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COBORDISM GROUPS OF IMMERSIONS

FUICHI UCHIDA

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1. Introduction

In the previous paper [3] we have introduced the cobordism groups \( I(n, k) \), \( E(n, k) \) and \( G(n, k) \) of immersions, embeddings and generic immersions of \( n \)-manifolds into \((n+k)\)-manifolds respectively. Mainly we have considered the cobordism group \( G(n, k) \), and we have obtained the following exact sequence:

\[
0 \to E(n, k) \to G(n, k) \to B(n-k, k) \to 0
\]

where the group \( B(n, k) \) is the cobordism group of bundles over manifolds with involution defined in [3].

In general \( E(n, k) \) is isomorphic to the bordism group \( \mathfrak{n}_{n+k}(MO(k)) \), so we have studied the cobordism group \( B(n, k) \) in [3], [4], in order to study the cobordism group \( I(n, k) \), since \( I(n, k) \) is canonically isomorphic to \( G(n, k) \) in the meta-stable range (i.e. \( 2k>n+1 \)). Especially, as one of the consequence of the previous paper, the forgetting homomorphism

\[
\alpha_* : E(n, k) \to I(n, k)
\]

is injective in the meta-stable range.

One of the results of the present paper is that the homomorphism

\[
\alpha_* : E(n, k) \to I(n, k)
\]

is injective without restriction of the meta-stable range.

First we will introduce the notion "completely regular \((p)\)-immersion" and study some properties of completely regular \((p)\)-immersions in sections 2, 3. We will define the cobordism group \( C(n, k; p) \) of completely regular \((p)\)-immersions in section 4, and we will show, in section 5, that the forgetting homomorphism

\[
\alpha_p : C(n, k; p) \to C(n, k; p+1)
\]

is injective. Therefore the homomorphism

\[
\alpha_* : E(n, k) \to I(n, k)
\]
is injective.

Next, in section 6, we will study the oriented cobordism groups of completely regular \((p)\)-immersions. In the final section 7, we will return to the unoriented case and study the cokernel of the homomorphism \(\alpha_p\).

2. Definitions and notations

Let \(M, N\) be \(C^\infty\)-differentiable manifolds.

2.1. First we will give some definitions.

**DEFINITION 2.1.** A \(C^\infty\)-differentiable mapping \(f: M \to N\) is proper if \(\partial N - f(\partial M) = \partial M\) and there are collar neighborhoods 
\[ c : \partial M \times [0, 1) \to M \]
\[ c' : \partial N \times [0, 1) \to N \]
for which the following diagram is commutative:

\[
\begin{array}{ccc}
\partial M \times [0, 1) & \xrightarrow{c} & M \\
\downarrow (f|\partial M) \times 1 & & \downarrow f \\
\partial N \times [0, 1) & \xrightarrow{c'} & N \\
\end{array}
\]

**DEFINITION 2.2.** Subspaces \(V_1, V_2, \ldots, V_p\) of a vector space \(V\) are in general position if

\[
\dim (V_{i_1} \cap \cdots \cap V_{i_k}) = \sum_{i=1}^k \dim (V_{i_r}) - (k-1) \dim V
\]

for \(1 \leq i_1 < i_2 < \cdots < i_k \leq p\).

**DEFINITION 2.3.** A proper immersion \(f: M \to N\) is completely regular if the subspaces \(df(M_{x_1}), \ldots, df(M_{x_p})\) of \(N_x\) are in general position for \(x_1, \ldots, x_p\) in \(M\) such that \(y = f(x_1) = \cdots = f(x_p)\), where \(M_x\) is a tangent space of \(M\) at \(x\), and \(df: M_x \to N_{f(x)}\) is a differential of \(f\).

**DEFINITION 2.4.** A completely regular immersion \(f: M \to N\) is completely regular \((p)\)-immersion if \(f^{-1}(f(x))\) has at most \(p\)-elements for any point \(x\) in \(M\).

**Lemma 2.5.** Let \(V_1, \ldots, V_p\) be subspaces of a vector space \(V\), then the following conditions are equivalent:

(a) \(V_1, \ldots, V_p\) are in general position,

(b) \(V_{i_0} + (V_{i_1} \cap \cdots \cap V_{i_k}) = V\) for mutually distinct indices \(i_0, i_1, \ldots, i_k\),

(c) orthogonal complements of \(V_1, \ldots, V_p\) are linearly independent for some
inner product of $V$.

2.2. Next we will fix some notations.

$E(M, N)$: the set of all proper embeddings from $M$ to $N$,
$I(M, N)$: the set of all proper immersions from $M$ to $N$,
$CR(M, N)$: the set of all proper completely regular immersions from $M$ to $N$,
$CR_{c,p}(M, N)$: the set of all proper completely regular ($p$)-immersions from $M$ to $N$,
$C^\omega(M, N)$: the set of all proper $C^\omega$-differentiable mappings from $M$ to $N$.

Then

$E(M, N) = CR_{c,1}(M, N) \subset CR_{c,2}(M, N) \subset \cdots \subset CR_{c,p}(M, N) \subset CR_{c,p+1}(M, N) \subset \cdots$,
$CR(M, N) = \bigsqcup_{p \geq 1} CR_{c,p}(M, N)$, $CR(M, N) \subset I(M, N) \subset C^\omega(M, N)$,
$CR_{c,p}(M, N) = CR(M, N)$ for $p(\dim N - \dim M) > \dim M$,
$CR(M, N) = I(M, N)$ for $\dim M = \dim N$.

3. Completely regular immersions

Let $X$ be a set. Denote by $X^{(p)}$ the $p$-fold cartesian product of $X$, $\Delta^p X$ the diagonal set of $X^{(p)}$ and

$\Delta_{c,p} X = \{(x_1, \ldots, x_p) \in X^{(p)} | x_i = x_j \text{ for some } i < j \}$.

Let $f: X \to Y$ be a mapping. Denote by

$f^{(p)}: X^{(p)} \to Y^{(p)}$

the mapping defined by $f^{(p)}(x_1, \ldots, x_p) = (f(x_1), \ldots, f(x_p))$.

Then we have the following result from the definition of the transverse regularity condition.

**Lemma 3.1.** Let $M, N$ be $C^\omega$-differentiable manifolds without boundary and $f:M \to N$ be a $C^\omega$-differentiable mapping. Then the $C^\omega$-differentiable mapping $f^{(p)}: M^{(p)} - \Delta_{c,p} M \to N^{(p)}$ is transverse regular over the diagonal $\Delta p N$, if and only if

$$df_{x_1} + (df_{x_2} \cap \cdots \cap df_{x_p}) = N_y$$

for $(x_1, \ldots, x_p) \in M^{(p)} - \Delta_{c,p} M$ such that $y = f(x_1) = \cdots = f(x_p)$.

As a corollary of this lemma we have the following result.

**Theorem 3.2.** Let $M, N$ be $C^\omega$-differentiable manifolds without boundary and $f:M \to N$ be an immersion. Then
(a) \( f \) is completely regular if and only if \( f^{(p)} : M^{(p)} - \Delta_{(p)} M \to N^{(p)} \) is transverse regular over the diagonal \( \Delta^p N \) for all \( p \geq 2 \),

(b) \( f \) is completely regular \((p)\)-immersion if and only if \( f^{(k)} : M^{(k)} - \Delta_{(k)} M \to N^{(k)} \) is transverse regular over the diagonal \( \Delta^p N \) for \( 2 \leq k \leq p \) and \( f^{(p+1)} (M^{(p+1)} - \Delta_{(p+1)} M) \) does not meet the diagonal \( \Delta^{p+1} N \).

**Corollary 3.3.** \( CR(M, N) \) and \( CR_{<p}(M, N) \) are open subsets of \( I(M, N) \) with respect to the fine \( C^1 \)-topology.

**Corollary 3.4.** Let \( f : M \to N \) be a completely regular \((p)\)-immersion and \( X = \{ x \in M \mid f^{-1}(f(x)) \text{ has just } p \text{-elements} \} \). Then \( X \) is a closed submanifold of \( M \) with dimension \( \dim N - p(\dim N - \dim M) \).

**Theorem 3.5.** Let \( M \) be a compact \( C^\infty \)-differentiable manifold and \( N \) a \( C^\infty \)-differentiable manifold. Then

(a) the set \( CR(M, N) \) is a dense open subset of \( I(M, N) \) with respect to the fine \( C^1 \)-topology,

(b) let \( A \) be a closed subset of \( M \) and \( f : M \to N \) be an immersion, if the restriction of \( f \) over \( A \) is completely regular and \( f(A) \) does not intersect \( f(M - A) \), then, as an arbitrarily closed \( C^1 \)-approximation of \( f \), there is a completely regular immersion \( g : M \to N \) such that \( g = f \) on \( A \).

Proof. This is an immediate corollary of Theorem 3.2 and the generalized transversality theorem (Theorem 1.10[1]).

**Corollary 3.6.** Any immersion \( f : M \to N \) is differentiably homotopic to a completely regular immersion.

Proof. This follows from Theorem 3.5 and the fact that \( I(M, N) \) is locally contractible with respect to the fine \( C^1 \)-topology (cf. [2]).

### 4. Cobordism of immersions

#### 4.1. A completely regular \((p)\)-immersion of dimension \((n, k)\) is a triple \((f, M, N)\) consisting of two closed \( C^\infty \)-differentiable manifolds \( M, N \) of dimensions \( n, n+k \) respectively and a completely regular \((p)\)-immersion \( f : M \to N \). We identify \((f, M, N)\) with \((f', M', N')\) if and only if there are diffeomorphisms \( \varphi : M \to M' \) and \( \psi : N \to N' \) for which \( \psi f = f' \varphi \).

A completely regular \((p)\)-immersion \((f, M, N)\) of dimension \((n, k)\) will be said to be **cobordant to zero** if there exists a triple \((F, V, W)\) where:

1. \( V \) and \( W \) are compact \( C^\infty \)-differentiable manifolds of dimensions \( n+1, n+k+1 \) respectively, and
2. \( F : V \to W \) is a proper completely regular \((p)\)-immersion such that \( (F|\partial V, \partial V, \partial W) = (f, M, N) \).
Then we denote $\partial (F, V, W) = (f, M, N)$. Two completely regular $(p)$-immersions $(f_0, M_0, N_0)$ and $(f_1, M_1, N_1)$ of dimension $(n, k)$ will be said to be cobordant if and only if the disjoint union $(f_0, N_0, M_0) + (f_1, M_1, N_1)$ is cobordant to zero.

This cobordism relation is an equivalence relation and denote by $C(n, k; p)$ the set of equivalence classes under this relation of completely regular $(p)$-immersions of dimension $(n, k)$. As usual, an abelian group structure is imposed on $C(n, k; p)$ by disjoint union, which is called the cobordism group of completely regular $(p)$-immersions of dimension $(n, k)$, and every element of $C(n, k; p)$ is its own inverse.

4.2. In the above definition, if the term “completely regular $(p)$-immersion” is replaced by “embedding”, “immersion” and “completely regular immersion”, one may define the cobordism groups of embeddings $E(n, k)$, immersions $I(n, k)$ and completely regular immersions $C(n, k)$, of dimension $(n, k)$ respectively.

By definition $E(n, k) = C(n, k; 1)$ and there are natural forgetting homomorphisms

$$
\alpha_p : C(n, k; p) \to C(n, k; p+1)
$$

$$
\alpha_p : C(n, k; p) \to I(n, k)
$$

$$
\alpha_* : E(n, k) \to I(n, k)
$$

such that $\alpha_* = \alpha_p \circ \alpha_{p-1} \circ \cdots \circ \alpha_2 \circ \alpha_1$.

Remark. (a) $\alpha_p$ is an isomorphism for $kp > n+1$,

(b) $\alpha_1$ is injective, since $C(n, k; 2) = G(n, k)$ the cobordism group of generic immersions defined in the previous paper ([3], Section 4).

In the next section, we will prove that the homomorphism $\alpha_p$ is injective for all $p \geq 1$.

5. Splitting homomorphisms

5.1. Let $f : M \to N$ be a proper completely regular $(p)$-immersion where $M, N$ are compact $C^\infty$-differentiable manifolds of dimensions $n, n+k$ respectively. Let

$$
X = \{ x \in M \mid f^{-1}(f(x)) \text{ has just } p \text{-elements} \}
$$

and $Y = f(X)$. Then $X$ and $Y$ are closed submanifolds of $M, N$ respectively, and $\dim X = \dim Y = n - (p-1)k$. Moreover $f \mid X : X \to Y$ is a $p$-fold covering. Then

Lemma 5.1. There are Riemannian metrics on $M, N$ such that
(a) the differential \( df : M \rightarrow N \) is isometric for \( x \in X \),
(b) the orthogonal complements of \( df_{M_x}, \ldots, df_{M_p} \) in \( N_y \) are mutually orthogonal if \( y = f(x) = \cdots = f(x_p) \).

Proof. Firstly, define a Riemannian metric satisfying (b) on a neighborhood in \( N \) of each point of \( Y \). Define a Riemannian metric on \( N \) satisfying the condition (b) by making use of a \( C^\infty \)-differentiable partition of unity on \( N \). Next, define a Riemannian metric on \( M \) as the induced metric by \( df \).

5.2. Let \( \nu(X) \) and \( \nu(Y) \) be the normal bundles of the embeddings \( X \subset M, Y \subset N \) respectively, with respect to the above Riemannian metrics. Denote by \( E(\nu(X)), E(\nu(Y)) \) the total spaces of these normal bundles, and \( E_\epsilon(\nu(X)), E_\epsilon(\nu(Y)) \) the set of all normal vectors with length \( \leq \epsilon \).

Then the differential \( df \) maps \( E(\nu(X)) \) into \( E(\nu(Y)) \) and the following diagram is commutative:

\[
\begin{array}{ccc}
E(\nu(X)) & \xrightarrow{df} & E(\nu(Y)) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f \mid X} & Y \\
\end{array}
\]

where the vertical mappings are bundle projections. Then we have the following result by an elementary method.

Lemma 5.2. There is a differentiably homotopic approximation \( g \) of \( f \) such that
(a) \( g = f \) on \( X \),
(b) \( dg = df \) on \( E(\nu(X)) \),
(c) the following diagram is commutative for some \( \epsilon > 0 \):

\[
\begin{array}{ccc}
E_\epsilon(\nu(X)) & \xrightarrow{dg} & E_\epsilon(\nu(Y)) \\
\downarrow \exp & & \downarrow \exp \\
M & \xrightarrow{g} & N \\
\end{array}
\]

where \( \exp \) is the exponential mapping.

5.3. Under these notations, let
\[
\begin{align*}
M_0 &= M - \exp (\text{int } E_\epsilon(\nu(X))), \\
N_0 &= N - \exp (\text{int } E_\epsilon(\nu(Y))), \\
\partial_1 M_0 &= \exp (\partial E_\epsilon(\nu(X))), \\
\partial_1 N_0 &= \exp (\partial E_\epsilon(\nu(Y))),
\end{align*}
\]

Then \( \partial_1 M_0 \) and \( \partial_1 N_0 \) have fixed point free \( C^\infty \)-differentiable involutions \( a, b \).
induced from the bundle involutions of $E(\nu(X)), E(\nu(Y))$ respectively, and the following diagram is commutative:

$$
\begin{array}{ccc}
\partial_1 M_0 & \xrightarrow{g|\partial_1 M_0} & \partial_1 N_0 \\
\downarrow a & & \downarrow b \\
\partial_1 M_0 & \xrightarrow{g|\partial_1 M_0} & \partial_1 N_0
\end{array}
$$

Let $M_1$ be the quotient space of $M_0$ by the relation $x = a(x)$ for $x \in \partial_1 M_0$. Let $N_1$ be the quotient space of $N_0$ by the relation $y = b(y)$ for $y \in \partial_1 N_0$. Then $M_1, N_1$ have naturally $C^\infty$-differentiable structures and a mapping $g_1: M_1 \to N_1$ is induced from the mapping $g|M_0: M_0 \to N_0$ which is proper completely regular $(p-1)$-immersion.

**Theorem 5.3.** There is a homomorphism.

$$
\gamma_p : C(n, k; p) \to C(n, k; p+1)
$$

such that $\gamma_{p+1} \circ \alpha_p = \text{identity}$.

**Proof.** By the above notations, the correspondence from $(f, M, N)$ to $(g, M_1, N_1)$ is cobordism invariant, and this defines a desired homomorphism.

**Corollary 5.4.** The homomorphism $\alpha_p : C(n, k; p) \to C(n, k; p+1)$ is injective for all $p \geq 1$ and the image of $\alpha_p$ is a direct summand.

**Corollary 5.5.** The homomorphism $\alpha_\ast: E(n, k) \to I(n, k)$ is injective.

**Proof.** If $k > 0$, then this follows from the above corollary. If $k = 0$, then this follows directly from the definition of $E(n, k)$ and $I(n, k)$.

### 6. Oriented cobordism of immersions

6.1. By similar argument to the unoriented case one may define the oriented cobordism groups of completely regular $(p)$-immersions $C^\circ(n, k; p)$, embeddings $E^\circ(n, k)$, immersions $I^\circ(n, k)$ and completely regular immersions $C^\circ(n, k)$, of dimension $(n, k)$ respectively, where we consider orientation preserving mappings if $k = 0$.

By definition $E^\circ(n, k) = C^\circ(n, k; 1)$ and there are forgetting homomorphisms

$$
\begin{align*}
\alpha_p^0 : & \ C^\circ(n, k; p) \to C^\circ(n, k; p+1) \\
\alpha_p^0 : & \ C^\circ(n, k; p) \to I^\circ(n, k) \\
\alpha_\ast^0 : & \ E^\circ(n, k) \to I^\circ(n, k)
\end{align*}
$$

such that $\alpha_\ast^0 = \alpha_p^0 \circ \alpha_{p-1}^0 \circ \cdots \circ \alpha_2^0 \circ \alpha_1^0$. 

REMARK. (a) \( \alpha^p_0 \) is an isomorphism for \( kp > n+1 \),
(b) \( C^0(n, k; 2) = G^0(n, k) \) the oriented cobordism group of generic immersions defined in the previous paper ([3], Section 10).

We could not find such a homomorphism as \( \gamma_p \), so we do not know in general whether the homomorphisms \( \alpha^p_0 \) and \( \alpha^k_0 \) are injective or not. In the following we give some partial results.

6.2. First, we consider the case of low codimensions. Let \( s \) be a point of \( k \)-sphere \( S^k \). Let \( M, N \) be oriented closed \( C^\infty \)-differentiable manifolds of dimensions \( n, n+k \) respectively, and define a mapping

\[
f : M \to N + M \times S^k
\]

by \( f(x) = (x, s) \), then \( f \) is an embedding. The function

\[
\iota : \Omega_{n+k} \to \mathbb{E}^0(n, k)
\]

defined by \( \iota([M], [N]) = [f, M, N + M \times S^k] \) is a well-defined homomorphism.

**Lemma 6.1.** The homomorphism

\[
\alpha^0_{\#} \circ \iota : \Omega_{n+k} \to \mathbb{I}^0(n, k)
\]
is injective and the image of \( \iota \) is a direct summand of \( \mathbb{E}^0(n, k) \).

**Proof.** Let \( \pi : \mathbb{I}^0(n, k) \to \Omega_{n+k} \) be a homomorphism defined by \( \pi([f, M, N]) = ([M], [N]) \), then \( \pi \circ \alpha^0_{\#} \circ \iota = \text{identity} \). Therefore we have the desired result.

**Proposition 6.2.** The homomorphism

\[
\alpha^0_\#: \mathbb{E}^0(n, k) \to \mathbb{I}^0(n, k)
\]
is injective for \( k = 0 \) and \( k = 1 \).

**Proof.** In general \( \mathbb{E}^0(n, k) \) is isomorphic to \( \Omega_{n+k}(MSO(k)) \). If \( k = 1 \), then \( MSO(1) \) is homotopy equivalent to the circle and hence \( \Omega_{*}(MSO(1)) \) is isomorphic to the tensor product \( H_*(MSO(1); Z) \otimes \Omega_* \). Therefore \( \mathbb{E}^0(n, 1) \) is isomorphic to \( \Omega_* \otimes \Omega_{*+1} \), and \( \alpha^0_{\#} \) is injective by Lemma 6.1. If \( k = 0 \), then the results follows directly from the definition of \( \mathbb{E}^0(n, 0) \) and \( \mathbb{I}^0(n, 0) \).

6.3. If \( k > n+1 \), then the homomorphism

\[
\alpha^0_\#: \mathbb{E}^0(n, k) \to \mathbb{I}^0(n, k)
\]
is isomorphic by Remark in 6.1. We consider the case \( k = n+1 \) and \( k = n \).

**Proposition 6.3.**
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(a) $\alpha_\ast^0 : E^n(n, n+1) \to I^n(n, n+1)$ is an isomorphism,
(b) $\alpha_\ast^0 : E^n(n, n) \to I^n(n, n)$ is injective.

Proof. Since the homomorphism

$$\alpha_\ast^0 : C^n(n, k; 2) \to I^n(n, k)$$

is an isomorphism in both case except $n=k=1$, and $E^n(1, 1)=0$, it is sufficient to consider the homomorphism

$$\alpha_0^0 : E^n(n, k) \to C^n(n, k; 2)=G^n(n, k).$$

There are exact sequences [3]:

$$G^n(2s-1, 2s-1) \to B^+(0, 2s-1) \to E^n(2s-2, 2s-1) \to G^n(2s-2, 2s-1) \to 0,$$

$$G^n(2s-2, 2s) \to B^+(0, 2s) \to E^n(2s-1, 2s) \to G^n(2s-1, 2s) \to 0.$$

Now define a mapping

$$f : S^n_1 + S^n_2 \to S^n \times S^n$$

by $f(x)=(x, s)$ for $x \in S^n_1$ and $f(y)=(s, y)$ for $y \in S^n_2$, where $S^n_1$ is a copy of $S^n$ and $s \in S^n$ is a base point. Then $f$ is a completely regular $(2)$-immersion with unique double point and therefore the homomorphisms $\beta$ are onto in the above sequences. Consequently the homomorphism $\alpha_\ast^0$ is an isomorphism.

Next, since $B^-(1, 2s-1)=0$ in the exact sequence

$$B^-(1, 2s-1) \to E^n(2s-1, 2s-1) \to \alpha_\ast^0 G^n(2s-1, 2s-1) \to B^-(0, 2s-1) \to 0,$$

the homomorphism $\alpha_\ast^0 : E^n(2s-1, 2s-1) \to G^n(2s-1, 2s-1)$ is injective and not onto since $B^-(0, 2s-1)=Z_2$.

Lastly, $B^+(1, 2s)=Z_2$ in the following exact sequence

$$G^n(2s+1, 2s) \to B^+(1, 2s) \to E^n(2s, 2s) \to \alpha_\ast^0 G^n(2s, 2s).$$

We will prove that $\beta$ is onto, and it is sufficient to show the existence of a completely regular $(2)$-immersion $(f, M, N)$ of dimension $(2s+1, 2s)$ such that

$$X_f = \{x \in M \mid f^{-1}(f(x)) \text{ has 2-elements}\}$$

is diffeomorphic to the circle $S^1$. Let $CP^n$ be the complex projective space and

$$f : S^1 \times CP^n \to S^1 \times CP^{2n}$$

a mapping defined by
\[ f(e^{\theta}, \langle z_0, z_1, \cdots, z_s \rangle) = (e^{\theta}, \langle z_0 e^{\theta}, z_1 \cos \theta, \cdots, z_s \cos \theta, z_1 \sin \theta, \cdots, z_s \sin \theta \rangle) \]

where \( \langle z_0, z_1, \cdots, z_s \rangle \) is a homogeneous coordinate of \( \mathbb{C}P^s \). Then \( f \) is a completely regular \((2)\)-immersion and

\[ X_f = S^1 \times \{ \langle 1, 0, \cdots, 0 \rangle \} \]

**Remark.** By direct calculation, if \( n+k \leq 7 \) but \( (n, k) \neq (4, 2) \), then the homomorphism

\[ \alpha^0_\#: E^0(n, k) \to I^0(n, k) \]

is injective, and the homomorphism

\[ \alpha^0_\#: E^0(4, 2) \to C^0(4, 2; 2) = G^0(4, 2) \]

is injective.

### 7. Bundles over covering spaces

**7.1.** Now we return to the unoriented case. The homomorphism

\[ \alpha_\#: C(n, k; p) \to C(n, k; p+1) \]

is injective and the image of \( \alpha_\# \) is a direct summand by Corollary 5.4, so we study now the cokernel of \( \alpha_\# \). For this purpose, we introduce new cobordism groups as follows.

**7.2.** Let \( k, p \) be fixed non-negative integers. A pair of bundles over a covering space is a quadruple \((\xi, \eta, h, \bar{h})\), where

\[ \xi : E(\xi) \to B(\xi), \quad \eta : E(\eta) \to B(\eta) \]

are \( C^\infty \)-differentiable vector bundles over compact \( C^\infty \)-differentiable manifolds with fibre dimensions \( pk, (p+1)k \) respectively,

\[ \bar{h} : B(\xi) \to B(\eta) \]

is a \((p+1)\)-fold covering which is a proper \( C^\infty \)-differentiable mapping, and

\[ h : E(\xi) \to E(\eta) \]

is a \( C^\infty \)-differentiable mapping covering \( \bar{h} \). The following must be satisfied:

1. \( h \) maps each fibre \( \xi_x \) over \( x \in B(\xi) \) linearly one to one into a fibre \( \eta_{\bar{h}(x)} \),
2. for each \( y \) in \( B(\eta) \) and \( x_0, x_1, \cdots, x_p \) in \( B(\xi) \) such that \( y = \bar{h}(x_0) = \bar{h}(x_1) = \cdots = \bar{h}(x_p) \), subspaces \( h(\xi_{x_0}), \cdots, h(\xi_{x_p}) \) of a vector space \( \eta_y \) are in general position.
7.3. A quadruple \((\xi, \eta, h, \bar{h})\) is identified with a quadruple \((\xi', \eta', h', \bar{h}')\) if and only if there is a quadruple \((a, \bar{a}, b, \bar{b})\) of \(C^\infty\)-diffeomorphisms

\[
a : E(\xi) \to E(\xi'), \quad \bar{a} : B(\xi) \to B(\xi'), \\
b : E(\eta) \to E(\eta'), \quad \bar{b} : B(\eta) \to B(\eta'),
\]
such that \(\bar{b} \circ h = h' \circ a\) and \(a, b\) are bundle mappings covering \(\bar{a}, \bar{b}\) respectively.

For a quadruple \((\xi, \eta, h, \bar{h})\), denote by \(\partial(\xi, \eta, h, \bar{h})\) a quadruplet consisting of the restrictions

\[
\xi|\xi^{-1}(\partial B(\xi)) : \xi^{-1}(\partial B(\xi)) \to \partial B(\xi), \\
\eta|\eta^{-1}(\partial B(\eta)) : \eta^{-1}(\partial B(\eta)) \to \partial B(\eta), \\
h|\xi^{-1}(\partial B(\xi)) : \xi^{-1}(\partial B(\xi)) \to \eta^{-1}(\partial B(\eta)), \\
\bar{h}|\partial B(\bar{h}) : \partial B(\bar{h}) \to \partial B(\bar{h}).
\]

7.4. The cobordism group \(B(n, k; p)\) of pairs of bundles over a covering spaces of \(n\)-manifold may be now defined. If \(B(\xi_o)\) and \(B(\xi_i)\) are closed \(n\)-manifolds, then a quadruple \((\xi_o, \eta_o, h_o, \bar{h}_o)\) is cobordant to a quadruple \((\xi_i, \eta_i, h_i, \bar{h}_i)\) if and only if there is a quadruple \((f, \gamma, \Lambda, h)\) such that

\[
\partial(\xi, \eta, h, \bar{h}) = (\xi_o, \eta_o, h_o, \bar{h}_o) + (\xi_i, \eta_i, h_i, \bar{h}_i)
\]

where the symbol \(+\) denotes disjoint union. Then this cobordism relation is an equivalence relation. Denote by \(B(n, k; p)\) the set of all cobordism classes. As usual an abelian group structure is imposed on \(B(n, k; p)\) by disjoint union, then every element is its own inverse.

**Remark.** \(B(n, k; 1)\) is naturally isomorphic with the cobordism group \(B(n, k)\) of bundles over manifolds with involution defined in the previous paper ([3], Section 3).

7.5. Now we define homomorphisms

\[
\beta_p : C(n, k; p+1) \to B(n-pk, k; p), \\
\pi_p : B(n, k; p) \to C(n+pk, k; p+1).
\]

(7.5.1) Let \(a \in C(n, k; p+1)\) be represented by a completely regular \((p+1)\)-immersion \(f : M \to N\). Let

\[
X = \{x \in M \mid f^{-1}(f(x)) \text{ has just } (p+1)\text{-elements}\}
\]

and \(Y = f(X)\). Then there are Riemannian metrics on \(M, N\) satisfying the conditions of Lemma 5.1. Let \(\nu(X)\), \(\nu(Y)\) be the normal bundles of the embeddings \(X \subseteq M\), \(Y \subseteq N\) respectively, with respect to these Riemannian metrics, and then the differential \(df\) maps \(E(\nu(X))\) into \(E(\nu(Y))\). Define \(\beta_p(a)\) the cobordism class.
of the quadruple \((\nu(X), \nu(Y), df | E(\nu(X)), f | X)\).

(7.5.2.) Let \(b \in B(n, k; p)\) be represented by a quadruple \((\xi, \eta, h, \bar{h})\). Define \(\pi_p(b)\) the cobordism class of a completely regular \((p+1)\)-immersion defined by the mapping

\[ P(h \oplus 1) : P(\xi \oplus \theta^\ell) \to P(\eta \oplus \theta^\ell) \]

where \(\theta^\ell\) is the trivial line bundle, \(P(\xi \oplus \theta^\ell)\) and \(P(\eta \oplus \theta^\ell)\) are the total spaces of the associated projective space bundles, and \(P(h \oplus 1)\) is a mapping canonically induced from the mapping \(h\).

**Theorem 7.1.** There is an exact sequence:

\[ 0 \to C(n, k; p) \xrightarrow{\alpha_p} C(n, k; p+1) \xrightarrow{\beta_p} B(n-pk, k; p) \to 0. \]

Proof. The homomorphism \(\alpha_p\) is injective by Corollary 5.4, and the homomorphism \(\beta_p\) is surjective since \(\beta_p \circ \pi_p = \text{identity}\) by definition (cf. [3] Theorem A'). The exactness at \(C(n, k; p+1)\) is proved by the handle attaching construction (cf. [3], Section 5), so we omit the details.

**References**


