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## ON THE HIGHER DIMENSIONAL MORDELL CONJECTURE OVER FUNCTION FIELDS

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### Introduction

The purpose of this note is to give a partial answer to the following conjecture which is a function theoretic analogue of Mordell conjecture and was formulated by S. Lang, E. Bombieri and J. Noguchi ([6], [10], [11]):

Let  $K$  be a function field over the complex number field  $\mathbf{C}$ . Let  $V$  be a projective variety defined over  $K$ ,  $\Omega_{V/K}$  the sheaf of regular differential 1-forms  $\omega_V$  the canonical invertible sheaf. Recall that  $V$  is called a variety of general type if the rational mapping associated with the  $l$ -th pluri-canonical system  $|\omega_V^l|$  for an integer  $l > 0$  is birational. We say that  $V$  is isotrivial if there exist a projective variety  $V_0$  defined over  $\mathbf{C}$  and a finite extension  $K'$  of  $K$  such that  $V \otimes_K K'$  is birationally equivalent to  $V_0 \otimes_{\mathbf{C}} K'$ .

**Conjecture M.** *Let  $V$  be a projective variety of general type defined over  $K$ . Suppose that  $V$  is not isotrivial. Then the set of  $K$ -rational points of  $V$  cannot be Zariski dense in  $V$ .*

(i) Mordell conjectured that any curve of genus  $\geq 2$  defined over a number field  $\mathbb{R}$  does not admit an infinite number of  $\mathbb{R}$ -rational points, which is proved by G. Faltings. An analogue of Mordell conjecture over function fields was proved by Y. Manin and H. Grauert ([2], [3], [6]).

In this case a curve is assumed to be not isotrivial over the definition function field.

(ii) J. Noguchi ([11]) and M. Deschamps ([1]) proved Conjecture *M* under the assumption that  $\Omega_{V/K}$  is ample, in other words the fundamental sheaf  $\mathcal{O}_{P(\Omega_{V/K})}(1)$  of the projective bundle  $P(\Omega_{V/K})$  is ample. Note that if  $\mathcal{O}_{P(\Omega_{V/K})}(1)$  is ample then  $\mathcal{O}_{P(\Omega_{V/K})}(\alpha) \otimes \mathcal{O}_{P(\Omega_{V/K})}^{-1}$  for some  $\alpha > 0$  is ample, which turns out to be nef and big (for the definition see §1).

(iii) A compact analytic space  $X$  is said to be hyperbolic if any holomorphic map from  $\mathbf{C}$  into  $X$  is constant, i.e.,  $X$  does not contain any singular elliptic curve as well as any rational curve. It is conjectured that a hyperbolic variety is a

variety of general type.

(iv) D. Riebeschl ([12]) proved Conjecture  $M$  under the hypothesis of negative curvature and the assumption that all the fibres have negative curvature. Further J. Noguchi ([10]) proved it under the hypothesis that  $V$  is hyperbolic with the Chern class  $c(\omega_V)$  represented by a semipositive  $(1, 1)$ -form.

(v) Conjecture  $M$  follows from the boundeness hypothesis to the effect that the intersection number  $(\Gamma, \omega_X)$  is bounded above for any non-singular curve  $\Gamma$  with fixed genus contained in a given variety ([9]).

The main result of this paper is the following:

Let  $K$  be a function field over  $\mathbf{C}$  and let  $V$  be a projective non-singular variety over  $K$ .

**Theorem.** *Assume that  $V$  is of general type and that the fundamental sheaf  $\mathcal{O}_{P(\Omega_{V/K})}(1)$  of the projective bundle  $P(\Omega_{V/K})$  is  $K$ -nef and  $K$ -big and that there exists  $\alpha > 0$  such that  $\mathcal{O}_{P(\Omega_{V/K})}(\alpha) \otimes \omega_{P(\Omega_{V/K})}^{-1}$  is  $K$ -nef. The set of  $K$ -rational points  $\{s_\lambda(K)\}$  is not dense in  $V$  provided that  $V$  is not isotrivial over  $K$ .*

REMARKS. (a) Under the same assumption as above,  $V$  does not contain any rational curve but may contain a singular elliptic curve.

(b) In the previous paper ([9]), the same result was proved under the assumption that  $\mathcal{O}(\alpha) \otimes p^* \omega_X^{-1}$  is  $f \circ p$ -nef over the whole  $X$ , not only over the generic fibre.

### 1. Notation

We recall the following

DEFINITION ([4]). Let  $f: X \rightarrow S$  be a proper morphism onto a variety  $S$  and  $L$  an invertible sheaf on  $X$ . Let  $\eta$  be the generic point of  $S$  and  $L_\eta$  denote the restriction of  $L$  to the generic fibre  $X_\eta$ . An invertible sheaf  $L$  is  $f$ -ample if for any coherent sheaf  $\mathcal{F}$ , the natural homomorphisms  $f^* f_*(\mathcal{F} \otimes L^m) \rightarrow \mathcal{F} \otimes L^m$  for some  $m_0$  and any  $m \geq m_0$  are surjective. An invertible sheaf  $L$  is said to be  $f$ -big, if for any invertible sheaf  $M$  on  $X$ , the natural homomorphism  $f^* f_*(M \otimes L^m) \rightarrow M \otimes L^m$  for some  $m > 0$  is not zero, in other words  $f_*(M \otimes L^m) \neq 0$ . An invertible sheaf  $L$  is said to be  $f$ -nef if  $\deg_D L_D \geq 0$  for every curve  $D$  which is mapped to a point on  $S$  by  $f$ . When  $S = \text{Spec } K$ ,  $f$ -big and  $f$ -nef are said to be  $K$ -big and  $K$ -nef, respectively.

Let  $f: X \rightarrow S$  be a proper surjective morphism of projective complex manifolds. Let  $K$  be the function field of  $S$  and  $V$  the generic fibre of  $f$ . We let  $\Omega_{V/K}$  denote the sheaf of the Kähler differential on  $V$ , let  $P(\Omega_{V/K})$  denote the projective bundle associated to  $\Omega_{V/K}$  over  $V$  and let  $\mathcal{O}_{P(\Omega_{V/K})}(1)$  denote the funda-

mental sheaf over  $\mathbf{P}(\Omega_{V/K})$ . We denote by  $\omega_V$  the canonical invertible sheaf, i.e.,  $\det \Omega_{V/K}$ . We have the exact sequence

$$0 \rightarrow f^*\Omega_K \rightarrow \Omega_V \rightarrow \Omega_{V/K} \rightarrow 0.$$

Then  $\mathbf{P}(\Omega_{V/K}) \subset \mathbf{P}(\Omega_V)$ . We have  $\Omega_V = \mathcal{O}_V \otimes_{\mathcal{O}} (\Omega_X|_V)$  and  $\Omega_K = K \otimes_{\mathcal{O}} \Omega_S$ ;

$$\begin{array}{ccc} \mathbf{P}(\Omega_{X/S}) \supset \mathbf{P}(\Omega_{V/K}) & & \mathbf{P}(\Omega_X) \supset \mathbf{P}(\Omega_V) \\ \downarrow \square \downarrow & & \downarrow \square \downarrow \\ X \supset V & & X \supset V \\ F \downarrow \square \downarrow f & & \\ S \supset \text{Spec } K & & \end{array}$$

Here  $\square$  means that the diagram is cartesian.

### 2. Proof of the main theorem

In order to prove the theorem, we first consider the case in which  $\text{tr. deg } K/C = 1$ . In this case, we denote  $S$  by  $C$ .

**Lemma 1.** *Some power  $\mathcal{O}(\beta)$  of the fundamental sheaf  $\mathcal{O}(1)$  on  $\mathbf{P}(\Omega_V)$  is generated by its global sections for any  $\beta \gg 0$ .*

*Proof.* We will use the following Kawamata-Shoklov's base point free theorem (see [4], Base Point Free Theorem):

*Let  $X$  be a compact manifold and  $f: X \rightarrow S$  a proper surjective morphism onto a variety. Assume that  $L^\alpha \otimes \omega_X^{-1}$  is  $f$ -nef and  $f$ -big for some  $\alpha > 0$  and that  $L$  is  $f$ -nef. Then there exists a positive integer  $m_0$  such that  $f_* f^* L^m \rightarrow L^m$  is surjective for any  $m \geq m_0$ .*

We return to the proof.

Observing the exact sequence  $0 \rightarrow \mathcal{O}_V \rightarrow \Omega_V \rightarrow \Omega_{V/K} \rightarrow 0$ , one sees that  $\mathbf{P}(\Omega_{V/K})$  is identified with a member  $D$  of the complete linear system  $|\mathcal{O}(1)|$  on  $\mathbf{P}(\Omega_V)$ . One has the following exact sequence:

$$0 \rightarrow \mathcal{O}(\beta-1) \rightarrow \mathcal{O}(\beta) \rightarrow \mathcal{O}_D(\beta) \rightarrow 0.$$

By the assumption of the theorem, one has  $H^1(\mathcal{O}_D(\beta)) = 0$  for  $\beta > \alpha$  using Kawamata-Viehweg vanishing theorem ([4]). Hence  $\dim H^1(\mathcal{O}(\beta))$  is a monotonous decreasing function in  $\beta$  if  $\beta \gg 0$ . Thus  $H^0(\mathcal{O}(\beta)) \rightarrow H^0(\mathcal{O}_D(\beta))$  is surjective for sufficiently large number  $\beta$ . On the other hand, applying Kawamata-Shoklov's base point free theorem [4] to  $\mathcal{O}_D(1)$ , one sees that  $\mathcal{O}_D(\beta)$  is base point free for  $\beta > \beta_0 \gg 0$ . Combining these observations, one proves the lemma.  $\square$

Set  $g = f \circ p$ . Then the surjection  $g^* g_* \mathcal{O}(l) \rightarrow \mathcal{O}(l)$  for  $l \gg 0$  gives a  $g$ -birational morphism  $\varphi: \mathbf{P}(\Omega_V) \rightarrow \mathbf{P}(g_* \mathcal{O}(l))$ . Thus one obtains the following diagram:

$$\begin{array}{ccc}
 \mathbf{P}(\Omega_V) & & \\
 \mathfrak{p} \downarrow & \searrow \varphi & \\
 V & & \mathbf{P}(g_*\mathcal{O}(l)) \\
 f \downarrow & \swarrow & \\
 K & & 
 \end{array}$$

Let  $\mathcal{F}$  be a coherent sheaf over  $V$  and let  $T \rightarrow V$  be a map such that there exists a surjection  $\mathcal{F}_T \rightarrow \mathcal{L}$ , where  $\mathcal{L}$  is an invertible sheaf over  $T$ . Then there exists a unique map  $T \rightarrow \mathbf{P}(\mathcal{F})$  over  $V$  such that  $\mathcal{F}_T \rightarrow \mathcal{L}$  is the pull-back to  $T$  of the fundamental surjection  $\mathcal{F}_{\mathbf{P}(\mathcal{F})} \rightarrow \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ . Applying this to the natural surjections  $\Omega_V|_{s_\lambda(K)} \rightarrow \Omega_{s_\lambda(K)}$ , we have the Gauss maps  $\sigma_\lambda: s_\lambda(K) \rightarrow \mathbf{P}(\Omega_V)$ . Let  $Z$  be a component of the Zariski closure of the set of  $K$ -rational points  $\{\sigma_\lambda(s_\lambda(K))\}$  defined by Gauss map such that  $\mathfrak{p}(Z) = V$ . For each  $l$ , the  $l$  multiple of the divisor  $D = \mathbf{P}(\Omega_{V/K})$  is the pull-back of a hyperplane  $\Sigma$  of  $\mathbf{P}(g_*\mathcal{O}(l))$ . We denote  $\varphi(Z)$  by  $W$ . We divide into two cases:

- (i)  $\dim W = 0$ ,
- (ii)  $\dim W > 0$ .

We prove some preliminary lemmas.

**Lemma 2.** *Let  $U$  denote  $\mathbf{P}(\Omega_V) - \mathbf{P}(\Omega_{V/K})$ . Put  $\sigma_\lambda(s_\lambda(K)) =$  the rational point defined by the natural surjection  $\Omega_V|_{s_\lambda(K)} \rightarrow \Omega_{s_\lambda(K)}$  defining  $\sigma_\lambda: s_\lambda(K) \rightarrow \mathbf{P}(\Omega_V)$ . Then  $\sigma_\lambda$  factors through  $U$ . Let  $T$  be any scheme over  $V$  such that there exist an invertible sheaf  $L$  and a surjection  $\Omega_V|_T \rightarrow L$ . Then we have a  $V$ -morphism  $\phi: T \rightarrow \mathbf{P}(\Omega_V)$ . We have the following diagram:*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{O}_V|_T & \longrightarrow & \Omega_V|_T & \longrightarrow & \Omega_{V/K}|_T \longrightarrow 0 \\
 & & a(T) & \searrow & \downarrow & & \\
 & & & & \mathcal{O}(1)|_T & & 
 \end{array}$$

Let  $t$  be a point of  $T$ . If  $\phi(t) \in D$ , we have  $a(t) = 0$  and if  $\phi(t) \in U$ ,  $a(t)$  is bijective. Hence if  $T \subset U$ ,  $a(T)$  is bijective and the exact sequence above splits over  $T$ .

Proof. Since  $f^*\Omega_K = \sigma_\lambda^*\mathcal{O}(1)$ , the result follows. (cf. [1])

**Lemma 3.** *Let  $u: M \rightarrow N$  be a proper surjective morphism between varieties. Suppose that  $N$  is a normal variety. Then the exact sequence  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0$  of locally free sheaves of finite rank on  $N$  splits if and if the pull back of this sequence splits on  $M$ .*

Proof. It follows from the injectivity of the natural map  $H^1(L) \rightarrow H^1(u^*L)$  for any locally free coherent sheaf  $L$ . □

Case (i).

Note that  $\varphi(Z)$  consists of a single point. From Lemmas 2 and 3, one has

the splitting of the exact sequence  $0 \rightarrow f^* \Omega_K \rightarrow \Omega_V \rightarrow \Omega_{V/K} \rightarrow 0$ . We take a projective non-singular model of  $f: V \rightarrow \text{Spec } K$ , denoted by  $f: X \rightarrow C$ . Thus  $f: X \rightarrow C$  is locally trivial in the sense of étale topology.

Case (ii).

Note that  $Z \cap D \neq \emptyset$ .

**Lemma 4.** *The  $K$ -rational points  $\{\sigma_\lambda \circ s_\lambda(K)\}$  on  $P(\Omega_V)$  are not contained in  $\text{Bs}|\mathcal{O}(\beta) \otimes \omega_{\bar{F}}^k|$  for general  $\lambda$  nad some  $\beta$  and  $k > 0$ .*

Proof. Observing the exact sequence  $0 \rightarrow \mathcal{O}_V \rightarrow \Omega_V \rightarrow \Omega_{V/K} \rightarrow 0$ , one sees that  $P(\Omega_{V/K})$  on  $P(\Omega_V)$  is a divisor of the complete linear system  $|\mathcal{O}(1)|$ . One has the following exact sequence:

$$0 \rightarrow \mathcal{O}(\beta - 1) \otimes \omega_{\bar{F}}^{-k} \rightarrow \mathcal{O}(\beta) \otimes \omega_{\bar{F}}^{-k} \rightarrow \mathcal{O}_D(\beta + k) \otimes \omega_{\bar{D}}^{-k} \rightarrow 0.$$

By the assumption of the theorem we can apply Kawamata-Viehweg's vanishing theorem to obtain  $H^1(\mathcal{O}_D(\beta + k) \otimes \omega_{\bar{D}}^{-k}) = 0$ , if  $\beta > \alpha(k + 1) - k$ . Hence  $\dim H^1(\mathcal{O}(\beta) \otimes \omega_{\bar{F}}^{-k})$  is a monotonous decreasing function in  $\beta$  if  $\beta \gg 0$ . Thus  $H^0(\mathcal{O}(\beta) \otimes \omega_{\bar{F}}^{-1}) \rightarrow H^0(\mathcal{O}_D(\beta + k) \otimes \omega_{\bar{D}}^{-k})$  is surjective for sufficiently large number  $\beta$ . By the hypothesis of the theorem, applying Kawamata's base point free theorem [4] to  $\mathcal{O}_D(\alpha') \otimes \omega_{\bar{D}}^{-1}$  for  $\alpha' > 2\alpha$ , one concludes that  $\mathcal{O}_D(k\alpha') \otimes \omega_{\bar{D}}^{-k}$  is base point free for sufficiently large  $k \gg 0$ . On the other hand some power of  $\mathcal{O}_D(1)$  is generated by its global sections by Kawamata's theorem. Thus  $\mathcal{O}_D(\beta + k) \otimes \omega_{\bar{D}}^{-k}$  is generated by its global sections for sufficiently large  $\beta$  and  $k \gg 0$ . Hence  $\text{Bs}|\mathcal{O}(\beta) \otimes \omega_{\bar{F}}^{-k}| \cap D = \emptyset$ . Since  $Z \cap D \neq \emptyset$ , we conclude that  $\text{Bs}|\mathcal{O}(\beta) \otimes \omega_{\bar{F}}^{-k}|$  does not include  $Z$ .

Considering  $f: X \rightarrow C$ , we have some ample invertible sheaf  $L$  on  $C$  such that the natural map

$$\mathcal{O}_{\sigma_\lambda \circ s_\lambda(C)} \otimes H^0(\sigma_\lambda \circ s_\lambda(C), \mathcal{O}(\beta) \otimes \mathcal{O}(\omega_{\bar{F}}^{-k}) \otimes p^* f^* L) \rightarrow \mathcal{O}(\beta) \otimes \mathcal{O}(\omega_{\bar{F}}^{-k}) \otimes p^* f^* L|_{\sigma_\lambda \circ s_\lambda(C)}$$

is generically surjective for suitable  $\beta, k > 0$ . Hence we have a dense set of curves  $\{\sigma_\lambda(s_\lambda(C))\}$  in  $Z$  such that the intersection  $(\mathcal{O}(\beta) \otimes \omega_{\bar{F}}^{-k} \otimes p^* f^* L, \sigma_\lambda \circ s_\lambda(C)) \geq 0$ . recalling that

$$(\mathcal{O}(1), \sigma_\lambda \circ s_\lambda(C)) = 2g - 2, \quad \omega_{P/X} = \mathcal{O}(-n - 1) \otimes p^* \det \Omega_X,$$

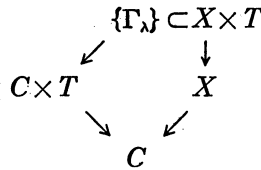
one has

$$\deg_{\sigma_\lambda(s_\lambda(C))} p^* \omega_X^k = (\sigma_\lambda(s_\lambda(C)), p^* \omega_X^k) \leq (g(C) - 1)(\beta + n - 1) + \frac{1}{2} \deg_C L.$$

By the projection formula, one obtains

$$(s_\lambda(C), \omega_X) \leq \frac{\beta+n-1}{k} (g(C)-1) + \frac{1}{2k} \deg_C L.$$

By the Viehweg formula ([14]), one has  $\kappa(\omega_X \otimes f^*L) = \kappa(\omega_V) + 1$ . Hence for any ample invertible sheaf  $H$  over  $X$  there exist a positive integer  $\nu$  and an effective divisor  $F$  such that  $(\omega_X \otimes f^*L)^\nu = H + F$ . Thus we can bound the degree of sections  $C_\lambda = \sigma_\lambda(C)$  which are not contained in  $F$  of  $X$  and we have at most a finite number of Hilbert polynomials of the graphs  $\Gamma_\lambda$  of sections  $C_\lambda$  in  $C \times X$ . Thus we let  $H$  be a Hilbert scheme parametrizing proper subschemes in  $C \times X$  with the Hilbert polynomials mentioned above. Thus we have a subvariety  $T^0$  which parametrizes the graphs  $\Gamma_\lambda$  of sections  $C_\lambda$ , whose set is dense in  $X$ . Let  $T$  be a compactification of  $T^0$ . Hence we have the following commutative diagram:



Thus  $f: X \rightarrow C$  is birationally trivial over  $C$  from the lemma ([7], section 5(p. 115), Appendix (p. 119)):

*Let  $T$  be a complete variety and  $\phi: T \times S \rightarrow X$  be a dominant  $S$ -rational map. Then  $X$  is birationally trivial over  $S$ .*

We can easily reduce the general case to the case when  $\text{tr. deg } K/C = 1$ . Considering the pluri- $S$ -canonical mapping  $X/S \rightarrow \mathbf{P}_S(f_*\omega_X^{\otimes k}/S)$  for  $k \gg 0$  and noting that varieties of general type have no infinitesimal automorphisms except for finite automorphisms, we have a dense open  $S^0$  in  $S$  such that every fibre of  $X/S$  is birational, since we can join any two points in  $S^0$  by a non singular curve in  $S^0$ . Hence one can find etale covering  $S'$  over  $S$  such that the pull-back of the pluri- $S$ -canonical mapping  $X/S \rightarrow \mathbf{P}_S(f_*\omega_X^{\otimes k}/S)$  is trivial. Q.E.D.

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