\mu} = 0$. In fact if $\lambda \pm \mu$ ($\mu \in \Sigma^+(0), \mu \neq 0$) is a root, then $(\lambda \pm \mu)(H_p) = \pi\sqrt{-1}$ from $\mu \in \Sigma^+(0)$. This means $\lambda \pm \mu \in \Sigma^+(\pi)$. However, this contradicts to $\Sigma^+(\pi) = \{\lambda\}$. Thus, $\lambda \pm \mu$ is not a root and $\mathfrak{m}_{\lambda\pm\mu} = \{0\}$. Therefore, we obtain $[\mathfrak{k}^+, \mathfrak{m}_\lambda] \subset \mathfrak{a}_\lambda + \mathfrak{m}_\lambda$. Since it follows that $[\mathfrak{k}^+, \mathfrak{m}_p] \subset \mathfrak{m}_p$, we see $\mathrm{Ad}(K(o, p))H_p \subset \mathfrak{m}_p$. Then, $\mathrm{Ad}(K(o, p))H_p$ is a compact submanifold of the round sphere $S(0, \|H_p\|)$ centered at 0 in $\mathfrak{m}_p$ with radius $\|H_p\|$. Moreover, the tangent space $T_{H_p}(\mathrm{Ad}(K(o, p))H_p)$ of $\mathrm{Ad}(K(o, p))H_p$ at $H_p$ is $[\mathfrak{k}^+, H_p] = \mathfrak{m}_\lambda$. Thus, we obtain $\mathrm{Ad}(K(o, p))H_p = S(0, \|H_p\|)$. We see $\exp\mathfrak{m}_p \cdot o = \{\exp tX \cdot o\ ; t \in \mathbb{R}, X \in S(0, \|H_p\|)\}$. Therefore,
  $$K(o, p)\gamma = \{\exp t\mathrm{Ad}(K(o, p))H_p \cdot o; t \in \mathbb{R}\} = \{\exp tX \cdot o\ ; t \in \mathbb{R}, X \in S(0, \|H_p\|)\} = M_p.$$
  In this case, $F(s_o, M) = \{p\}$, so the open ball $B(0, \|H_p\|)$ in $\mathfrak{m}_p$ centerd at 0 with radius $\|H_p\|$ is diffeomorphic to $M_p - \{p\}$. Hence, $M_p$ is a sphere. Thus, $M_p$ is a totally geodesic sphere of $M$. □

Summarizing Section 2 and 3, we obtain the following theorem.

**Theorem 3.10.** *Let $o, p \in M$ be anitpodal two points. Suppose that there is a shortest closed geodesic of $M$ through $o$ and $p$. Then, there is a totally geodesic sphere $M_p$ satisfying the following properties:*

(1) *Any shortest closed geodesic of $M$ through $o$ and $p$ is included in $M_p$.*
(2) *If $K(o, p)$ is the identity component of $K_o \cap K_p$ and $\gamma$ is a shortest closed geodesic of $M$ through $o$ and $p$, then $M_p = (K_o \cap K_p)\gamma = K(o, p)\gamma$.*

## 4. Expansion of antipodal sets and homogeneous antipodal sets

In this section, introducing the connectedness of antipodal sets and some subgroup $G_W$ of the isometry group of $M$ we construct the method to make a bigger antipodal set from a given antipodal set. Moreover, we consider a sufficient condition that a maximal antipodal set is homogeneous.

**4.1. Preparations.** In this subsection, we introduce the connectedness of antipodal sets and the subgroup $G_W$.

DEFINITION 4.1. Let $p$ and $q$ $(p \neq q)$ be two antipodal points of $M$. If there is a shortest closed geodesic on $M$ through $p, q$, then we say that $p$ is *connected* to $q$ or $p, q$ are *connected*.

Let $S$ be an antipodal set of $M$ and $o \in S$. We set $S_o = \{x \in S\,; x \text{ is connected to } o.\}$. In the following, we suppose $S_o \neq \phi$ and denote $M_p$ by $M_{o,p}$ for any $p \in S_o$.

**Proposition 4.2.** *Let $p \in S_o$. Then, there is a shortest closed geodesic of $M$ through $o$ and $p$ which is invariant under every $s_q$ $(q \in S)$.*

Proof. It is sufficient to show that there is a closed geodesic of $M_{o,p}$ through $o$ and $p$ which is invariant under every $s_q$ $(q \in S)$. In this proof, $M_{o,p}$ is denoted by $N$. Since $N$ is invariant under the action of $K_o \cap K_p$ by Theorem 3.10, $s_q(N) \subset N$ for any $q \in S$. We can regard every $s_q|_N$ $(q \in S)$ as an isometry of $N$. We consider the subgroup $Z$ of the isometry group of $N$ which is generated by $\{s_q|_N; q \in S\}$.

If $N$ is a closed geodesic, the statement follows from $s_q(N) \subset N$ $(q \in S)$. Suppose $N \cong S^{n-1}$ $(n \geq 3)$. Let $\phi : S^{n-1} \cong N$ be an automorphism such that

$$\phi(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}) = o, \ \phi(\begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}) = p.$$

Let $\bar{Z}$ be the subgroup of the isometry group $O(n)$ of $S^{n-1}$ corresponding to $Z$ by $\phi$. Then, $\bar{Z} \subset 1 \times O(n-1)$. It follows that $(s_q)^2 = \mathrm{id}_M$ and $s_q s_r = s_r s_q$ for any $q, r \in S$. Therefore, $\bar{Z}$ is a 2-subgroup of $1 \times O(n-1)$. It is known that any 2-subgroup of $1 \times O(n-1)$ is conjugate to a subgroup of

$$I = \left\{ \begin{pmatrix} 1 & & & \\ & \epsilon_1 & & \\ & & \ddots & \\ & & & \epsilon_{n-1} \end{pmatrix} ; \epsilon_i = \pm 1 \ (1 \leq i \leq n-1) \right\}.$$

Hence, there is some $a \in 1 \times O(n-1)$ such that $a\bar{Z}a^{-1} \subset I$. We set $\gamma_i(t)$ $(0 \leq t \leq 2\pi, 2 \leq t \leq n)$ of $S^{n-1}$ as follows:

$$\gamma_i(t) = (\cos t)e_1 + (\sin t)e_i = \begin{pmatrix} \cos(t) \\ 0 \\ \vdots \\ \sin(t) \\ 0 \\ \vdots \end{pmatrix} \quad (0 \leq t \leq 2\pi),$$

where $\{e_i \ ; \ i \in \{1, \cdots, n\}\}$ is the standard basis of $\mathbb{R}^n$. Then, $\gamma(t)$ is a big circle of $S^{n-1}$ through the north pole and the south pole and invariant under the action of $I$. In pariticular, $\gamma(t)$ is invariant under the action of $a\bar{Z}a^{-1}$. Therefore, $a^{-1}\gamma(t)$ is a geodesic through the north pole and the south pole which is invariant under the action of $\bar{Z}$. Hence, $\phi(a^{-1}\gamma(t))$ is a closed geodesic of $N$ through $o$ and $p$ which is invariant under every $s_q$ $(q \in S)$. □

We introduce a concept of connectedness of antipodal sets.

Definition 4.3. Let $S$ be an antipodal set.

(1) If a point series $\{p_i\}_{i=1}^{l}$ of $S$ satisfies that $p_i$ is connected to $p_{i+1}$, then we say this point series is a *connected point series*.
(2) If $S$ satisfies the following condition, we say that $S$ is *connected*: for any $p, q \in S$, there is a connected point series $\{p_i\}_{i=1}^{l}$ of $S$ containing $p$ and $q$.
(3) Let $S$ be connected. If there are no connected anitpodal sets containing $S$ properly, then we say that $S$ is a *maximally connected* antipodal set.
(4) Let $S$ be not necessarily connected and $T$ be a connectd subset of $S$. If there are no connected antipodal subsets of $S$ containing $T$ properly, we say that $T$ is a *connected component* of $S$.

Remark 4.4. It is true that any connected maximal antipodal set is maximally connected. However, any maximally connected antipodal set is not necessarily maximal.

We introduce some notations to use later.

Notation 4.5. Let $S$ be an antipodal set and $o \in S$.

- $\bar{S_o} := S_o \cup \{o\}$.
- $L(o, p, S)$ $(p \in S_o)$ : the set of all shortest closed geodesics through $o, p$ invariant under all $s_q$ $(q \in S)$.
- $L(o, S) := \bigcup_{p \in S_o} L(o, p, S)$.
- $L(S) := \bigcup_{p,q \in S, p,\, q \text{ are connected}} L(p, q, S)$.
- $CL(o, p, S)$ $(p \in S_o)$ : all middle points between $o$ and $p$ on every closed geodesic in $L(o, p, S)$.
- $CL(o, S) := \bigcup_{p \in S_o} CL(o, p, S)$.
- $CL(S) := \bigcup_{p,q \in S, p,\, q \text{ are connected}} CL(p, q, S)$.
- $G_{o,S}$ : the group generated by $\{s_x; x \in CL(o, S)\}$.
- $G_S$ : the group generated by $\{s_y; y \in CL(S)\}$.

For any subset $W$ of $CL(S)$, let $G_W$ be the group generated by $\{s_q; q \in$W$\}$. Each of $G_{o,S}$ and $G_W$ is a subgroup of $G_S$.

**4.2. Expansions of antipodal sets.** In this subsection, we construct a big antipodal set from a given antipodal set using $G_W$. For any antipodal set $S$, remark that $CL(S) \neq \phi$ is equivalent to that $S$ contains connected two points by Proposition 4.2. In the following, we often use the notation

$$x = \begin{cases} a, \\ b. \end{cases}$$

This means $x = a$ or $x = b$.

**Lemma 4.6.** *Let $S$ be an antipodal set. Suppose that $S$ has connected two points. Then the followings are true.*

(1) $s_q(x) = x$ *or* $s_q(x) = s_p(x)$ *for any* $q \in S$ *and* $x \in CL(p, S)$ $(p \in S)$. *Hence,*

$$s_q s_x = \begin{cases} s_x s_q, \\ s_p s_x s_p s_q. \end{cases}$$

(2) *Let $m \in M$ be antipodal to all points of $S$. Then, for any $x \in CL(p, S)$ $(p \in S)$*

$$s_p s_x s_p(m) = s_x(m).$$

Proof. Let $x \in CL(p, r, S)$ $(r \in S_p)$ and $\gamma(t) \in L(p, r, S)$ such that $\gamma(0) = \gamma(2) = p, \gamma(1) = r$ and $\gamma(\frac{1}{2}) = x$. Firstly we will show (1). We see $s_q(\gamma(t)) = \gamma(t)$ or $s_q(\gamma(t)) = \gamma(-t)$ since $s_q(\gamma) \subset \gamma$, $s_q^2 = \mathrm{id}_M$ and $s_q$ fixes $p = \gamma(0)$ and $r = \gamma(1)$. In the former case, we obtain $s_q(x) = s_q(\gamma(\frac{1}{2})) = \gamma(\frac{1}{2}) = x$. In the latter case, $s_q(x) = s_q(\gamma(\frac{1}{2})) = \gamma(-\frac{1}{2}) = s_p(x)$. We consider (2). We see $r = s_x(p)$ is antipodal to $m$ by the definitions of $x$ and $m$. Hence, $p$ is antipodal to $s_x(m)$. Therefore, $s_p s_x s_p(m) = s_p s_x(m) = s_x(m)$. □

**Proposition 4.7.** *Let $S$ be an antipodal set containing connected two points . Let $W$ be any subset of $CL(S)$ and $g \in G_W$. Then, $S \cup gS$ is an antipodal set.*

Proof. Since each of $S$ and $gS$ is an antipodal set, it is sufficient to show that any $r \in S$ is antipodal to any $g(q) \in gS$ $(q \in S)$. From the definition of $G_W$, we may write $g \in G_W$ as $g = s_{x_l} \cdots s_{x_2} s_{x_1}$ $(x_1, x_2, \cdots, x_l \in W)$. Let $x_i \in CL(p_i, S)$ $(p_i \in S, 1 \leq i \leq l)$. We will prove the statement by induction for $l$.

By Lemma 4.6, for $x_1 \in CL(p_i, S)$

$$s_r(s_{x_1}(q)) = \begin{cases} s_{x_1} s_r(q) = s_{x_1}(q), \\ s_{p_1} s_{x_1} s_{p_1} s_r(q) = s_{p_1} s_{x_1} s_{p_1}(q) = s_{x_1}(q). \end{cases}$$

Hence, $r$ is antipodal to $s_{x_1}(q)$. We assume that the statement is true until $l-1$. Then, by using Lemma 4.6 we obtain

$$\begin{aligned} s_r\Big(s_{x_l} \cdots s_{x_1}(q)\Big) &= (\epsilon_l s_{x_l} \epsilon_l)(\epsilon_{l-1} s_{x_{l-1}} \epsilon_{l-1}) \cdots (\epsilon_1 s_{x_1} \epsilon_1) s_r(q) \\ &= (\epsilon_l s_{x_l} \epsilon_l)\Big(s_{x_{l-1}} \cdots s_{x_1}(q)\Big) \\ &= s_{x_l} s_{x_{l-1}} \cdots s_{x_1}(q), \end{aligned}$$

where $\epsilon_i$ $(1 \leq i \leq l)$ is $s_{p_i}$ or $\mathrm{id}_M$. Therefore $r$ is antipodal to $s_{x_l} \cdots s_{x_1}(q)$. □

**Theorem 4.8.** *Let $S$ be an antipodal set containing connected two points. Let $W$ be any subset of $CL(S)$. Then, $G_W(S) = \bigcup_{g \in G_W} gS$ is an antipodal set.*

Proof. It is sufficient to prove that $g_1(S) \cup g_2(S)$ is an antipodal set for any $g_1, g_2 \in G_W$. However, since we see that $S \cup g_1^{-1} g_2(S)$ is an antipodal set, $g_1(S) \cup g_2(S)$ is an antipodal set. Thus, $G_W(S)$ is an antipodal set. □

Definition 4.9. Let $S$ be an antipodal set containing connected two points. Let $W$ be any subset of $CL(S)$. Then, we call the antipodal set $G_W(S)$ the *$G_W$-expanded* set of $S$. It is obvious that $S \subset G_W(S)$.

The next proposition is obvious from the definition of maximal antipodal sets.

**Corollary 4.10.** *Let $S$ be a maximal antipodal set containing connected two points. Let $W$ be any subset of $CL(S)$. Then, $G_W(S) \subset S$.*

We use the following lemma later.

**Lemma 4.11.** *Let $S$ be an antipodal set containing connected two points. Let $W$ be any subset of $CL(S)$ and set $T = G_W(S)$. Then, it follows that $L(p,q,T) = L(p,q,S)$ for any connected two points $p, q \in T$.*

Proof. Since $S \subset T$, it is obvious that $L(p,q,T) \subset L(p,q,S)$. We will show $L(p,q,S) \subset L(p,q,T)$ in the followings. Let $\gamma \in L(p,q,S)$. It is sufficient to show $s_{g(r)}(\gamma) \subset \gamma$ for any $g \in G_W$ and $r \in S$. By the definition of $G_W$, there are $x_1, x_2, \cdots, x_l \in W$ such that $g = s_{x_l} \cdots s_{x_2} s_{x_1}$. Let $x_i \in CL(p_i, q_i, S)$ ($p_i, q_i \in S, p_i, q_i$ are connected., $1 \leq i \leq l$). We prove $s_{g(r)}(\gamma) \subset \gamma$ by induction for $l$.

For $x_1 \in CL(p_1, q_1, S)$,

$$s_{s_{x_1}(r)}(\gamma) = s_{x_1} s_r s_{x_1}(\gamma) = \begin{cases} s_{x_1} s_{x_1} s_r(\gamma) = s_r(\gamma) \subset \gamma, \\ s_{x_1} s_{p_1} s_{x_1} s_{p_1} s_r(\gamma) = s_{s_{x_1}(p_1)} s_{p_1} s_r(\gamma) = s_{q_1} s_{p_1} s_r(\gamma) \subset \gamma, \end{cases}$$

by Lemma 4.6. Hence $s_{s_{x_1}(r)}(\gamma) \subset \gamma$. We assume that it is true until $l-1$. We see

$$s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l} = \begin{cases} s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}, \\ s_{p_l} s_{x_l} s_{p_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}, \end{cases}$$

for $x_1, \cdots, x_l \in W$ as follows. Let $\delta(t) \in L(p_l, q_l, S)$ $(0 \leq t \leq 2)$ satisfy $\delta(0) = \delta(2) = p_l, \delta(1) = q_l$ and $\delta(\frac{1}{2}) = x_l$. By the assumption of induction, it follows that $s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\delta) \subset \delta$. We obtain that

$$\begin{aligned} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\delta(0)) &= s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(p_l) = p_l = \delta(0), \\ s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\delta(1)) &= s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(q_l) = q_l = \delta(1) \end{aligned}$$

from Proposition 4.7. Hence it follows that $s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\delta(t)) = \delta(t)$ or $s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\delta(t)) = \delta(-t)$. In the former case, it is true that

$$\begin{aligned} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(x_l) = x_l &\Rightarrow s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)} = s_{x_l} \\ &\Rightarrow s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l} = s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}. \end{aligned}$$

In the latter case, it is true that

$$\begin{aligned} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(x_l) = s_{p_l}(x_l) &\Rightarrow s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)} = s_{p_l} s_{x_l} s_{p_l} \\ &\Rightarrow s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l} = s_{p_l} s_{x_l} s_{p_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}. \end{aligned}$$

From above arguments, we obtain

$$s_{s_{x_l} \cdots s_{x_1}(r)}(\gamma) = s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l}(\gamma) = \begin{cases} s_{x_l} s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\gamma) \subset \gamma, \\ s_{x_l} s_{p_l} s_{x_l} s_{p_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\gamma) \subset s_{q_l} s_{p_l}(\gamma) \subset \gamma. \end{cases}$$

Hence, it follows that $s_{s_{x_l} \cdots s_{x_1}(r)}(\gamma) \subset \gamma$. By induction, we proved $s_{g(r)}(\gamma) \subset \gamma$ that is $\gamma \in L(p,q,T)$. □

From the proof of Lemma 4.11, we obtain the following lemma which will be used later.

**Lemma 4.12.** *Let $S$ be an antipodal set containing connected two points. Let $x_1, \cdots, x_l \in CL(S)$ and $x_l \in CL(p_l, q_l, S)$ ($p_l, q_l \in S$, $p_l, q_l$ are connected). Then, for any $r \in S$*

$$s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l} = \begin{cases} s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}, \\ s_{p_l} s_{x_l} s_{p_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}. \end{cases}$$

Let $S = S_1$ be an antipodal set containing connected two points. We consider an expanded-series of $S$

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset S_{k+1} \subset \cdots ,$$

where $S_{k+1} = G_{W_k}(S_k)$ for some subset $W_k$ of $CL(S_k)$. Then we see that the following proposition follows immediately from Lemma 4.11.

**Proposition 4.13.** *Let $p, q$ be connected points of $S$. Then, $L(p, q, S_k) = L(p, q, S_{k-1}) = \cdots = L(p, q, S_1)$ for any $k \geq 1$.*

**4.3. Orbits of $G_W$.** Let $S$ be an antipodal set containing connected two points. Let $W$ be any subset of $CL(S)$. Then $G_W(S)$ is an antipodal set. We study $G_W(p)$ for each $p \in S$.

**Proposition 4.14.** *Let $S$ be an antipodal set containing connected two points. Let $x \in CL(S)$ and $x \in CL(r_1, r_2, S)$ ($r_1, r_2 \in S$, $r_1, r_2$ are connected). Suppose that $\gamma \in L(r_1, r_2, S)$ satisfies $\gamma(0) = \gamma(2) = r_1, \gamma(1) = r_2$ and $\gamma(\frac{1}{2}) = x$. Let $m \in M$ be antipodal to every point of $S$ and $s_m(\gamma) \subset \gamma$. If $m \neq s_x(m)$, then $m$ is antipodal and connected to $s_x(m)$.*

Proof. Firstly we will show that $m$ is antipodal to $s_x(m)$. We see $s_m(\gamma(0)) = s_m(r_1) = r_1 = \gamma(0)$ and $s_m(\gamma(1)) = s_m(r_2) = r_2 = \gamma(1)$ since $m$ is antipodal to every point of $S$. Hence $s_m(\gamma(t)) = \gamma(t)$ or $s_m(\gamma(t)) = \gamma(-t)$ since $s_m(\gamma) \subset \gamma$. We obtain $s_m(x) = x$ in the former case and $s_m(x) = s_{r_1}(x)$ in the latter case. Therefore,

$$s_{s_x(m)}(m) = s_x s_m s_x(m) = \begin{cases} s_x s_x s_m(m) = m, \\ s_x s_{r_1} s_x s_{r_1} s_m(m) = s_{s_x(r_1)} s_{r_1}(m) = s_{r_2} s_{r_1}(m) = m. \end{cases}$$

We showed that $m$ is antipodal to $s_x(m)$.

Secondly we will show that $m$ is connected to $s_x(m)$. From the homogeneity of $M$, we may let $o = r_1$ and denote $r_2$ by $r$ simply. There is some $X \in \mathfrak{m}$ such that $\gamma(t) = \exp tX \cdot o$. We consider the map $\iota : M \to G; p \mapsto s_p s_o$. Since it is known that $\iota$ maps geodesics of $M$ to geodesics of $G$, $s_{\gamma(t)} s_o$ ($t \in \mathbb{R}$) is a geodesic of $G$ through unit element of $G$. In particular, $s_{\gamma(t)} s_o = \exp 2tX$. We will show that $s_{\gamma(t)} s_o(m) = \exp t2X \cdot m$ is a geodesic of $M$. It is sufficient to prove $X \in \mathrm{Ad}(g)\mathfrak{m}$, where $m = g \cdot o$ ($g \in G$). We obtain

$$s_m(\exp 2tX) s_m = s_m s_{\gamma(t)} s_o s_m = s_m s_{\gamma(t)} s_m s_o = s_{s_m \gamma(t)} s_o = \begin{cases} s_{\gamma(t)} s_o = \exp 2tX, & \text{(A)} \\ s_{\gamma(-t)} s_o = \exp 2(-t)X, & \text{(B)} \end{cases}$$

from the first part of this proof. Since $\sigma_m = \mathrm{Ad}(g)\sigma_o \mathrm{Ad}(g^{-1})$ in $\mathfrak{g}$, we see (A) $\Rightarrow X \in \mathrm{Ad}(g)\mathfrak{k}$ and (B) $\Rightarrow X \in \mathrm{Ad}(g)\mathfrak{m}$.

If $X \in \mathrm{Ad}(g)\mathfrak{k}$, then

$$s_{\gamma(t)}(m) = s_{\gamma(t)} s_o(g \cdot o) \subset (gKg^{-1})(g \cdot o) = g \cdot o = m.$$

This contradicts to $s_x(m) \neq m$. Therefore $X \in \mathrm{Ad}(g)\mathfrak{m}$, so we showed that $s_{\gamma(t)}(m) = s_{\gamma(t)}s_o(m)$ ($t \in \mathbb{R}$) is a geodesic of $M$. In particular, $s_{\gamma(t)}(m)$ ($0 \leq t \leq 1$) is a closed geodesic since $s_{\gamma(0)}(m) = s_{r_1}m = m = s_{r_2}(m) = s_{\gamma(1)}(m)$. Moreover, since the length of $\gamma(t)$ ($0 \leq t \leq 2$) is $|2X|$ and the length of $s_{\gamma(t)}(m)$ ($0 \leq t \leq 1$) is $|\mathrm{Ad}(g^{-1})(2X)|$, we see that these two closed geodesics have the same length. In particular, $s_{\gamma(t)}(m)$ ($0 \leq t \leq 1$) is a shortest closed geodesic. Hence, we showed that $m$ and $s_x(m) = s_{\gamma(\frac{1}{2})}(m)$ are connected. □

By Proposition 4.14, we obtain the following theorem immediately.

**Theorem 4.15.** *Let $S$ be an antipodal set containing connected two points. Let $W$ be any subset of $CL(S)$. Then, $G_W(p)$ is a connected antipodal set for any $p \in S$.*

We obtain the following corollaries from the definition of connected antipodal sets and Theorem 4.15.

**Corollary 4.16.** *Let $S$ be a connected antipodal set and $W$ be any subset of $CL(S)$. Then, $G_W(S)$ is a connected antipodal set.*

**Corollary 4.17.** *Let $S$ be a maximally connected antipodal set and $W$ be any subset of $CL(S)$. Then, $G_W(S) \subset S$.*

**4.4. $G_{o,S}$-homogeneous antipodal sets.** In this section, we consider some homogeneous antipodal sets.

Definition 4.18. Let $S$ be an antipodal set and $o \in S$. If $G_{o,S}(o) = S$, we say that $S$ is *$G_{o,S}$-homogeneous*.

We see that $G_{o,S}$-homogeneous antipodal set is connected from Theorem 4.15.

**Theorem 4.19.** *Let $S$ be a connected antipodal set, $o \in S$ and $G_{o,S}(S) \subset S$. Then, $S$ is $G_{o,S}$-homogenoeus.*

Proof. $G_{o,S}(o) \subset S$ is obvious, so it is sufficient to show $S - G_{o,S}(o) = \phi$. We see $S_o \subset G_{o,S}(o)$. In fact, for any $p \in S_o$ there is some $x \in CL(o, p, S)$ and $p = s_x(o) \in G_{o,S}(o)$. We assume that $S - G_{o,S}(o) \neq \phi$. Let $p_0 \in G_{o,S}(o)$ and $p_l \in S - G_{o,S}(o)$. From the connectedness of $S$, there is a connected point series $\{p_i\}_{i=0}^{l}$ containing $p_0$ and $p_l$. We see that there is some $0 \leq i \leq l-1$ such that $p_i \in G_{o,S}(o)$ and $p_{i+1} \in S - G_{o,S}(o)$. Then there is some $g \in G_{o,S}$ such that $p_i = g(o)$. In particular $g$ is an isometry of $M$, so $g^{-1}(p_{i+1}) \in S_o$. From the above remark we obtain $p_{i+1} \in G_{o,S}(o)$. However, this is a contradiction. Therefore, $S - G_{o,S}(o) \neq \phi$ is wrong, so $S - G_{o,S}(o) = \phi$. □

From Corollary 4.10 and Theorem 4.19, we obtain the following theorem immediately.

**Theorem 4.20.** *Let $S$ be a maximal antipodal set and $o \in S$. If $S$ is connected, then $S$ is $G_{o,S}$-homogeneous.*

From Corollary 4.17 and Theorem 4.19, we obtain the following theorem similarly.

**Theorem 4.21.** *If $S$ is a maximally connected antipodal set and $o \in S$, then $S$ is $G_{o,S}$-homogeneous.*

In the followings, we study $G_{o,S}$-homogeneous sets.

**Lemma 4.22.** *Let $S$ be $G_{o,S}$-homogeneous. Then, it follows that $L(p, S) = g(L(o, S))$ for any $p = g(o) \in S$ ($g \in G_{o,S}$).*

Proof. We remark $S_p = g(S_o)$ since $g$ is an isometry of $M$. Let $r \in S_o$ and $\gamma \in L(o, r, S)$. Then, $g(\gamma)$ is a shortest closed geodesic through $p = g(o)$ and $g(r)$. Let $q$ be any point of $S$. Because there is $u \in S$ such that $q = g(u)$, we obtain

$$s_q(g(\gamma)) = s_{g(u)}(g(\gamma)) = g s_u g^{-1} g(\gamma) = g s_u(\gamma) \subset g(\gamma).$$

Therefore $g(\gamma) \in L(p, g(r), S)$, so $g(L(o, r, S)) \subset L(p, g(q), S)$. Thus $g(L(o, S)) \subset L(p, S)$. Repeating the above argument replacing $o$ by $p$ we obtain $g^{-1}(L(p, S)) \subset L(o, S)$. Hence, $L(p, S) \subset g(L(o, S))$. Thus, we conclude $L(p, S) = g(L(o, S))$. □

Let $S$ be $G_{o,S}$-homogeneous. For any connected points $p_1, p_2$ of $S$ and any point $y \in CL(p_1, p_2, S)$, there are $x \in CL(o, S)$ and $g \in G_{o,S}$ such that $y = g(x)$, so $s_y = s_{g(x)} = g s_x g^{-1}$. In particular, $s_y \in G_{o,S}$. This argument gives the following proposition.

**Proposition 4.23.** *If $S$ is $G_{o,S}$-homogeneous, $G_S = G_{o,S}$.*

Next, we study that $G_{o,S}$ is decided by only $\bar{S_o} = S_o \cup \{o\} \subset S$.

**Proposition 4.24.** *Let $S$ be an antipodal set and $o \in S$. Suppose that $S$ is a $G_{o,S}$-homogeneous set. Then, $G_{o,S} = G_{o,\bar{S_o}}$. Hence $S = G_{o,\bar{S_o}}(o)$.*

Proof. It is sufficient to prove $L(o, S) = L(o, \bar{S_o})$. Since $\bar{S_o} \subset S$, we see $L(o, S) \subset L(o, \bar{S_o})$ immediately. We show $L(o, \bar{S_o}) \subset L(o, S)$. Because $S = G_{o,S}(o)$, for every $p \in S$ there are $x_1, \cdots, x_l \in CL(o, S)$ such that $s_{x_l} \cdots s_{x_1}(o) = p$. Let $x_i \in CL(o, p_i, S)$ ($p_i \in S_o, 1 \leq i \leq l$). We prove $s_p(\gamma) \subset \gamma$ by induction for $l$.

In $l = 1$, $s_{s_{x_1}(o)}(\gamma) \subset \gamma$ since $s_{x_l}(o) \subset \bar{S_o}$. We assume that it is true until $l-1$. From Lemma 4.12, we see

$$s_{s_{x_l} \cdots s_{x_1}(o)}(\gamma) = s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(o)} s_{x_l}(\gamma) = \begin{cases} s_{x_l} s_{x_1} s_{s_{x_{l-1}} \cdots s_{x_1}(o)}(\gamma) \subset \gamma, \\ s_{x_l} s_o s_{x_l} s_o s_{s_{x_{l-1}} \cdots s_{x_1}(o)}(\gamma) \subset s_{p_l} s_o(\gamma) \subset \gamma. \end{cases}$$

Hence, it follows that $s_{s_{x_l} \cdots s_{x_1}(o)}(\gamma) \subset \gamma$, so we showed $s_{g(o)}(\gamma) \subset \gamma$ for any $g \in G_{o,S}$ by induction. Therefore, $L(o, \bar{S_o}) \subset L(o, S)$ and we conclude $L(o, \bar{S_o}) = L(o, S)$. □

Using Proposition 4.24, we may construct a maximally connected antipodal set easily.

**Proposition 4.25.** *Let $M_1^+, \cdots, M_k^+$ be polars of $o$ in $M$ of which for every point there is a shortest closed geodesic of $M$ through $o$ and it. Let $S_o$ be an antipodal set of $M$ and $S_o \subset M_1^+ \cup \cdots \cup M_k^+$. Then, $G_{o,\bar{S_o}}(o)$ is a connected antipodal set of $M$. Moreover, set $\mathcal{A} = \{S\ ;\ S$ is an antipiodal set and $S \subset M_1^+ \cup \cdots \cup M_k^+\}$ and let $T_o \in \mathcal{A}$ be maximal with respect to the inclusion relation in $\mathcal{A}$. Then, $G_{o,\bar{T_o}}(o)$ is a maximally connected antipodal set.*

Proof. It is obvious that $\bar{S_o}$ is a connected antipodal set of $M$. Hence, $G_{o,\bar{S_o}}(o)$ is a connected antipodal set by Theorem 4.16. We prove the later half of the statement. Let $U$ be a maximally connected antipodal subset of $M$ containing $G_{o,\bar{T_o}}(o)$. Then, $T_o \subset U_o$. However, $T_o = U_o$ by the definition of $T_o$. Since $U$ is a maximally conneted antipodal set,

$U$ is $G_{o,U}$-homogeneous and $U = G_{o,U}(o) = G_{o,\bar{U}_o}(o)$ by Proposition 4.24. Hence, we obtain $G_{o,\bar{T}_o}(o) = G_{o,\bar{U}_o}(o) = U$. □

REMARK 4.26. In Proposition 4.25, if $k = 1$, then considering $T_o$ is equivalent to considering a maximal antipodal set of $M_1^+$.

Next we consider a connected component of a maximal antipodal set.

**Theorem 4.27.** *Let $T$ be a maximal antipodal set and not connected. Let $S$ be a connected component of $T$ and $o \in S$. Then, $S$ is $G_{o,S}$-homogeneous.*

Proof. Firstly, we show that every point of $S$ is antipodal to evey point of $g(T)$ for any $g \in G_{o,S}$. It is sufficient to prove $s_q(g(r)) = g(r)$ for any $q \in S, r \in T$. By the definition, we may write $g = s_{x_l} \cdots s_{x_1}$, where $x_1, \cdots, x_l \in CL(o, S)$. Let $x_k \in CL(o, p_k, S)$ $(p_k \in S_o, 1 \leq k \leq l)$. We prove it by induction for $l$.

In $l = 1$, we obtain from Lemma 4.6

$$s_q(s_{x_1}(r)) = \begin{cases} s_{x_1} s_q(r) = s_{x_1}(r), \\ s_o s_{x_1} s_o s_q(r) = s_o s_{x_1} s_o(r) = s_{x_1}(r). \end{cases}$$

We assume that it is true until $l - 1$. Then, from Proposition 4.6 again we obtain

$$\begin{aligned} s_q(s_{x_l} s_{x_l} \cdots s_{x_1}(r)) &= (\epsilon_l s_{x_l} \epsilon_l)(\epsilon_{l-1} s_{x_{l-1}} \epsilon_{l-1}) \cdots (\epsilon_1 s_{x_1} \epsilon_1) s_q(r) \\ &= (\epsilon_l s_{x_l} \epsilon_l)(s_{x_{l-1}} \cdots s_{x_1})(r) \\ &= s_{x_l} s_{x_{l-1}} \cdots s_{x_1}(r), \end{aligned}$$

where every $\epsilon_i$ $(1 \leq i \leq l)$ is $s_o$ or $\mathrm{id}_M$. Therefore, we proved $s_q(g(r)) = g(r)$ by induction. Thus, we showed that every point of $S$ is antipodal to evey point of $g(T)$. Then, it follows that the every point of $g(S)$ is antipodal to every point of $T$ and $g(S) \subset T$ because of the maximality of $T$. Thus, $G_{o,S}(S) \subset T$. Then $G_{o,S}(S)$ is connected and $S \subset G_{o,S}(S)$. Since $S$ is a connected component of $T$, $G_{o,S}(S) \subset S$. Thus we conclude that $S$ is $G_{o,S}$-homogeneous by Theorem 4.19. □

**4.5. Properties of $G_W$-expansions.** In this section, let $S$ be an antipodal set containing connected two points and $W$ be any subset of $CL(S)$.

Let $S_1 = S$ and we consider a series of antipodal sets

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset S_{k+1} \subset \cdots,$$

where $S_{k+1} = G_{S_k}(S_k)$ $(k \in \mathbb{N})$. Then, there is a natural number $m \in \mathbb{N}$ such that $S_m = S_{m+1} = \cdots$ since $\#_2 M$ is finite. If $S_i = S_{i+1}$ for some natural number $i < m$, then $S_i = S_{i+1} = S_{i+2} = \cdots$ because $S_{i+2} = G_{S_{i+1}}(S_{i+1}) = G_{S_i}(S_i) = S_{i+1}$. Hence, we can rewrite the above sequence as follows;

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subsetneq S_{k+1} \subsetneq \cdots \subsetneq S_m = S_{m+1} = \cdots.$$

Let $X = S_m$. Then $G_{o,X}(X) \subset X$. If $S$ is connected, then every $S_k$ is connected by Corollary 4.16. Hence, $X = G_{o,X}(o)$ by Theorem 4.19.

On the other hand, let $T_1 = S$ and we consider a series of antipodal sets

$$T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_k \subsetneq T_{k+1} \subsetneq \cdots,$$

where $T_{k+1} = G_{W_k}(T_k)$ for some subset $W_k$ of $CL(T_k)$ such that $T_k \subsetneq T_{k+1}$. By the finiteness of $\#_2 M$, there is a natural number $n \in \mathbb{N}$ such that $G_W(T_n) \subset T_n$ for any subset $W \subset CL(T_n)$. Thus, we rewrite the sequence as follows:

$$T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_k \subsetneq T_{k+1} \subsetneq \cdots \subsetneq T_n.$$

Let $Y = T_n$. Then, $G_{o,Y}(Y) \subset Y$. If $S$ is connected, then every $T_k$ $(1 \leq k \leq n)$ is connected by Corollary 4.16, so $Y = G_{o,Y}(o)$ by Theorem 4.19.

**Theorem 4.28.** *In above setting, $X = Y$.*

Proof. Firstly we will prove $T_k \subset S_k$ for any $k \leq \min(m, n)$. In $k = 1$, it is obvious by $T_1 = S = S_1$. We assume that it is true until $k - 1$. We see that $W_{k-1}$ is a subset of $CL(S_{k-1})$ by Lemma 4.11, so $G_{W_{k-1}}$ is a subgroup of $G_{S_{k-1}}$. Hence, $T_k = G_{W_{k-1}}(T_{k-1}) \subset G_{S_{k-1}}(T_{k-1}) \subset G_{S_{k-1}}(S_{k-1}) = S_k$. Therefore, it is true that $T_k \subset S_k$ for any $1 \leq k \leq n$.

Dividing the problem into two cases we show $Y \subset X$ : (i) $n < m$ (ii) $m < n$. In (i),

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subsetneq S_{k+1} \subsetneq \cdots \subsetneq S_n \subsetneq \cdots \subsetneq S_m = X,$$
$$T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_k \subsetneq T_{k+1} \subsetneq \cdots \subsetneq T_n = Y.$$

It is obvious that $Y \subset X$ since $Y = T_n \subset S_n \subsetneq S_m = X$. In (ii),

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset S_{k+1} \subset \cdots \subset S_m = X,$$
$$T_1 \subset T_2 \subset \cdots \subset T_k \subset T_{k+1} \subset \cdots \subset T_m \subset \cdots T_n = Y.$$

We prove $T_{m+a} \subset S_m$ for any $a$ $(0 \leq a \leq n-m)$ by induction for $a$. It is obvious that $T_m \subset S_m$ from above arguments. We assume that it is true until $a$. Then, $G_{W_{m+a}}$ is a subgroup of $G_{T_{m+a}}$ since $W_{m+a}$ is a subset of $CL(T_{m+a})$. By Proposition 4.13, $G_{T_{m+a}}$ is a subgroup of $G_{S_m}$ since $T_{m+a} \subset S_m$. Thus, $G_{W_{m+a}}$ is a subgroup of $G_{S_m}$. Hence $T_{m+a+1} = G_{W_{m+a}}(T_{m+a}) \subset G_{W_{m+a}}(S_m) \subset G_{S_m}(S_m) = S_m$. Therefore, we proved $T_{m+a} \subset S_m = X$ for any $0 \leq a \leq n - m$ by induction and $Y = T_n \subset S_m = X$.

Next we will show $X \subset Y$. For the sake of this, we prove $S_k \subset Y$ $(1 \leq k \leq m)$ by induction for $k$. In $k = 1$, this is obvious. We assume that it is true until $k - 1$. By Proposition 4.13, $G_{S_{k-1}}$ is a subgroup of $G_Y$ since $S_{k-1} \subset Y$. Hence, $S_k = G_{S_{k-1}}(S_{k-1}) \subset G_Y(Y) = Y$, so $S_k \subset Y$ for any $1 \leq k \leq m$. Therefore, $X = S_m \subset Y$. Thus, we conclude $X = Y$. □

Let $S$ be a connected antipodal set and $o \in S$, then we see that $X(= Y)$ is obtained by $G_{o,S}$. Let $U_1 = S$ and consider a series of antipodal sets

$$U_1 \subset U_2 \subset \cdots \subset U_k \subset U_{k+1} \subset \cdots \subset U_l \subset U_{l+1} \subset \cdots,$$

where $U_k = G_{o,U_k}(U_k)$. Then, there is a natural number $l$ such that $U_{l+1} = U_l$. If there is some $i$ $(i < l)$ such that $U_i = U_{i+1} = G_{o,U_{i+1}}(U_{i+1})$, then $U_i = U_{i+1} = U_{i+2} = \cdots$ since $U_{i+2} = G_{o,U_{i+1}}(U_{i+1}) = G_{o,U_i}(U_i) = U_{i+1}$. Hence, we may rewrite the above sequence as follows:

$$U_1 \subset U_2 \subset \cdots \subset U_k \subset U_{k+1} \subset \cdots \subset U_l = U_{l+1} = \cdots.$$

Let $Z = U_l$.

**Corollary 4.29.** *In above setting $Z = X$.*

Proof. It is sufficient to show $G_Z(Z) \subset Z$. By Corollary4.16, $Z$ is connected. Moreover, $G_{o,Z}(Z) \subset Z$ by the definition of $Z$. Hence, $Z$ is $G_{o,Z}$-homogeneous. Therefore, $G_{o,Z} = G_Z$ by Proposition 4.23. Thus, $G_Z(Z) \subset Z$, so $Z = Y = X$. □

## 5. An example of $G_W$-expansions

In this section, we apply the $G_W$-expansion for an antipodal set of the oriented real Grassmannian $\tilde{G}_5(\mathbb{R}^{10}) \cong SO(10)/SO(5) \times SO(5)$. Let $v_1, \cdots, v_k \in \mathbb{R}^n$ be linearly independent. We denote the subspace spanned by $v_1, \cdots, v_k$ with the positive orientation or the negative orientation by

$$\pm V = \pm[v_1 \wedge v_2 \wedge \cdots \wedge v_k].$$

Let us denote

$$v_i = \begin{pmatrix} v_{i1} \\ \vdots \\ v_{in} \end{pmatrix} \in \mathbb{R}^n.$$

Moreover, we write $\pm V$ as follows:

$$\pm V = \pm \begin{bmatrix} v_{11} & \cdots & v_{kn} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ v_{1n} & \cdots & v_{kn} \end{bmatrix}.$$

Let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{R}^n$. We recall some results of anitpodal sets of oriented real Grassmannians $\tilde{G}_k(\mathbb{R}^n)$ from the work of Tasaki [3].

**Proposition 5.1** ([3]). *Let $S$ be any antipodal set of $\tilde{G}_k(\mathbb{R}^n)$. Then, there is an orthonormal basis $v_1, \cdots, v_n$ of $\mathbb{R}^n$ satisfying the following condition:*

$$S \subset \{\pm[v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)}];\ \alpha \in \mathrm{Inc}_k(n)\},$$

*where* $\mathrm{Inc}_k(n) = \{\alpha : \{1, \ldots, k\} \to \{1, \ldots, n\}\ ;\ 1 \leq i < j \leq k \Rightarrow \alpha(i) < \alpha(j)\}$.

For $\alpha, \beta \in \mathrm{Inc}_k(n)$, we denote $\beta - \alpha = \{b \in \mathrm{Im}\beta;\ b \notin \mathrm{Im}\alpha\}$.

**Proposition 5.2** ([3]). *Let $V_\alpha = [v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)}]$ and $V_\beta = [v_{\beta(1)} \wedge \cdots \wedge v_{\beta(k)}] \in \tilde{G}_k(\mathbb{R}^n)$ $(\alpha, \beta \in \mathrm{Inc}_k(n))$. Then, following two conditions are equivalent. Moreover, this is true for any pair of $(\pm V_\alpha, \pm V_\beta)$.*

(1) *$V_\alpha$ is antipodal to $V_\beta$.*
(2) *The cardinality of $\beta - \alpha$ is even.*

We consider the condition that $V_\alpha$ and $V_\beta$ are connected.

**Proposition 5.3.** *Let $V_\alpha = [v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)}]$ and $V_\beta = [v_{\beta(1)} \wedge \cdots \wedge v_{\beta(k)}] \in \tilde{G}_k(\mathbb{R}^n)$ $(\alpha, \beta \in$ $\mathrm{Inc}_k(n))$. Suppose that $V_\alpha$ is antipodal to $V_\beta$. Then, following two conditions are equivalent. Moreover, this is true for any pair of $(\pm V_\alpha, \pm V_\beta)$.*

(1) *$V_\alpha$ is connected to $V_\beta$.*
(2) *The cardinality of $(\beta-\alpha)$ is 2.*

Proof. By the homogeneity of $\tilde{G}_k(\mathbb{R}^n)$, we may assume $V_\alpha, V_\beta \in \{\pm[e_{\sigma(1)}\wedge\cdots\wedge e_{\sigma(k)}];\ \sigma \in \mathrm{Inc}_k(n)\}$ and $V_\alpha = o = +[e_1\wedge\cdots\wedge e_k]$.

In a general compact Riemannian symmetric space $M$, the following two conditions are equivalent: (i) $p_1, p_2 \in M$ are antipodal and connected. (ii) $p_2$ is included in the polar $N$ of $p_1$ whose each point is contained in some shortest closed geodesic on $M$ through $p_1$. In $\tilde{G}_k(\mathbb{R}^n)$, the such polar $N$ of $o$ is given as follows:

$$N = \left\{ \pm\left[ \left( \begin{array}{ccc|cc} * & \cdots & * & & \\ \vdots & \ddots & * & & \\ \hline & & & * & * \\ & & & \vdots & \vdots \\ & & & * & * \end{array} \right) \right] \begin{array}{l} \}k \\ \in \tilde{G}_k(\mathbb{R}^n) \\ \}n-k \end{array} \right\},$$

where the first $k-2$ columns and the last $2$ columns are indicated, and where the component of blank parts is 0. Denote $V_\beta = [e_{\sigma(1)}\wedge\cdots\wedge e_{\sigma(k)}]$. Then $V_\beta \in N$ holds if and only if $\#(\mathrm{Im}(\sigma)\cap\{1,\cdots,k\}) = k-2$. Therefore, the statement follows. □

In the following, denote $\pm[e_{i_1}\wedge\cdots\wedge e_{i_k}]$ $(1 \le i_1,\cdots,i_k \le n)$ by $\pm[i_1\wedge\cdots\wedge i_k]$. Let $E_{i,j}$ be the $10\times 10$ matrix whose $(i,j)$-componet is 1 and other componets are 0 and $F_{i,j} = E_{i,j} - E_{j,i}$ for any $i \ne j$. From Proposition 5.2, the following set $S$ is an antipodal set of $\tilde{G}_5(\mathbb{R}^{10})$:

$$S = \left\{ \begin{array}{rr} \pm o = \pm[1\wedge 2\wedge 3\wedge 4\wedge 5],\ \pm p_1 = \pm[1\wedge 2\wedge 3\wedge 6\wedge 7], & \pm p_2 = \pm[2\wedge 3\wedge 4\wedge 6\wedge 8], \\ \pm p_3 = \pm[1\wedge 3\wedge 4\wedge 6\wedge 9], & \pm p_4 = \pm[5\wedge 7\wedge 8\wedge 9\wedge 10] \end{array} \right\}.$$

Denote $+o$ by $o$ simply. We consider $G_{o,S}(S)$. We see $S_o = \{\pm p_1, \pm p_2, \pm p_3\}$ by Proposition 5.3. Firstly, we consider $M_{o,p_1}$ for $o$ and $p_1$. For example, the following $\delta(\theta)$ is a shortest closed geodesic on $M$ through $o$ and $p_1$:

$$\delta(\theta) = \exp\theta(F_{4,6} + F_{5,7})\cdot o.$$

$K_o \cap K_{p_1}$ is given as follows:

$$K_o \cap K_{p_1} = \left\{ g = \begin{pmatrix} A & & & \\ & B & & \\ & & C & \\ & & & D \end{pmatrix} \in SO(10);\ \begin{array}{r} A, D \in O(3), B, C \in O(2), \\ (\det A)(\det B) = 1, (\det A)(\det C) = 1, \\ (\det B)(\det D) = 1, (\det C)(\det D) = 1 \end{array} \right\}.$$

Hence, the identity component $K(o, p_1)$ of $K_o \cap K_{p_1}$ is as follows:

$$K(o,p_1) = \left\{ g = \begin{pmatrix} A & & & \\ & B & & \\ & & C & \\ & & & D \end{pmatrix} \in SO(10);\ \begin{array}{r} A, D \in SO(3), \\ B, C \in SO(2) \end{array} \right\}.$$

Set $T(\phi) = (\cos\phi)(F_{4,6} + F_{5,7}) + (\sin\phi)(F_{4,7} - F_{5,6})$ $(\phi \in \mathbb{R})$. By Theorem 3.10, we obtain

$$\begin{aligned} M_{o,p_1} &= K(o,p)\delta(\theta) \\ &= \{\exp\theta T(\phi)\cdot o\ ;\ \theta,\phi\in\mathbb{R}\}. \end{aligned}$$

Next, we consider $L(o,p_1,S)$. Then $s_{\pm p_2}, s_{\pm p_3}, s_{\pm p_4}$ are as follows:

$$\begin{aligned} s_{\pm p_2} &= \sum_{i=2,3,4,6,8,} E_{i,i} - \sum_{j=1,5,7,9,10} E_{j,j}, \\ s_{\pm p_3} &= \sum_{i=1,3,4,6,9,} E_{i,i} - \sum_{j=2,5,7,8,10} E_{j,j}, \\ s_{\pm p_4} &= \sum_{i=5,7,8,9,10} E_{i,i} - \sum_{j=1,2,3,4,6} E_{j,j}. \end{aligned}$$

To consider closed geodesics on $M_{o,p_1}$ through $o$, $p_1$ which are invariant under every $s_{\pm p_i}$ ($i = 2,3,4$) we study $T(\phi)$ satisfying $s_{p_i}T(\phi)s_{p_i} = \pm T(\phi)$ ($i = 2,3,4$). By caliculations, we obtain

$$\begin{aligned} L(o,p_1,S) &= \left\{\exp\,\theta T(\phi)\cdot o\ ;\ \phi\equiv 0,\frac{\pi}{2}\ \mathrm{mod}\pi\right\}, \\ CL(o,p_1,S) &= \left\{\exp\,\theta T(\phi)\cdot o\ ;\ \theta\equiv\frac{\pi}{4},\frac{3\pi}{4}\ \mathrm{mod}\pi,\ \ \phi\equiv 0,\frac{\pi}{2}\ \mathrm{mod}\pi\right\}. \end{aligned}$$

We consider the geodesic symmetry of every point of $CL(o,p_1,S)$. For $t = \exp(\theta T(\phi))\cdot o \in CL(o,p_1,S)$, since $s_t = \exp(\theta T(\phi))s_o(\exp(\theta T(\phi)))^{-1}$ and

$$s_o = \sum_{i=1,2,3,4,5} E_{i,i} - \sum_{j=6,7,8,9,10} E_{j,j},$$

we see that the geodesic symmetry of every point of $CL(o,p_1,S)$ is given by one of the following matrices :

$$\sum_{i\in\{1,2,3\}} E_{ii} \pm (G_{46}+G_{57}) - \sum_{i\in\{8,9,10\}} E_{ii}, \qquad \sum_{i\in\{1,2,3\}} E_{ii} \pm (G_{47}-G_{56}) - \sum_{i\in\{8,9,10\}} E_{ii},$$

where $G_{ij} = E_{ij} + E_{ji}$ ($i \neq j$). The geodesic symmetry of every point of $CL(o,\pm p_i,S)$ is also given by the similar way. The geodesic symmetry of every point of $CL(o,-p_1,S)$ is given by following matrices:

$$\sum_{i\in\{1,2,3\}} E_{ii} \pm (G_{46}-G_{57}) - \sum_{i\in\{8,9,10\}} E_{ii}, \qquad \sum_{i\in\{1,2,3\}} E_{ii} \pm (G_{47}+G_{56}) - \sum_{i\in\{8,9,10\}} E_{ii}.$$

The geodesic symmetry of every point of $CL(o,p_2,S)$ is given by followings:

$$\sum_{i\in\{2,3,4\}} E_{ii} \pm (G_{18}+G_{56}) - \sum_{i\in\{7,9,10\}} E_{ii}, \qquad \sum_{i\in\{2,3,4\}} E_{ii} \pm (G_{16}-G_{58}) - \sum_{i\in\{7,9,10\}} E_{ii}.$$

The geodesic symmetry of every point of $CL(o,-p_2,S)$ is given by followings:

$$\sum_{i\in\{2,3,4\}} E_{ii} \pm (G_{18}-G_{56}) - \sum_{i\in\{7,9,10\}} E_{ii}, \qquad \sum_{i\in\{2,3,4\}} E_{ii} \pm (G_{16}+G_{58}) - \sum_{i\in\{7,9,10\}} E_{ii}.$$

The geodesic symmetry of every point of $CL(o,p_3,S)$ is given by followings:

$$\sum_{i\in\{1,3,4\}} E_{ii} \pm (G_{26}+G_{59}) - \sum_{i\in\{7,8,10\}} E_{ii}, \qquad \sum_{i\in\{1,3,4\}} E_{ii} \pm (G_{29}-G_{56}) - \sum_{i\in\{7,8,10\}} E_{ii}.$$

The geodesic symmetry of every point of $CL(o,-p_3,S)$ is given by followings:

$$\sum_{i\in\{1,3,4\}} E_{ii} \pm (G_{26} - G_{59}) - \sum_{i\in\{7,8,10\}} E_{ii}, \qquad \sum_{i\in\{1,3,4\}} E_{ii} \pm (G_{29} + G_{56}) - \sum_{i\in\{7,8,10\}} E_{ii}.$$

By the definition, $G_{o,S}$ is the group generated by all above permutation matices. We obtain $T = G_{o,S}(S)$ as follows:

$$T = \left\{ \pm[i_1 \wedge i_2 \wedge i_3 \wedge i_4 \wedge i_5];\ \begin{array}{r} i_1 \in \{1,8\}, i_2 \in \{2,9\}, i_3 \in \{3,10\} i_4 \in \{4,7\}, i_5 \in \{5,6\}, \\ \#\{i_k\ ;\ k = 1,\cdots,5 \text{ and } i_k \geq 6\} = 2 \text{ or } 4 \end{array} \right\}.$$

We see that $T$ is connected easily. Moreover, we see $G_{o,T}(T) \subset T$. Hence, $T$ is a $G_{o,T}$-homogeneous set.

Remark 5.4. It is known that $T$ is a maximal antipodal set. The following $E_{v_{10}}$ is known as a maximal antipodal set [4].

$$E_{v_{10}} = \left\{ \pm[j_1 \wedge j_2 \wedge j_3 \wedge j_4 \wedge j_5];\ \begin{array}{r} j_1 \in \{1,2\}, j_2 \in \{3,4\}, j_3 \in \{5,6\} j_4 \in \{7,8\}, j_5 \in \{9,10\}, \\ \#\{j_k\ ;\ k = 1,\cdots,5 \text{ and } j_k \text{ is even}\} = 2 \text{ or } 4 \end{array} \right\}.$$

Then, $T$ is conjugate to $E_{v_{10}}$.

## 6. Decision of the homogeneity of maximal antipodal sets

In this section, we decide whether a given maximal antipodal set is homogeneous in some compact symmetric spaces considering the connectedness of antipodal sets. From Theorem 4.20 we see that if a maximal antipodal set $S$ is connected, then $S$ is homogeneous. In the case where $S$ is not connected, we obtain the following proposition obviously since the connectedness is invariant under isometries.

**Proposition 6.1.** *Let $S$ be a maximal antipodal set of $M$ and not connected. Suppose that $S = T_1 \sqcup \cdots \sqcup T_k$ is the decomposition of $S$ by connected components. If there are no isometries $g$ of $M$ such that $T_i = g(T_j)$ for some $T_i, T_j$, then $S$ is not homogeneous.*

Hence, we see that a maximal antipodal set $S$ can be homogeneous if

(1) $S$ is connected, or
(2) $S$ is not connected and for any two connected components $T_i, T_j$ of $S$ there is some isometry $g$ such that $g(S) \subset S$ and $g(T_i) = T_j$.

**6.1. Oriented Real Grassmannians.** In oriented real Grassmannians $\tilde{G}_k(\mathbb{R}^n)$, any maximal antipodal set is not necessarily great. Moreover, any two maximal antipodal sets are not necessarily congruent to each other. In the following, we list out maximal antipodal sets in each oriented real Grassmannian which are already known. When $k = 3, 4$, maximal antipodal sets are classified completely. However, when $k \geq 5$, the classification is incomplete. These results are works of Tasaki [3], [4], [6], [5].

- $\tilde{G}_3(\mathbb{R}^n)$

| $n$ | $3,4$ | $5$ | $6$ | $7,8$ | $9 \leq n$ |
|---|---|---|---|---|---|
| | $A(3,3)$ | $A(3,5)$ | $B(3,6)$ | $B(3,7)$ | $A(3, 2[\frac{n-1}{2}]+1), B(3,7)$ |

- $\tilde{G}_4(\mathbb{R}^n)$

| $n$ | $4,5$ | $6$ | $7$ | $8,9$ | $10$ |
|---|---|---|---|---|---|
| | $A(4,4)$ | $A(4,6)$ | $B(4,7)$ | $B(4,8)$ | $A(4,10), B(4,8)$ |

| $n$ | $11 \leq n$ |
|---|---|
| | $A(4, 2[\frac{n}{2}])$, $B(4,7) \cup \{X+7\ ;\ X$ is a maximal antipodal set of $\tilde{G}_4(\mathbb{R}^{n-7})$ in this list$\}$ $B(4,8) \cup \{Y+8\ ;\ Y$ is a maximal antipodal set of $\tilde{G}_4(\mathbb{R}^{n-8})$ in this list $\}$ |

- $\tilde{G}_{2k}\ (\mathbb{R}^{2l}), \tilde{G}_{2k}(\mathbb{R}^{2l+1})\ (l \geq 3k-1) : A(2k, 2l)$
- $\tilde{G}_{2k+1}(\mathbb{R}^{2l+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2l+2})(k \geq 2) : A(2k+1, 2l+1)$
- $\tilde{G}_{4m+k}(\mathbb{R}^n)\ (k = 0,1,2,3,\ m \geq 1)$

| $n$ | $8m$ | $8m+1$ | $8m+2$ | $8m+3$ | $8m+4$ | $8m+5$ | $8m+6$ | $8m+7$ |
|---|---|---|---|---|---|---|---|---|
| $k=0$ | $E^+_{v_{8m}}$ | $E^+_{v_{8m}}$ | $E^+_{v_{8m}}$ | $E^+_{v_{8m}}$ | | | | |
| $k=1$ | | | $E_{v_{8m+2}}$ | $E_{v_{8m+2}}$ | $E_{v_{8m+2}}$ | $E^+_{v_{8m+2}}$ | | |
| $k=2$ | | | | | $E_{v_{8m+4}}$ | $E_{v_{8m+4}}$ | $E^+_{v_{8m+4}}$ | |
| $k=3$ | | | | | | | $E_{v_{8m+6}}$ | $E^+_{v_{8m+6}}$ |

Above symbols imply followings. We use some notations in section 5.

$$B(3,6) = \{\ \pm[1\wedge 2\wedge 3], \pm[1\wedge 4\wedge 5], \pm[2\wedge 4\wedge 6], \pm[3\wedge 5\wedge 6]\ \}(= E_{v_6}),$$

$$B(3,7) = \{\ \pm[1\wedge 2\wedge 3], \pm[1\wedge 4\wedge 5], \pm[2\wedge 4\wedge 6], \pm[3\wedge 5\wedge 6], \pm[1\wedge 6\wedge 7], \pm[2\wedge 5\wedge 7], \pm[3\wedge 4\wedge 7]\ \}(= E^+_{v_6}),$$

$$B(4,7) = \left\{\begin{matrix} \pm[4\wedge 5\wedge 6\wedge 7], & \pm[2\wedge 3\wedge 6\wedge 7], & \pm[1\wedge 3\wedge 5\wedge 7], & \pm[1\wedge 2\wedge 4\wedge 7], & \pm[2\wedge 3\wedge 4\wedge 5], \\ \pm[1\wedge 3\wedge 4\wedge 6], & \pm[1\wedge 2\wedge 5\wedge 6] & & & \end{matrix}\right\}(= E^+_{v_6}),$$

$$B(4,8) = \left\{\begin{matrix} \pm[4\wedge 5\wedge 6\wedge 7], & \pm[2\wedge 3\wedge 6\wedge 7], & \pm[1\wedge 3\wedge 5\wedge 7], & \pm[1\wedge 2\wedge 4\wedge 7], & \pm[2\wedge 3\wedge 4\wedge 5], \\ \pm[1\wedge 3\wedge 4\wedge 6], & \pm[1\wedge 2\wedge 5\wedge 6], & & & \\ \pm[1\wedge 2\wedge 3\wedge 8], & \pm[1\wedge 4\wedge 5\wedge 8], & \pm[2\wedge 4\wedge 6\wedge 8], & \pm[3\wedge 5\wedge 6\wedge 8], & \pm[1\wedge 6\wedge 7\wedge 8], \\ \pm[2\wedge 5\wedge 7\wedge 8], & \pm[3\wedge 4\wedge 7\wedge 8] & & & \end{matrix}\right\}(= E^+_{v_8}),$$

$$A(2k,2l) = \left\{\ \pm\Big[\big(\alpha(1)\wedge(\alpha(1)+1)\big)\wedge\cdots\wedge\big(\alpha(k)\wedge(\alpha(k)+1)\big)\Big]\ ;\ \begin{matrix} 1\leq\alpha(1)<\cdots<\alpha(k)\leq 2l-1, \\ \alpha(i)\ (1\leq i\leq k) \text{ is odd.}\end{matrix}\right\},$$

$$A(2k+1,2l+1) = \left\{\ \pm\Big[\big(\alpha(1)\wedge(\alpha(1)+1)\big)\wedge\cdots\wedge\big(\alpha(k)\wedge(\alpha(k)+1)\big)\wedge(2l+1)\Big]\ ;\ \begin{matrix} 1\leq\alpha(1)<\cdots<\alpha(k)\leq 2l-1, \\ \alpha(i)\ (1\leq i\leq k) \text{ is odd.}\end{matrix}\right\},$$

$$E_{v_{2m}} = \left\{\ \pm\big[\alpha(1)\wedge\alpha(2)\wedge\cdots\wedge\alpha(m)\big]\ ;\ \begin{matrix}\alpha(i)\in\{2i-1,2i\}\ (1\leq i\leq m), \\ \#\{\alpha(i)\ ;\ \alpha(i) \text{ is even.}\ \}\in 2\mathbb{Z}\end{matrix}\right\},$$

$$E^+_{v_{8m}} = E_{v_{8m}} \cup A(4m, 8m),$$

$$E^+_{v_{8m+2}} = E_{v_{8m+2}} \cup \Big\{\ \pm\big[v\wedge(8m+3)\wedge(8m+4)\wedge(8m+5)\big]\ ; v\in A(4m-2, 8m+2)\Big\},$$

$$E^+_{v_{8m+4}} = E_{v_{8m+4}} \cup \Big\{\ \pm\big[v\wedge(8m+5)\wedge(8m+6)\big]\ ; v\in A(4m, 8m+4)\Big\},$$

$$E^+_{v_{8m+6}} = E_{v_{8m+6}} \cup \Big\{\ \pm\big[v\wedge(8m+7)\big]\ ; v\in A(4m+2, 8m+6)\Big\}.$$

Let $A$ be a subset of $\{(i_1,\cdots,i_k)\ ;\ 1\leq i_1<\cdots<i_k\leq n\}$. For $X = \{\pm[i_1\wedge\cdots\wedge i_k]\ ;\ (i_1,\cdots,i_k)\in A\}\subset\tilde{G}_k(\mathbb{R}^n)$, set $X+m = \{\pm[(i_1+m)\wedge\cdots\wedge(i_k+m)]\ ;\ \pm[i_1\wedge\cdots\wedge i_k]\in X\}$ for $m\in\mathbb{N}$.

It is known that any maximal antipodal set of $\tilde{G}_3(\mathbb{R}^n)$ and $\tilde{G}_4(\mathbb{R}^n)$ is congruent to some maximal antipodal set in above list [3]. In the following, we consider the connectedness and the homogeneity of each maximal antipodal set in the above list.

**Proposition 6.2.** *$B(3,6), B(3,7), B(4,7)$ and $B(4,8)$ are connected.*

Proof. By the definition of the connectedness, we fix $o$ and it is sufficient to show that for any point $p$ there is a connected point series $\{p_i\}_{i=0}^l$ containing $o$ and $p$.

In $B(3,6)$ and $B(3,7)$, let $o = [1 \wedge 2 \wedge 3]$. Then we see that any point except for $-o$ is connected to $o$. Let $p_1 = [1 \wedge 4 \wedge 5]$. Then $\{p_0 = o, p_1, p_2 = -o\}$ is a connected point series containing $o$ and $-o$. We see that $B(4,7)$ is connected by the similar way. In $B(4,8)$, let $o = [4 \wedge 5 \wedge 6 \wedge 7]$. Then we see that any point except for $-o, \pm p = \pm[1 \wedge 2 \wedge 3 \wedge 8]$ is connected to $o$. Let $p_1 = [2 \wedge 3 \wedge 6 \wedge 7]$ and $q_1 = [3 \wedge 4 \wedge 7 \wedge 8]$. Then $\{p_0 = o, p_1, p_2 = -o\}$ is a connected point series containing $o, -o$ and $\{q_0 = o, q_1, q_2 = \pm p\}$ is a connected point series containing $o, p$ and $\{r_0 = o, r_1 = q_1, r_2 = -p\}$ is a connected point series containing $o, -p$. □

**Proposition 6.3.** *$A(2k, 2l) \subset \tilde{G}_{2k}(\mathbb{R}^{2l}), \tilde{G}_{2k}(\mathbb{R}^{2l+1})$ is connected.*

Proof. Fix $o = [(1 \wedge 2) \wedge \cdots \wedge (2k-1 \wedge 2k)]$. It is sufficient to show that for any point $p \in A(2k, 2l)$ $(p \neq o)$ there is a connected point series containing $o$ and $p$. Let

$$p = \pm[\big(\alpha(1) \wedge (\alpha(1)+1)\big) \wedge \cdots \wedge \big(\alpha(k) \wedge (\alpha(k)+1)\big)] \neq -o,$$

where $1 \leq \alpha(1) < \cdots < \alpha(k) \leq 2l-1$ and every $\alpha(i)$ is odd. We see $\alpha(i) \geq 2i-1$ for $1 \leq i \leq k$ obviously. Let $\{p_i\}_{i=0}^{k}$ be as follows:

$$\begin{aligned}
&p_0 = o,\\
&p_1 = [(1 \wedge 2) \wedge (3 \wedge 4) \wedge \cdots \wedge (2k-3 \wedge 2k-2) \wedge \big(\alpha(k) \wedge (\alpha(k)+1)\big)],\\
&p_2 = [(1 \wedge 2) \wedge (3 \wedge 4) \wedge \cdots \wedge \big(\alpha(k-1) \wedge (\alpha(k-1)+1)\big) \wedge \big(\alpha(k) \wedge (\alpha(k)+1)\big)],\\
&\vdots\\
&p_{k-1} = [(1 \wedge 2) \wedge \big(\alpha(2) \wedge (\alpha(2)+1)\big) \wedge \cdots \wedge \big(\alpha(k-1) \wedge (\alpha(k-1)+1)\big) \wedge \big(\alpha(k) \wedge (\alpha(k)+1)\big)],\\
&p_k = p.
\end{aligned}$$

We can take a connected subseries of $\{p_i\}_{i=0}^{k}$ containing $o, p$ because $p_i = p_{i+1}$ or $p_i$ is connectd to $p_{i+1}$ for $1 \leq i \leq k-1$. Moreover, for $-o$ we consider a point series $\{q_1 = o, q_2 = p_1, q_3 = -o\}$. This point series is a connected point series of $A(2k, 2l)$ containing $o$ and $-o$. Thus, we conclude that $A(2k.2l)$ is connected. □

We can prove the following proposition by the similar way.

**Proposition 6.4.** *$A(2k+1, 2l+1) \subset \tilde{G}_{2k+1}(\mathbb{R}^{2l+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2l+2})$ is connected.*

Summarizing above results we obtain the following theorem.

**Theorem 6.5.** *In $\tilde{G}_3(\mathbb{R}^n)$ $(n \geq 6)$, any maximal antipodal set is homogeneous.*

In $\tilde{G}_4(\mathbb{R}^n)$ $(n \geq 11)$, any maximal antipodal set is congruent to some antipodal set contained in

$$\left\{\begin{matrix} A(4, 2[\frac{n}{2}]), & I(4,7) = B(4,7) \cup \{X+7\ ;\ X \text{ is a maximal antipodal set of } \tilde{G}_4(\mathbb{R}^{n-7}) \text{ in the list.}\}, \\ & I(4,8) = B(4,8) \cup \{Y+7\ ;\ Y \text{ is a maximal antipodal set of } \tilde{G}_4(\mathbb{R}^{n-8}) \text{ in the list.}\} \end{matrix}\right\}.$$

By Propositions 5.3 we see that any antipodal set in $I(4,7)$ and $I(4,8)$ is not connected. Moreover, we see that any connected component of any maximal antipodal set in $I(4,7)$ and $I(4,8)$ is congruent to one of $B(4,7), B(4,8)$ and $A(4,2l)$ $(l \in \mathbb{N})$.

REMARK 6.6. It is true that any connected maximal antipodal set is maximally connected, but any maximally connected antipodal set is not necessarily maximal. $B(4,7)$ and $B(4,8)$ is maximally connected in $\tilde{G}_4(\mathbb{R}^n)$ $(n \geq 11)$, but they are not maximal.

**Proposition 6.7.** *Let A be a maximal antipodal set in* $I(4,7)$*. If there is some connected component of A which is not congrurent to* $B(4,7)$*, then A is not homogeneous. Similarly, let B be a maximal antipodal set in* $I(4,8)$*. If there is some connected component of B which is not congrurent to* $B(4,8)$*, then B is not homogeneous.*

Proof. We prove the former part of the statement. The latter part is proved by the similar way. Let $A_0$ be a connected component of $A$ and $A_0 = B(4,7)$. By Proposition 6.1, it is sufficient to show that if there is a connected component $A_1$ of $A$ which is not congruent to $B(4,7)$, then there are no isometries $g$ such that $g(A_0) = A_1$. Then, $A_1$ is congruent to $B(4,8)$ or $A(4,2l)$ $(i \in \mathbb{N})$. We see $\#B(4,7) = 14$ and $\#B(4,8) = 28$. Moreover, we see $\#A(4,2l)$ increases as $l$ increases and $12 = \#A(4,8) < \#B(4,7) < \#A(4,10) = 20 < \#B(4,8) < \#A(4,12) = 30$. Therefore, we conclude that there are no isometries $g$ such that $g(A_0) = A_1$. □

**Proposition 6.8.** *Let A be a maximal antipodal set in* $I(4,7)$*. If every connected component of A is congruent to* $B(4,7)$ *that is* $A = B(4,7) \cup (B(4,7)+7) \cup \cdots$*, then A is homogeneous. Similarly, let B be a maximal antipodal set in* $I(4,8)$*. If every connected component of B is congruent to* $B(4,8)$ *that is* $B = B(4,8) \cup (B(4,8)+8) \cup \cdots$*, then B is homogeneous.*

Proof. We consider the former part of the statement. The latter part is proved by the similar way. Then $A$ is the following maximal antipodal set

$$\bigcup_{m=0}^{k-1} \Big\{ \pm [(n_1+7m) \wedge (n_2+7m) \wedge (n_3+7m) \wedge (n_4+7m)] \ ; \ \pm[n_1 \wedge n_2 \wedge n_3 \wedge n_4] \in B(4,7) \Big\},$$

where $7k \leq n \leq 7k+3$ and every connected component of $A$ is

$$B(4,7)_m = \Big\{ \pm [(n_1+7m) \wedge (n_2+7m) \wedge (n_3+7m) \wedge (n_4+7)] \ ; \ \pm[n_1 \wedge n_2 \wedge n_3 \wedge n_4] \in B(4,7) \Big\},$$

where $0 \leq m \leq k-1$. Then, we see that any $g \in G_{B(4,7)_m}$ $(0 \leq m \leq k-1)$ fixes every point of $B(4,7)_l$ $(l \neq m)$ by caliculations. Moreover, we consider permutation matrices corresponding to following permutations : for $0 \leq m \leq k-1$,

$$\sigma_m : \{1, \cdots, n\} \to \{1, \cdots, n\}; \sigma_m(a) = \begin{cases} a+7m & (1 \leq a \leq 7), \\ a-7m & (1+7m \leq a \leq 7+7m), \\ a & (a \text{ is otherwise}). \end{cases}$$

We denote the permutation matrix corresponding to $\sigma_m$ by the same letter. Then, we obtain

$$\sigma_m(B(4,7)_0) = B(4,7)_m, \ \sigma_m(B(4,7)_m) = B(4,7)_0, \ \sigma_m|_{B(4,7)_l} = \mathrm{Id}|_{B(4,7)_l} \ (l \neq 0, m).$$

We consider the subgroup of the isometry group generated by every element of $G_{B(4,7)_0}$ and $\sigma_m$ $(0 \leq m \leq k-1)$. Then, we see that this group acts on $A$ and this action is transitive. Thus, $A$ is homogeneous. □

We obtain the following theorem summarizing above propositions.

**Theorem 6.9.** *In $\tilde{G}_4(\mathbb{R}^n)$, the followings are true.*

(i) *In $4 \leq n \leq 10$, any maximal antipodal set is homogeneous.*

(ii) *In $11 \leq n$, a maximal antipodal set $A$ is homogeneous if and only if $A$ satisfies either of the following three conditions:*
  (1) $A = A(4, 2[\frac{n}{2}])$.
  (2) *Each connected component of $A$ is congruent to $B(4, 7)$.*
  (3) *Each connected component of $A$ is congruent to $B(4, 8)$.*

Next, we consider $E_{v_{2m}}$-type antipodal sets.

**Proposition 6.10.** *$E_{v_{2m}}$ is connected.*

Proof. Let $o = +[1 \wedge 3 \wedge \cdots \wedge 2m-3 \wedge 2m-1] \in E_{v_{2m}}$. It is sufficient to show that for any $p \in E_{v_{2m}}$ there is a connected point series containing $o$ and $p$. Let

$$p = \pm[\alpha(1) \wedge \cdots \wedge \alpha(k) \wedge \beta(k+1) \wedge \cdots \wedge \beta(m)] \neq -o \begin{pmatrix} \alpha(i)\ (1 \leq i \leq k) \text{ is even and } k \text{ is even.} \\ \beta(j)\ (k+1 \leq i \leq m) \text{ is odd.} \end{pmatrix}.$$

We define the point series $\{p_i\}_{i=0}^{\frac{k}{2}}$ as follows:

$$\begin{aligned}
p_0 &= p, \\
p_1 &= [\big(\alpha(1)-1\big) \wedge \big(\alpha(2)-1\big) \wedge \alpha(3) \wedge \cdots \wedge \alpha(k) \wedge \beta(k+1) \wedge \cdots \wedge \beta(m)], \\
p_2 &= [\big(\alpha(1)-1\big) \wedge \big(\alpha(2)-1\big) \wedge \big(\alpha(3)-1\big) \wedge \big(\alpha(4)-1\big) \cdots \wedge \alpha(k) \wedge \beta(k+1) \wedge \cdots \wedge \beta(m)], \\
&\vdots \\
p_{\frac{k}{2}-1} &= [\big(\alpha(1)-1\big) \wedge \cdots \wedge \big(\alpha(k-2)-1\big) \wedge \alpha(k-1) \wedge \alpha(k) \wedge \beta(k+1) \wedge \cdots \wedge \beta(m)], \\
p_{\frac{k}{2}} &= o = \pm[\big(\alpha(1)-1\big) \wedge \cdots \wedge \big(\alpha(k)-1\big) \wedge \beta(k+1) \wedge \cdots \wedge \beta(m)],
\end{aligned}$$

where we add $\pm$ to the last term so that $p_{\frac{k}{2}}$ becomes $o$. Then, we see that $\{p_i\}_{i=0}^{\frac{k}{2}}$ is a connected point series containing $o$ and $p$. For $-o$, we consider the point series $\{q_1 = o, q_2 = p_{\frac{k}{2}-1}, q_3 = -o\}$. This point series becomes a connecetd series containing $o$ and $-o$. Hence, we conclude that $E_{v_{2m}}$ is a connected antipodal set. □

We can prove the following proposition by the similar way.

**Proposition 6.11.** *$E_{v_{8m+2}}$ is connected in $\tilde{G}_{4m+1}(\mathbb{R}^{8m+2})$, $\tilde{G}_{4m+1}(\mathbb{R}^{8m+3})$, $\tilde{G}_{4m+1}(\mathbb{R}^{8m+4})$. $E_{v_{8m+4}}$ is connecetd in $\tilde{G}_{4m+2}(\mathbb{R}^{8m+4})$, $\tilde{G}_{4m+2}(\mathbb{R}^{8m+5})$. $E_{v_{8m+6}}$ is connected in $\tilde{G}_{4m+3}(\mathbb{R}^{8m+6})$.*

Next we consider $E^+_{v_{8m}}$-type maximal antipodal sets.

**Proposition 6.12.** *Let $m \geq 2$. Then $E^+_{v_{8m}} \subset \tilde{G}_{4m}(\mathbb{R}^{8m})$, $\tilde{G}_{4m}(\mathbb{R}^{8m+1})$, $\tilde{G}_{4m}(\mathbb{R}^{8m+2})$, $\tilde{G}_{4m}(\mathbb{R}^{8m+3})$ is not homogeneous.*

Proof. We recall $E^+_{v_{8m}} = E_{v_{8m}} \cup A(4m, 8m)$. Firstly, we will show that $E^+_{v_{8m}}$ is not connected and $E_{v_{8m}}$ and $A(4m, 8m)$ are connected components of $E^+_{v_{8m}}$. We see that $E_{v_{8m}}$ and $A(4m, 8m)$ are connected respectively from Proposition 6.3 and Proposition 6.10. Let $p = [n_1 \wedge \cdots \wedge n_{4m}] \in E_{v_{8m}}$ and $q = [k_1 \wedge \cdots \wedge k_{4m}] \in A(4m, 8m)$. Then, we see that the cardinality of the set difference $\{n_1, \cdots, n_{4m}\} - \{k_1, \cdots, k_{4m}\} = \{n_i \,;\, n_i \notin \{k_1, \cdots, k_{4m}\}, 1 \leq i \leq 4m\}$ is $2m$ by definitions of $E_{v_{8m}}$ and $A(4m, 8m)$. Thus $p$ is not connected to $q$ because of $m \geq 2$. Hence, $E^+_{v_{8m}}$ is not connected and $E_{v_{8m}}$ and $A(4m, 8m)$ are connected components of $E^+_{v_{8m}}$.

Secondly, we will show that there are no isomeries $g$ such that $g(E_{v_{8m}}) = A(4m, 8m)$. We see $\#(E_{v_{8m}})_p = \frac{1}{2}4m(4m-1) = 2m(4m-1)$ for any $p \in E_{v_{8m}}$ and $\#A(4m, 8m)_q = 4m^2$ for any $q \in A(4m, 8m)$. It is true that $\#(E_{v_{8m}})_p \neq \#A(4m, 8m)_q$ for any $m \geq 2$. Since the connectedness is invariant under isometries, there are no isometries $g$ such that $g(E_{v_{8m}}) = A(4m, 8m)$. Therefore, $E^+_{v_{8m}}$ is not homogeneous by Proposition 6.1. □

Remark 6.13. When $m = 1$, then $E^+_{v_8} = B(4, 8)$. In paricular, $E^+_{v_8}$ is connected and homogeneous.

We obtain the following proposition by the similar way.

**Proposition 6.14.** *$E^+_{v_{8m+2}} \subset \tilde{G}_{4m+1}(\mathbb{R}^{8m+5})$ $(m \geq 1)$, $E^+_{v_{8m+4}} \subset \tilde{G}_{4m+2}(\mathbb{R}^{8m+6})$ $(m \geq 1)$ and $E^+_{v_{8m+6}} \subset \tilde{G}_{4m+3}(\mathbb{R}^{8m+7})$ $(m \geq 1)$ are not homogeneous.*

The following list is the summary of this subsection.

- $\tilde{G}_3(\mathbb{R}^n)$

| $n$ | $3,4$ | $5$ | $6$ | $7,8$ | $9 \leq n$ | |
|---|---|---|---|---|---|---|
| | $A(3,3)$ | $A(3,5)$ | $B(3,6)$ | $B(3,7)$ | $A(3,2l+1)$ $(l = [\frac{n-1}{2}])$ | $B(3,7)$ |
| great | ○ | ○ | ○ | ○ | ○ | × |
| connectedness | ○ | ○ | ○ | ○ | ○ | ○ |
| homogeneity | ○ | ○ | ○ | ○ | ○ | ○ |

- $\tilde{G}_4(\mathbb{R}^n)$

| $n$ | $4,5$ | $6$ | $7$ | $8,9$ | $10$ | |
|---|---|---|---|---|---|---|
| | $A(4,4)$ | $A(4,6)$ | $B(4,7)$ | $B(4,8)$ | $A(4,10)$ | $B(4,8)$ |
| great | ○ | ○ | ○ | ○ | × | ○ |
| connectedness | ○ | ○ | ○ | ○ | ○ | ○ |
| homogeneity | ○ | ○ | ○ | ○ | ○ | ○ |

| $n$ | $11 \leq n$ | | | |
|---|---|---|---|---|
| | $A(4, 2[\frac{n}{2}])$ | $B(4,7) \sqcup \cdots \sqcup B(4,7)$ | $B(4,8) \sqcup \cdots \sqcup B(4,8)$ | otherwise |
| great | ○ | × | × | × |
| connectedness | ○ | × | × | × |
| homogeneity | ○ | ○ | ○ | × |

- $\tilde{G}_{2k}(\mathbb{R}^{2l}), \tilde{G}_{2k}(\mathbb{R}^{2l+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2l+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2l+2})$

| | $\tilde{G}_{2k}(\mathbb{R}^{2l}), \tilde{G}_{2k}(\mathbb{R}^{2l+1})$ $(l \geq 3k-1)$ | $\tilde{G}_{2k+1}(\mathbb{R}^{2l+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2l+2})$ $(k \geq 2)$ |
|---|---|---|
| | $A(2k, 2l)$ | $A(2k+1, 2l+1)$ |
| connectedness | ○ | ○ |
| homogeneity | ○ | ○ |

- $\tilde{G}_{4m}(\mathbb{R}^n)$

| $n$ | $8m$ | $8m+1$ | $8m+2$ | $8m+3$ | $8m+4$ | $8m+5$ | $8m+6$ | $8m+7$ |
|---|---|---|---|---|---|---|---|---|
| | $E^+_{v_{8m}}$ | $E^+_{v_{8m}}$ | $E^+_{v_{8m}}$ | $E^+_{v_{8m}}$ | | | | |
| connectedness | × | × | × | × | | | | |
| homogeneity | × | × | × | × | | | | |

- $\tilde{G}_{4m+1}(\mathbb{R}^n)$

| $n$ | $8m$ | $8m+1$ | $8m+2$ | $8m+3$ | $8m+4$ | $8m+5$ | $8m+6$ | $8m+7$ |
|---|---|---|---|---|---|---|---|---|
| | | | $E_{v_{8m+2}}$ | $E_{v_{8m+2}}$ | $E_{v_{8m+2}}$ | $E^+_{v_{8m+2}}$ | | |
| connectedness | | | ○ | ○ | ○ | × | | |
| homogeneity | | | ○ | ○ | ○ | × | | |

- $\tilde{G}_{4m+2}(\mathbb{R}^n)$

| $n$ | $8m$ | $8m+1$ | $8m+2$ | $8m+3$ | $8m+4$ | $8m+5$ | $8m+6$ | $8m+7$ |
|---|---|---|---|---|---|---|---|---|
| | | | | | $E_{v_{8m+4}}$ | $E_{v_{8m+4}}$ | $E^+_{v_{8m+4}}$ | |
| connectedness | | | | | ○ | ○ | × | |
| homogeneity | | | | | ○ | ○ | × | |

- $\tilde{G}_{4m+3}(\mathbb{R}^n)$

| $n$ | $8m$ | $8m+1$ | $8m+2$ | $8m+3$ | $8m+4$ | $8m+5$ | $8m+6$ | $8m+7$ |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | $E_{v_{8m+6}}$ | $E^+_{v_{8m+6}}$ |
| connectedness | | | | | | | ○ | × |
| homogeneity | | | | | | | ○ | × |

It is known that if $k = 5$ and $n \geq 87$, then $A(5, 2\lfloor \frac{n-1}{2} \rfloor + 1) \subset \tilde{G}_5(\mathbb{R}^n)$ is a great antipodal set of $\tilde{G}_5(\mathbb{R}^n)$ [5]. Moreover, if $n$ is sufficiently larger than $k$, then $A(2k, 2\lfloor \frac{n}{2} \rfloor + 1)$ is a great antipodal set in $\tilde{G}_{2k}(\mathbb{R}^n)$ and $A(2k+1, 2\lfloor \frac{n-1}{2} \rfloor + 1)$ is a great antipodal set in $\tilde{G}_{2k+1}(\mathbb{R}^n)$ [9]. From the above list, we see that there are not-homogeneous maximal antipodal sets and not-connected homogeneous maximal antipodal sets. However, we see that great antipodal sets which are already known are connected and homogeneous.

**6.2. Compact symmetric spaces having one polar except for the trivial pole.** If a compact Riemannian symmetric space $M$ has one polar except for the trivial pole, we can decide the homogeneity of maximal antipodal sets of $M$ by the connectedness.

**Theorem 6.15.** *If $M$ has only one polar except for the trivial pole, then any antipodal set is connected.*

Proof. Let $o \in M$ and $M^+_o$ be the polar of $o$. Then for any point $p$ of $M^+_o$ there is some shortest closed geodesic through $o$ and $p$ since the number of polars is one. Hence, any two antipodal points are connected. Therefore, any antipodal set of $M$ is connected. □

In particular, any maximal antipodal set in $M$ is connected and homogeneous. By the classification of polars [1][2][11], we obtain the following example.

EXAMPLE 6.16. Any maximal antipodal set of $E_6/F_4$, $(E_6/F_4)^*$, $F_4/\mathrm{Spin}(9)$ and $G_2/SO(4)$ is connected and homogeneous, where $(E_6/F_4)^*$ is the bottom space of $E_6/F_4$.

REMARK 6.17. $F_4/\mathrm{Spin}(9)$ is a symmetric $R$-space, so it has been known the homogeneity of their maximal antipodal sets [7].

**6.3. Symmetric *R*-spaces.** In above two subsections we study the homogeneity and the connectedness of maximal antipodal sets in some compact symmetric spaces. In symmetric $R$-spaces it is known that all maximal antipodal sets are congruent to each other and any maximal antipodal set is great and homogeneous [7]. We will study the connectedness of great antipodal sets in symmetric $R$-spaces.

Let $M$ be an irreducible symmetric $R$-space. The followings are known. Let $(G, K)$ be some compact simple Riemannian symmetric pair and $\mathfrak{g}$ and $\mathfrak{k}$ be Lie algebras of $G$ and $K$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the standard decomposition of $\mathfrak{g}$ with respect to $(G, K)$. Then, there is $E \in \mathfrak{m}$ such that $N \cong \mathrm{Ad}(K)E$. The metric of $N$ is induced by the $K$-inavariant inner product of $\mathfrak{m}$ which is the restriction of a negative constant multiple of the Killing form of $\mathfrak{g}$. Let $\mathfrak{h}$ be a maximal abelian subspace of $\mathfrak{m}$ containing $E$ and $W$ be the Weyl group of $\mathfrak{h}$. Then, it is known that $A = W(E)$ is a great antipodal set of $M$ and any great antipodal set of $M$ is congruent to $A$.

In these setting, it is known that following lemmas are true.

**Lemma 6.18** ([8])**.** *Let $T$ be a maximal flat torus of $M$ through $E$. Then $T$ satisfies the following two natures.*

(1) *Let $T_E(T)$ be the tangent space at $E$ of $T$. Then, there is a basis $X_1, \cdots, X_r$ ($r$ = rank($M$)) of $T_EM$ such that $|X_1| = \cdots = |X_r|$, $\langle X_i, X_j\rangle = 0$ ($i \neq j$) and $\{X \in T_E(T)\ ;\ \mathrm{Ad}(\exp X)E = E\} = \{X_1, \cdots, X_r\}_{\mathbb{Z}}$, where $\langle , \rangle$ is the inner product of $T_E(T)$ induced by the metric of $N$ and $|\cdot|$ is the norm induced by $\langle , \rangle$.*

(2) $\{\mathrm{Ad}(\exp tX_i)E;\ 0 \leq t \leq 1\}$ *is a shortest closed geodesic in $M$.*

**Lemma 6.19** ([8])**.** *For the great antipodal set $A$, there is a maximal flat torus $T$ of $M$ through $E$ satisfying the following conditions.*

(1) *There is a basis $X_1, \cdots, X_r$ of $T_E(T)$ satisfying properties of Lemma 6.18 and*

$$A \cap T = \{\mathrm{Ad}(\exp(\epsilon_1 X_1 + \cdots + \epsilon_r X_r))E;\ \epsilon_i = 0 \text{ or } \frac{1}{2}\ (1 \leq i \leq r)\}.$$

(2) *Let $W_0 = \{s \in W;\ s(E) = E\}$. Then any point of $A$ is congruent to some point of $A \cap T$ by the action of $W_0$.*

We obtain the following proposition by above two lemmas.

**Proposition 6.20.** *A is connected.*

Proof. It is sufficient to prove that for any $p \in A$ there is a connected point series containing $E$ and $p$. Accroding to Lemma 6.19, there is some $w \in W_0$ such that

$$q = w(p) = \mathrm{Ad}(\exp(\frac{1}{2}X_{i_1} + \cdots + \frac{1}{2}X_{i_k}))E \in T,$$

where $1 \leq i_1 < \cdots i_k \leq r$. We define $\{q_j\}_{j=0}^{k} \subset A \cap T$ as follows:

$$
\begin{aligned}
q_0 &= E,\\
q_1 &= \mathrm{Ad}(\exp(\frac{1}{2}X_{i_1}))E,\\
q_2 &= \mathrm{Ad}(\exp(\frac{1}{2}X_{i_1} + \frac{1}{2}X_{i_2}))E,\\
&\vdots\\
q_{k-1} &= \mathrm{Ad}(\exp(\frac{1}{2}X_{i_1} + \frac{1}{2}X_{i_2} + \cdots + \frac{1}{2}X_{i_{k-1}}))E,\\
q_{k-1} &= \mathrm{Ad}(\exp(\frac{1}{2}X_{i_1} + \frac{1}{2}X_{i_2} + \cdots + \frac{1}{2}X_{i_{k-1}} + \frac{1}{2}X_{i_k}))E.
\end{aligned}
$$

Then, we see that $q_i$ is connected to $q_{i+1}$ for $0 \leq j \leq k-1$ by Lemma 6.18 and Lemma 6.19. Therefore, $\{q_j\}_{j=0}^k$ is a connected point series in $A \cap T$ containing $E$ and $q$. Let $p_j = w^{-1}(q_j)$ $(0 \leq j \leq k)$. Then, $\{p_j\}_{j=0}^k$ is included in $A$ and becomes a connected point series containing $E$ and $p$. Hence, $A$ is connected. □

Summarizing this subsection and results of Tanaka and Tasaki [7] we obtain the following theorem.

**Theorem 6.21.** *Let $M$ be an irreducible symmetric R-space. Then, any great antipodal set of $M$ is connected and homogeneous.*

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