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Author(s)	Matumoto, Takao; Nakagawa, Noriaki
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Matumoto, T. and Nakagawa, N.
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COMPLEMENT TO EXPLICIT DESCRIPTION OF HOPF SURFACES AND THEIR AUTOMORPHISM GROUPS

TAKAO MATUMOTO and NORIAKI NAKAGAWA

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Abstract

In the previous paper we determined $\widetilde{Aut}(X)$ of each Hopf surface $X = W/G$ with $W = \mathbf{C}^2 - (0, 0)$ so that its holomorphic automorphism group is given by $Aut(X) = \widetilde{Aut}(X)/G$. We calculate the group of connected components $\pi_0(Aut(X))$ by reviewing the classification.

A Hopf surface X is a compact complex surface whose universal covering space is $W = \mathbf{C}^2 - (0, 0)$. So, $X = W/G$ by denoting its covering transformation group by G. A Hopf surface is called primary if its fundamental group G is isomorphic to the group $\mathbf{Z}$ of integers, and secondary otherwise.

In Theorem 1 of [3] we determined $\widetilde{Aut}(X)$ of each secondary Hopf surface $X = W/G$ so that its holomorphic automorphism group is given by $Aut(X) = \widetilde{Aut}(X)/G$, where $\widetilde{Aut}(X)$ is the normalizer of G in the holomorphic automorphism group of $\mathbf{C}^2$ fixing (0, 0). Moreover, in the following cases (2) and (3) it coincides with the normalizer of G in $GL(2, \mathbf{C})$. We calculate $\pi_0(Aut(X))$ including primary Hopf surfaces as a continuation by correcting some parts of [3].

Before stating Theorem 2 we review the classification of Hopf surfaces. Except the special case (0) that G is not given by any subgroup of $GL(2, \mathbf{C})$, we may assume that $G \subset GL(2, \mathbf{C})$ and G is an extension of $H = \{g \in G;\ |\det g| = 1\}$ by $\mathbf{Z}$. Let a be a primitive m-th root of 1, $\rho_n = \exp(\pi i/n)$, $\zeta = \rho_4$, $\epsilon = \exp(2\pi i/5)$ and $\alpha, \beta, \gamma \in \mathbf{C}$ with $0 < |\alpha|, |\beta|, |\gamma| < 1$. Also, we define $K = \{g \in G;\ \det g = 1\} \subset H$.

The case (1) that $G \subset GL(2, \mathbf{C})$ and G is abelian is divided into two cases:
(A) G is generated by $g(z_1, z_2) = (\alpha z_1, \beta z_2)$ and $h(z_1, z_2) = (a z_1, a^n z_2)$ with $(m, n) = 1$ and
(B) G is generated by $g(z_1, z_2) = (\alpha z_1 + z_2, \alpha z_2)$ and $h(z_1, z_2) = (a z_1, a z_2)$.

The case (2) that G is not abelian and decomposable, that is, isomorphic to $\mathbf{Z} \times H$ is divided into six cases.

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(C1) $G = \langle\gamma I\rangle \times H_1$ where $H_1 = \langle aI\rangle \times B'_{2^k(2l+1)}$ and $K = A_{2(2l+1)}$ with $(2^k(2l+1), m) = 1$, $2l+1 \geq 3$ and $k \geq 3$. Note that $B'_{2^k(2l+1)} = \left\langle h' = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, h = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}\right\rangle$ such that s and d have finite orders $2l+1$ and 2^k respectively. Note also that $K = A_{2(2l+1)}$ is generated by $-h'$.
(C2) $G = \langle\gamma I\rangle \times H_2$ where $H_2 = \langle aI\rangle \times B_n$ and $K = B_n$ with $(m, 4n) = 1$ and $n \geq 2$. Here $B_n = \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} \rho_n & 0 \\ 0 & \rho_n^{-1} \end{pmatrix}\right\rangle$ is the binary dihedral group of order $4n$.
(C3) $G = \langle\gamma I\rangle \times H_3$ where $H_3 = \langle aI\rangle \times C$ and $K = C$ with $(m, 6) = 1$. Here $C = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, (1/\sqrt{2})\begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix}\right\rangle$ is the binary tetrahedral group of order 24.
(C4) $G = \langle\gamma I\rangle \times H_4$ where $H_4 = \langle aI\rangle \times C'_{8\cdot 3^k}$ and $K = B_2$ with $(m, 6) = 1$ and $k \geq 2$. Here $C'_{8\cdot 3^k} = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, (\omega/\sqrt{2})\begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix}\right\rangle$ is a group of order $8 \cdot 3^k$ and ω is a primitive 3^k-th root of 1.
(C5) $G = \langle\gamma I\rangle \times H_5$ where $H_5 = \langle aI\rangle \times D$ and $K = D$ with $(m, 6) = 1$. Here $D = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, (1/\sqrt{2})\begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix}\right\rangle$ is the binary octahedral group of order 48.
(C6) $G = \langle\gamma I\rangle \times H_6$ where $H_6 = \langle aI\rangle \times E$ and $K = E$ with $(m, 30) = 1$. Here $E = \left\langle \begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (1/\sqrt{5})\begin{pmatrix} \epsilon^4 - \epsilon & \epsilon^2 - \epsilon^3 \\ \epsilon^2 - \epsilon^3 & \epsilon - \epsilon^4 \end{pmatrix}\right\rangle$ is the binary icosahedral group of order 120.

The case (3) that G is indecomposable, that is, not isomorphic to the product $\mathbf{Z} \times H$: G is given as

$$G = G_0 \cup gG_0, \quad G_0 = \langle\gamma^2 I\rangle \times H \quad \text{and} \quad g = \gamma u$$

in the following cases from (D1) to (D6) and in the exceptional case (D7)

$$G = G_0 \cup gG_0 \cup g^2 G_0, \quad G_0 = \langle\gamma^3 I\rangle \times H \quad \text{and} \quad g = \gamma u,$$

where H is a finite cyclic group or one of H in the case (2) and $u \in GL(2, \mathbf{C})$.
(D1) The case H is abelian and $K = \left\langle \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}\right\rangle$ is of order $m_K \geq 3$: We can take $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the case is divided into the following three cases.
(D1-1) $H = \langle aI\rangle \times K$ with $K = \left\langle \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}\right\rangle$ where s has the finite order $m_K \geq 3$ and $(m, m_K) = 1$.
(D1-2) $H = \langle aI\rangle \times \left\langle \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\right\rangle \times \left\langle \begin{pmatrix} c & 0 \\ 0 & -c^{-1} \end{pmatrix}\right\rangle$ where b and c have finite orders $2l+1 \geq 1$ and 2^k with $k \geq 3$ respectively. Moreover, we have $(m, 2l+1) = (m, 2) = 1$, $m_K = 2^{k-1}(2l+1)$ and $s = bc^2$.
(D1-3) $H = \langle aI\rangle \times \left\langle \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\right\rangle \times \left\langle \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}\right\rangle$ where b and c have finite orders $2l+1 \geq 3$ and 2^k with $k \geq 3$ respectively. Moreover, $(m, 2l+1) = (m, 2) = 1$, $m_K = 2(2l+1)$ and $s = -b$.

(D2) The case H is abelian and $K = \{\pm I\}$: $H = \langle aI \rangle \times \left\langle \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \right\rangle$ and $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where c is a primitive 2^k-th root of 1 with $k \geq 3$ and $(2, m) = 1$.
(D3) $H = H_1$ as in (C1) and $u = \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$. We have two conjugate classes: $t = 1$ and $t = i$. Their transformation groups are not isomorphic to each other.
(D4 and D5) $H = \langle aI \rangle \times B_n$, $K = B_n$ as in (C2) and $u = \begin{pmatrix} \rho_{2n} & 0 \\ 0 & \rho_{2n}^{-1} \end{pmatrix}$ where (D4) is the case $n \geq 3$ and (D5) is the case $n = 2$.
(D6-1)[1] $H = \langle aI \rangle \times C$, $K = C$ as in (C3) and $u = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$.
(D6-2) $H = \langle aI \rangle \times C'_{8\cdot 3^k}$ with $k \geq 2$, $K = B_2$ as in (C4) and $u = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$.
(D7) $H = \langle aI \rangle \times B_2$, $K = B_2$ as in (D5) and $u = (1/\sqrt{2})\begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix}$.

Since we did not give a proof of the classification for the indecomposable cases (D1) to (D7), we will give an outline of proof. We may assume $K \neq \{I\}$, since G is abelian otherwise (cf. [1] Proposition 8).

We consider the case that H is abelian at first. Since H operates on S^3 freely, H is a cyclic group of order m_H. We may assume that the generator is $\begin{pmatrix} d & 0 \\ 0 & d^n \end{pmatrix}$. The matrix u is also determined as $u = \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$ in this case by [1] Lemma 6 and Proposition 8. The condition $n^2 \equiv 1 \bmod m_H$ should be satisfied because the conjugation of the generator by u is contained in H. Since u does not commute with the generator of H, we get the condition $n \not\equiv 1 \bmod m_H$. Note that $n^2 \equiv 1 \bmod p^k$ implies $n \equiv \pm 1 \bmod p^k$ for odd prime p. In case $p = 2$, $n^2 \equiv 1 \bmod 2^k$ implies $n \equiv \pm 1 \bmod 2^k$ for $k = 1, 2$ and $n \equiv \pm 1$ or $n \equiv \pm 1 + 2^{k-1} \bmod 2^k$ for $k \geq 3$. Let $m_H = 2^k p_1^{k_1} \cdots p_q^{k_q}$ be the prime decomposition.

(D1) Assume first that $n \equiv \pm 1 \bmod 2^k$. Then, we get the case (D1-1). When $k \geq 3$ and $n \equiv -1 + 2^{k-1} \bmod 2^k$, we get the case (D1-2). When $n \equiv 1 + 2^{k-1} \bmod 2^k$ with $k \geq 3$, it is easy to see that $K = \{\pm I\}$ if and only if $n \equiv 1 \bmod p_j^{k_j}$ for every odd prime p_j. Since we treat the case $K \neq \{\pm I\}$ in (D1), the case when $n \equiv 1 + 2^{k-1} \bmod 2^k$ with $k \geq 3$ and $n \not\equiv 1 \bmod p_j^{k_j}$ for some odd prime p_j is named (D1-3). We see also that the matrix u above with any $t \in \mathbf{C}^*$ is conjugate to u with $t = 1$ in these three cases.

(D2) This is the remaining case when $n \equiv 1 + 2^{k-1} \bmod 2^k$ with $k \geq 3$ and $n \equiv 1 \bmod p_j^{k_j}$ for every odd prime p_j. Note that this case $K = \{\pm I\}$ is studied separately in [1] p. 229 and u is determined as above in the indecomposable case. For the matrix u we can take $t = 1$ in the same way as in the case (D1).

When K is not abelian, [1, 2] Lemma 7 and Lemma 7′ imply that K and u are uniquely determined as in the cases (D4) to (D7) including the case (D6-2).

[1] The case (D6-1) is denoted by (D6) in [3] and (D6-2) was missing there.

Finally when K is abelian and H is not abelian, we may assume that $H = H_1$ as in (C1). Since $K = A_{2(2l+1)}$, we are concerned only with Step 3 of [1] pp. 235–236. Hence, the group G is conjugate in $GL(2, \mathbf{C})$ to one of the non-isomorphic groups $\langle h', h, \gamma u_1\rangle$ and $\langle h', h, \gamma u_2\rangle$, where $h' = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$, $h = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$, $u_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. We get the case (D3) and no others. This is the end of the outline of proof.

The case (D3) is subdivided and remarkable since the diffeomorphic classes can be distinguished by $\pi_0(Aut(X))$.

Theorem 2. *The group of connected components* $\pi_0(Aut(X))$ *of the holomorphic automorphism group* $Aut(X) = \widetilde{Aut}(X)/G$ *of each Hopf surface* $X = \{\mathbf{C}^2 - (0, 0)\}/G$ *is described as follows.*

(0) *The case that G is not given by any subgroup of* $GL(2, \mathbf{C})$ $(\pi_1(X) \cong \mathbf{Z} \times \mathbf{Z}_m)$:
$\pi_0(Aut(X)) = 0$.

(1) *The case that G is contained in* $GL(2, \mathbf{C})$ *and abelian* $(\pi_1(X) \cong \mathbf{Z} \times \mathbf{Z}_m)$:
$\pi_0(Aut(X)) = 0$,
except the case (A2) *when* $\alpha = \beta$ *and* $n \not\equiv n^2 \equiv 1 \bmod m$ *where*
$\pi_0(Aut(X)) \cong \mathbf{Z}_2$.

(2) *The case that G is decomposable and not abelian*:
$\pi_0(Aut(X)) = 0$ *in the cases* (C5) *and* (C6).
$\pi_0(Aut(X)) \cong \mathbf{Z}_2$ *in the cases* (C1), (C2) *with* $n \geq 3$, (C3) *and* (C4).
$\pi_0(Aut(X)) \cong D_3$ *in the case* (C2) *with* $n = 2$, *where* D_k *is the dihedral group of order* $2k$.

(3) *The case that G is indecomposable*:
$\pi_0(Aut(X)) = 0$ *in the cases* (D1-1) *with odd* m_K, (D1-2), (D1-3), (D2), (D3) *with* $t = i$, (D4), (D5), (D6-1), (D6-2) *and* (D7).
$\pi_0(Aut(X)) \cong \mathbf{Z}_2$ *in the cases* (D1-1) *with even* m_K *and* (D3) *with* $t = 1$.

Corollary. *For any primary Hopf surface X we see that* $\pi_0(Aut(X)) = 0$.

It is quite interesting to see that any decomposable Hopf surface with non-trivial $\pi_0(Aut(X))$ is a double or triple covering of the corresponding indecomposable Hopf surface: (A2) $\to$ (D1) or (D2), (C1) $\to$ (D3) with $t = i$, (C2) with $n \geq 3 \to$ (D4), (C2) with $n = 2 \to$ (D5) or (D7), (C3) $\to$ (D6-1) and (C4) $\to$ (D6-2). In fact, the covering transformation element u in these cases is non-trivial in $\pi_0(Aut(X))$. Similarly (D1-1) with $m_K = 2l$ is a double covering of (D1-1) with $m_K = 4l$. But as is well-known, the natural operation of $\left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\rangle \Big/ \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \cong \mathbf{Z}_2$ on the Hopf surface in the case (C1) is not fixed point free, and the same for the case (D3) with $t = 1$.

The proof of Theorem 2 follows easily from Theorem 1 of [3]. In the cases (0) and (1) except (A2), we have $\pi_0(\widetilde{Aut}(X)) = 0$ and hence $\pi_0(Aut(X)) = 0$. In the case (A2)

$$\pi_0(Aut(X)) = \mathbf{C}^* I \cdot \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \Big/ \mathbf{C}^* I \cong \mathbf{Z}_2.$$

Now note that $\pi_0((\mathbf{C}^* I \cdot G')/G) = \pi_0((\mathbf{C}^* I \cdot G')/(\mathbf{C}^* I \cdot G)) = \pi_0((\mathbf{C}^* I \cdot G')/(\mathbf{C}^* I \cdot G'')) = G'/G''$ for a normal subgroup G'' of a finite subgroup G' of $GL(2, \mathbf{C})$, provided that $\mathbf{C}^* I \cdot G'' = \mathbf{C}^* I \cdot G$ and $\pi_0(G') = \pi_0(\mathbf{C}^* I \cdot G')$.

Then, in the case (2) we get the result by noticing that $B_{2n}/B_n \cong \mathbf{Z}_2$, $D/B_2 \cong S_4/D_2 \cong D_3$, $D/C \cong \mathbf{Z}_2$ and $\mathbf{C}^* I \cdot C'_{8\cdot 3^k} = \mathbf{C}^* I \cdot C$. In fact, we take $G' = B_{2(2l+1)}$ and $G'' = B_{2l+1}$ in the case (C1). Here, $B_m = \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} \rho_m & 0 \\ 0 & \rho_m^{-1} \end{pmatrix} \right\rangle$ and the fact $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \notin B_{2l+1}$ is essential. In the other cases we take $G' = B_{2n}$ and $G'' = B_n$ in the case (C2) with $n \geq 3$, $G' = D$ and $G'' = B_2$ in the case (C2) with $n = 2$, $G' = D$ and $G'' = C$ in the cases (C3) and (C4), $G' = G'' = D$ in the case (C5) and $G' = G'' = E$ in the case (C6).

The proof for the case (3) is a little more delicate. In the case (D1) we have $\widetilde{Aut}(X) = \mathbf{C}^* I \cdot B_{m_K}$. Moreover, in the case (D1-1) if m_K is odd then $-I \in \mathbf{C}^* I \cdot G$ implies $\mathbf{C}^* I \cdot G = \mathbf{C}^* I \cdot B_{m_K}$, otherwise $\mathbf{C}^* I \cdot G = \mathbf{C}^* I \cdot B_{m_K/2}$. In the case (D1-2) $\begin{pmatrix} c & 0 \\ 0 & -c^{-1} \end{pmatrix} \in G$ implies $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in G$ and hence $\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathbf{C}^* I \cdot G$. On the other hand we can take $bc = \rho_{m_K}$ for $m_K = 2^{k-1}(2l + 1)$. So, we have $\mathbf{C}^* I \cdot G = \mathbf{C}^* I \cdot B_{m_K}$. In the cases (D1-3) and (D2) $\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \in \mathbf{C}^* I \cdot G$ implies $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathbf{C}^* I \cdot G$ and hence $\mathbf{C}^* I \cdot G = \mathbf{C}^* I \cdot B_{2(2l+1)}$, where $l \geq 1$ in the case (D1-3) and $l = 0$ in the case (D2). We used the correction $\widetilde{Aut}(X) = \mathbf{C}^* I \cdot B_2$ below in the case (D2) with $t = 1$. Note that $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ is contained in $\mathbf{C}^* I \cdot G$ and $\mathbf{C}^* I \cdot G = \mathbf{C}^* I \cdot B_{2(2l+1)}$ in the case (D3) with $t = i$. In the case (D3) with $t = 1$ the same element is not contained in $\mathbf{C}^* I \cdot G$ and hence $\mathbf{C}^* I \cdot G = \mathbf{C}^* I \cdot B_{2l+1}$. So, we get the result in these cases. For the other cases we have only to note that $\mathbf{C}^* I \cdot G = \mathbf{C}^* I \cdot B_{2n}$ in the case (D4), $\mathbf{C}^* I \cdot G = \mathbf{C}^* I \cdot B_4$ in the case (D5), $\mathbf{C}^* I \cdot G = \mathbf{C}^* I \cdot D$ in the case (D6-1) and $\mathbf{C}^* I \cdot G = \mathbf{C}^* I \cdot C$ in the case (D7). Also, it is not difficult to see that $\widetilde{Aut}(X) = \mathbf{C}^* I \cdot D = \mathbf{C}^* I \cdot G$ and $\pi_0(Aut(X)) = 0$ in the case (D6-2) which was missing in Theorem 1 of [3]. This completes the proof of Theorem 2.

Finally we note that there are six errors in [3] to be corrected. First 'splits' in the 7-th line from the bottom of the p. 1 should read 'decomposes'. Second shift the condition from $k \geq 1$ to $k \geq 2$ in the case (C4); the action C'_{24} on S^3 has fixed points. Third in the case (D2) where $K = \{\pm I\}$ and H is abelian we should correct $m = 2(2l + 1) \geq 6$ to $m = 2^k(2l + 1)$ with $k \geq 3$, where m denotes the order of H. (In the new description given in this paper $m = 2l + 1$ and the order of H is $2^k(2l + 1)$.) Fourth we should not restrict ourselves to $t \neq 1$ in the case (D3); there are just two conjugate classes as stated above. Fifth there is one more case (D6-2).

Sixth in the case (D2) of Theorem 1 we should correct the conclusion as follows:
(D2) $\widetilde{Aut}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix} \right\rangle \cong \mathbf{C}^* I \cdot B_2$.

The proof which is located in the last five lines of p. 422 of [3] should read: In the case (D2) we have $N_{GL(2,\mathbf{C})}(H) = \left\{ \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix} \;\middle|\; a', b', c', d' \in \mathbf{C}^* \right\}$. The commutator of u and $\begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix}$ is $\begin{pmatrix} d'/a' & 0 \\ 0 & a'/d' \end{pmatrix}$ and that of u and $\begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix}$ is $\begin{pmatrix} c't^{-2}/b' & 0 \\ 0 & b't^2/c' \end{pmatrix}$. Since they should be contained in K, we have $a' = \pm d'$ and $c' = \pm b't^2$. So, we get the result.

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Takao Matumoto
16-7-1002 Osuga-cho
Minami-ku, Hiroshima 732–0821
Japan
e-mail: matumoto@math.sci.hiroshima-u.ac.jp

Noriaki Nakagawa
Tohnan High School
Minami-ku, Kyoto 601–8348
Japan
e-mail: wp-n-nkgw@s4.dion.ne.jp