Myung, H. C.
Osaka J. Math.
9 (1972), 41-47

ON PRIME IDEALS AND PRIMARY DECOMPOSITIONS IN A NONASSOCIATIVE RING

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(Received May 10, 1971)

1. Introduction

Prime ideals in nonassociative rings have been defined by a number of authors in different terms (for example, [1], [4], and [5]). In [5], the Brown-McCoy type prime ideals and radical have been defined for Jordan rings by using the quadratic operation. In [4], using $*$-operation defined on the family of ideals and a function defined on the ring, the results in [5] have been extended to weakly W-admissible rings which generalize many of the well known nonassociative rings, in particular, alternative and Jordan rings.

For the associative case, the concept of prime ideals in the sense of McCoy [2] was generalized in [3] by defining f-systems which generalize the m-systems of McCoy. Also, f-primary ideals are defined in [3] and an analogue of the uniqueness theorem of the Lasker-Noether decomposition in the commutative case is proved for arbitrary associative rings in terms of f-primary ideals.

The essential purpose of this paper is to extend some of the results in [3] for associative rings to arbitrary nonassociative rings. Using the same function f as in [3] and the $*$-operation, we give a definition of f-prime ideals for arbitrary nonassociative rings. Under certain choices of the $*$-operation and the function f, our present f-prime ideals coincide with the prime ideals in [5] for Jordan rings and those in [3] for associative rings. We also obtain analogous results of f-primary decomposition in [3] for nonassociative rings.

2. f-prime ideals

Let R be an arbitrary nonassociative ring and let $\mathcal{I}(R)$ denote the family of (two-sided) ideals in R.

DEFINITION 2.1. We define a $*$-operation as a mapping of $\mathcal{I}(R)\times\mathcal{I}(R)$ into the family of additive subgroups of R such that

AMS 1970 Subject Classification. Primary 17A99, 16A21.
Key words and phrases. f-system, f-prime ideal, f-radical, f-quotient, f-primary ideal, f-primary decomposition.

(i) for A, B, C, and D in $\mathcal{J}(R)$, if $A\subseteq C$ and $B\subseteq D$, then $A*B\subseteq C*D$,
(ii) $A*B\subseteq A\cap B$ for all A, B in $\mathcal{J}(R)$,
(iii) $(A+C)*(B+C)\subseteq A*B+C$ for A, B, C in $\mathcal{J}(R)$.

A similar operation to this defined in [4] and is a special case of the present definition. Let R be a Jordan ring and let $yU_x=2(yx)x-yx^2$ for x, y in R. Let BU_A denote the additive subgroup (in fact, an ideal) of R generated by yU_x, $x\in A$, $y\in B$, for A, B in $\mathcal{J}(R)$. Then the U-operation satisfies the conditions above. More generally, for any positive integers m, n, let $p(x_1, \cdots, x_m, y_1, \cdots, y_n)$ be an element in the free nonassociative ring generated by the set $\{x_1, \cdots, x_m, y_1, \cdots, y_n\}$. Let $p(x_1, \cdots, x_m, y_1, \cdots, y_n)$ be one such that each monomial in $p(x_1, \cdots, x_m, y_1, \cdots, y_n)$ is of degree ≥ 1 in at least one of x_i's and of y_i's. For A, B in $\mathcal{J}(R)$, if we define $A*B$ as the additive subgroup $p(A, \cdots, A, B, \cdots, B)$ generated by the elements $p(a_1, \cdots, a_m, b_1, \cdots, b_n)$, $a_i\in A$, $b_j\in B$, then $A*B$ satisfies the conditions in Definition 2.1.

As in [4] or [5] (or as one can easily verify), we have the following lemma:

Lemma 2.2. *Let R be an arbitrary nonassociative ring where the $*$-operation is defined. For an ideal P of R, the following are equivalent:*

(i) *If $A*B\subseteq P$ for A, B in $\mathcal{J}(R)$, then either $A\subseteq P$ or $B\subseteq P$.*

(ii) *If $A\cap c(P)\neq\emptyset$ and $B\cap c(P)\neq\emptyset$, then $A*B\cap c(P)\neq\emptyset$ for A, B in $\mathcal{J}(R)$, where $c(P)$ is the complement of P in R.*

(iii) *If a and b are in $c(P)$, then $(a)*(b)\cap c(P)\neq\emptyset$, where (a) is the principal ideal generated by a in R.*

DEFINITION 2.3. An ideal P of R is called a $*$-prime (or simply prime) ideal if it satisfies any one of Lemma 2.2. A nonempty subset M of R is called a $*$-system if, for A, B in $\mathcal{J}(R)$, $A\cap M\neq\emptyset$ and $B\cap M\neq\emptyset$ imply $A*B\cap M\neq\emptyset$.

Hence an ideal P in R is prime if and only if $c(P)$ is a $*$-system. The prime ideals in this definition coincide with the f^*-prime ideals in [4] in case $f(a)=(a)$ for all a in R. Let R be a Jordan ring. If we define $A*B$ as AU_B, the $*$-prime ideals and $*$-systems coincide with the prime ideals and Q-systems in [5], respectively. In particular, if $p(x_1, y_1)=x_1y_1$ and $A*B=p(A, B)\equiv AB$, the $*$-prime ideals are the prime ideals of McCoy in the associative case and the u-prime ideals in [1] with $u=x_1x_2$.

DEFINITION 2.4. Let R be any nonassociative ring. Following [3], we define f as a function of R into $\mathcal{J}(R)$ such that, for every element a in R,

(i) $a\in f(a)$, and
(ii) $x\in f(a)+A$ implies $f(x)\subseteq f(a)+A$ for any ideal A in R.

A similar function to f has been defined in [4]. The principal ideal (a) generated by a in R is an example of $f(a)$. More generally, for a subset S of R, if we let $f(a)=(a, S)$, the ideal generated by a and S in R, then f satisfies the

conditions. We note that $f(a)=(a)$ for all a in R if and only if $f(0)=(0)$.

DEFINITION 2.5. A nonempty subset M of R is called an f-system if M contains a $*$-system M' such that $f(a)\cap M'\neq\emptyset$ for every element a in M, where M' is called a kernel of M.

For the associative case, the f-systems are defined in [3] by using the m-systems. Any $*$-system is clearly an f-system. If $f(a)=(a)$ for all a in R, the f-systems coincide with the $*$-systems. In fact, let A and B be ideals in R with $A\cap M\neq\emptyset$ and $B\cap M\neq\emptyset$. For $a\in A\cap M$ and $b\in B\cap M$, we have $(a)\cap M'\neq\emptyset$ and $(b)\cap M'\neq\emptyset$, and so $A*B\cap M\supseteq(a)*(b)\cap M'\neq\emptyset$ since M' is a $*$-system. Hence M is a $*$-system.

DEFINITION 2.6. An ideal P is said to be f-prime if either $c(P)$ is an f-system in R, or $P=R$.

As before, if $f(a)=(a)$ for all a in R, the f-prime ideals coincide with the $*$-prime ideals. But in general, f-prime ideals may not be $*$-prime for certain choices of the $*$-operation and the function f. In the associative case, an example for this may be found in [3]. We now give an example in nonassociative case.

EXAMPLE 2.7. Let R be the free Jordan ring with an identity on free generators x and y. If we take $A*B=AU_B$ for the U-operation in R, it is shown that $AU_A=AA^2$ and AU_A is an ideal of R for every ideal A in R (see [5]). Let $P=(x)U_{(x)}=(x)(x)^2$, then P is not Q-prime since (x) is not contained in the ideal P. For a fixed positive integer k, we define $f(a)=(a, y^k)$ for all a in R. Let $M'=\{y, y^2, y^3, \cdots\}$, then M' is a Q-system and $M'\cap P=\emptyset$ since R is free. Hence $c(P)$ is an f-system with kernel M' and P is f-prime. This example shows that an f-system may have different kernels.

Lemma 2.8. *For any f-prime ideal P, $f(a)*f(b)\subseteq P$ implies that $a\in P$ or $b\in P$.*

Proof. If a and b are in $c(P)$, $f(a)\cap c(P)'\neq\emptyset$ and $f(b)\cap c(P)'\neq\emptyset$ for a kernel $c(P)'$ of $c(P)$. Since $c(P)'$ is a $*$-system, $f(a)*f(b)\cap c(P)'\neq\emptyset$ and $f(a)*f(b)$ is not contained in P.

DEFINITION 2.9. Let A be an ideal in R and let $r(A)=\{r\in R\mid$ any f-system containing r meets $A\}$. Following [3], we call $r(A)$ the f-radical of A.

We now prove the usual characterization of f-radicals.

Theorem 2.10. *For an ideal A in R, the f-radical of A is the intersection of all f-prime ideals in R containing A.*

Proof. Let $\bigcap_i P_i$ be the intersection of all the f-prime ideals containing A.

If $b \in c(P_i)$ for some i, $b \notin r(A)$ since $A \cap c(P) = \emptyset$. Hence $r(A) \subseteq \bigcap_i P_i$.

Conversely, if $b \notin r(A)$, there exists an f-system M such that $b \in M$ but $M \cap A = \emptyset$. By Zorn's lemma, we find a maximal ideal P in R such that $P \supseteq A$ but $P \cap M = \emptyset$. We show that $c(P)$ is an f-system; that is, P is f-prime. Let M' be a kernel of the f-system M. Since $P \cap M' \neq \emptyset$, $M' + P \subseteq c(P)$. First, to show $M' + P$ is a $*$-system, let A, B be ideals in R such that $A \cap (M' + P) \neq \emptyset$ and $B \cap (M' + P) \neq \emptyset$. Then $(A + P) \cap M' \neq \emptyset$ and $(B + P) \cap M' \neq \emptyset$. Since M' is a $*$-system, $\emptyset \neq (A + P) * (B + P) \cap M' \subseteq (A * B + P) \cap M'$. This implies that $A * B \cap (M' + P) \neq \emptyset$, and hence $M' + P$ is a $*$-system. To see that $M' + P$ is a kernel of $c(P)$, let $a \in c(P)$. By the maximality of P, $f(a) + P$ contains an element u in M. Then $f(u) \cap M' \neq \emptyset$ and let $u' \in f(u) \cap M'$. Since $u \in f(a) + P$, $f(u) \subseteq f(a) + P$ and so $u' = c + d$ for $c \in f(a)$ and $d \in P$, thus $c = u' - d$ is in $M' + P$ and $f(a) \cap (M' + P) \neq \emptyset$. Hence $c(P)$ is an f-system. Recalling that $b \in c(P)$ and $P \supseteq A$, this proves $r(A) = \bigcap_i P_i$.

3. f-primary decompositions

In this section, we consider an analogue for nonassociative rings of primary ideals and the uniqueness theorem of Lasker-Noether decomposition of ideals in commutative, associative rings.

DEFINITION 3.1. A $*$-operation is said to be left (or right) additive if $(A + B) * C \subseteq A * C + B * C$ (or $A * (B + C) \subseteq A * B + A * C$) for all ideals A, B, C, in R.

The U-operation in a Jordan ring is left-additive. More generally, let $p(x, y_1, \cdots, y_n)$ be an element in the free nonassociative ring on free generators $x, y_1, \cdots, y_n$. If each monomial in $p(x, y_1, \cdots, y_n)$ is of degree ≥ 1 in at least one of y_i's and is of degree 1 in x, then $A * B \equiv p(A, B, \cdots, B)$ defines a left additive $*$-operation. Examples for right additive ones are similarly obtained. In fact, if we define $A *' B \equiv B * A$ for ideals A, B, then $*'$ is a right additive operation.

Henceforth we assume that all $*$-operations are left-additive. By the remark above, results for a right additive operation are entirely similar.

DEFINITION 3.2. For an ideal A in R and an element b in R, the (left) f-quotient $A:b$ of A by b is defined as the subset of all elements x in R such that $f(x) * f(b)$ is contained in A. If B is an ideal of R, we let $A:B = \bigcap_{b \in B} (A:b)$.

$A:b$ may be empty. For instance, let $A * B = AB$ and $f(a) = R$ for all a in R. If R is a ring with $R^2 = R$, then, for any proper ideal A in R, we have $A:b = \emptyset$. However, if $A:b \neq \emptyset$, $A:b$ is an ideal containing A. In fact, let $x \in A:b$ and $a \in A$, then $x + a \in f(x) + A$ and so $f(x + a) \subseteq f(x) + A$. Recalling that $*$ is left-additive and Definition 2.1, we have $f(x + a) * f(b) \subseteq (f(x) + A) * f(b) \subseteq f(x) * f(b) + A * f(b) \subseteq A$. Hence $(A:b) + A \subseteq A:b$. If x and y are in $A:b$, then $f(x + y) * f(b) \subseteq (f(x) + f(y)) * f(b) \subseteq A$. This proves that $A:b$ is an ideal of R containing A. We note

that if $f(a)=(a)$ for all a in R, $A:b\neq\emptyset$ for every ideal A and b since $(0)*f(b)=(0)$.

DEFINITION 3.3. Let M be an f-system of R. Then a kernel M' of M is said to be densed in M if, for any ideal A in R, $M\cap A\neq\emptyset$ implies $M'\cap A\neq\emptyset$.

If $f(a)=(a)$ for all a in R and M is an f-system, every kernel of M is clearly densed in M. But, in general, this is not the case. Let R be the Jordan ring in Example 2.7. For the ideal $P=(x)(x)^2$, $c(P)$ is an f-system and $M'=\{y, y^2, y^3, \cdots.\}$ is a kernel of $c(P)$, but we note that $c(P)\cap(x)\neq\emptyset$ since $(x)\not\subseteq P$, while $(x)\cap M'=\emptyset$.

Lemma 3.4. *Let A and B be ideals in R. Then we have*

(i) *if $A\subseteq B$, $r(A)\subseteq r(B)$,*

(ii) *$r(r(A))=r(A)$,*

(iii) *if every f-system M in R has a kernel densed in M, $r(A\cap B)=r(A)\cap r(B)$.*

Proof. (i) and (ii) are immediate from the definition of the radical. For the proof of (iii), we note that we always have $r(A\cap B)\subseteq r(A)\cap r(B)$. Let x now be in $r(A)\cap r(B)$ and M be any f-system containing x. If M' is a kernel of M densed in it, $M'\cap A\neq\emptyset$ and $M'\cap B\neq\emptyset$. Since M' is a $*$-system, $A*B\cap M'\neq\emptyset$ and $A*B\cap M'\subseteq A\cap B\cap M'\neq\emptyset$. Hence $A\cap B\cap M\neq\emptyset$ and $x\in r(A\cap B)$. This proves (iii).

DEFINITION 3.5. An ideal Q in R is called (left) f-primary if $f(a)*f(b)\subseteq Q$ and $a\notin Q$ imply $b\in r(Q)$.

By Lemma 2.8 we see that every f-prime ideal in R is f-primary.

Lemma 3.6. *Suppose that every f-system in R has a kernel densed in it. If Q and Q' are f-primary ideals in R such that $r(Q)=r(Q')$, then $Q\cap Q'$ is also f-primary and $r(Q\cap Q')=r(Q)=r(Q')$.*

Proof. By definition, $Q\cap Q'$ is clearly f-primary. By Lemma 3.4 (iii), $r(Q\cap Q')=r(Q)\cap r(Q')=r(Q)=r(Q')$.

Theorem 3.7. *Suppose that the function f satisfies the condition:*

(I) *For any ideals A and B in R, $B\not\subseteq r(A)$ implies $A:B\neq\emptyset$.*

Then an ideal Q is f-primary if and only if $Q:B=Q$ for all ideals B not contained in $r(Q)$.

Proof. Let Q be f-primary and B be an ideal not contained in $r(Q)$. Let $a\in B$ but $a\notin r(Q)$, then by (I) $Q:b\neq\emptyset$. For any element a in $Q:b$, $f(a)*f(b)\subseteq Q$, and since $b\notin r(Q)$, $a\in Q$. Thus $Q:b$ is contained in Q and this implies that $Q=Q:B$ since $Q:B$ is an ideal containing Q and $Q:B\subseteq Q:b$.

Conversely, suppose that $f(a)*f(b)\subseteq Q$ and $b\notin r(Q)$. Then $f(b)\not\subseteq r(Q)$ and

$Q:f(b)=Q$. Hence, for any element c in $f(b)$, $f(a)*f(c)\subseteq f(a)*f(b)\subseteq Q$ and so $a\in Q:f(b)=Q$. Therefore, Q is f-primary.

Let A be an ideal of R. Suppose that A is represented as a finite intersection $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ of f-primary ideals Q_i (called an f-primary decomposition of A). As in the associative case, an f-primary decomposition of A is said to be irredundant if it satisfies the conditions:

(a) No Q_i contains the intersection of the other ones.

(b) The Q_i have distinct f-prime radicals $r(Q_i)$.

If every f-system has a kernel densed in it, then, in view of Lemma 3.6, each f-primary decomposition can be refined to irredundant one.

For the main theorem, we need the conditions (I) and

(II) for any f-primary ideal Q in R, $Q:Q=R$.

Conditions (I) and (II) hold in case $f(a)=(a)$ for all a in R. In fact, let A be any ideal in R, then for x in R and a in A, $(x)*(a)\subseteq(x)\cap(a)\subseteq A$ and so $A:A=R$, and hence (II). Condition (I) is trivial. Conditions (I) and (II) are not true in general.

EXAMPLE 3.8. Let R be the free Jordan ring in Example 2.7. Let $f(a)=(a, y)$ for all a in R. Then $P=(x)(x)^2$ is f-prime and so f-primary. To see $P:0=\emptyset$, suppose that $P:0\neq\emptyset$, then $P:0$ is an ideal and so $f(0)U_{f(0)}=(y)(y)^2\subseteq P$, a contradiction. Hence $P:P=\emptyset$. Also $P:(y)=\emptyset$ and $(y)\not\subseteq r(P)=P$.

We now state an analogue of the uniqueness theorem of Lasker-Noether decomposition in commutative, associative rings.

Theorem 3.9. *Let R be a nonassociative ring where a left additive $*$-operation and the function f are defined and (I) and (II) hold. Suppose that every f-system in R has a kernel densed in it. If an ideal A admits an f-primary decomposition and*

$$A=Q_1\cap Q_2\cap\cdots\cap Q_m=Q_1'\cap Q_2'\cap\cdots\cap Q_n'$$

are two irredundant f-primary decompositions for A, then $m=n$ and for a suitable ordering of the Q_i and Q_j' we have $r(Q_i)=r(Q_i')$ for all i.

The proof depends on Lemma 3.6, Theorem 3.7, (II), and on the following easy properties:

(1) if $A\subseteq B$, $A:C\subseteq B:C$,

(2) if $B\subseteq C$, $A:B\supseteq A:C$,

(3) $(A\cap B):C=(A:C)\cap(B:C)$

for ideals A, B, C in R. Hence the proof is essentially the same as in [3] and we shall omit it.

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