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# A HOMOTOPY GROUP OF THE SYMMETRIC SPACE $SO(2n)/U(n)$

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In [1] B. Harris calculated some homotopy groups of the symmetric space $\Gamma_n = SO(2n)/U(n)$. He determined $\pi_{2n+r}(\Gamma_n)$ for $-1 \leqq r \leqq 1$ and for $r=3$, $n \equiv 0 \pmod 4$ except for $r=1$, $n \equiv 2 \pmod 4$. For the last case he made a group extension

$$0 \to Z_2 \to \pi_{2n+1}(\Gamma_n) \to Z_{n!/2} \to 0 \tag{1}$$

from the homotopy exact sequence of the fibration $\Gamma_n \to \Gamma_{n+1} \to S^{2n}$. The purpose of this note is to show that this extension splits.

**Theorem.** $\pi_{2n+1}(\Gamma_n) = Z_2 \oplus Z_{n!/2}$ *if* $n \equiv 2 \pmod 4$.

Proof. If $n=2$, then the conclusion is obvious, by (1). Thus we will always assume that $n \equiv 2 \pmod 4$ and $n \geqq 6$.

The rotation group $SO(m)$ and the unitary group $U(m)$ are embedded, respectively, in $SO(m+1)$ and $U(m+1)$ as the upper left hand blocks. We embed $U(m)$ in $SO(2m)$ as the subset of matrices consisting of $2\times 2$ blocks

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

The natural map $SO(2n-1)/U(n-1) \to SO(2n)/U(n) = \Gamma_n$ is a homeomorphism and will be used to identify these spaces. The inclusion map $SO(2n-2) \to SO(2n-1)$ then induces a map between the fibrations:

$$\begin{tikzcd}
U(n-1) \arrow[r, equal] \arrow[d, "j"] & U(n-1) \arrow[d] \\
SO(2n-2) \arrow[r] \arrow[d] & SO(2n-1) \arrow[d, "p"] \\
\Gamma_{n-1} \arrow[r, "i"] & \Gamma_n .
\end{tikzcd}$$

Applying the homotopy functor $\pi_*(-)$ to this, we obtain a commutative diagram with exact columns:

$$\begin{tikzcd}
\pi_{2n+1}(\Gamma_{n-1}) \arrow[r, "i_*"] \arrow[d, "\partial"] & \pi_{2n+1}(\Gamma_n) \arrow[d, "\Delta"] \\
\pi_{2n}(U(n-1)) \arrow[r, equal] \arrow[d, "j_*"] & \pi_{2n}(U(n-1)) \arrow[d] \\
\pi_{2n}(SO(2n-2)) \arrow[r] \arrow[d] & \pi_{2n}(SO(2n-1)) \arrow[d, "p_*"] \\
\pi_{2n}(\Gamma_{n-1}) \arrow[r] \arrow[d] & \pi_{2n}(\Gamma_n) \arrow[d] \\
\pi_{2n-1}(U(n-1)) \arrow[r, equal] & \pi_{2n-1}(U(n-1)) .
\end{tikzcd}$$

We already know all the groups except $\pi_{2n+1}(\Gamma_n)$ in the above diagram, as follows:

(2) $\pi_{2n+1}(\Gamma_{n-1})=Z_{n!(24,n-2)/48}$, *by* (8.2) *of* [2];
(3) $\pi_{2n}(U(n-1))=Z_{n!/2}$, *by* Lemma 1.6 *of* [3];
(4) $\pi_{2n}(SO(2n-2))=Z_{12}$, *by* [3];
(5) $\pi_{2n}(SO(2n-1))=Z_2$, *by* [3];
(6) $\pi_{2n}(\Gamma_{n-1})=Z_{(24,n-2)/2}$, *by* (6.2) *of* [2];
(7) $\pi_{2n}(\Gamma_n)=Z_2$, *by* [1];
(8) $\pi_{2n-1}(U(n-1))=0$, *by* Lemma 1.4 *of* [3].

Here $(a, b)$ denotes the greatest common divisor of $a$ and $b$.

We use the following notations. For a finite abelian group $G$, $G_{ev}$ and $G_{od}$ denote the 2-primary and the odd components of $G$, respectively. For a homomorphism $f: G \to H$, $f_{ev}: G_{ev} \to H_{ev}$ and $f_{od}: G_{od} \to H_{od}$ are the restrictions of $f$ to the appropriate busgroups.

By (5), (7) and (8), $p_*$ is an isomorphism, so $\Delta$ is an epimorphism. It follows that $\Delta_{od}$ is an isomorphism, from (1) and (3), and that (1) splits if and only if $\Delta$ has a right inverse. Therefore (1) splits if and only if $\Delta_{ev}$ has a right inverse.

Let $n \equiv 2 \pmod 8$. By (4), (6) and (8), the image of $j_*$, *Image*$(j_*)$, is $Z_3$ or 0, so $\partial_{ev}$ is an epimorphism. Hence $\partial_{ev}$ is an isomorphism, by (2) and (3). It follows that $(i_*)_{ev} \circ \partial_{ev}^{-1}$ is a right inverse of $\Delta_{ev}$, so that (1) splits.

Let $n \equiv 6 \pmod 8$. By (4), (6) and (8), $(\mathit{Image}(j_*))_{ev}=Z_2$. Hence, by (2) and (3), we have a commutative diagram with exact columns:

$$\begin{tikzcd}
(Z_{n!/4})_{ev} \arrow[r, "(i_*)_{ev}"] \arrow[d, "\partial_{ev}"] & (\pi_{2n+1}(\Gamma_n))_{ev} \arrow[d, "\Delta_{ev}"] \\
(Z_{n!/2})_{ev} \arrow[r, equal] \arrow[d] & (Z_{n!/2})_{ev} \arrow[d] \\
Z_2 \arrow[d] & 0 . \\
0 &
\end{tikzcd}$$

Suppose that (1) does not split, that is, $\pi_{2n+1}(\Gamma_n)=Z_{n!}$. Then we can choose generators $\alpha$, $\beta$ and $\gamma$ of $(Z_{n!/4})_{ev}$, $(\pi_{2n+1}(\Gamma_n))_{ev}$ and $(Z_{n!/2})_{ev}$, respectively, such that $\partial_{ev}(\alpha)=2\gamma$ and $\Delta_{ev}(\beta)=\gamma$. Since we can write $(i_*)_{ev}(\alpha)=4x\beta$ for some integer $x$, we have

$$2\gamma=\partial_{ev}(\alpha)=(\Delta_{ev}\circ(i_*)_{ev})(\alpha)=\Delta_{ev}(4x\beta)=4x\gamma\,.$$

Hence $2(2x-1)\gamma=0$. But this is impossible, because the order of $\gamma$ is a multiple of 8. Therefore (1) splits. This completes the proof.

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