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RANK RIGIDITY, CONES, AND CURVATURE-HOMOGENEOUS HADAMARD MANIFOLDS

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1. Introduction

The rank of a geodesic in a Riemannian manifold is the dimension of the real vector space of all parallel Jacobi vector fields along it. The rank of the manifold is the minimum of the ranks of all its geodesics. A manifold is said to have higher rank if its rank is greater than one. Higher rank seems to be rather exceptional. For instance, any locally irreducible, compact, Riemannian manifold with nonpositive sectional curvature and higher rank is a locally symmetric space ([1], [3], [4]). The curvature assumption is essential ([8]). It is an open problem whether the compactness assumption can be weakened to completeness (*Rank Rigidity Conjecture*, [2]).

In the present paper we approach this problem via infinitesimal geometry. We give examples of connected, simply connected, locally irreducible Riemannian manifolds with nonpositive sectional curvature and higher rank but which are not locally symmetric. These examples are not complete, but they show that the local Rank Rigidity Conjecture does not hold and that rank rigidity is a global phenomenon. We also study Riemannian manifolds which are almost flat in some infinitesimal sense. Among them are some irreducible, curvature-homogeneous, inhomogeneous, Hadamard manifolds. In view of rank rigidity for homogeneous Hadamard manifolds [6] it is natural to investigate the rank of these manifolds. We show that these manifolds have indeed rank one.

We now describe the contents of this paper in more detail. The *infinitesimal rank* of a Riemannian manifold M is the largest integer k such that for every $p \in M$ and $v \in T_pM$ there exists a k-dimensional subspace F (*infinitesimal k-flat*) of T_pM with $v \in F$ and $R(X,Y)Z = 0$ for all $X, Y, Z \in F$, where R is the curvature tensor of M. The n-dimensional Riemannian manifolds with infinitesimal rank n are obviously the flat Riemannian manifolds. In Section 2 we classify the n-dimensional Riemannian manifolds with infinitesimal rank $n-1$. Since the null space of the Jacobi operator of these manifolds has codimension one these manifolds are good candidates for having many parallel Jacobi vector fields. Among them are certain cones and certain Hadamard manifolds which can be realized as twisted products. In Section 3 we present some formulas for the curvature of twisted products. In Section 4 we show that

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certain cones over Riemannian manifolds with sectional curvature bounded from above provide counterexamples to the local Rank Rigidity Conjecture. These cones have rank two and are never complete. In Section 5 we prove that the rank of the above mentioned curvature-homogeneous manifolds is equal to one. More generally, we prove that each locally irreducible Riemannian manifold whose curvature tensor is that of the Riemannian product of a Euclidean space and a surface of constant curvature has rank one. This result implies that rank rigidity holds for curvature-homogeneous semi-symmetric spaces.

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2. Riemannian n-manifolds of infinitesimal rank $n-1$

The infinitesimal rank of a Riemannian manifold is bigger or equal than its rank. It is clear that an n-dimensional Riemannian manifold M has infinitesimal rank n precisely if M is flat. We now consider n-dimensional Riemannian manifolds with infinitesimal rank equal to $n-1$.

Proposition 1. *Let M be an n-dimensional Riemannian manifold. Then M has infinitesimal rank $n-1$ if and only if M is nonflat and at each point $p \in M$ its Riemannian curvature tensor is that of $\mathbb{R}^{n-2} \times M^2(\kappa(p))$ for some $\kappa(p) \in \mathbb{R}$, where $M^2(\kappa(p))$ is a surface of constant curvature $\kappa(p)$.*

Proof. Let M be an n-dimensional Riemannian manifold with infinitesimal rank $n-1$. We fix some point $p \in M$. Let F be some $(n-1)$-dimensional infinitesimal flat in T_pM and $X_n \in T_pM$ some unit vector which is perpendicular to F. By assumption, there exists an $(n-1)$-dimensional infinitesimal flat F' in T_pM with $X_n \in F'$. Let $X_1, \ldots, X_{n-2}$ be an orthonormal basis of $F \cap F'$ and X_{n-1} be a unit vector in F which is perpendicular to $F \cap F'$. Then $X_1, \ldots, X_n$ is an orthonormal basis of T_pM and, by assumption, we have $g_p(R_p(X_i, X_j)X_k, X_l) = 0$ whenever at least three indices are in $\{1, \ldots, n-2, n-1\}$ or in $\{1, \ldots, n-2, n\}$. Therefore the only possible non-vanishing expression of this form is $g_p(R_p(X_{n-1}, X_n)X_n, X_{n-1})$ modulo the algebraic curvature identities. This shows that R_p has the form of the curvature tensor of the Riemannian product $\mathbb{R}^{n-2} \times M^2(\kappa(p))$, where $M^2(\kappa(p))$ is a standard space of constant curvature $\kappa(p) \in \mathbb{R}$. We must have $\kappa(p) \neq 0$ at some $p \in M$, because otherwise R vanishes everywhere and M is flat and hence has infinitesimal rank n, which is a contradiction. Therefore M is nonflat.

Conversely, let M be an n-dimensional nonflat Riemannian manifold whose curvature tensor at any point $p \in M$ is that of $\mathbb{R}^{n-2} \times M^2(\kappa(p))$ for some $\kappa(p) \in \mathbb{R}$. Clearly every $v \in T_pM$ is contained in a subspace $F = \mathbb{R}^{n-2} \oplus \operatorname{span}\{w\} \subset T_pM = \mathbb{R}^{n-2} \oplus$

$T_pM^2(\kappa(p))$ for a suitable $w \in T_pM^2(\kappa(p))$, and F is an infinitesimal $(n-1)$-flat. $\square$

REMARK. Riemannian manifolds for which at each point the Riemannian curvature tensor is that of some Riemannian symmetric space are known as semi-symmetric spaces. The local classification of semi-symmetric spaces has been achieved by Szabó [9]. From his classification we get the following list of Riemannian manifolds whose curvature tensor at each point p is that of $\mathbb{R}^{n-2} \times M^2(\kappa(p))$ for some $\kappa(p) \in \mathbb{R}$:
1. Riemannian manifolds which are locally isometric to the Riemannian symmetric space $\mathbb{R}^{n-2} \times M^2(\kappa)$ for some $\kappa \in \mathbb{R}$. These manifolds are locally symmetric and locally reducible.
2. Riemannian manifolds which are foliated by Euclidean leaves of codimension two. This class of manifolds has been thoroughly studied in [5]. The value of κ does depend on the point in general, but when M is connected and nonflat it is either negative or positive everywhere. For κ constant these are the so-called curvature-homogeneous semi-symmetric spaces, for which a full classification can be found in Chapter 4 of [5]. An interesting oberservation is that there exist connected, complete, irreducible, curvature-homogeneous semi-symmetric spaces which are diffeomorphic to $\mathbb{R}^n$ and modelled on the curvature tensor of $\mathbb{R}^{n-2} \times M^2(\kappa)$, $\kappa < 0$, $n \geq 3$. These manifolds have obviously nonpositive curvature and hence provide counterexamples to the following infinitesimal version of the Rank Rigidity Conjecture: If M is a connected, complete, locally irreducible Riemannian manifold of nonpositive curvature and infinitesimal rank greater than one, then M is locally symmetric. In Section 5 we will investigate these spaces more thoroughly in relation with the Rank Rigidity Conjecture.
3. If $n = 3$: Euclidean cones, elliptic cones and hyperbolic cones. These particular cones are cones over surfaces of constant curvature. There is an obvious higher-dimensional generalization of these cones, which we shall discuss in Section 4 also in relation with the Rank Rigidity Conjecture.
4. If $n = 2$: any Riemannian manifold.

3. Twisted products

In Sections 4 and 5 we will focus on the manifolds which we encountered in the classification of the n-dimensional Riemannian manifolds with infinitesimal rank $n-1$. Many of them can be realized as twisted products. For this reason we summarize in this section some basic material concerning twisted products.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and $f : M_1 \times M_2 \to \mathbb{R}$ be some smooth function with positive values. The *twisted product* $M_1 \times_f M_2$ of M_1 and M_2 is the smooth manifold $M := M_1 \times M_2$ equipped with the Riemannian metric

$$g(X, Y) := g_1(X_1, Y_1) + f g_2(X_2, Y_2)$$

where X_1 and X_2 denote the component of X which is tangent to M_1 and M_2 at the corresponding points, respectively. If f is a function which is defined only on M_1 then M is a so-called *warped product*, and if f is constant equal to one then M is the Riemannian product $M_1 \times M_2$ of M_1 and M_2. If M_1 is some open interval in $\mathbb{R}$ and if $f(t) = (at + b)^2$ with $a \neq 0$ then M is a *cone* over M_2. We denote by ∇, $\nabla^\times$ the Levi Civita covariant derivative and by R, $R^\times$ the Riemannian curvature tensor of M, $M_1 \times M_2$, respectively.

Proposition 2. *Let $M_1 \times_f M_2$ be a twisted product and $F := \ln(\sqrt{f})$. Then*

$$\nabla_X Y = \nabla^\times_X Y + dF(X)Y_2 + dF(Y)X_2 - g(X_2, Y_2)\,\mathrm{grad}^g F$$

and

$$\begin{aligned} R(X, Y)Z = R^\times(X, Y)Z & \\ &+ \|\mathrm{grad}^g F\|_g^2[g(Y_2, Z_2)X_2 - g(X_2, Z_2)Y_2] \\ &- [(\mathrm{hess}^g F)(Y, Z)X_2 - (\mathrm{hess}^g F)(X, Z)Y_2] \\ &- dF(Z)[dF(Y)X_2 - dF(X)Y_2] \\ &- [g(Y_2, Z_2)(\mathrm{Hess}^g F)X - g(X_2, Z_2)(\mathrm{Hess}^g F)Y] \\ &- [dF(X)g(Y_2, Z_2) - dF(Y)g(X_2, Z_2)]\,\mathrm{grad}^g F\ , \end{aligned}$$

where $\mathrm{grad}^g F$, $\mathrm{Hess}^g F$ *and* $\mathrm{hess}^g F$ *are defined by*

$$g(\mathrm{grad}^g F, X) = dF(X)$$

and

$$\mathrm{hess}^g F(X, Y) = g((\mathrm{Hess}^g F)X, Y) = d(dF(Y))(X) - dF(\nabla_X Y)$$

for all vector fields X, Y tangent to $M_1 \times_f M_2$.

Proof. The expression for ∇ follows from the Koszul formula for the Levi Civita covariant derivative by a lengthy but straightforward calculation. The expression for R follows from the one for ∇ by an even lengthier, but still straightforward, calculation. □

We next derive some formulas for the sectional curvature on twisted products. For this we denote by K, K_1 and K_2 the sectional curvature function of $M = M_1 \times_f M_2$, M_1 and M_2, respectively. For each $p = (p_1, p_2)$ we identify T_pM and $T_{p_1}M_1 \oplus T_{p_2}M_2$ in the usual way.

Proposition 3. *Let $M = M_1 \times_f M_2$ be a twisted product, $F = \ln(\sqrt{f})$, $p = (p_1, p_2) \in M$ and σ some two-dimensional linear subspace of T_pM.*

(i) *If* $\sigma \subset T_{p_1}M_1$ *then*

$$K(\sigma) = K_1(\sigma) \ .$$

(ii) *If* $\sigma \subset T_{p_2}M_2$ *then*

$$K(\sigma) = \frac{1}{f(p)} K_2(\sigma) + \| \mathrm{pr}^g_{\sigma^\perp} \ \mathrm{grad}^g_p F \|^2_g - \mathrm{tr}_g(\mathrm{hess}^g_p F \mid \sigma) \ ,$$

where $\mathrm{pr}^g_{\sigma^\perp}$ *denotes the* g*-orthogonal projection onto the* g*-orthogonal complement* $\sigma^\perp$ *of* σ *in* T_pM *and* tr_g *denotes the trace with respect to* g*. If* M *is a warped product this expression reduces to*

$$K(\sigma) = \frac{1}{f(p_1)} K_2(\sigma) - \| \mathrm{grad}^{g_1}_{p_1} F \|^2_{g_1} \ .$$

(iii) *If* $\sigma \cap T_{p_1}M_1$ *and* $\sigma \cap T_{p_2}M_2$ *are both one-dimensional then there exists some orthonormal basis* X_1, Y_2 *of* σ *with* $X_1 \in T_{p_1}M_1$ *and* $Y_2 \in T_{p_2}M_2$*, and we have*

$$K(\sigma) = -(\mathrm{hess}^g_p F)(X_1, X_1) - d_pF(X_1)^2 \ .$$

Proof. (i) follows from the fact that $M_1 \times \{p_2\}$ is totally geodesic in M, (ii) and (iii) are straightforward calculations using the explicit expression for R as in Proposition 2. □

4. Counterexamples to the local Rank Rigidity Conjecture

We start with a result about totally geodesic submanifolds in warped products.

Proposition 4. *Let* $M_1 \times_f M_2$ *be a warped product.*
(i) *If* N_2 *is a totally geodesic submanifold of* M_2 *then* $M_1 \times_f N_2$ *is a totally geodesic submanifold of* $M_1 \times_f M_2$*.*
(ii) *Let* $\gamma\colon J \to M_1 \times_f M_2$ *be a geodesic in* $M_1 \times_f M_2$*. Then* γ *is contained in the totally geodesic submanifold* $M_1 \times_f N_2$ *of* $M_1 \times_f M_2$*, where* N_2 *is the totally geodesic submanifold of* M_2 *which is determined by* $\dot\gamma(t_o)_2$ *for some* $t_o \in J$*. Note that* N_2 *has dimension less or equal than one, and the dimension of* N_2 *is equal to zero if and only if* γ *is a geodesic in* $M_1 \times_f \{p_2\}$ *for some* $p_2 \in M_2$*.*

Proof. The first statement follows easily from the expression for the Levi Civita covariant derivative of $M_1 \times_f M_2$ according to Proposition 2. The second statement is a consequence of (i). □

We will now be interested in warped products of the form $M = I \times_f M_2$, particularly cones. Here, I is some open interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is a smooth function with positive values. The Riemannian metric on I is the one which is induced from the canonical Riemannian metric on $\mathbb{R}$. We denote by t the parameter on I.

Let $a, b \in \mathbb{R}$ with $a \neq 0$ and $at + b > 0$ for all $t \in I$. Then the warped product $M = I \times_f M_2$ with $f(t) = (at + b)^2$ is a cone over M_2. Note that, by construction, a cone is never complete as a metric space. Let $\gamma\colon J \to M$ be a geodesic in M and $s_o \in J$. If $\dot\gamma(s_o)_2 = 0$, then γ is a geodesic in $I \times_f \{p\}$, where p is determined by $\gamma(s_o) = (t, p)$. Let N_2 be any one-dimensional totally geodesic submanifold of M_2 with $p \in N_2$. Then $I \times_f N_2$ is a two-dimensional totally geodesic submanifold of M which contains γ. Now assume that $\dot\gamma(s_o) \neq 0$. Let N_2 be the one-dimensional totally geodesic submanifold of M_2 which is determined by $\dot\gamma(s_o)_2$. As we have seen in Proposition 4, γ is contained in the two-dimensional totally geodesic submanifold $N := I \times_f N_2$ of M. We thus conclude that each geodesic in M is contained in some two-dimensional totally geodesic submanifold of M of the form $I \times_f N_2$, where N_2 is some one-dimensional totally geodesic submanifold of M_2.

Let $N = I \times_f N_2$ be such a totally geodesic submanifold, $(t_o, q) \in N$ and $\sigma = T_{(t_o,q)}N$. Since N is totally geodesic in M, Proposition 3 (iii) tells us that the sectional curvature of N at (t_o, q) is equal to

$$K(\sigma) = -F''(t_o) - F'^2(t_o) \ .$$

Since $F(t) = \ln(at + b)$, a simple calculation yields $K(\sigma) = 0$. We summarize this in the following proposition.

Proposition 5. *If M is a cone over any Riemannian manifold, then each geodesic in M is contained in some two-dimensional totally geodesic flat submanifold of M. In particular, both the rank and the infinitesimal rank of M are greater or equal than two.*

We now calculate the sectional curvature function K of the cone M. If M_2 is one-dimensional the above argument shows that M is flat. We therefore assume $\dim M_2 \geq 2$. Let $(t, q) \in M$, $T = (\partial/\partial t)(t) \in T_t I$, and $\sigma \subset T_{(t,q)}M$ be a two-dimensional linear subspace. Then there exist some $\alpha \in [0, \pi/2]$ and some orthonormal vectors $X_2, Y_2 \in T_q M_2$ such that

$$\sigma = \operatorname{span}\{\cos(\alpha)T + \sin(\alpha)X_2, Y_2\} \ .$$

We denote by σ_{T,Y_2} and σ_{X_2,Y_2} the two-planes spanned by T, Y_2 and X_2, Y_2, respectively. Then

$$K(\sigma) = \cos^2(\alpha)K(\sigma_{T,Y_2}) + \sin^2(\alpha)K(\sigma_{X_2,Y_2}) + 2\sin(\alpha)\cos(\alpha)g(R(T, Y_2)Y_2, X_2) \ .$$

We already know that

$$K(\sigma_{T,Y_2}) = 0 \ .$$

By means of Proposition 3 (ii) we get for $K(\sigma_{X_2,Y_2})$ the expression

$$K(\sigma_{X_2,Y_2}) = \frac{1}{f(t)}K_2(\sigma_{X_2,Y_2}) - F'^2(t) = \frac{1}{(at+b)^2}(K_2(\sigma_{X_2,Y_2}) - a^2) \ .$$

And from Proposition 2 we get

$$g(R(T,Y_2)Y_2, X_2) = 0 \ .$$

Altogether we therefore have

$$K(\sigma) = \frac{\sin^2(\alpha)}{(at+b)^2}(K_2(\sigma_{X_2,Y_2}) - a^2) \ .$$

We therefore conclude

Proposition 6. *Let $M = I \times_{(at+b)^2} M_2$ be a cone over some Riemannian manifold M_2. Then M has nonpositive sectional curvature if and only if $K_2 \leq a^2$.*

Let M_2 be some Riemannian manifold whose sectional curvature function K_2 satisfies $K_2 < a^2$. According to Proposition 6 the cone $M = I \times_{(at+b)^2} M_2$ has nonpositive sectional curvature function K. Moreover, we have $K(\sigma) = 0$ if and only if $\alpha = 0$, which shows that M is locally irreducible. We therefore conclude:

Theorem 1. *The local Rank Rigidity Conjecture does not hold. More precisely, let M_2 be a connected, simply connected, irreducible, Riemannian manifold whose sectional curvature function K_2 satisfies $K_2 < a^2$ for some non-zero $a \in \mathbb{R}$. Let $b \in \mathbb{R}$ and $I = (-\infty, -b/a)$ or $I = (-b/a, \infty)$. Then the cone $I \times_{(at+b)^2} M_2$ has nonpositive sectional curvature, and any such cone is connected, simply connected, locally irreducible and has rank equal to two.*

We finish this section with a remark why it is natural in our context to study cones within the class of warped products of the form $M = I \times_f M_2$. Suppose that $K(\sigma) = 0$ for all two-planes σ of the form $\sigma = \mathrm{span}\{T, Y_2\}$ with some unit vector Y_2 tangent to M_2. Then we get the differential equation $F'' + F'^2 = 0$. Since we assume f and hence also F to be smooth this implies that $F(t) = \ln(at+b)$ for some $a, b \in \mathbb{R}$. If $a = 0$ then M is the Riemannian product of I and M_2 equipped with some Riemannian metric which is homothetic to the orginial one. Otherwise M is a cone.

5. Rank rigidity of curvature-homogeneous semi-symmetric spaces

We have seen in Proposition 1 that an n-dimensional Riemannian manifold M has infinitesimal rank $n-1$ if and only if M is a nonflat Riemannian manifold whose curvature tensor at each point p is that of $\mathbb{R}^{n-2} \times M^2(\kappa(p))$ for some $\kappa(p) \neq 0$. Since such manifolds have, apart from flat spaces, the highest possible infinitesimal rank, it is natural to investigate what their rank is. Since some of them are connected, simply connected, complete, irreducible and have nonpositive curvature, one might view them as excellent candidates for complete counterexamples to the Rank Rigidity Conjecture. We will investigate this here for the case that the value of κ does not vary from point to point.

We first recall that each locally homogeneous semi-symmetric space is locally symmetric (see e.g. [5, p. 69]). The locally inhomogeneous, curvature-homogeneous, semi-symmetric and locally irreducible Riemannian manifolds have the curvature of $\mathbb{R}^{n-2} \times M^2(\kappa)$ for some fixed $\kappa \neq 0$ (see [5, p. 50]) and can be described as follows (see [5, p. 65], and [7]). There exists some dense open subset U of M such that in a neighborhood of every point $p \in U$ there exist local coordinates $(x_1, \ldots, x_{n-1}, t)$ around p and an orthonormal coframe field of the form

$$\omega_0 = h(x,t)dt \ , \ \ \omega_i = dx_i + (A(t)x^T)dt \ \ (i = 1, \ldots, n-1) \ ,$$

where A is some smooth function with values in the Lie algebra of all skew-symmetric real $(n-1) \times (n-1)$-matrices and the function $h \neq 0$ is given either by

$$h(x,t) = a(t)e^{kx_1} + b(t)e^{-kx_1}$$

or by

$$h(x,t) = a(t)\cos(kx_1) + b(t)\sin(kx_1)$$

with some smooth functions a and b and some non-zero constant k. Conversely, any local metric of this form is curvature-homogeneous and its curvature tensor is that of $\mathbb{R}^{n-2} \times M^2(-k^2)$ and $\mathbb{R}^{n-2} \times M^2(k^2)$, respectively. Generically these metrics are also inhomogeneous and locally irreducible. By a suitable choice of A, a, b the first type defines a connected, simply connected, complete, irreducible, inhomogeneous Riemannian manifold of non-positive curvature (see [7, p. 488], for details). The second type never leads to complete metrics.

We will now investigate metrics of the above form. First of all, by a suitable coordinate transformation we may rewrite the metric as

$$g = \sum_{i=1}^{n-1} dx_i \otimes dx_i + f(x,t)dt \otimes dt$$

with

$$f(x,t) := h^2 \left(\sum_{j=1}^{n-1} b_j(t)x_j, x_2, \ldots, x_{n-1}, t \right)$$

and some smooth functions $b_j(t)$ (see [7, Proposition 11.2]). This is clearly the metric of some twisted product and we hence may apply Proposition 2.

Before we continue we need a simple lemma from linear algebra.

Lemma 1. *Let $b = (b_1, \ldots, b_{n-1}) \in \mathbb{R}^{n-1}$ be some non-zero vector and B the $(n-1) \times (n-1)$-matrix defined by $b_{\nu\mu} = b_\nu b_\mu$, that is, $B = b^T b$. Then B is positive semidefinite and the kernel of B is equal to the orthogonal complement of the one-dimensional linear subspace of $\mathbb{R}^{n-1}$ which is spanned by b.*

Proof. For all $w \in \mathbb{R}^{n-1}$ we have

$$wBw^T = w(b^T b)w^T = (wb^T)(bw^T) = (w \cdot b)^2 \ ,$$

where $w \cdot b$ is the usual dot product in $\mathbb{R}^{n-1}$. From this the assertion follows easily. □

Now let M be a twisted product as described above.

Lemma 2. *Let γ be some geodesic in M and V a Jacobi vector field along γ such that $R(V, \dot\gamma)\dot\gamma = 0$. We suppose that γ depends on the real parameter s and write $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ according to the factors of M at $\gamma(s)$. Furthermore, let T be the unit tangent vector field along γ defined by*

$$\sqrt{f(\gamma(s))}T(s) = \frac{\partial}{\partial t}(\gamma(s))$$

for all s. Define a vector field W along γ by

$$W := g(V, T)\dot\gamma - g(\dot\gamma, T)V \ .$$

Then $W(s)$ is tangent to the first factor of M at $\gamma(s)$ and orthogonal to the vector $(b_1, \ldots, b_{n-1})(\gamma_2(s))$ for all s.

Proof. Taking inner product of W with T shows immediately that $W(s)$ is tangent to the first factor of M at $\gamma(s)$ for all s. Next, the expression for the curvature tensor R of M according to Proposition 2 implies

$$0 = g(R(V, \dot\gamma)\dot\gamma, V) = -(\mathrm{hess}^g F)(W, W) - dF(W)^2 = -e^{-F}(\mathrm{hess}^g e^F)(W, W)$$

and hence

$$0 = (\mathrm{hess}^g e^F)(W, W) \ .$$

Recall the two possible choices for the function h. In the first case we have

$$\exp\{F(x,t)\} = a(t)\exp\left\{k\sum b_j(t)x_j\right\} + b(t)\exp\left\{-k\sum b_j(t)x_j\right\}$$

and in the second case it is

$$e^{F(x,t)} = a(t)\cos\left(k\sum b_j(t)x_j\right) + b(t)\sin\left(k\sum b_j(t)x_j\right) \ .$$

We denote by $B(s)$ the $(n-1)\times(n-1)$-matrix whose entry in the ν-th row and μ-th column is $b_\nu(\gamma_2(s))b_\mu(\gamma_2(s))$. A simple calculation shows that

$$(\mathrm{hess}^g\, e^F)(W,W) = k^2 e^{F\circ\gamma} W^T B W$$

in the first case and

$$(\mathrm{hess}^g\, e^F)(W,W) = -k^2 e^{F\circ\gamma} W^T B W$$

in the second case. Using Lemma 1, we now see that in both cases $W(s)$ is orthogonal to $(b_1,\ldots,b_{n-1})(\gamma_2(s))$ for all s. This concludes the proof of Lemma 2. □

Lemma 3. *Let γ be a geodesic in M and V a parallel vector field along γ. Let T and W be defined as in* Lemma 2. *Then we have*

$$0 = g(V,T)\gamma'' - g(\dot\gamma,T)V' + g(\dot\gamma,T)dF(W)T \ ,$$

where $'$ denotes the standard covariant differentiation in $\mathbb{R}^n$.

Proof. By means of Proposition 2, γ satisfies the differential equation

$$0 = \gamma'' + 2dF(\dot\gamma)g(\dot\gamma,T)T - g(\dot\gamma,T)^2(\mathrm{grad}^g\, F)\circ\gamma \ .$$

Next, using again Proposition 2, the differential equation for the parallel vector field V along γ is

$$0 = V' + dF(V)g(\dot\gamma,T)T + dF(\dot\gamma)g(V,T)T - g(\dot\gamma,T)g(V,T)(\mathrm{grad}^g\, F)\circ\gamma \ .$$

We multiply the first equation with $g(V,T)$, the second one with $g(\dot\gamma,T)$, and then subtract the resulting equations from each other, which gives the result. □

We will now prove that rank rigidity holds for curvature-homogeneous semi-symmetric spaces.

Theorem 2. *Let M be an n-dimensional locally irreducible Riemannian manifold whose curvature tensor is that of $\mathbb{R}^{n-2}\times M^2$ at each point, where M^2 is a surface of nonzero constant curvature. Then the rank of M is one.*

Proof. If M is locally homogeneous then M is locally symmetric as already mentioned above. If M is locally inhomogeneous, we may represent M on some dense open subset locally by twisted products of the form as described above. Let γ be a geodesic in such a twisted product and V a parallel Jacobi vector field along γ. Since V is parallel we have in particular $R(V,\dot\gamma)\dot\gamma = 0$. Let W and T be as in Lemma 2. According to Lemma 2, $W(s)$ is tangent to the first factor of M at $\gamma(s)$ and orthogonal to the vector $(b_1,\ldots,b_{n-1})(\gamma_2(s))$ for all s.

On the other hand, a simple calculation shows that $(\operatorname{grad}^g F)\circ\gamma(s)$ is a multiple of

$$(b_1(\gamma_2(s)),\ldots,b_{n-1}(\gamma_2(s)),\lambda(s))$$

with some suitable $\lambda(s)\in\mathbb{R}$. This implies that

$$dF(W) = 0\ ,$$

and from Lemma 3 we therefore see that V satisfies the differential equation

$$g(\dot\gamma,T)V' = g(V,T)\gamma''\ .$$

Since the first factor is totally geodesic in M everywhere we see that $g(\dot\gamma,T)(s) = 0$ for some s if and only if $g(\dot\gamma,T)(s) = 0$ for all s. We now assume that γ is a geodesic which is not tangent to the first factor somewhere, and hence everywhere. Then $g(\dot\gamma,T)\neq 0$ everywhere and we get

$$V' = \frac{g(V,T)}{g(\dot\gamma,T)}\gamma''\ .$$

This is a first order equation and we easily see that all solutions are given by $V(t) = \rho\gamma'(t)$ for some constant $\rho\in\mathbb{R}$. This shows that the rank of the geodesic γ is one, and we conclude that M has rank one. $\square$

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