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NECESSARY CONDITIONS FOR THE WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR HYPERBOLIC SYSTEMS

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1. Introduction

In the present paper we study necessary conditions for the well-posedness of the Cauchy problem for hyperbolic systems of arbitrary order with multiple characteristics.

In the scalar case the seminal paper of Ivrii and Petkov [6] has shown that the correctness of the Cauchy problem implies that for a given hyperbolic differential operator, near a multiple characteristic point, a set of vanishing conditions on the homogeneous parts of the lower order terms must be satisfied.

Evidently in the case of hyperbolic differential systems the situation is more complex, the vector structure playing a relevant role. As a consequence the above mentioned result may not be true any more as we shall see in some examples in the second section of the paper.

Before stating our main result we would like to recall, as a motivation, some results in the case of hyperbolic systems already existing in the literature.

Let us introduce the notation and the definition of well-posedness for a differential equation (system of differential equations). We work in an open subset Ω of $\mathbb{R}^{n+1}$, with coordinates $x = (x_0, x_1, \dots, x_n) = (x_0, x')$, and assume that the origin belongs to Ω; let $\Omega^t = \{x \in \Omega \mid x_0 < t\}$, $\Omega_t = \{x \in \Omega \mid x_0 > t\}$.

DEFINITION 1.1. The Cauchy problem for a differential system $P(x, D)$ is said to be well-posed in Ω^t (Ω_t, respectively) if
i) For every $f \in (C_0^\infty(\Omega))^N$ there is a $u \in (\mathcal{E}'(\Omega))^N$ such that $P(x, D)u = f$ in Ω^t (Ω_t, respectively). Here N denotes the size of the system.
ii) For every $u \in \left(\mathcal{E}'(\Omega)\right)^N$ with $P(x, D)u = 0$ in Ω^t (Ω_t, respectively) we have $u = 0$ in Ω^t (Ω_t).

In [7] Nishitani proved that if the system has real analytic coefficients defined in Ω and the Cauchy problem is well-posed in Ω^t and Ω_t for every t small independently of the lower order terms (i.e. P is strongly hyperbolic) then the cofactor matrix of the principal symbol of P vanishes of order $r - 2$ at a characteristic point of multiplicity

r. In [7] as well as in the other papers quoted here P is assumed to be a first order differential system.

On the other hand in [8] it has been proved under some conditions on h, where $h = \det P_1$ is the (operator whose principal symbol is the) determinant of the principal symbol of P, that if there exists a pseudodifferential operator $M(x, D)$, of order $N-1$, whose principal symbol is the cofactor matrix of P_1, such that

$$PM = hI_N + H_{N-1} + \cdots + H_{N-j} + \cdots$$

where $H_{N-j}(x, \xi)$ vanishes at a characteristic point of multiplicity r of order $r - 2j$, then the Cauchy problem for P is well-posed.

For the non-strongly hyperbolic case, in [2] it is supposed that the rank of the principal symbol of P at a given characteristic point is maximal, i.e. $N-1$. Then it is proved that a certain differential polynomial of order N must satisfy the well known Ivrii–Petkov conditions if the Cauchy problem for P is correctly posed. In [1] a more precise Levi condition is found for a more particular case.

We point out that the rank of the principal symbol of P at a characteristic point plays a crucial role and that its maximality allows everything to come down to the scalar case.

Our purpose here is to study a truly vectorial case and, when possible, to drop the assumption about the rank of the principal symbol. Moreover, from what has been said above, it is clear that there is a strong connection between first order systems and higher order systems possibly of reduced size.

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2. Notation and Statement of the Result

Let m, r, N be positive integers, $r \leq m$ and Ω an open subset of $\mathbb{R}^{n+1}$ containing the origin. We denote by $P(x, D)$ a differential operator of order m with coefficients in $C^\infty(\Omega, M_N(\mathbb{C}))$, the space of all $N \times N$ matrices depending smoothly on the variable $x \in \Omega$. We shall also write

$$P(x, D) = P_m(x, D) + P_{m-1}(x, D) + \cdots + P_0(x), \tag{2.1}$$

where $P_{m-j}(x, D)$ is the homogeneous part of order $m - j$. We shall always assume that

$$P \quad \text{is hyperbolic} \tag{2.2}$$

i.e. the polynomial $h(x, \xi_0, \xi') = \det P_m(x, \xi_0, \xi')$ has only real roots with respect to the variable ξ_0.

We are interested in the well-posedness of the Cauchy problem for P in the sense of Definition 1.1. Without loss of generality we may think that the hyperplane $\{x_0 = 0\}$ is non characteristic for P, i.e. it is non characteristic for h. The following definition will be useful in the sequel:

DEFINITION 2.1. Let $f(x,\xi)$ be a smooth function defined on $T^*\Omega\setminus 0$ and assume that at a point $\rho \in T^*\Omega\setminus 0$, f vanishes of finite order. Then by f_ρ we denote the first non vanishing term in the Taylor expansion of f around ρ;

$$f(\rho + \lambda^{-1}\delta z) = \lambda^{-s_f}\left[f_\rho(\delta z) + O(\lambda^{-1})\right]$$

so that f_ρ is a homogeneous polynomial of degree s_f, where λ is a large positive parameter and $\delta z \in T^*\mathbb{R}^{n+1}$.

We are now ready to state our result. Let $\rho \in T^*\Omega\setminus 0$ be a characteristic point of P_m of multiplicity r. For sake of simplicity we shall assume that $\rho = (0, e_n)$. Then we have

Theorem 2.1. *Let P be as in* (2.1) *and $\rho = (0, e_n)$ be a characteristic point of P_m such that $d^j P_m(0, e_n) = 0$ for $j = 0, 1, \dots, r-1$. Denote by s_j the degree of the homogeneous polynomial $P_{m-j,(0,e_n)}$, $j = 1, \dots, [r/2]$. Assume that*
i) *There exists at least one $j \in \{1, \dots, [r/2]\}$ for which $s_j < r - 2j$. Define then*

$$\theta_0 = \min_{\substack{j\in\{1,\dots,[r/2]\}\\ s_j<r-2j}} \frac{j}{r-s_j}, \qquad \hat{P}(x,\xi) = \sum_{\substack{m-r\theta_0=m-j-s_j\theta_0\\ s_j<r \text{ when } j>0}} P_{m-j,(0,e_n)}(x,\xi). \tag{2.3}$$

ii) *The polynomial $\xi_0 \longmapsto \det \hat{P}(x,\xi_0,\xi')$ has non real roots at least one of which has multiplicity at most* 3.
Then the Cauchy problem for P is not well-posed.

Let us now look at an example in which Theorem 2.1 can be applied. For the sake of simplicity we consider a case of matrix dimension 6, but this can be easily generalized. Consider the following first order differential system

$$L_1(x,D)u + B(x)u = f, \tag{2.4}$$

where L_1 denotes the first order matrix operator

$$L_1(x,D) = \begin{bmatrix} D_0 - a_1(x_0)D_n & D_n & 0 \\ 0 & D_0 - a_2(x_0)D_n & D_n \\ 0 & 0 & D_0 - a_3(x_0)D_n \end{bmatrix}$$

$$\oplus \begin{bmatrix} D_0 - a_4(x_0)D_n & D_n & 0 \\ 0 & D_0 - a_5(x_0)D_n & D_n \\ 0 & 0 & D_0 - a_6(x_0)D_n \end{bmatrix} \tag{2.5}$$

where we assume $a_i(0) = 0$, $1 \le i \le 6$. Here B is a smooth 6×6 matrix valued function defined in a neighborhood of, say, the origin in the x-variable. Let us look at the characteristic point $(0, e_n)$ and denote by $L_1(x, \xi) = (l_{ij}(x, \xi))$ the symbol of the principal part. Let $M(x, \xi) = (m_{ij}(x, \xi))$ be the cofactor matrix of $L_1(x, \xi)$. Define

$$\begin{aligned} M_4(x, \xi) = & \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(D_{x_0} m_{12})\xi_n^{-1} & -(D_{x_0} m_{13})\xi_n^{-1} \\ 0 & 0 & -(D_{x_0} m_{23})\xi_n^{-1} + l_{22}(D_{x_0} m_{13})\xi_n^{-2} \end{bmatrix} \\ & \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(D_{x_0} m_{45})\xi_n^{-1} & -(D_{x_0} m_{46})\xi_n^{-1} \\ 0 & 0 & -(D_{x_0} m_{56})\xi_n^{-1} + l_{55}(D_{x_0} m_{46})\xi_n^{-2} \end{bmatrix} \end{aligned}$$

and

$$M_3(x, \xi) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (D_{x_0}^2 m_{13})\xi_n^{-2} \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (D_{x_0}^2 m_{46})\xi_n^{-2} \end{bmatrix}.$$

Note that $M_4(x, \xi)$ vanishes of order 2 at $(0, e_n)$. We now study

$$\begin{aligned} P(x, D) &= (L_1(x, D) + B(x))(M(x, D) + M_4(x, D) + M_3(x, D)) \\ &= h(x, D)I_6 + H_5(x, D) + H_4(x, D) + H_3(x, D) \end{aligned}$$

where $H_j(x, \xi)$ is of homogeneous of order j in ξ. It is easy to see that

$$H_5(x, \xi) \equiv B(x)\bar{M}(x, \xi), \quad H_4(x, \xi) \equiv 0$$

modulo terms vanishing of order 4 and 2 at $(0, e_n)$ respectively where

$$\bar{M}(x, \xi) = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with

$$\alpha = \prod_{i=4}^{6} (\xi_0 - a_i(x_0)\xi_n), \quad \beta = \prod_{i=1}^{3} (\xi_0 - a_i(x_0)\xi_n).$$

Then with $\theta_0 = 1/3$ we have

$$\hat{P}(x,\xi) = \hat{h}(x,\xi)I_6 + B(0)\hat{M}(x,\xi)$$

where $\hat{h}(x,\xi) = h_{(0,e_n)}(x,\xi)$ and

$$\hat{M}(x,\xi) = \bar{M}_{(0,e_n)}(x,\xi) = \begin{bmatrix} 0 & 0 & \hat{\alpha} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & \hat{\beta} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with

$$\hat{\alpha} = \prod_{i=4}^{6}(\xi_0 - a_i'(0)x_0), \quad \hat{\beta} = \prod_{i=1}^{3}(\xi_0 - a_i'(0)x_0).$$

Thus it follows that

$$\det \hat{P} = \det(\hat{h}I_6 + B(0)\hat{M}) = \hat{h}^4 \begin{vmatrix} \hat{h} + \hat{\alpha}b_{31}(0) & \hat{\beta}b_{34}(0) \\ \hat{\alpha}b_{61}(0) & \hat{h} + \hat{\beta}b_{64}(0) \end{vmatrix}.$$

Since $\hat{\alpha}\hat{\beta} = \hat{h}$ this gives

$$\det \hat{P} = \hat{h}^5 \begin{vmatrix} \hat{\beta} + b_{31}(0) & b_{34}(0) \\ b_{61}(0) & \hat{\alpha} + b_{64}(0) \end{vmatrix} = \hat{h}^5 \hat{g}.$$

Taking $x_0 = 0$, it is clear that

$$\hat{g} = \xi_0^6 + (\operatorname{tr} K)\xi_0^3 + \det K \tag{2.6}$$

where

$$K = \begin{bmatrix} b_{31}(0) & b_{34}(0) \\ b_{61}(0) & b_{64}(0) \end{bmatrix}.$$

If K is not nilpotent then $\hat{g} = 0$ has a non real root of multiplicity at most 2. Then by Theorem 2.1 the nilpotency of K is necessary in order that the Cauchy problem for P is well posed. Since $M + M_4 + M_3$ is upper triangular and hence the Cauchy problem for $M + M_4 + M_3$ is well posed (microlocally near $(0, e_n)$) the nilpotency of K is also necessary for the well posedness of the Cauchy problem for the original $L_1 + B$.

On the other hand, if the first and the fourth columns of $B(x)$ vanish near the origin and $a_i'(0)$ are different from each other then the Cauchy problem for P, and hence for $L_1 + B$ is well posed which follows from Theorem 1.2 in [8].

3. Inductive Lemmas

The method of proof is just to construct an asymptotic solution to the equation $Pu \equiv 0$ depending on a large parameter λ and violating an a priori estimate (Proposition 3.1 below) which follows from

$$\|u\|_{C^0(K)} \leq C\|Pu\|_{C^p(K)}, \tag{3.1}$$

where $C > 0$ is a suitable constant and $K \Subset \Omega$ is a compact set containing the origin and $u \in \left(C_0^\infty(K)\right)^N$. The estimate (3.1) is deduced if the Cauchy problem for P is well-posed (see e.g. [6] and [3]).

Let now θ_0 be defined by (2.3). Due to assumption i) in the statement of Theorem 2.1, we obtain that

$$\theta_0 < \frac{1}{2}. \tag{3.2}$$

Define

$$\sigma_0 = 1 - 2\theta_0. \tag{3.3}$$

Let us compute

$$\begin{aligned} P_\lambda(x, D) &= \sum_{j=0}^{m} P_{m-j}\left(\lambda^{-\theta_0}x, \lambda e_n + \lambda^{\theta_0}D\right) \\ &= \sum_{j=0}^{m} \lambda^{m-j} P_{m-j}\left(\lambda^{-\theta_0}x, e_n + \lambda^{-\theta_0-\sigma_0}D\right) \\ &= \sum_{j=0}^{m} \sum_{\substack{k\geq 0 \\ m-j-s_j\theta_0-k\theta_0>-M}} \lambda^{m-j-s_j\theta_0-k\theta_0} \\ &\quad \times \sum_{|\alpha+\beta|=s_j+k} \frac{1}{\alpha!\beta!} P^{(\alpha)}_{m-j(\beta)}(0, e_n)\, x^\beta \left(\lambda^{-\sigma_0}D\right)^\alpha + O\left(\lambda^{-M}\right), \end{aligned} \tag{3.4}$$

where M is an arbitrarily large positive integer and by $O(\lambda^{-M})$ we denoted a differential operator whose coefficients are bounded by λ^{-M} on any bounded preassigned open set U in $\mathbb{R}^{n+1}$. Here we note that the a priori estimate (3.1) implies an a priori estimate for P_λ (see Proposition 2.2 in [7]).

Proposition 3.1. *Assume that $0 \in \Omega$ and the Cauchy problem for $P(x, D)$ is well posed in both Ω^t and Ω_t for every small t. Then for every compact set $W \subset \mathbb{R}^{n+1}$ and for every positive $T > 0$ we can find $C > 0$, $\bar{\lambda} > 0$ and $p \in \mathbb{N}$ such that*

$$\|u\|_{C^0(W^t)} \leq C\lambda^{(\theta_0+1)p}\|P_\lambda u\|_{C^p(W^t)},$$

$$\|u\|_{C^0(W_t)} \leq C\lambda^{(\theta_0+1)p}\|P_\lambda u\|_{C^p(W_t)}$$

for every $u \in C_0^\infty(W)$, $\lambda \geq \bar{\lambda}$, $|t| < T$.

We point out explicitly that the sum in the above formula (3.4) is a finite sum since $\theta_0 > 0$. Moreover the terms obtained when $k = 0$ correspond to $P_{m-j,\ (0,e_n)}(x, \lambda^{-\sigma_0}D)$.

Let

$$\tilde{G}^{(0)}(x,\xi;\lambda) = \sum_{j=0}^{m} \sum_{\substack{k\geq 0 \\ m-j-s_j\theta_0-k\theta_0>-M}} \lambda^{-j+(r-s_j)\theta_0-k\theta_0} \times \sum_{|\alpha+\beta|=s_j+k} \frac{1}{\alpha!\beta!} P_{m-j\ \binom{(\alpha)}{(\beta)}}(0,e_n)x^\beta\xi^\alpha \tag{3.5}$$

then it is clear that

$$P_\lambda(x,D) = \lambda^{m-r\theta_0}\tilde{G}^{(0)}(x,\lambda^{-\sigma_0}D;\lambda) + O(\lambda^{-M}).$$

It is useful to rewrite $\tilde{G}^{(0)}(x,\xi;\lambda)$ in the following way

$$\tilde{G}^{(0)}(x,\xi;\lambda) = \sum_{j\geq 0} \lambda^{-\delta_j(\tilde{G}^{(0)})}\tilde{G}_j^{(0)}(x,\xi), \tag{3.6}$$

where

$$0 = \delta_0(\tilde{G}^{(0)}) < \delta_1(\tilde{G}^{(0)}) < \cdots < \delta_j(\tilde{G}^{(0)}) < \cdots \tag{3.7}$$

and it is understood that the sum in (3.6) is finite. Furthermore from (3.5) we obtain

$$\tilde{G}_0^{(0)}(x,\xi) = \sum_{\substack{j\geq 0,\ s_j<r \\ j-(r-s_j)\theta_0=0}} P_{m-j,\ (0,e_n)}(x,\xi) \tag{3.8}$$

and that all the $\delta_j(\tilde{G}^{(0)})$ are multiples of the same rational number θ_0. As a consequence $\delta_1(\tilde{G}^{(0)}) \geq \theta_0$ and we may find a positive integer $k(0)$ such that σ_0, θ_0 and $\delta_j(\tilde{G}^{(0)})$, for $j \geq 0$, can be expressed as fractions whose denominator is $k(0)$ and whose numerator is a non-negative integer.

DEFINITION 3.1. We say that a differential operator $P(x,D;\lambda)$, depending on a large positive paramenter λ is in the class $\mathcal{R}_U$ if there exists a positive rational number κ and differential operators $P_j(x,D)$ whose coefficients are in $C^\infty(U)$, $j =$

$0, 1, \dots, L$, for some $L \in \mathbb{N}$, such that

$$P(x, D; \lambda) = \sum_{j=0}^{L} \lambda^{-\kappa j} P_j(x, D).$$

Our next step is to prove the following general purpose lemma.

Lemma 3.1. *Let $G(x, D)$ be a differential operator with smooth coefficients defined in U and let σ, θ be rational numbers with $\sigma \geq \theta > 0$. Denote by $\varphi(x)$ a function in $C^\infty(U)$. Then*
i) $e^{-i\lambda^\theta \varphi(x)} G(x, \lambda^{-\sigma} D) e^{i\lambda^\theta \varphi(x)} = G(x, \lambda^{-(\sigma-\theta)}(\varphi_x(x) + \lambda^{-\theta} D)) + \lambda^{-\theta} r(x, \lambda^{-\theta} D; \lambda)$, *where $r \in \mathcal{R}_U$.*
ii) *If $G(x, \xi) = O(|\xi|^q)$ as $|\xi| \to 0$, uniformly with respect to x, for some positive integer q, then*

$$\begin{aligned} e^{-i\lambda^\theta \varphi(x)} G(x, \lambda^{-\sigma} D) e^{i\lambda^\theta \varphi(x)} \\ = G(x, \lambda^{-(\sigma-\theta)}(\varphi_x(x) + \lambda^{-\theta} D)) + \lambda^{-(\sigma-\theta)q-\theta} r(x, \lambda^{-\theta} D; \lambda), \end{aligned} \tag{3.9}$$

where again $r \in \mathcal{R}_U$.

REMARK 3.1. It is important to remark that in the notation above the quantity $G(x, \lambda^{-(\sigma-\theta)}(\varphi_x(x) + \lambda^{-\theta} D))$ does not contain the terms in which the derivatives land on $\varphi_x(x)$, as will be clear from the proof; those terms are pushed into the "error" term r and thus $G(x, \lambda^{-(\sigma-\theta)}(\varphi_x(x) + \lambda^{-\theta} D))$ is to be thought of as a commutative expression.

Proof. Denote by $\varphi_2(x, y) = \varphi(y) - \varphi(x) - \langle y - x, \varphi_x(x) \rangle$. Then, if $u(x)$ denotes a smooth function,

$$\begin{aligned} A(x, D; \lambda) &= e^{-i\lambda^\theta \varphi(x)} G(x, \lambda^{-\sigma} D) e^{i\lambda^\theta \varphi(x)} u(x) \\ &= \sum_{\alpha \geq 0} \frac{1}{\alpha!} G^{(\alpha)}(x, \lambda^{-\sigma+\theta} \varphi_x(x)) (\lambda^{-\sigma} D_y)^\alpha \left[e^{i\lambda^\theta \varphi_2(x,y)} u(y) \right]_{|y=x} \\ &= \sum_{\alpha \geq 0} \frac{1}{\alpha!} G^{(\alpha)}(x, \lambda^{-\sigma+\theta} \varphi_x(x)) (\lambda^{-\sigma} D_x)^\alpha u(x) \\ &\quad + \sum_{\alpha \geq 0} \frac{1}{\alpha!} G^{(\alpha)}(x, \lambda^{-\sigma+\theta} \varphi_x(x)) \\ &\qquad \times \sum_{2 \leq |\beta| \leq |\alpha|} \binom{\alpha}{\beta} \left[(\lambda^{-\sigma} D_y)^\beta e^{i\lambda^\theta \varphi_2(x,y)} (\lambda^{-\sigma} D_y)^{\alpha-\beta} u(y) \right]_{|y=x}. \end{aligned}$$

The first sum is "by definition" what has been called $G(x, \lambda^{-(\sigma-\theta)}(\varphi_x(x) + \lambda^{-\theta} D))$ (see

the remark above). Let us take a closer look at the second sum. Due to the vanishing of $\varphi_2(x,y)_{|y=x}$ and of $\nabla_y\varphi_2(x,y)_{|y=x}$, the quantity $D^\beta e^{i\lambda^\theta\varphi_2(x,y)}{}_{|y=x}$ is a polynomial in the variable λ^θ of degree less than or equal to $[|\beta|/2]$. Factoring out $\lambda^{\theta(|\beta|/2)}$ we obtain a polynomial of the same degree in the variable $\lambda^{-\theta}$. Thus the second sum above can be rewritten as

$$\begin{aligned}
&\sum_{\alpha\geq 0}\frac{1}{\alpha!}G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\varphi_x(x))\lambda^{-(\sigma-\theta)|\alpha|}\\
&\qquad\qquad\cdot\sum_{2\leq|\beta|\leq|\alpha|}\lambda^{-\theta|\alpha|+\theta(|\beta|/2)}P_{\alpha,\beta,\varphi}(x;\lambda^{-\theta})D^{\alpha-\beta}u(x)\\
&=\sum_{\alpha\geq 0}\frac{1}{\alpha!}G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\varphi_x(x))\lambda^{-(\sigma-\theta)|\alpha|}\\
&\qquad\qquad\cdot\sum_{\nu=1}^{[|\alpha|/2]}\lambda^{-\theta\nu}\sum_{|\beta|=2\nu}P_{\alpha,\beta,\varphi}(x;\lambda^{-\theta})(\lambda^{-\theta}D)^{\alpha-\beta}u(x).
\end{aligned}$$

Since G is a fixed differential operator we may assume that $0\leq|\alpha|\leq M$, for some M. Then the above quantity can be rewritten as

$$\begin{aligned}
&\lambda^{-\theta}\sum_{k=0}^{M}\sum_{\nu=1}^{[k/2]}\lambda^{-(\sigma-\theta)k}\sum_{|\alpha|=k}\frac{1}{\alpha!}G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\varphi_x(x))\\
&\qquad\qquad\cdot\sum_{|\beta|=2\nu}\lambda^{-\theta(\nu-1)}P_{\alpha,\beta,\varphi}(x;\lambda^{-\theta})(\lambda^{-\theta}D)^{\alpha-\beta}u(x).
\end{aligned}$$

Hence the second term can be rewritten as

$$\lambda^{-\theta}r(x,\lambda^{-\theta}D;\lambda),\quad r(x,D;\lambda)=\sum_{k=0}^{M}\sum_{\nu=1}^{[k/2]}b_{k,\nu}(x,D;\lambda),$$

where

$$\begin{aligned}
b_{k,\nu}(x,D;\lambda)=\lambda^{-(\sigma-\theta)k}&\sum_{|\alpha|=k}\frac{1}{\alpha!}G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\varphi_x(x))\\
&\cdot\sum_{|\beta|=2\nu}\lambda^{-\theta(\nu-1)}P_{\alpha,\beta,\varphi}(x;\lambda^{-\theta})D^{\alpha-\beta}.
\end{aligned}$$

hence $r\in\mathcal{R}_U$. The first assertion of the lemma is proved.

Let us now turn to the second part. It is obvious that nothing is changed in the first term, so that all we have to do is just look at the second sum. Now two cases may occur:

a) $k \geq q$. Then trivially

$$\lambda^{-(\sigma-\theta)k} \leq \lambda^{-(\sigma-\theta)q}.$$

b) $k < q$. In that case our assumption implies that

$$G^{(\alpha)}(x,\xi) = O(|\xi|^{q-|\alpha|}) = O(|\xi|^{q-k})$$

and hence $G^{(\alpha)}(x, \lambda^{-(\sigma-\theta)}\varphi_x(x)) = O(\lambda^{-(\sigma-\theta)(q-k)})$.

Plugging this information into the expression for $b_{k,\nu}$ we have

$$b_{k,\nu}(x, D; \lambda) = \lambda^{-(\sigma-\theta)q}\tilde{b}_{k,\nu}(x, D; \lambda),$$

where, again, $\tilde{b}_{k,\nu} \in \mathcal{R}_U$. This concludes the proof of the lemma. □

In order to prove Theorem 2.1 we prove first a more general inductive lemma in the following subsections.

From the assumption we may start off assuming that (see also Equation (2.3))

$$\xi_0 \longmapsto \det \tilde{G}_0^{(0)}(x, \xi_0, \xi') \text{ has a non real root of multiplicity } q_0, \text{ i.e.} \tag{3.10}$$

$$\det \tilde{G}_0^{(0)}(x,\xi) = (\xi_0 - \tau_0(x,\xi'))^{q_0}\Delta_0(x,\xi), \quad \Delta_0(x, \tau_0(x,\xi'), \xi') \neq 0 \tag{3.11}$$

in some open set $U \times V$ in $\mathbb{R}^{n+1}_x \times \mathbb{R}^n_{\xi'}$. In the sequel U and V stands for an open set in $\mathbb{R}^{n+1}$ and in $\mathbb{R}^n$ respectively which may differ from line to line but the subsequent one will be contained in the preceding one.

Denote by $\varphi^{(0)}(x)$ a complex-valued smooth (i.e. real analytic) function in U such that

$$\partial_{x_0}\varphi^{(0)}(x) = \tau_0(x, \partial_{x'}\varphi^{(0)}(x)). \tag{3.12}$$

Then

$$\begin{aligned} e^{-i\lambda^{\sigma_0}\varphi^{(0)}(x)}\tilde{G}^{(0)}(x, \lambda^{-\sigma_0}D; \lambda)e^{i\lambda^{\sigma_0}\varphi^{(0)}(x)} \\ = \tilde{G}^{(0)}(x, \varphi_x^{(0)}(x) + \lambda^{-\sigma_0}D; \lambda) + \lambda^{-\sigma_0}R^{(0)}(x, \lambda^{-\sigma_0}D; \lambda), \end{aligned} \tag{3.13}$$

where $R^{(0)}$ is a symbol in the class $\mathcal{R}_U$.

In order to construct an asymptotic solution for the $N \times N$ matrix-valued operator $\tilde{G}^{(0)}(x, \lambda^{-\sigma_0}D; \lambda)$ we first prove a general inductive lemma enabling us to construct the phase functions required.

Lemma 3.2 (First inductive step). *Let σ_p, $\theta_p \in \mathbb{Q}^+$, $r_p \leq N$, $\varphi^{(p)}(x)$ be a smooth function defined in U and consider the $r_p \times r_p$ matrix-valued differential operator*

$$\tilde{G}^{(p)}(x, \varphi_x^{(p)}(x) + \lambda^{-\sigma_p}D; \lambda) + \lambda^{-\sigma_p}R^{(p)}(x, \lambda^{-\sigma_p}D; \lambda) \tag{3.14}$$

where $R^{(p)} \in \mathcal{R}_U$ *and*

i)$_p$
$$\tilde{G}^{(p)}(x,\xi;\lambda) = \sum_{j\geq 0} \lambda^{-\delta_j(\tilde{G}^{(p)})} \tilde{G}_j^{(p)}(x,\xi),$$

the sum being finite and $\tilde{G}_j^{(p)}$*denoting differential operators with real analytic coefficients. Furthermore*

$$\begin{aligned} &0 = \delta_0(\tilde{G}^{(p)}) < \delta_1(\tilde{G}^{(p)}) < \cdots \\ &\textit{and } \delta_j(\tilde{G}^{(p)}) \in \mathbb{Q}^+, \ \ j \geq 1. \end{aligned} \tag{3.15}$$

ii)$_p$ $\det \tilde{G}_0^{(p)}(x,\xi) = (\xi_0 - \tau_p(x,\xi'))^{q_p} \Delta_p(x,\xi), \quad \Delta_p(x,\tau_p(x,\xi'),\xi') \neq 0$
in some open set $U \times V$ *in* $\mathbb{R}_x^{n+1} \times \mathbb{R}_{\xi'}^n$.
iii)$_p$ *The function* $\varphi^{(p)}(x)$*, defined in* U*, satisfies the eikonal equation*

$$\partial_{x_0}\varphi^{(p)}(x) = \tau_p(x, \partial_{x'}\varphi^{(p)}(x)). \tag{3.16}$$

iv)$_p$

$$\operatorname{rank} \tilde{G}_0^{(p)}(x, \varphi_x^{(p)}(x)) = r_p - r_{p+1} \tag{3.17}$$

in U *for a suitable positive integer* $r_{p+1} \leq r_p$.
v)$_p$ *There exists a positive integer* $k(p)$ *such that*

$$\sigma_p, \theta_p, \delta_j(\tilde{G}^{(p)}), j \geq 1, \ \textit{belong to } \frac{\mathbb{N}}{k(p)}$$

(*i.e. are positive rational numbers with the same denominator* $k(p)$).
Then we can find a $r_{p+1} \times r_{p+1}$ *matrix-valued differential operator*

$$F^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) \tag{3.18}$$

such that

$$F^{(p)}(x,\xi;\lambda) = \sum_{j\geq 0} \lambda^{-\delta_j(F^{(p)})} F_j^{(p)}(x,\xi), \tag{3.19}$$

the sum being finite and

$$\begin{aligned} &0 = \delta_0(F^{(p)}) < \delta_1(F^{(p)}) < \cdots \\ &\textit{and } \delta_j(F^{(p)}) \in \mathbb{Q}^+, \ \ j \geq 1. \end{aligned} \tag{3.20}$$

$$\sigma_p, \theta_p, \delta_j(F^{(p)}), j \geq 1, \ \textit{belong to } \frac{\mathbb{N}}{k'(p)} \tag{3.21}$$

for a suitable positive integer $k'(p)$.

$$F_0^{(p)}(x,0) = 0. \tag{3.22}$$

$$\det F_0^{(p)}(x, \lambda^{-\sigma_p}\xi) = \frac{1}{E(x, \lambda^{-\sigma_p}\xi)} \det \tilde{G}_0^{(p)}(x, \varphi_x^{(p)}(x) + \lambda^{-\sigma_p}\xi) + O(\lambda^{-(M+1)\delta_p^*}), \tag{3.23}$$

where M is an arbitrarily large positive integer and $\delta_p^* > 0$, suitable and $E(x, \xi)$ is an elliptic symbol. Moreover the construction of an asymptotic solution for the operator (3.14) *is reduced to the construction of an asymptotic solution for the operator* (3.18).

3.1. Rank reduction Set

$$G^{(p)}(x, \xi; \lambda) = \tilde{G}^{(p)}(x, \varphi_x^{(p)}(x) + \xi; \lambda) + \lambda^{-\sigma_p} R^{(p)}(x, \xi; \lambda). \tag{3.1.1}$$

Then $G^{(p)}$ can be written as a finite sum of differential operators:

$$G^{(p)}(x, \xi; \lambda) = \sum_{j \geq 0} \lambda^{-\delta_j(G^{(p)})} G_j^{(p)}(x, \xi); \tag{3.1.2}$$

we remark that in the above (finite) sum

$$\delta_0(G^{(p)}) = \delta_0(\tilde{G}^{(p)}) = 0, \tag{3.1.3}$$

$$G_0^{(p)}(x, \xi) = \tilde{G}_0^{(p)}(x, \varphi_x^{(p)} + \xi). \tag{3.1.4}$$

Due to Assumption v)$_p$ and Definition 3.1 it is then obvious that θ_p, σ_p and the $\delta_j(G^{(p)})$, $j \geq 1$, satisfy a condition of the same type as v)$_p$, with possibly a different $k(p)$. For our present purpose we shall continue to denote this new number by the same symbol.

By (3.17) we have

$$\operatorname{rank} G_0^{(p)}(x, 0) = r_p - r_{p+1}. \tag{3.1.5}$$

Thus we can find two non singular smooth matrices, $M_p(x)$, $N_p(x)$, defined in U, such that

$$M_p(x) G_0^{(p)}(x, 0) N_p(x) = \left[\begin{array}{c|c} I_{r_p - r_{p+1}} & 0 \\ \hline 0 & 0 \end{array}\right], \tag{3.1.6}$$

where $I_{r_p - r_{p+1}}$ denotes the $(r_p - r_{p+1}) \times (r_p - r_{p+1})$ identity matrix. Now we set

$$\begin{aligned} \hat{G}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) &= M_p(x) G^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) N_p(x) \\ &= \sum_{j \geq 0} \lambda^{-\delta_j(\hat{G}^{(p)})} \hat{G}_j^{(p)}(x, \lambda^{-\sigma_p} D). \end{aligned} \tag{3.1.7}$$

We point out that the $\delta_j(\hat{G}^{(p)})$, $j \geq 1$, satisfy the same assumption v)$_p$ as the $\delta_j(G^{(p)})$, $j \geq 1$, do, even though they are not the same, due to the following lemma whose proof is straightforward.

Lemma 3.1.1. *Let*

$$A(x, \lambda^{-\sigma} D; \lambda) = \sum_{j \geq 0} \lambda^{-\delta_j(A)} A_j(x, \lambda^{-\sigma} D)$$

and

$$B(x, \lambda^{-\sigma} D; \lambda) = \sum_{l \geq 0} \lambda^{-\delta_l(B)} B_l(x, \lambda^{-\sigma} D)$$

be operators of the form (3.1.2). *Then*

$$(A \circ B)(x, \lambda^{-\sigma} D; \lambda) = \sum_{j \geq 0} \lambda^{-\delta_j(C)} C_j(x, \lambda^{-\sigma} D),$$

where $\delta_j(C) = \delta_i(A) + \delta_l(B) + k\sigma$ *for suitable* i, l *and* k.

Since M_p and N_p are non singular matrices then the construction of an asymptotic solution for $G^{(p)}(x, \lambda^{-\sigma_p} D; \lambda)$ is equivalent to the construction of an asymptotic solution for $\hat{G}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda)$ and moreover we have that

$$\hat{G}_0^{(p)}(x, \xi) = M_p(x) G_0^{(p)}(x, \xi) N_p(x) \tag{3.1.8}$$

so that

$$\hat{G}_0^{(p)}(x, 0) = \left[\begin{array}{c|c} I_{r_p - r_{p+1}} & 0 \\ \hline 0 & 0 \end{array}\right]. \tag{3.1.9}$$

Here $\hat{G}^{(p)}(x, \xi; \lambda)$ has the same properties as $G^{(p)}(x, \xi; \lambda)$ where (3.1.5) has to be replaced by (3.1.9).

From now on we switch back to the $G^{(p)}$ notation, dropping the hat sign. Let us write $G^{(p)}$ in block form:

$$G^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) = \begin{bmatrix} G_{11}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) & G_{12}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) \\ G_{21}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) & G_{22}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) \end{bmatrix}, \tag{3.1.10}$$

where the blocks have the same size as those in the block partition of (3.1.9). We have

$$\begin{aligned} G_{11}^{(p)}(x, \xi; \lambda) &= \sum_{j \geq 0} \lambda^{-\delta_j(G^{(p)})} G_{11,j}^{(p)}(x, \xi) \\ &= I - \sum_{j \geq 0} \lambda^{-\delta_j(G^{(p)})} B_j^{(p)}(x, \xi) \\ &= I - B^{(p)}(x, \xi; \lambda) \end{aligned} \tag{3.1.11}$$

where $B_j^{(p)}(x,\xi)$ are $(r_p - r_{p+1}) \times (r_p - r_{p+1})$ matrices and

$$B_0^{(p)}(x,0) = 0. \tag{3.1.12}$$

Define

$$\mathcal{R}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda) = \sum_{k=0}^{M} \left(B^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\right)^k, \tag{3.1.13}$$

where M is a positive integer that, in the sequel, will be chosen suitably large. We may then write

$$\mathcal{R}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda) = \sum_{j\geq 0} \lambda^{-\delta_j(\mathcal{R}^{(p)})}\mathcal{R}_j^{(p)}(x,\lambda^{-\sigma_p}D), \tag{3.1.14}$$

where the sum is a finite sum whose number of terms depends on M and, using again Lemma 3.1.1, the $\delta_j(\mathcal{R}^{(p)})$, $j \geq 1$, are an increasing sequence of rational numbers whose denominator is the same, i.e. $k(p)$. From (3.1.13) we obtain that

$$G_{11}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\mathcal{R}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda) = I - \left(B^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\right)^{M+1}. \tag{3.1.15}$$

We want to show that $\left(B^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\right)^{M+1}$ becomes negligible provided M is chosen large enough.

Lemma 3.1.2. *Let $B^{(p)}$ be defined as in* (3.1.11). *Then*

$$\partial_\xi^\gamma \sigma\left(\left[B^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\right]^{M+1}\right) = O(\lambda^{-(M+1)\delta_p^*}),$$

where $\delta_p^* = \min\{\sigma_p, \delta_1(G^{(p)})\} > 0$ and $\sigma(\cdot)$ denotes the symbol of a given differential operator.

Proof. Since

$$\begin{aligned}\left(B^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\right)^{M+1} &= \left(\sum_{j\geq 0} \lambda^{-\delta_j(G^{(p)})} B_j^{(p)}(x,\lambda^{-\sigma_p}D)\right)^{M+1} \\ &= \sum_{j_1,\dots,j_{M+1}\geq 0} \lambda^{-\delta_{j_1}(G^{(p)})-\cdots-\delta_{j_{M+1}}(G^{(p)})} B_{j_1}^{(p)}(x,\lambda^{-\sigma_p}D)\cdots B_{j_{M+1}}^{(p)}(x,\lambda^{-\sigma_p}D)\end{aligned} \tag{3.1.16}$$

we can easily compute (a derivative of) its symbol:

$$\begin{aligned}
&\partial_\xi^\gamma \sigma\left(\left(B^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\right)^{M+1}\right)\\
&= \sum_{j_1,\dots,j_{M+1}\geq 0}\ \sum_{\substack{\alpha_1,\dots,\alpha_M\geq 0\\ \sum_{k=j+1}^{M+1}|\beta_{jk}|=|\alpha_j|,\\ j=1,\dots,M}}\ \sum_{|\gamma_1|+\dots+|\gamma_{M+1}|=|\gamma|} C_{\alpha,\beta,\gamma}\lambda^{-\sum_{l=1}^{M+1}\delta_{j_l}(G^{(p)})}\\
&\quad\cdot \partial_\xi^{\alpha_1+\gamma_1}B_{j_1}^{(p)}(x,\lambda^{-\sigma_p}\xi)\cdots\partial_\xi^{\alpha_M+\gamma_M}D_x^{\beta_{1M}+\dots+\beta_{M-1M}}B_{j_M}^{(p)}(x,\lambda^{-\sigma_p}\xi)\\
&\quad\cdot \partial_\xi^{\gamma_{M+1}}D_x^{\beta_{1M+1}+\dots+\beta_{MM+1}}B_{j_{M+1}}^{(p)}(x,\lambda^{-\sigma_p}\xi).
\end{aligned}$$

Now remark that if $j_l \neq 0$ then $\delta_{j_l}(G^{(p)}) \geq \delta_1(G^{(p)})$ by definition. Moreover, because of (3.1.12), if $j_l = 0$ we collect a contribution of size $O(\lambda^{-\sigma_p})$ from that factor, no matter how many derivatives are landing on it. Thus, in a generic summand in the above formula, let $L = \sharp\{j_l \mid j_l > 0\}$ then

$$\sum_{l=1}^{M+1}\delta_{j_l}(G^{(p)}) \geq \sum_{j_l\neq 0}\delta_1(G^{(p)}) = L\delta_1(G^{(p)}).$$

On the other hand, again no matter how many derivatives land on them, the terms corresponding to $j_l = 0$ yield a contribution of size $O(\lambda^{-(M+1-L)\sigma_p})$. Since

$$L\delta_1(G^{(p)}) + (M+1-L)\sigma_p \geq (M+1)\delta_p^*$$

we have proved the assertion. □

REMARK 3.1.1. From the above proof we can also deduce that

$$\partial_\xi^\gamma \sigma\left(\left(B^{(p)}(x,\lambda^{-\theta}\xi;\lambda)\right)^{M+1}\right) = O\left(\lambda^{-(M+1)\theta}\right),$$

where $0 < \theta \leq \min\{\sigma_p, \delta_1(G^{(p)})\}$.

Now define

$$\Lambda^{(p)}(x,\lambda^{-\sigma_p}D;\lambda) = \begin{bmatrix} I & -\mathcal{R}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)G_{12}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\\ 0 & I\end{bmatrix} \tag{3.1.17}$$

in block form notation, the blocks corresponding to those in Equation (3.1.10). We have

$$\begin{aligned}
&G^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\Lambda^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\\
&= \begin{bmatrix} G_{11}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda) & \left(I - G_{11}^{(p)}\mathcal{R}^{(p)}\right)(x,\lambda^{-\sigma_p}D;\lambda)G_{12}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)\\ G_{21}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda) & \left(G_{22}^{(p)} - G_{21}^{(p)}\mathcal{R}^{(p)}G_{12}^{(p)}\right)(x,\lambda^{-\sigma_p}D;\lambda)\end{bmatrix}
\end{aligned} \tag{3.1.18}$$

and define

$$F^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) = \left(G_{22}^{(p)} - G_{21}^{(p)} \mathcal{R}^{(p)} G_{12}^{(p)} \right) (x, \lambda^{-\sigma_p} D; \lambda). \tag{3.1.19}$$

Furthermore we have that

$$\left(I - G_{11}^{(p)} \mathcal{R}^{(p)} \right) (x, \lambda^{-\sigma_p} D; \lambda) G_{12}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) \\ = \left(B^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) \right)^{M+1} G_{12}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda).$$

Here $S^{(p)} T^{(p)}(x, \lambda^{-\sigma_p} D; \lambda)$ stands for $S^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) T^{(p)}(x, \lambda^{-\sigma_p} D; \lambda)$. The proof of the following proposition is then a straightforward consequence of Lemma 3.1.2.

Proposition 3.1.1. *The* $(1, 2)$*-block of the matrix in* (3.1.18)*, as a differential operator, is* $O(\lambda^{-(M+1)\delta_p^*})$*. In particular if* M *is chosen suitably large, since* $\Lambda^{(p)}$ *is non singular, the construction of an asymptotic solution for the* $r_p \times r_p$ *matrix of differential operators* $G^{(p)}(x, \lambda^{-\sigma_p} D; \lambda)$ *reduces to the construction of an asymptotic solution for the* $r_{p+1} \times r_{p+1}$ *matrix of differential operators* $F^{(p)}(x, \lambda^{-\sigma_p} D; \lambda)$*, defined in* (3.1.19).

We write

$$F^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) = \sum_{j \geq 0} \lambda^{-\delta_j(F^{(p)})} F_j^{(p)}(x, \lambda^{-\sigma_p} D). \tag{3.1.20}$$

We still have to prove (3.20)–(3.23).

First remark that (3.22) is immediate because of the construction of $F^{(p)}$. From (3.1.19) we may analyze the sequence $\left(\delta_j(F^{(p)})\right)_{j \geq 1}$, $\delta_0(F^{(p)})$ being 0. Using the proof of Lemma 3.1.2 we can see that the exponents $\delta_j(F^{(p)})$ are obtained summing a number of $\delta_j(G^{(p)})$ to integer multiples of σ_p. This proves (3.20), (3.21), because the sequence $\left(\delta_j(G^{(p)})\right)_{j \geq 1}$ satisfies assumption v)$_p$. Due to Lemma 3.1.2 we can see that the (operator-valued) matrix in (3.1.18) is lower triangular modulo terms that are $O(\lambda^{-(M+1)\delta_p^*})$ and therefore

$$F_0^{(p)}(x, \xi) = G_{22,\ 0}^{(p)}(x, \xi) - G_{21,\ 0}^{(p)}(x, \xi) \mathcal{R}_0^{(p)}(x, \xi) G_{12,\ 0}^{(p)}(x, \xi) \tag{3.1.21}$$

$$\det F_0^{(p)}(x, \lambda^{-\sigma_p} \xi) \\ = \frac{1}{\det G_{11,\ 0}^{(p)}(x, \lambda^{-\sigma_p} \xi)} \det G_0^{(p)}(x, \lambda^{-\sigma_p} \xi) + O\left(\lambda^{-(M+1)\delta_p^*} \right). \tag{3.1.22}$$

This proves (3.23), since $\det G_{11,\ 0}^{(p)}(x, 0) \neq 0$.

3.2. End of the inductive step The purpose of this section is to complete, under an additional technical assumption, the inductive step whose first part is Lemma 3.2.

Let us consider the operator $F^{(p)}$ as given by (3.1.20), defined in U, and denote by $s_j^{(p)}$ the vanishing order with respect to the variable ξ as $|\xi| \to 0$ of $F_j^{(p)}$, $j = 0, 1, \ldots$:

$$F_j^{(p)}(x, \lambda^{-\theta}\xi) = \lambda^{-\theta s_j^{(p)}} \left[\hat{F}_j^{(p)}(x, \xi) + O(\lambda^{-\theta})\right], \tag{3.2.1}$$

where θ is any positive real number. Here $\hat{F}_j^{(p)}$ is an $r_{p+1} \times r_{p+1}$ matrix valued homogeneous polynomial in the variable ξ of degree $s_j^{(p)}$. Define

$$\theta_{p+1} = \min_{\substack{j \geq 1 \\ s_j^{(p)} < s_0^{(p)}}} \left\{ \frac{\delta_j(F^{(p)})}{s_0^{(p)} - s_j^{(p)}}, \theta_p \right\} \tag{3.2.2}$$

so that, in particular, $\theta_{p+1} \leq \theta_p$, and let

$$\sigma_{p+1} = \sigma_p - \theta_{p+1}. \tag{3.2.3}$$

For our present purpose we shall assume that $\sigma_{p+1} > 0$. If $\sigma_{p+1} \leq 0$ we make a different argument in the following.

Let now $\varphi^{(p+1)}(x)$ be a real analytic function defined in some open set U; in the following we shall precise this function. Applying Lemma 3.1 we compute

$$\begin{aligned} & e^{-i\lambda^{\sigma_{p+1}}\varphi^{(p+1)}(x)} F^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) e^{i\lambda^{\sigma_{p+1}}\varphi^{(p+1)}(x)} \\ &= \sum_{j \geq 0} \lambda^{-\delta_j(F^{(p)})} F_j^{(p)}(x, \lambda^{-\theta_{p+1}}(\varphi_x^{(p+1)}(x) + \lambda^{-\sigma_{p+1}} D)) \\ &\qquad + \sum_{j \geq 0} \lambda^{-\delta_j(F^{(p)}) - \theta_{p+1} s_j^{(p)} - \sigma_{p+1}} \tilde{R}_j^{(p+1)}(x, \lambda^{-\sigma_{p+1}} D; \lambda), \end{aligned}$$

where $\tilde{R}_j^{(p+1)} \in \mathcal{R}_U$. Defining $\tilde{G}^{(p+1)}(x, \xi; \lambda)$ and $R^{(p+1)}(x, \xi; \lambda)$ by

$$\begin{aligned} F^{(p)}(x, \lambda^{-\theta_{p+1}}\xi; \lambda) &= \lambda^{-\theta_{p+1} s_0^{(p)}} \tilde{G}^{(p+1)}(x, \xi; \lambda) \\ &= \lambda^{-\theta_{p+1} s_0^{(p)}} \sum_{j \geq 0} \lambda^{-\delta_j(\tilde{G}^{(p+1)})} \tilde{G}_j^{(p+1)}(x, \xi), \\ \sum_{j \geq 0} \lambda^{-\delta_j(F^{(p)}) - \theta_{p+1} s_j^{(p)}} \tilde{R}_j^{(p+1)}(x, \xi; \lambda) &= \lambda^{-\theta_{p+1} s_0^{(p)}} R^{(p+1)}(x, \xi; \lambda), \end{aligned} \tag{3.2.4}$$

the right-hand side of the above equality can be written as

$$\begin{aligned} & \lambda^{-\theta_{p+1} s_0^{(p)}} \left[\tilde{G}^{(p+1)}(x, \varphi_x^{(p+1)}(x) + \lambda^{-\sigma_{p+1}} D) + \lambda^{-\sigma_{p+1}} R^{(p+1)}(x, \lambda^{-\sigma_{p+1}} D; \lambda)\right] \\ &= \lambda^{-\theta_{p+1} s_0^{(p)}} \left[\sum_{j \geq 0} \lambda^{-\delta_j(\tilde{G}^{(p+1)})} \tilde{G}_j^{(p+1)}(x, \varphi_x^{(p+1)}(x) + \lambda^{-\sigma_{p+1}} D)\right. \end{aligned} \tag{3.2.5}$$

$$+ \lambda^{-\sigma_{p+1}} R^{(p+1)}(x, \lambda^{-\sigma_{p+1}} D; \lambda) \Bigg] .$$

We want to show that, provided an additional assumption is made, the operator $\tilde{G}^{(p+1)}$ satisfies the conditions i)$_{p+1}$–v)$_{p+1}$ of Lemma 3.2, thus enabling us to start over again the induction process.

i)$_{p+1}$ is obvious. By definition θ_{p+1} is a positive rational number and so is σ_{p+1}, therefore also v)$_{p+1}$ is satisfied. From (3.2.4) we obtain

$$\tilde{G}_0^{(p+1)}(x, \xi) = \sum_{\substack{\theta_{p+1} s_0^{(p)} = \theta_{p+1} s_j^{(p)} + \delta_j(F^{(p)}) \\ s_j^{(p)} < s_0^{(p)} \text{ if } j>0}} \hat{F}_j^{(p)}(x, \xi), \tag{3.2.6}$$

each $\hat{F}_j^{(p)}$ being homogeneous of degree $s_j^{(p)}$ with respect to the variable ξ.

We make the following assumption:

$$q_p = r_{p+1} s_0^{(p)}. \tag{3.2.7}$$

From (3.1.22) and Remark 3.1.1 we have

$$\det F_0^{(p)}(x, \lambda^{-\theta}\xi) = \frac{1}{\det G_{11,\ 0}^{(p)}(x, \lambda^{-\theta}\xi)} \det G_0^{(p)}(x, \lambda^{-\theta}\xi) + O(\lambda^{-(M+1)\theta}) \tag{3.2.8}$$

for a suitable $\theta > 0$. But

$$F_0^{(p)}(x, \lambda^{-\theta}\xi) = \lambda^{-\theta s_0^{(p)}} \left[\hat{F}_0^{(p)}(x, \xi) + O(\lambda^{-\theta}) \right]$$

so that

$$\begin{aligned} \det G_0^{(p)}(x, \lambda^{-\theta}\xi) &= \det G_{11,\ 0}^{(p)}(x, \lambda^{-\theta}\xi) \lambda^{-\theta q_p} \left[\det \hat{F}_0^{(p)}(x, \xi) + O(\lambda^{-\theta}) \right] \\ &\quad + O(\lambda^{-(M+1)\theta}) \\ &= \lambda^{-\theta q_p} \left[E_p(x) + O(\lambda^{-\theta}) \right] \left[\det \hat{F}_0^{(p)}(x, \xi) + O(\lambda^{-\theta}) \right] \\ &\quad + O(\lambda^{-(M+1)\theta}). \end{aligned} \tag{3.2.9}$$

On the other hand

$$\begin{aligned} \det G_0^{(p)}(x, \lambda^{-\theta}\xi) &= c_p(x) \det \tilde{G}_0^{(p)}(x, \varphi_x^{(p)}(x) + \lambda^{-\theta}\xi) \\ &= \lambda^{-q_p\theta} c_p(x) \sum_{|\alpha| = q_p} \frac{1}{\alpha!} \left(\det \tilde{G}_0^{(p)} \right)^{(\alpha)} (x, \varphi_x^{(p)}(x)) \xi^\alpha + o(\lambda^{-q_p\theta}), \end{aligned} \tag{3.2.10}$$

where c_p denotes a non zero smooth function defined in U. Thus from (3.2.9) and (3.2.10) we obtain

$$\det \hat{F}_0^{(p)}(x,\xi) = c_p(x) \sum_{|\alpha|=q_p} \frac{1}{\alpha!} \left(\det \tilde{G}_0^{(p)}\right)^{(\alpha)} (x, \varphi_x^{(p)}(x))\xi^\alpha. \tag{3.2.11}$$

The above relation allows us to conclude that:
a) The right-hand side of (3.2.11) is a non characteristic polynomial with respect to the variable ξ_0 with real analytic coefficients. Then one can find a real analytic $\tau_{p+1}(x,\xi')$ defined in $U \times V$ such that

$$\begin{aligned} \det \tilde{G}_0^{(p+1)}(x,\xi) &= (\xi_0 - \tau_{p+1}(x,\xi'))^{q_{p+1}} \Delta_{p+1}(x,\xi), \\ &\qquad \Delta_{p+1}(x, \tau_{p+1}(x,\xi'), \xi') \neq 0 \end{aligned} \tag{3.2.12}$$

where $q_{p+1} \leq q_p$.
b) We can find a real analytic function $\varphi^{(p+1)}(x)$, defined in U such that

$$\partial_{x_0} \varphi^{(p+1)}(x) = \tau_{p+1}(x, \partial_{x'} \varphi^{(p+1)}(x)). \tag{3.2.13}$$

c) There exists a non negative integer r_{p+2} such that

$$\operatorname{rank} \tilde{G}_0^{(p+1)}(x, \varphi_x^{(p+1)}(x)) = r_{p+1} - r_{p+2} \tag{3.2.14}$$

in U.
These statements easily imply conditions ii)$_{p+1}$–iv)$_{p+1}$. Thus far we proved the following

Lemma 3.2.1 (Second half of the induction step). *Let $F^{(p)}(x,\xi;\lambda)$ be as in* Lemma 3.2 *and assume that hypothesis* (3.2.7) *holds. Then we can find a positive rational number $\theta_{p+1} \leq \theta_p$ with $\sigma_{p+1} = \sigma_p - \theta_{p+1} > 0$, a real analytic function $\varphi^{(p+1)}(x)$ defined in U and $r_{p+1} \times r_{p+1}$ matrix valued differential operators $\tilde{G}^{(p+1)}(x,D;\lambda)$, $R^{(p+1)}(x,D;\lambda)$ with $R^{(p+1)} \in \mathcal{R}_U$ such that the construction of an asymptotic solution for the operator* (3.18) *is reduced to the construction of an asymptotic solution for*

$$\tilde{G}^{(p+1)}(x, \varphi_x^{(p+1)}(x) + \lambda^{-\sigma_{p+1}} D; \lambda) + \lambda^{-\sigma_{p+1}} R^{(p+1)}(x, \lambda^{-\sigma_{p+1}} D; \lambda). \tag{3.2.15}$$

Furthermore conditions i)$_{p+1}$–v)$_{p+1}$ *hold.*

As a consequence of Lemma 3.2 and Lemma 3.2.1 we can state the following

Lemma 3.2.2. *Assume* (3.2.7) *holds. Then the construction of an asymptotic solution for* (3.14) *verifying* i)$_p$–v)$_p$ *is reduced to the construction of an asymptotic solution for* (3.2.15) *verifying* i)$_{p+1}$–v)$_{p+1}$.

4. Proof of Theorem 2.1

4.1. The case $q_0 \leq 2$ This section is devoted to the proof of Theorem 2.1. We start our argument in the case $q_0 \leq 2$ and in the next subsection we show how to modify it to prove Theorem 2.1 in the case $q_0 = 3$.

By (3.11), (3.12) and (3.13) we may start our induction process with $\tilde{G}^{(0)}$, σ_0, θ_0 and apply Lemma 3.2.2.

We point out explicitly here that (3.2.7) is not assumed to hold. However by (3.2.12) for each p either $q_p = 2$ or $q_p = 1$ and $q_p \geq r_{p+1}s_0^{(p)}$ in general. This implies that if $q_p = 1$ necessarily $r_{p+1} = s_0^{(p)} = 1$, i.e. we are in a scalar case and (3.2.7) is verified.

If the former case holds, i.e. if $q_p = 2$, then either $r_{p+1} = 2$ and (3.2.7) holds or $r_{p+1} = 1$. But in this case we are again in a scalar case and it cannot occur then that $q_p > r_{p+1}s_0^{(p)}$.

Summing up if $q_p \leq 2$ then necessarily (3.2.7) holds true. Next we show that the induction process for computing the phase function ends after finitely many steps.

Proposition 4.1.1. *Under the assumptions of* Theorem 2.1 *the iteration procedure of* Lemma 3.2.2 *occurs only a finite number of times before reaching a point where*

$$\sigma_{\bar{p}+1} = \sigma_0 - \sum_{i=1}^{\bar{p}+1} \theta_i \leq 0 \tag{4.1.1}$$

for a suitable integer $\bar{p}$.

In order to prove Proposition 4.1.1 we need a preliminary lemma:

Lemma 4.1.1. *Assume that there exists a* $\bar{p} \in \mathbb{N}$ *such that*

$$q_{\bar{p}} = q_{\bar{p}+1} = \cdots = q,$$
$$r_{\bar{p}} = r_{\bar{p}+1} = \cdots = r.$$

Under the hypotheses of Theorem 2.1 *there exists a* $k = k(\bar{p})$ *such that*

$$\sigma_p, \theta_p, \delta_j(\tilde{G}^{(p)}), j \geq 1, \quad \textit{belong to } \frac{\mathbb{N}}{k} \tag{4.1.2}$$

for every $p \geq \bar{p}$.

Proof. By (3.2.12) we have $q \leq 2$. Let us start considering the case $q = 2$. If $q = 1$ the argument is of the same kind and easier.

By what has been said above if $q = 2$ then either $s_0^{(p)} = 1$ or $r = 1$.

i) $s_0^{(p)} = 1$. The fact that $q_{p+1} = q_p$ implies that there are no roots of the equation $\det \tilde{G}_0^{(p+1)}(x,\xi) = 0$ with respect to ξ_0 with uniform multiplicity less than q_p. $\tilde{G}_0^{(p+1)}(x,\xi)$ is given by (3.2.6). Two cases may occur: either the sum in (3.2.6) has $\hat{F}_0^{(p)}$ as the only summand or there are also other summands. In the former case we have that $\theta_{p+1}s_0^{(p)} < \theta_{p+1}s_j^{(p)} + \delta_j(F^{(p)})$, for every $j \geq 1$, which implies, if $s_j^{(p)} < s_0^{(p)}$, that

$$\theta_{p+1} < \frac{\delta_j(F^{(p)})}{s_0^{(p)} - s_j^{(p)}}$$

or, because of (3.2.2),

$$\theta_{p+1} = \theta_p. \tag{4.1.3}$$

Assume now that there are terms other than $\hat{F}_0^{(p)}$, corresponding to $j > 0$, in the sum in (3.2.6). Since $s_0^{(p)} = 1$ the condition defining the sum implies that

$$\delta_1(F^{(p)}) = \theta_{p+1}. \tag{4.1.4}$$

ii) $r = 1$. We are then in a scalar case. Again considering the sum in (3.2.6) we conclude (4.1.3) if $\hat{F}_0^{(p)}$ is the only summand. Let us assume that there are also other summands different from $\hat{F}_0^{(p)}$.
Now $s_0^{(p)} = q$ and the assumption of the lemma implies that there is a $j \geq 1$ such that

$$\delta_j(F^{(p)}) = \theta_{p+1} \tag{4.1.5}$$

because of the following lemma.

Lemma 4.1.2. *Let $f_j(x,\xi)$ be homogeneous polynomials with respect to the variable $\xi$of degree q_j, $0 = q_0 < q_1 < \cdots < q_s$. Assume that*
i) *there exists a point $\bar{x}$ such that $f_0(\bar{x}) \neq 0$,*
ii) *f_s is non-characteristic with respect to ξ_0.*
Then the roots of $p(x,\xi) = \sum_{i=0}^{s} f_i(x,\xi)$ with respect to ξ_0 have multiplicity at most s near $(x,\xi') = (\bar{x},0)$.

We skip the proof of Lemma 4.1.2 and go back to the proof of Lemma 4.1.1. Summing up in both cases we conclude that either (4.1.3) or (4.1.4) hold. In particular this implies that $k(p+1) = k(p)$, since the $\delta_j(\tilde{G}^{(p+1)})$ are obtained summing and multiplying rational numbers whose denominator is $k(p)$. This ends the proof of Lemma 4.1.1. □

Proof of Proposition 4.1.1. By contradiction. If one could go through infinitely many iteration steps then necessarily the assumption of Lemma 4.1.1 must hold. But

in that case it is impossible for the series $\sum_{i=1}^{\infty} \theta_i$ to be convergent. Thus after a finite number of iteration steps we get a negative σ_t, for a suitable positive integer t. $\square$

In order to complete the proof of Theorem 2.1, by Proposition 4.1.1, we may assume that for a certain positive integer t

$$\sigma_t > 0 \tag{4.1.6}$$

but

$$\sigma_{t+1} = \sigma_t - \theta_{t+1} \leq 0. \tag{4.1.7}$$

Therefore

$$\theta_{t+1} = \min_{\substack{j>0,\\ s_j^{(t)} < s_0^{(t)}}} \left\{ \frac{\delta_j(F^{(t)})}{s_0^{(t)} - s_j^{(t)}}, \theta_t \right\} \geq \sigma_t. \tag{4.1.8}$$

Our purpose is to construct an asymptotic null solution for the operator

$$\tilde{G}^{(t)}(x, \varphi_x^{(t)}(x) + \lambda^{-\sigma_t} D; \lambda) + \lambda^{-\sigma_t} R^{(t)}(x, \lambda^{-\sigma_t} D; \lambda), \tag{4.1.9}$$

where $R^{(t)} \in \mathcal{R}_U$, in a neighborhood of the origin.

At this stage of the construction we can still apply Lemma 3.2 in order to possibly reduce the rank of the matrix in Equation (4.1.9). Hence we wind up with the construction of an asymptotic solution for the operator

$$F^{(t)}(x, \lambda^{-\sigma_t} D; \lambda) = \sum_{j \geq 0} \lambda^{-\delta_j(F^{(t)})} F_j^{(t)}(x, \lambda^{-\sigma_t} D) \tag{4.1.10}$$

of size $r_{t+1} \times r_{t+1}$. If

$$F_j^{(t)}(x, \lambda^{-1}\xi) = \lambda^{-s_j^{(t)}} \left[\hat{F}_j^{(t)}(x, \xi) + O(\lambda^{-1}) \right] \tag{4.1.11}$$

when $j \geq 0$ and $\lambda \to +\infty$, the operator in (4.1.10) can be written as

$$F^{(t)}(x, \lambda^{-\sigma_t} D; \lambda) = \sum_{j \geq 0} \lambda^{-\delta_j(F^{(t)}) - s_j^{(t)} \sigma_t} \left[\hat{F}_j^{(t)}(x, D) + O(\lambda^{-\sigma_t}) \right], \tag{4.1.12}$$

where $O(\lambda^{-\sigma_t})$ stands for a (matrix-valued) differential operator of order $\leq s_j^{(t)}$ whose coefficients are $O(\lambda^{-\sigma_t})$ uniformly in U. The fact that $\theta_{t+1} \geq \sigma_t$ implies that for every $j \geq 1$

$$\delta_j(F^{(t)}) + \sigma_t s_j^{(t)} \geq \sigma_t s_0^{(t)} \tag{4.1.13}$$

so that

$$F^{(t)}(x, \lambda^{-\sigma_t} D; \lambda) = \lambda^{-\sigma_t s_0^{(t)}} \left[\sum_{\sigma_t s_0^{(t)} = \sigma_t s_j^{(t)} + \delta_j(F^{(t)})} \hat{F}_j^{(t)}(x, D) + \lambda^{-\varepsilon_t} \sum_{j \geq 0} \lambda^{-\tilde{\delta}_j(F^{(t)})} \tilde{F}_j^{(t)}(x, D) \right], \tag{4.1.14}$$

where ε_t is a positive rational number whose denominator can be chosen to be the same as the denominator of σ_t, $\delta_j(F^{(t)})$, $j \geq 1$. Moreover $0 = \tilde{\delta}_0(F^{(t)}) < \tilde{\delta}_1(F^{(t)}) < \cdots$ and the terms with $j > 0$ in the first sum have order $s_j^{(t)}$, $j > 0$, with $s_j^{(t)} < s_0^{(t)}$.

Arguing as in the proof of Lemma 3.2.1 we can show that the principal part of the differential operator in the first sum, $\hat{F}_0^{(t)}$, is non characteristic.

Disposing of the power of λ in front of the operator in square brackets we are left, in the end, with the task of constructing an asymptotic solution for an operator of the form

$$P_{s_0^{(t)}}(x, D) + \tilde{P}(x, D) + \lambda^{-l/k} \sum_{j \geq 0} \lambda^{-j/k} P_j(x, D), \tag{4.1.15}$$

where $k \in \mathbb{N}$, $l \in \mathbb{N}$, $\operatorname{ord} \tilde{P} < s_0^{(t)}$ and $P_{s_0^{(t)}}$ is a non characteristic homogenous differential operator of order $s_0^{(t)}$. One can then seek an asymptotic solution for (4.1.15) in the form

$$\sum_{j \geq 0} \lambda^{-j/k} u_j(x)$$

and this is a well-known procedure. This ends the proof of Theorem 2.1.

4.2. The case $q_0 = 3$ This subsection is devoted to the proof of Theorem 2.1 in the case $q_0 = 3$. Actually we argue for a generic iteration step for which $q_p = 3$ and show how to modify the above argument to prove the theorem.

If $r_{p+1} s_0^{(p)} = q_p = 3$ then the argument in the previous section can be applied without modification; so we stick to the case $r_{p+1} s_0^{(p)} < q_p = 3$. Again if $r_{p+1} = 1$ we are in a scalar case, so that necessarily (3.2.7) holds and the previous proof can be applied. The only case left out is when $r_{p+1} = 2$ and then necessarily $s_0^{(p)} = 1$ and this is the case considered in the present section.

First we need some more precise notation; formula (3.2.1) can be written in the form

$$F_j^{(p)}(x, \lambda^{-\theta} \xi) = \lambda^{-\theta s_j^{(p)}} \left[\hat{F}_{j0}^{(p)}(x, \xi) + \lambda^{-\theta} \hat{F}_{j1}^{(p)}(x, \xi) + O(\lambda^{-2\theta}) \right], \tag{4.2.1}$$

θ being a positive real number, $\hat{F}_{j0}^{(p)}(x, \xi)$ vanishes of order $s_j^{(p)}$ as $|\xi| \to 0$ and $\hat{F}_{j1}^{(p)}(x, \xi)$ vanishes of order $s_j^{(p)} + 1$ as $|\xi| \to 0$.

By Lemma 3.2, since $\det F_0^{(p)}$ vanishes at $\xi = 0$ of the third order and $s_0^{(p)} = 1$ we conclude that $\det \hat{F}_{00}^{(p)}(x,\xi) \equiv 0$, $\hat{F}_{00}^{(p)}(x,\xi)$ being in general a non zero matrix depending linearly on ξ. Let us use a simpler notation for $\hat{F}_{00}^{(p)}(x,\xi)$:

$$\hat{F}_{00}^{(p)}(x,\xi) = \begin{bmatrix} l_{11}(x,\xi) & l_{12}(x,\xi) \\ l_{21}(x,\xi) & l_{22}(x,\xi) \end{bmatrix}, \tag{4.2.2}$$

where the $l_{ij}(x,\xi)$ are linear forms in the variable ξ with real analytic coefficients depending on the variable x defined in U. We may always assume that the linear form $l_{11}(x,\xi)$ is not identically zero.

Then we have the following lemma, whose proof we omit:

Lemma 4.2.1. *Using the above notation and assuming that $l_{11}(x,\xi) \not\equiv 0$ we can find two real analytic non singular matrices, $M(x)$, $N(x)$, defined in U, such that $\hat{F}_{00}^{(p)}(x,\xi)$ can be written in one of the following forms*:

i) *If $l_{11}(x,\xi)$ and $l_{12}(x,\xi)$ are linearly independent then*

$$M(x)\hat{F}_{00}^{(p)}(x,\xi)N(x) = \begin{bmatrix} l_{11}(x,\xi) & l_{12}(x,\xi) \\ 0 & 0 \end{bmatrix};$$

ii) *If $l_{11}(x,\xi)$ and $l_{21}(x,\xi)$ are linearly independent then*

$$M(x)\hat{F}_{00}^{(p)}(x,\xi)N(x) = \begin{bmatrix} l_{11}(x,\xi) & 0 \\ l_{21}(x,\xi) & 0 \end{bmatrix};$$

iii)

$$M(x)\hat{F}_{00}^{(p)}(x,\xi)N(x) = \begin{bmatrix} l_{11}(x,\xi) & 0 \\ 0 & 0 \end{bmatrix}$$

otherwise.

Using this lemma we want to modify the argument in Subsection 3.2 in order to complete the inductive step. Obviously (3.2.7) no longer holds.

The first and second cases in the above lemma are essentially the same, since the forms differ only by a transposition. We focus first on the second case, i.e. when $l_{11}(x,\xi)$ and $l_{21}(x,\xi)$ are linearly independent linear forms with respect to the variable ξ. Denote by

$$\tilde{F}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) = M(x)F^{(p)}(x, \lambda^{-\sigma_p} D; \lambda)N(x)\Gamma_\lambda, \tag{4.2.3}$$

where $M(x)$ and $N(x)$ are the matrices in Lemma 4.2.1 and Γ_λ is the matrix

$$\Gamma_\lambda = \begin{bmatrix} \lambda^{-\theta/2} & 0 \\ 0 & \lambda^{\theta/2} \end{bmatrix}. \tag{4.2.4}$$

Since $M(x)$ and $N(x)$ are non singular matrices, the construction of an asymptotic solution for $F^{(p)}(x, \lambda^{-\sigma_p} D; \lambda)$ is reduced to the construction of an asymptotic solution for $\tilde{F}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda)$. On the other hand

$$\tilde{F}^{(p)}(x, \lambda^{-\sigma_p}\xi; \lambda) = \sum_{\alpha\geq 0}\sum_{j\geq 0}\frac{1}{\alpha!}\lambda^{-\delta_j(F^{(p)})-\sigma_p|\alpha|}M(x)F_j^{(p)(\alpha)}(x, \lambda^{-\sigma_p}\xi)N_{(\alpha)}(x)\Gamma_\lambda,$$

so that, if θ is a positive real number, $\theta < \sigma_p$, we have

$$\begin{aligned}\tilde{F}^{(p)}(x, \lambda^{-\theta}\xi; \lambda) &= \sum_{\alpha\geq 0}\sum_{j\geq 0}\frac{1}{\alpha!}\lambda^{-\delta_j(F^{(p)})-\sigma_p|\alpha|}M(x)F_j^{(p)(\alpha)}(x, \lambda^{-\theta}\xi)N_{(\alpha)}(x)\Gamma_\lambda \\ &= \sum_{j\geq 0} S_j(x, \lambda^{-\theta}\xi; \lambda).\end{aligned} \tag{4.2.5}$$

Let us take a look at the S_j's above when $j \geq 1$. We have:

$$\begin{aligned}S_j(x, \lambda^{-\theta}\xi; \lambda) &= \sum_{|\alpha|\leq s_j^{(p)}}\frac{1}{\alpha!}\lambda^{-\delta_j(F^{(p)})-s_j^{(p)}\theta-(\sigma_p-\theta)|\alpha|}\left[K_{j\alpha}(x,\xi)+O(\lambda^{-\theta})\right]\Gamma_\lambda \\ &\quad + \sum_{|\alpha|\geq s_j^{(p)}+1}\frac{1}{\alpha!}\lambda^{-\delta_j(F^{(p)})-|\alpha|\theta-(\sigma_p-\theta)|\alpha|}\left[K_{j\alpha}(x,\xi)+O(\lambda^{-\theta})\right]\Gamma_\lambda \\ &= \lambda^{-\delta_j(F^{(p)})-s_j^{(p)}\theta+\theta/2}\left\{\begin{bmatrix}0 & c_j(x,\xi)\\ 0 & d_j(x,\xi)\end{bmatrix}+O(\lambda^{-\theta})\right\},\end{aligned} \tag{4.2.6}$$

where c_j and d_j have order $\leq s_j^{(p)}$ with respect to ξ.

Let us now consider $S_0(x, \lambda^{-\theta}\xi; \lambda)$; proceding as above we have

$$\begin{aligned}S_0(x, \lambda^{-\theta}\xi; \lambda) &= \sum_{\alpha\geq 0}\frac{1}{\alpha!}\lambda^{-\sigma_p|\alpha|}M(x)F_0^{(p)(\alpha)}(x, \lambda^{-\theta}\xi)N_{(\alpha)}(x)\Gamma_\lambda \\ &= \left\{\lambda^{-\theta}\begin{bmatrix}l_{11}(x,\xi) & 0\\ l_{21}(x,\xi) & 0\end{bmatrix}+\lambda^{-2\theta}\begin{bmatrix}m_{11}(x,\xi) & m_{12}(x,\xi)\\ m_{21}(x,\xi) & m_{22}(x,\xi)\end{bmatrix}+O(\lambda^{-3\theta})\right\}\Gamma_\lambda \\ &\quad + \lambda^{-\sigma_p}\left\{K_{01}(x)+O(\lambda^{-\theta})\right\}\Gamma_\lambda + O(\lambda^{-2\sigma_p})\Gamma_\lambda \\ &= \lambda^{-3\theta/2}\left\{\begin{bmatrix}l_{11}(x,\xi) & m_{12}(x,\xi)\\ l_{21}(x,\xi) & m_{22}(x,\xi)\end{bmatrix}+O(\lambda^{-\theta})\right\} \\ &\quad +\lambda^{-\sigma_p+\theta/2}\left\{\begin{bmatrix}0 & c_0^{(p)}(x)\\ 0 & d_0^{(p)}(x)\end{bmatrix}+O(\lambda^{-\theta})\right\}+O(\lambda^{-2\sigma_p+\theta/2}).\end{aligned} \tag{4.2.7}$$

Here we have used the fact that first order derivatives with respect to ξ of a linear form in ξ yield only a function of x.

Lemma 4.2.2. *Define* $j^* = \min\{j \mid s_j^{(p)} = 0\}$. *We understand that* $j^* = +\infty$ *if* $s_j^{(p)} \geq 1$ *for every* $j \geq 0$. *Choose*

$$\theta_{p+1} = \min\left\{\delta_1(F^{(p)}), \frac{\delta_{j^*}(F^{(p)})}{2}, \frac{\sigma_p}{2}\right\} \tag{4.2.8}$$

and

$$\theta_{p+1} = \min\left\{\delta_1(F^{(p)}), \frac{\sigma_p}{2}\right\}$$

if $j^* = +\infty$. *Then*

$$\begin{aligned}\tilde{F}^{(p)}(x, \lambda^{-\theta_{p+1}}\xi; \lambda) &= \lambda^{-(3/2)\theta_{p+1}} \sum_{j\geq 0} \lambda^{-\delta_j(\tilde{G}^{(p+1)})} \tilde{G}_j^{(p+1)}(x, \xi) \\ &= \lambda^{-(3/2)\theta_{p+1}} \tilde{G}^{(p+1)}(x, \xi; \lambda).\end{aligned}$$

In particular we have

$$\tilde{G}_0^{(p+1)}(x, \xi) = \begin{bmatrix} l_{11}(x, \xi) & m_{12}(x, \xi) + c(x, \xi) \\ l_{21}(x, \xi) & m_{22}(x, \xi) + d(x, \xi) \end{bmatrix},$$

where $\deg_\xi c$, $d \leq 1$ *and*

$$\det \begin{bmatrix} l_{11}(x, \xi) & m_{12}(x, \xi) \\ l_{21}(x, \xi) & m_{22}(x, \xi) \end{bmatrix} = e(x) \sum_{|\alpha|=q_p} \frac{1}{\alpha!} \left(\det \tilde{G}_0^{(p)}\right)^{(\alpha)} (x, \varphi_x(x))\xi^\alpha,$$

where $e(x)$ *is a non zero smooth function.*

Proof. Let us assume $j^* < +\infty$. Then if $s_j^{(p)} \geq 1$ we have that

$$\delta_j(F^{(p)}) + s_j^{(p)}\theta_{p+1} - \frac{\theta_{p+1}}{2} \geq \delta_1(F^{(p)}) + \frac{\theta_{p+1}}{2} \geq \frac{3\theta_{p+1}}{2},$$

where equality implies $s_j^{(p)} = 1$. On the other hand if $s_j^{(p)} = 0$ then

$$\delta_j(F^{(p)}) + s_j^{(p)}\theta_{p+1} - \frac{\theta_{p+1}}{2} \geq \delta_{j_*}(F^{(p)}) - \frac{\theta_{p+1}}{2} \geq \frac{3\theta_{p+1}}{2},$$

where equality implies that the corresponding terms in the expansion (4.2.6) are functions of x only. At last we obviously have $\sigma_p - \theta_{p+1}/2 \geq (3\theta_{p+1})/2$. We may therefore write

$$\tilde{F}^{(p)}(x, \lambda^{-\theta_{p+1}}\xi; \lambda) = \lambda^{-(3/2)\theta_{p+1}} \sum_{j\geq 0} \lambda^{-\delta_j(\tilde{G}^{(p+1)})} \tilde{G}_j^{(p+1)}(x, \xi) \tag{4.2.9}$$

and forget about the factor in front. The other relations are deduced from Lemma 3.2.2. This proves the lemma in the case $j^* < +\infty$. If $j^* = +\infty$ the argument is the same without any reference to the case $s_j^{(p)} = 0$. □

In the definition of θ_{p+1} in (4.2.8) a division by two may occur. This could hamper our technique of showing that only a finite number of phases is needed. In the next lemma we show that if we have a triple root of $\det F_0^{(p)}$, then this can occur only once in size 2×2 and then everything becomes scalar.

Lemma 4.2.3. *Assume that* $\det \tilde{G}_0^{(p+1)}(x, \xi)$ *has a root of uniform multiplicity* 3. *Then there is a real analytic phase function* $\varphi^{(p+1)}(x)$ *such that*

$$\det \tilde{G}_0^{(p+1)}(x, \varphi_x^{(p+1)}(x)) = 0,$$
$$\operatorname{rank} \tilde{G}_0^{(p+1)}(x, \varphi_x^{(p+1)}(x)) = 1$$

in some open set in $\mathbb{R}^{n+1}$.

Proof. Using Lemma 3.2.2 we see that

$$\det \tilde{G}_0^{(p+1)}(x, \xi) = c_{p+1}(x) \left(\xi_0 - \tau(x, \xi')\right)^3$$

for some non vanishing smooth function c_{p+1} and τ, a first order polynomial with respect to ξ'. We can then construct a function $\varphi^{(p+1)}$ such that $\varphi_{x_0}^{(p+1)} = \tau(x, \varphi_{x'}^{(p+1)})$ and, for some small t, $\varphi^{(p+1)}(t, x') = \langle x', \eta' \rangle$. Then the matrix $\tilde{G}_0^{(p+1)}(x, \varphi_x^{(p+1)}(x))$ has its first column with entries $l_{s1}(x, \varphi_x^{(p+1)}(x))$, $s = 1, 2$. We claim that $l_{s1}(x, \varphi_x^{(p+1)}(x))$ cannot both vanish. In fact if they both vanish at the same time identically, we could deduce that for $x_0 = t$ the linear forms l_{s1} are not linearly independent because η' is arbitrary, and this is a contradiction to the assumption of case ii) of Lemma 4.2.1. Hence the rank of $\tilde{G}_0^{(p+1)}(x, \varphi_x^{(p+1)}(x))$ is one and the following iteration reduces it to a scalar problem in which (3.2.7) holds. □

This completes the discussion in the case ii) of Lemma 4.2.1. Let us now consider case iii) of the same lemma. Let now

$$\Gamma_\lambda = \begin{bmatrix} \lambda^{-\theta/4} & 0 \\ 0 & \lambda^{\theta/4} \end{bmatrix} \tag{4.2.10}$$

and define

$$\tilde{F}^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) = \Gamma_\lambda M(x) F^{(p)}(x, \lambda^{-\sigma_p} D; \lambda) N(x) \Gamma_\lambda. \tag{4.2.11}$$

Arguing as in the proof of Lemma 4.2.2 we can easily show the following

Lemma 4.2.4. *Define* j^* *as in* Lemma 4.2.2 *and choose* θ_{p+1} *according to* Equation (4.2.8). *Then*

$$\begin{aligned}\tilde{F}^{(p)}(x, \lambda^{-\theta_{p+1}}\xi; \lambda) &= \lambda^{-(3/2)\theta_{p+1}} \sum_{j\geq 0} \lambda^{-\delta_j(\tilde{G}^{(p+1)})} \tilde{G}_j^{(p+1)}(x, \xi) \\ &= \lambda^{-(3/2)\theta_{p+1}} \tilde{G}^{(p+1)}(x, \xi; \lambda).\end{aligned}$$

In particular we have

$$\tilde{G}_0^{(p+1)}(x, \xi) = \begin{bmatrix} l_{11}(x, \xi) & 0 \\ 0 & m_{22}(x, \xi) + d(x, \xi) \end{bmatrix}, \tag{4.2.12}$$

where $\deg_\xi d \leq 1$ *and*

$$\det \begin{bmatrix} l_{11}(x, \xi) & 0 \\ 0 & m_{22}(x, \xi) \end{bmatrix} = e(x) \sum_{|\alpha|=q_p} \frac{1}{\alpha!} \left(\det \tilde{G}_0^{(p)}\right)^{(\alpha)} (x, \varphi_x(x))\xi^\alpha,$$

where $e(x)$ *is a non zero smooth function.*

We point out explicitly that if $d(x, \xi)$ in the above expression for $\tilde{G}_0^{(p+1)}(x, \xi)$ is not zero or $m_{22}(x, \xi)$ is not proportional, as a quadratic form in ξ, to $l_{11}(x, \xi)^2$, then $\det \tilde{G}_0^{(p+1)}(x, \xi)$ has roots at most double and we fall back to the case discussed in Subsection 4.1. Thus the worst case occurs when in (4.2.12) we have

$$d(x, \xi) = 0 \quad \text{and} \quad m_{22}(x, \xi) = \alpha(x) l_{11}^2(x, \xi) \tag{4.2.13}$$

for a non zero smooth function $\alpha(x)$.

Next we discuss case iii) of Lemma 4.2.1 when (4.2.13) occurs. Dividing out the factor $\lambda^{-(3\theta_{p+1})/2}$, and keeping into account that in this case no rank reduction is possible, thus preserving the 2×2 size, we reach the point where

$$\tilde{G}^{(p+1)}(x, \xi; \lambda) = F^{(p+1)}(x, \xi; \lambda) = \sum_{j\geq 0} \lambda^{-\delta_j(F^{(p+1)})} F_j^{(p+1)}(x, \xi), \tag{4.2.14}$$

where

$$\tilde{G}_0^{(p+1)}(x, \xi) = F_0^{(p+1)}(x, \xi) = \begin{bmatrix} l(x, \xi) & 0 \\ 0 & \alpha(x) l^2(x, \xi) \end{bmatrix}, \tag{4.2.15}$$

$l(x, \xi)$ being a linear form in ξ and α as above.

By (4.2.15) we do not need intertwining matrices any more and compute

$$\begin{aligned}\Gamma_\lambda F^{(p+1)}(x,\lambda^{-\theta}\xi;\lambda)\Gamma_\lambda &= \sum_{j\geq 0}\lambda^{-\delta_j(F^{(p+1)})}\Gamma_\lambda F_j^{(p+1)}(x,\lambda^{-\theta}\xi)\Gamma_\lambda \\ &= \sum_{j\geq 0}\lambda^{-\delta_j(F^{(p+1)})-\theta s_j^{(p+1)}}\Gamma_\lambda\left[\hat{F}_{j,0}^{(p+1)}(x,\xi)+\lambda^{-\theta}\hat{F}_{j,1}^{(p+1)}+\cdots\right]\Gamma_\lambda, \qquad (4.2.16)\end{aligned}$$

where the $\hat{F}_{j,h}^{(p+1)}(x,\xi)$ are homogeneous of order $s_j^{(p+1)}+h$ with respect to ξ as in (3.2.1).

The above sum can be written as

$$\begin{aligned}\Gamma_\lambda F^{(p+1)}(x,\lambda^{-\theta}\xi)\Gamma_\lambda = {}& \lambda^{-(3\theta)/2}\begin{bmatrix} l(x,\xi) & 0 \\ 0 & \alpha(x)l^2(x,\xi)\end{bmatrix} \\ &+\sum_{j\geq 1}\lambda^{-\delta_j(F^{(p+1)})-\theta s_j^{(p+1)}}\left\{\lambda^{\theta/2}\begin{bmatrix} 0 & 0 \\ 0 & \left(\hat{F}_{j,0}^{(p+1)}(x,\xi)\right)_{22}\end{bmatrix}\right. \\ &+\begin{bmatrix} 0 & \left(\hat{F}_{j,0}^{(p+1)}(x,\xi)\right)_{12} \\ \left(\hat{F}_{j,0}^{(p+1)}(x,\xi)\right)_{21} & 0\end{bmatrix} \\ &+\lambda^{-\theta/2}\begin{bmatrix} \left(\hat{F}_{j,0}^{(p+1)}(x,\xi)\right)_{11} & 0 \\ 0 & \left(\hat{F}_{j,1}^{(p+1)}(x,\xi)\right)_{22}\end{bmatrix} \\ &\left.+\lambda^{-\theta}\begin{bmatrix} 0 & \left(\hat{F}_{j,1}^{(p+1)}(x,\xi)\right)_{12} \\ \left(\hat{F}_{j,1}^{(p+1)}(x,\xi)\right)_{21} & 0\end{bmatrix}+\cdots\right\}.\end{aligned}$$

Now denote by $j^* = \min\{j \mid s_j^{(p+1)} = 0\}$ and assume that $j^* < \infty$. If $\left(\hat{F}_{j^*,0}^{(p+1)}(x)\right)_{22} \neq 0$ then we can choose

$$\theta_{p+2} = \min\left\{\frac{\delta_{j^*}(F^{(p+1)})}{2}, \sigma_{p+1}\right\}$$

and in such a case we can see using the same arguments as above that $\det \tilde{G}_0^{(p+2)}(x,\xi)$ no longer has a triple root where

$$\Gamma_\lambda F^{(p+1)}(x,\lambda^{-\theta_{p+2}}\xi;\lambda)\Gamma_\lambda = \lambda^{-(3/2)\theta_{p+2}}\tilde{G}^{(p+2)}(x,\xi;\lambda).$$

On the other hand if $\left(\hat{F}_{j^*,0}^{(p+1)}(x)\right)_{22} \equiv 0$ and

$$\begin{bmatrix} 0 & \left(\hat{F}_{j^*,0}^{(p+1)}(x,\xi)\right)_{12} \\ \left(\hat{F}_{j^*,0}^{(p+1)}(x,\xi)\right)_{21} & 0\end{bmatrix} \neq 0$$

we may choose

$$\theta_{p+2} = \min\left\{\frac{2}{3}\delta_{j^*}(F^{(p+1)}), \sigma_{p+1}\right\}$$

and see that either, as before, $\det \tilde{G}_0^{(p+2)}(x,\xi)$ no longer has a triple root or, if it still has one, then there is a phase function $\varphi^{(p+2)}(x)$ associated with the triple root such that $\operatorname{rank} \tilde{G}_0^{(p+2)}(x, \varphi_x^{(p+2)}(x)) = 1$. Therefore we wind up in a scalar case.

If $\left(\hat{F}_{j^*,0}^{(p+1)}(x)\right)_{22} = 0$ and

$$\begin{bmatrix} 0 & \left(\hat{F}_{j^*,0}^{(p+1)}(x,\xi)\right)_{12} \\ \left(\hat{F}_{j^*,0}^{(p+1)}(x,\xi)\right)_{21} & 0 \end{bmatrix} = 0$$

or $j^* = \infty$ we take

$$\theta_{p+2} = \min\left\{\delta_1(F^{(p+1)}), \sigma_{p+1}\right\}.$$

Then it is easy to see that

$$\tilde{G}_0^{(p+2)}(x,\xi) = \begin{bmatrix} l(x,\xi) + c(x) & 0 \\ 0 & \alpha(x)l^2(x,\xi) + d(x,\xi) \end{bmatrix}$$

where $\deg_\xi d \leq 1$. Note that $\det \tilde{G}_0^{(p+2)}(x,\xi) = 0$ has a root of multiplicity at most double if $c(x) \neq 0$ or $d(x,\xi) \neq 0$ by Lemma 4.1.2.

Summing up we proved the following

Lemma 4.2.5. *Assume that* $\tilde{G}^{(p+1)}(x,\xi;\lambda)$ *verifies* (4.2.15). *Then we can find* θ_{p+2} *such that one of the following cases takes place*:
(i) $\det \tilde{G}_0^{(p+2)}(x,\xi) = 0$ *has no longer triple root.*
(ii) *there is a phase function* $\varphi^{(p+2)}(x)$ *associated with the triple root such that*

$$\operatorname{rank} \tilde{G}_0^{(p+2)}(x, \varphi_x^{(p+2)}(x)) = 1.$$

(iii) $\theta_{p+2} = \min\left\{\delta_1(F^{(p+1)}), \sigma_{p+1}\right\}$ *and*

$$\tilde{G}_0^{(p+2)}(x,\xi) = \begin{bmatrix} l(x,\xi) & 0 \\ 0 & \alpha(x)l^2(x,\xi) \end{bmatrix}.$$

If (i) or (ii) in Lemma 4.2.5 occurs then the remaining part of the construction of asymptotic solutions is reduced to the case of double roots and we skip it. If (iii) occurs then one can apply again Lemma 4.2.5 to $\tilde{G}^{(p+2)}(x,\xi;\lambda)$. If (iii) happens repeatedly then (4.1.2) in Lemma 4.1.1 holds because the denominator of θ_{p+2} is the same as that of θ_{p+1} (and σ_{p+1} and the $\delta_j(F^{(p+1)})$'s). Thus the rest of the proof is just a repetition of the case of double roots.

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