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ON PROJECTIVE MODULES OVER DIRECTLY FINITE REGULAR RINGS SATISFYING THE COMPARABILITY AXIOM

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In [2], J. Kado has studied simple directly finite regular rings satisfying the comparability axiom, and completely determined the directly finiteness of projective modules over these rings. In this paper, without the assumption of simplicity in [2], we shall study projective modules over directly finite regular rings satisfying the comparability axiom. In Theorem 6, we shall give a criterion of the directly finiteness over these rings. Using this criterion, in Theorem 7, we shall show the following result: Let R be a directly finite regular ring satisfying the comparability axiom. If P and Q are directly finite projective R-modules, then so is $P\oplus Q$.

Throughout this paper, R is a ring with identity and R-modules are unitary right R-modules. If M and N are R-modules, then the notation $N\lesssim M$ (resp. $N\lesssim\oplus M$) means that N is isomorphic to a submodule of M (resp. N is isomorphic to a direct summand of M). For a cardinal number α and an R-module M, αM denotes a direct sum of α-copies of M.

First we recall some definitions and well-known results (cf. [1]).

DEFINITION. A ring R is *directly finite* if $xy=1$ implies $yx=1$, for all $x, y\in R$. An R-module M is *directly finite* if $End_R(M)$ is directly finite. A ring R (a module M) is *directly infinite* if it is not directly finite. It is well-known that M is directly finite if and only if M is not isomorphic to a proper direct summand of M itself. A regular ring R is said to *satisfy the comparability axiom* provided that, for any $x, y\in R$, either $xR\lesssim yR$ or $yR\lesssim xR$, or equivalently, for any finitely generated projective R-modules P and Q, either $P\lesssim Q$ or $Q\lesssim P$. A ring R is said to be *unit-regular* if, for each $x\in R$, there is a unit (i.e. an invertible element) u of R such that $xux=x$.

Lemma 1. (a) *Every directly finite regular ring satisfying the comparability axiom is unit-regular* (cf. [1, Theorem 8.12]).

(b) *Let R be a unit-regular ring. Then,*

(1) *Every finitely generated projective R-module is directly finite* ([1,

Proposition 5.2]).

(2) *Let B, A_1, A_2, $\cdots$ be projective R-modules. If each A_n is finitely generated and $A_1 \oplus \cdots \oplus A_n \lesssim B$ for all n, then $\oplus A_n \lesssim B$* ([1, Proposition 4.8]).

(3) *Let A be a finitely generated projective R-module. If B and C are any R-modules such that $A \oplus B \cong A \oplus C$, then $B \cong C$* ([1, Theorem 4.14]).

(4) *Let A, B and C be projective R-modules such that $A = B \oplus C$. If C is finitely generated, then A is directly finitely if and only if B is directly finite.*

An R-module M is said to *have the exchange property* if, for any direct decomposition $G = M' \oplus C = \oplus_{i \in I} D_i$ with $M' \cong M$ and the index set I, there are submodules $D_i' \leqq D_i$ ($i \in I$) such that $G = M' \oplus (\oplus_{i \in I} D_i')$.

Lemma 2. *Every projective module over a regular ring has the exchange property.*

Proof. This follows from [4, Theorem 3], [5, Proposition 3] and the proof of [3, Lemma 1].

Let R be a regular ring, and let P be a countably generated, but not finitely generated, projective R-module which has a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying the condition

(*) $P_i \gtrsim P_{i+1}$ for all i, and there exists no nonzero R-module X such that $X \lesssim P_i$ for all i.

Consider the following conditions on $\{P_i\}$;

(A) There exists a positive integer m such that

(1) for each $i \geqq m$, $P_i \lesssim t_i P_{i+1}$ for some positive integer t_i, and

(2) $\oplus_{i=m}^{\infty} P_i \lesssim t P_m$ for some positive integer t.

(B) There exists an increasing sequence $1 = i_1 < i_2 < \cdots$, of positive integers such that $P_{i_n} \gtrsim \aleph_0 P_{i_{n+1}}$ for $n = 1, 2, \cdots$.

(C) There exists a positive integer m for which the condition (1) of (A) holds.

Lemma 3. *Let R be a directly finite regular ring satisfying the comparability axiom. Then, for a countably generated, but not finitely generated, projective R-module P with a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfying* (*), *either* (B) *or* (C) *hold, but not both.*

Proof. It is clear from Lemma 1 that if $P = \oplus_{i=1}^{\infty} P_i$ does not satisfy (B), then $P = \oplus_{i=1}^{\infty} P_i$ satisfies (C). Assume that $P = \oplus_{i=1}^{\infty} P_i$ does not satisfy (C). Then there exists an increasing sequence $j_1 < j_2 < \cdots$, of positive integers such that, for each $n = 1, 2, \cdots$, $P_{j_n} \not\lesssim t P_{j_n+1}$ for all positive integers t, and so $P_{j_n} \gtrsim \aleph_0 P_{j_n+1} \gtrsim \aleph_0 P_{j_{n+1}}$ by Lemma 1. Therefore, taking $i_1 = 1$ and $i_n = j_n$ for $n = 2, 3, \cdots$, we see that (B) holds for $P = \oplus_{i=1}^{\infty} P_i$.

Proposition 4. *Let R be a directly finite regular ring satisfying the comparability axiom. For a countably generated, but not finitely generated, projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying* (*) *and* (C), *the following are equivalent*:

(a) *P is directly finite.*

(b) *There exists a positive integer $n>m$ such that $\oplus_{i=n}^{\infty}P_i\lesssim P_m$.*

(c) *There exists a positive integer $t>1$ such that $\oplus_{i=m}^{\infty}P_i\lesssim tP_m$.*

Proof. (a)$\rightarrow$(b). Assume that (b) does not hold. Then there exists an increasing sequence $m<m_1<m_2<\cdots$, of positive integers such that $P_{m_i+1}\oplus\cdots\oplus P_{m_{i+1}}\gtrsim P_m$ for $i=1, 2, \cdots$ by Lemma 1. Thus $\oplus_1^{\infty}P_m\lesssim\oplus P$ and so P is directly infinite. (b)$\rightarrow$(c) is clear. (c)$\rightarrow$(a). Assume, to the contrary, that P is directly infinite. Putting $Q=\oplus_{i=m}^{\infty}P_i$, in view of Lemma 1, we see that Q is directly infinite; so there exists a nonzero R-module X such that $\aleph_0X\lesssim Q$. Using (*), we can choose a positive integer $i>m$ such that $P_i\lesssim X$, and hence $tP_m\lesssim\aleph_0P_m\lesssim\aleph_0t_m\cdots t_{i-1}P_i\cong\aleph_0P_i\lesssim\aleph_0X\lesssim Q\lesssim tP_m$. This contradicts the directly finiteness of tP_m.

REMARK. Let R and P be as in Proposition 4. This P is not always directly finite in general. Take, for example, a field F and set $R_n=M_{2^n}(F)$ for all $n=1, 2, \cdots$. Map each $R_n\rightarrow R_{n+1}$ along the diagonal, i.e., map $x\rightarrow\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix}$, and set $R=\varinjlim R_n$ (see [1, Example 21.20]). Then R is a directly finite regular ring satisfying the comparability axiom and we can choose a cyclic projective R-module $Q_n=x_nR$ such that $N(x_n)=1/2^n$ for $n=1, 2, \cdots$, where N is a unique rank function on R. Put $Q_i=P_{2^i-1}=\cdots=P_{2^{i+1}-2}$ for $i=1, 2, \cdots$ and $P=\oplus_{i=1}^{\infty}P_i$. Then, using [2, Lemma 2.5], we see that $P\cong\aleph_0R$ is a countably generated, but not finitely generated, directly infinite projective R-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (C).

Proposition 5. *Let R be a directly finite regular ring satisfying the comparability axiom. Then, a countably generated, but not finitely generated, projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying* (*) *and* (B) *is directly finite.*

Proof. Putting $\bar{P}_{i_n}=P_{i_n+1}\oplus\cdots\oplus P_{i_{n+1}-1}$ for each n, we have $P=(\oplus_{n=1}^{\infty}P_{i_n})\oplus(\oplus_{n=1}^{\infty}\bar{P}_{i_n})$. Let $P=S\oplus T$ with $P\cong S$ and assume $T\neq 0$. We have decompositions

$$P_{i_n}=P'_{i_n}\oplus P''_{i_n} \text{ and } \bar{P}_{i_n}=\bar{P}'_{i_n}\oplus\bar{P}''_{i_n}$$

such that

$$S\cong(\oplus_{n=1}^{\infty}P'_{i_n})\oplus(\oplus_{n=1}^{\infty}\bar{P}'_{i_n}) \text{ and } T\cong(\oplus_{n=1}^{\infty}P''_{i_n})\oplus(\oplus_{n=1}^{\infty}\bar{P}''_{i_n})$$

for each $n=1, 2, \cdots$ (Lemma 2). Noting $T\neq 0$, we see by (*) that there exist a finitely generated nonzero submodule T' of T and a positive integer i_n such that $P_{i_n}\lesssim T'\leqq T$, and so that $P_{i_n}\lesssim U$, where $U=(\oplus_{i=1}^{m}P_{i_n}'')\oplus(\oplus_{n=1}^{m}\bar{P}_{i_n}'')$ for some positive integer $m>n$. Then $P\cong S$ shows that $U\lesssim(\oplus_{n=m+1}^{\infty}P_{i_n}')\oplus(\oplus_{n=m+1}^{\infty}\bar{P}_{i_n}')\lesssim\aleph_0 P_{i_m}\lesssim P_{i_n}\lesssim U$ by Lemma 1, which contradicts the directly finiteness of U. Therefore P is directly finite.

We are now in a position to prove the main theorem.

Theorem 6. *Let R be a directly finite regular ring satisfying the comparability axiom, and let P be a projective R-module. Then, P is directly finite if and only if*

(1) *P is finitely generated or*

(2) *P is a countably generated R-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying* (*) *and* (A), *or* (*) *and* (B). *If, in addition, R has the nonzero socle, then a projective R-module P is directly finite if and only if P is finitely generated.*

Proof. Put $P=\oplus_{i\in I}P_i$, where each P_i is a nonzero cyclic projective submodule of P. Denote by $|I|$ the cardinal number of I. If $|I|<\aleph_0$, then P is directly finite by Lemma 1. Assume that $|I|>\aleph_0$. Given $j\in I$, we set $I_j=\{i\in I\,|\,P_j\lesssim P_i\}$. Suppose that all I_j are finite sets. Then we can choose a countable sequence $\{i_1, i_2, \cdots\}$ of I such that $I_{i_1}\lneqq I_{i_2}\lneqq\cdots$ and $\cup_{n=1}^{\infty}I_{i_n}\neq I$. Hence there exists $i_0\in I-\cup_{n=1}^{\infty}I_{i_n}$ such that $I_{i_0}\not\supseteq\cup_{n=1}^{\infty}I_{i_n}$, which contradicts to the fact that I_{i_0} is a finite set. Thus there must exist an infinite set I_j such that $\aleph_0 P_j\lesssim\oplus\oplus_{i\in I_j}P_i\lesssim\oplus P$; so P is directly infinite. Now suppose that $|I|=\aleph_0$. Then, in order to observe that P is directly finite, we can assume that P has a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ with (*). Lemma 3, Propositions 4 and 5 show that P is directly finite if and only if $P=\oplus_{i=1}^{\infty}P_i$ satisfies (A) or (B). Thus the first part of the theorem has proved. Next, assume that the socle of R is nonzero. Then there exists a simple right ideal S of R such that $S\lesssim P_i$ for all $i\in I$. This, together with (1) and (2), shows that P is directly finite if and only if P is finitely generated.

REMARK 1. Let R be a nonzero simple directly finite regular ring satisfying the comparability axiom. Then, every non-finitely generated directly finite projective R-module P is a countably generated module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (A), because R has a strictly positive dimension function ([1, Corollary 16.15]).

REMARK 2. A directly finite regular ring R with the comparability axiom is classified into three cases:

(1) all non-finitely generated directly finite projective R-modules P have a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (A).

(2) all non-finitely generated directly finite projective R-modules P have a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (B).

(3) all directly finite projective R-modules are finitely generated.

Proof. Assume that there exist non-finitely generated directly finite projective R-modules P and Q with cyclic decompositions $P=\oplus_{i=1}^{\infty}P_i$ and $Q=\oplus_{i=1}^{\infty}Q_i$ satisfying the following:

(1) $P_i\gtrsim P_{i+1}$ and $Q_i\gtrsim Q_{i+1}$ for each i,

(2) if $X\lesssim P_i$ (resp. $X\lesssim Q_i$) for all i, then $X=0$ (resp. $X=0$),

(3) there exists a positive integer m such that, for each $i=m, m+1, \cdots$, $P_i\lesssim t_iP_{i+1}$ for some positive integer t_i, and $\oplus_{i=m}^{\infty}P_i\lesssim tP_m$ for some positive integer t, and

(4) there exists an increasing sequence $1=i_1<i_2< \cdots$, of positive integers such that $Q_{i_n}\gtrsim\aleph_0Q_{i_{n+1}}$ for $n=1, 2, \cdots$. Then we can choose positive integers i_n and $m_1>m$ such that $P_m\gtrsim Q_{i_n}\gtrsim Q_{i_{n+1}}\gtrsim P_{m_1}$, and so $t_{m_1-1}\cdots t_mP_{m_1}\gtrsim P_m\gtrsim Q_{i_n}\gtrsim\aleph_0Q_{i_{n+1}}\gtrsim\aleph_0P_{m_1}\gtrsim t_{m_1-1}\cdots t_mP_{m_1}$, which contradicts the directly finiteness of $t_{m_1-1}\cdots t_mP_{m_1}$.

From Lemma 1, Theorem 6 and Remark 2, we have the following result.

Theorem 7. *Let R be a directly finite regular ring satisfying the comparability axiom. If P and Q are directly finite projective R-modules, then so is $P\oplus Q$.*

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