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CURVATURES OF THE PRODUCT OF TWO 3-SPHERES WITH DEFORMED METRICS

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1. Introduction

Let (S^3, g) be the 3-sphere with the canonical metric of constant curvature 1 and let $(S^3 \times S^3, \tilde{g})$ be the Riemannian product of two (S^3, g), where $\tilde{g}$ denotes the product metric of two g. In §3 we consider Riemannian metrics which are left-invariant when we consider $S^3 \times S^3$ as a Lie group $SU(2) \times SU(2)$. In §4 we study special type of left invariant metrics. Let $\{\eta^1, \eta^2, \eta^3\}$ be a globally defined orthonormal coframe field on S^3 and $\{\eta^{\bar{1}}, \eta^{\bar{2}}, \eta^{\bar{3}}\}$ be one on the second S^3. Then the product metric $\tilde{g}$ on $S^3 \times S^3$ is expressed as $\tilde{g} = \sum_{u=1}^{3} \eta^u \otimes \eta^u + \sum_{v=1}^{3} \eta^{\bar{v}} \otimes \eta^{\bar{v}}$. We consider the following metric

$$\hat{g}(t) = \tilde{g} + t \sum_{u,v=1}^{3} r_{u\bar{v}} (\eta^u \otimes \eta^{\bar{v}} + \eta^{\bar{v}} \otimes \eta^u) \tag{1.1}$$

on $S^3 \times S^3$, where t is a real parameter $(-t_o < t < t_o)$ and $r = (r_{u\bar{v}}) = (r_{uv})$ is a constant real 3×3 matrix. If r is symmetric, then we can assume that r is diagonal $(r_u \delta_{uv})$ after some orthogonal change of frames if necessary.

The deformation given by (1.1) is natural. The purpose of this paper is to report that the phenomena of sectional curvatures for $t > 0$ and $t < 0$ are completely different in the most simplest case $r = (\delta_{uv})$.

Theorem A. *Suppose $r = (-\delta_{uv})$ in (1.1). Then there is a positive number t_* such that $\{\hat{g}(t),\ 0 \le t < t_*\}$ is a one parameter family of left invariant metrics on $S^3 \times S^3$ with non-negative sectional curvature. Here, the sections $\{\tilde{X}, \tilde{Y}\}$ with zero sectional curvature are of the form $\tilde{X} = (X, 0)$ and $\tilde{Y} = (0, X)$ for $t \in (0, t_*)$.*

Contrary to Theorem A, we have the following:

Theorem B. *Suppose $r = (\lambda_u \delta_{uv})$ with $1 = \lambda_1 \ge \lambda_2 \ge \lambda_3 > 0$. Then there is a positive number t'_* such that $\{\hat{g}(t),\ 0 \le t < t'_*\}$ is a one parameter family of left invariant metrics on $S^3 \times S^3$ with the following properties:*
(i) *There are planes of the form $\{\tilde{X}, \tilde{Y}\}$ with $\tilde{X} = (X, 0)$ and $\tilde{Y} = (0, \bar{Y})$ with*

zero sectional curvature with respect to each $\hat{g}(t)$. *If* $\lambda_1 > \lambda_2 > \lambda_3 > 0$, *then the number of such planes is three* (*at each point*).

(ii) *For any small positive number* t *there exist a plane* Π *and some positive number* $t_2 < t$ *such that the sectional curvature* $\hat{K}(\Pi)$ *is negative with respect to* $\hat{g}(t_2)$.

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2. An orthonormal frame field on (S^3, g)

Let (S^3, g) be the 3-sphere with the canonical metric of constant curvature 1. We have an orthonormal frame field $\{\xi_1, \xi_2, \xi_3\}$ on S^3 satisfying $[\xi_a, \xi_b] = 2\xi_c$ for $\varepsilon(a, b, c) = 1$, where $\varepsilon(a, b, c)$ denotes the sign of the permutation $(a, b, c) \to (1, 2, 3)$ (and $\varepsilon(a, b, c) = 0$ if the set $\{a, b, c\}$ is different from $\{1, 2, 3\}$). We denote the dual of $\{\xi_1, \xi_2, \xi_3\}$ by $\{\eta^1, \eta^2, \eta^3\}$. We define ϕ^a by $\phi^a = -\nabla\xi_a$ for $a = 1, 2, 3$, where ∇ denotes the Riemannian connection with respect to g. Then we have

$$\phi^a\phi^a X = -X + \eta^a(X)\xi_a, \tag{2.1}$$

$$g(\phi^a X, \phi^a Y) = g(X, Y) - \eta^a(X)\eta^a(Y), \tag{2.2}$$

$$d\eta^a(X, Y) = 2g(X, \phi^a Y), \tag{2.3}$$

$$(\nabla_X \phi^a)(Y) = g(X, Y)\xi_a - \eta^a(Y)X \tag{2.4}$$

for vector fields X and Y on S^3 and $a = 1, 2, 3$. Furthermore, $\xi_a = \phi^b\xi_c = -\phi^c\xi_b$ and

$$\phi^a = \phi^b\phi^c - \xi_b \otimes \eta^c = -\phi^c\phi^b + \xi_c \otimes \eta^b \tag{2.5}$$

hold for $\varepsilon(a, b, c) = 1$. For each a, $\{\eta^a, g\}$ is called a Sasakian structure on (S^3, g) and $\{\eta^1, \eta^2, \eta^3, g\}$ is called a Sasakian 3-structure (cf. Blair [1], Tanno [3], etc.).

Let $(\phi^{a}{}^{u}_{v})$ be the components of ϕ^a with respect to the frame field $\{\xi_1, \xi_2, \xi_3\}$. Then we have $\phi^{a}{}^{u}_{v} = -\varepsilon(a, u, v)$. Therefore, for example, we obtain

$$\phi^a{}_{uv} X^u Y^v = -(X \times Y)^a, \tag{2.6}$$

where $X \times Y$ denotes the vector product in $T_x S^3 \simeq E^3$ at each point $x \in S^3$. Furthermore, one may use $\phi^a{}_{uv} = -\phi^u{}_{av}$, etc. in the calculations, if necessary; for example, we have

$$A_u B_v \phi^{u}{}^{a}_{x} \phi^{v}{}^{x}_{c} X_a Y^c = -\langle A \times X, B \times Y \rangle, \tag{2.7}$$

where $\langle\ ,\ \rangle$ denotes the inner product defined by g. Here we recall the following relation:

$$\langle A \times B, C \times D\rangle = \langle A, C\rangle\langle B, D\rangle - \langle A, D\rangle\langle B, C\rangle,$$

which will be used in §4.

3. Riemannian metrics on $S^3 \times S^3$

We fix the range of indices as follows:

$$1 \leq i, j, k, l, x, y \leq 6, \qquad 1 \leq a, b, c, u, v \leq 3,$$

and we denote $\bar{a} = a + 3$ generally (i.e., if $\bar{a}$ is used in S^3 then $\bar{a}$ means simply a; while if $\bar{a}$ is used in $S^3 \times S^3$ then $\bar{a}$ means $a + 3$).

We have a globally defined orthonormal frame field $\{\xi_1, \xi_2, \xi_3, \xi_{\bar{1}}, \xi_{\bar{2}}, \xi_{\bar{3}}\}$ and its dual $\{\eta^1, \eta^2, \eta^3, \eta^{\bar{1}}, \eta^{\bar{2}}, \eta^{\bar{3}}\}$ on the Riemannian product $(S^3 \times S^3, \tilde{g})$. Here ξ_a ($\xi_{\bar{b}}$, resp.) is identified with $(\xi_a, 0)$ ($(0, \xi_{\bar{b}})$, resp.). The Riemannian connection with respect to $\tilde{g}$ is denoted by $\tilde{\nabla}$. Then we have $\tilde{\nabla}\xi_a = (\nabla\xi_a, 0)$ and $\tilde{\nabla}\xi_{\bar{b}} = (0, \nabla\xi_{\bar{b}})$, and hence we have $\phi^a = -\tilde{\nabla}\xi_a$ and $\phi^{\bar{a}} = -\tilde{\nabla}\xi_{\bar{a}}$ for $a = 1, 2, 3$. By $(\phi^{i}{}^{j}_{k})$ we denote the components of ϕ^i with respect to $\{\xi_a, \xi_{\bar{a}}\}$. One may notice that if one component $\phi^{i}{}^{j}_{k}$ has mixed indices $i \leq 3$ and $j \geq 4$ for example, then it vanishes.

Now we define Riemannian metrics $\hat{g}(t)$ on $S^3 \times S^3$ by

$$\hat{g}_{ij} = \tilde{g}_{ij} + th_{ij}, \tag{3.1}$$

where (and in many places below) we denote $\hat{g}(t)$ simply by $\hat{g}$, and

$$h_{ij} = s_u\eta^u_i\eta^u_j + r_{u\bar{v}}(\eta^u_i\eta^{\bar{v}}_j + \eta^u_j\eta^{\bar{v}}_i) + \bar{s}_{\bar{v}}\eta^{\bar{v}}_i\eta^{\bar{v}}_j, \qquad r_{\bar{u}v} = r_{v\bar{u}}, \tag{3.2}$$

where $r = (r_{u\bar{v}})$ is a constant real 3×3 matrix; and $s = (s_u)$, $\bar{s} = (\bar{s}_{\bar{v}})$ are constant 3-vectors. Here t is a sufficiently small real number so that $\hat{g} = (\hat{g}_{ij})$ is a Riemannian metric.

In the tensor calculus components of tensor fields are ones with respect to the natural frame of a local coordinate system. Otherwise, components are ones with respect to $\{\xi_a, \xi_{\bar{a}}\}$. This will be understood in the context.

Notice that (h_{ij}) given above is a general form of (h_{ij}) with constant coefficients. Indeed, let $h_{ij} = \beta_{kl}\eta^k_i\eta^l_j$. Then the first block (β_{ab}) of $(\beta_{ab}\eta^a_i\eta^b_j)$ is diagonalized to $(s_u\delta_{uv})$ so that $\beta_{ab}\eta^a_i\eta^b_j = s_u\eta'^u_i\eta'^u_j$ by some orthogonal transformation $\{\xi_a\} \to \{\xi'_a\}$. Similarly we have $(\bar{s}_{\bar{v}})$ so that $\beta_{\bar{a}\bar{b}}\eta^{\bar{a}}_i\eta^{\bar{b}}_j = \bar{s}_{\bar{v}}\eta'^{\bar{v}}_i\eta'^{\bar{v}}_j$. So we have (3.2). Moreover, $\hat{g}$ is a left invariant metric when we consider $S^3 \times S^3$ as a Lie group $SU(2) \times SU(2)$.

The inverse matrix of $\hat{g} = (\hat{g}_{ij})$ is denoted by $\hat{g}^{-1} = (\hat{g}^{is})$. Then, the difference $W^i_{jk} = \hat{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{jk}$ of the coefficients of the Riemannian connections with respect to $\hat{g}$ and $\tilde{g}$, and the Riemannian curvature tensor $\hat{R}^i_{jkl}$ are given by

$$W^i_{jk} = (t/2)\hat{g}^{is}(\tilde{\nabla}_j h_{sk} + \tilde{\nabla}_k h_{sj} - \tilde{\nabla}_s h_{jk}), \tag{3.3}$$

$$\hat{R}^i_{jkl} = \tilde{R}^i_{jkl} + \tilde{\nabla}_k W^i_{lj} - \tilde{\nabla}_l W^i_{kj} + W^s_{lj} W^i_{ks} - W^s_{kj} W^i_{ls}. \tag{3.4}$$

We denote components of a vector field $\tilde{X}$ on $S^3 \times S^3$ as

$$\tilde{X} = (\tilde{X}^i) = (X, \bar{X}) = (X^a, \bar{X}^{\bar{a}}) = (X^1, X^2, X^3; \bar{X}^{\bar{1}}, \bar{X}^{\bar{2}}, \bar{X}^{\bar{3}}),$$

where X ($\bar{X}$, resp.) is tangent to the first (second, resp.) S^3.

Lemma 3.1. *$\hat{g}(\hat{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X})$ is given by*

$$\begin{aligned} \hat{g}(\hat{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X}) &= \hat{g}_{hi} \tilde{R}^i_{jkl} \tilde{X}^h \tilde{X}^k \tilde{Y}^j \tilde{Y}^l \\ &+ [\tilde{\nabla}_k (\hat{g}_{hi} W^i_{lj}) - \tilde{\nabla}_l (\hat{g}_{hi} W^i_{kj})] \tilde{X}^h \tilde{X}^k \tilde{Y}^j \tilde{Y}^l \\ &- \hat{g}^{xy} [(\hat{g}_{xp} W^p_{kh})(\hat{g}_{yq} W^q_{lj}) - (\hat{g}_{xp} W^p_{lh})(\hat{g}_{yq} W^q_{kj})] \tilde{X}^h \tilde{X}^k \tilde{Y}^j \tilde{Y}^l. \end{aligned} \tag{3.5}$$

Proof. First we have

$$\hat{g}_{hi}[\tilde{\nabla}_k W^i_{lj} - \tilde{\nabla}_l W^i_{kj}] = \tilde{\nabla}_k (\hat{g}_{hi} W^i_{lj}) - \tilde{\nabla}_l (\hat{g}_{hi} W^i_{kj}) - t\tilde{\nabla}_k h_{hi} \cdot W^i_{lj} + t\tilde{\nabla}_l h_{hi} \cdot W^i_{kj}.$$

Next, using (3.3) we obtain $t\tilde{\nabla}_k h_{hi} = \hat{g}_{hs} W^s_{ki} + \hat{g}_{is} W^s_{kh}$ and

$$-t\tilde{\nabla}_k h_{hi} \cdot W^i_{lj} + \hat{g}_{hi} W^i_{ks} W^s_{lj} = -\hat{g}^{xy} (\hat{g}_{xp} W^p_{kh})(\hat{g}_{yq} W^q_{lj}).$$

Then applying these into (3.4), proof is completed. □

Lemma 3.2. *$\hat{g}_{is} W^s_{jk}$ is given by*

$$\begin{aligned} \hat{g}_{is} W^s_{jk} = -t[& s_u ({\phi^u}_{ij} \eta^u_k + {\phi^u}_{ik} \eta^u_j) + \bar{s}_{\bar{v}} ({\phi^{\bar{v}}}_{ij} \eta^{\bar{v}}_k + {\phi^{\bar{v}}}_{ik} \eta^{\bar{v}}_j) \\ &+ r_{u\bar{v}} ({\phi^u}_{ij} \eta^{\bar{v}}_k + {\phi^u}_{ik} \eta^{\bar{v}}_j + {\phi^{\bar{v}}}_{ij} \eta^u_k + {\phi^{\bar{v}}}_{ik} \eta^u_j)]. \end{aligned} \tag{3.6}$$

Proof. One may use relations; $\tilde{\nabla}_i \eta^u_j = {\phi^u}_{ij}$, etc. □

We continue some calculations to obtain the sectional curvature for a 2-plane determined by $\tilde{X}$ and $\tilde{Y}$. Here we assume that $\{\tilde{X}, \tilde{Y}\}$ is orthonormal with respect to $\tilde{g}$, i.e.,

$$\langle X, X \rangle + \langle \bar{X}, \bar{X} \rangle = 1, \quad \langle Y, Y \rangle + \langle \bar{Y}, \bar{Y} \rangle = 1, \quad \langle X, Y \rangle + \langle \bar{X}, \bar{Y} \rangle = 0.$$

Lemma 3.3. *Let $\{\tilde{X}, \tilde{Y}\}$ be an orthonormal pair with respect to $\tilde{g}$ at a point of $S^3 \times S^3$. Then we can assume $\langle X, Y \rangle = \langle \bar{X}, \bar{Y} \rangle = 0$.*

Proof. Assume $\langle X, Y\rangle \neq 0$ and consider $\tilde{Z} = \cos\theta\tilde{X} + \sin\theta\tilde{Y}$ and $\tilde{W} = -\sin\theta\tilde{X} + \cos\theta\tilde{Y}$. Then $\langle Z, W\rangle$ for $\tilde{Z} = (Z, \bar{Z})$ and $\tilde{W} = (W, \bar{W})$ is given by

$$\langle Z, W\rangle = \sin\theta\cos\theta(\|Y\|^2 - \|X\|^2) + (\cos^2\theta - \sin^2\theta)\langle X, Y\rangle.$$

If $\|Y\| = \|X\|$, then we may put $\theta = \pi/4$ to get $\langle Z, W\rangle = 0$. Then also $\langle \bar{Z}, \bar{W}\rangle = 0$ follows. If $\|Y\| \neq \|X\|$, then we can find θ such that $\langle Z, W\rangle = 0$. We have also $\langle \bar{Z}, \bar{W}\rangle = 0$. □

From now on we assume $\langle X, Y\rangle = \langle \bar{X}, \bar{Y}\rangle = 0$ for our orthonormal pair $\{\tilde{X}, \tilde{Y}\}$. Since $\tilde{g}$ is the product of Riemannian metrics of constant curvature 1, we obtain

$$\begin{aligned}(3.7)\qquad \hat{g}_{hi}\tilde{R}^i_{jkl}\tilde{X}^h\tilde{X}^k\tilde{Y}^j\tilde{Y}^l &= \|X \times Y\|^2 + \|\bar{X} \times \bar{Y}\|^2 \\ &\quad + t[r(X, \bar{X}) + \|Y\|^2 s(X, X) + \|\bar{Y}\|^2 \bar{s}(\bar{X}, \bar{X})],\end{aligned}$$

where s and $\bar{s}$ are considered as matrices $s = (s_u\delta_{uv})$ and $\bar{s} = (\bar{s}_{\bar{u}}\delta_{\bar{u}\bar{v}})$. By (2.4), (2.6) and (3.6), the second term of the right hand side of (3.5) is given by

$$\begin{aligned}(3.8)\qquad [\tilde{\nabla}_k(\hat{g}_{hi}W^i_{lj}) - \tilde{\nabla}_l(\hat{g}_{hi}W^i_{kj})]\tilde{X}^h\tilde{X}^k\tilde{Y}^j\tilde{Y}^l &= t[r(X, \bar{X}) \\ &\quad + 2r(Y, \bar{Y}) - 6r(X \times Y, \bar{X} \times \bar{Y}) \\ &\quad + 2\|X\|^2 s(Y, Y) + \|Y\|^2 s(X, X) - 3s(X \times Y, X \times Y) \\ &\quad + 2\|\bar{X}\|^2 \bar{s}(\bar{Y}, \bar{Y}) + \|\bar{Y}\|^2 \bar{s}(\bar{X}, \bar{X}) - 3\bar{s}(\bar{X} \times \bar{Y}, \bar{X} \times \bar{Y})].\end{aligned}$$

1-forms $(\hat{g}_{jp}W^p_{kh}\tilde{X}^k\tilde{X}^h)$ and $(\hat{g}_{jp}W^p_{lh}\tilde{X}^h\tilde{Y}^l)$ are expressed as follows:

$$\begin{aligned}(3.9)\qquad &(\hat{g}_{jp}W^p_{kh}\tilde{X}^k\tilde{X}^h) = 2t(U(\tilde{X})_u, \bar{U}(\tilde{X})_{\bar{u}}), \\ &U(\tilde{X}) = X \times (r(\bar{X}) + s(X)), \quad \bar{U}(\tilde{X}) = \bar{X} \times ({}^t r(X) + \bar{s}(\bar{X})),\end{aligned}$$

$$\begin{aligned}(3.10)\qquad &(\hat{g}_{jp}W^p_{lh}\tilde{X}^h\tilde{Y}^l) = t(V(\tilde{X}, \tilde{Y})_u, \bar{V}(\tilde{X}, \tilde{Y})_{\bar{u}}), \\ &V(\tilde{X}, \tilde{Y}) = X \times (r(\bar{Y}) + s(Y)) + Y \times (r(\bar{X}) + s(X)), \\ &\bar{V}(\tilde{X}, \tilde{Y}) = \bar{X} \times ({}^t r(Y) + \bar{s}(\bar{Y})) + \bar{Y} \times ({}^t r(X) + \bar{s}(\bar{X})),\end{aligned}$$

where ${}^t r$ denotes the transpose of r.

In the next Proposition we study some special type of sections for later use.

Proposition 3.4. *$\hat{g}(\hat{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X})$ for an orthonormal pair $\{\tilde{X} = (X, 0), \tilde{Y} = (0, \bar{Y})\}$ with respect to $\tilde{g}$ is given by*

$$\begin{aligned}&\hat{g}(\hat{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X}) \\ &= t^2\{\hat{g}^{uv}(X \times r(\bar{Y}))_u(X \times r(\bar{Y}))_v + \hat{g}^{\bar{u}\bar{v}}(\bar{Y} \times {}^t r(X))_{\bar{u}}(\bar{Y} \times {}^t r(X))_{\bar{v}} \\ &\quad + \hat{g}^{u\bar{v}}[2(X \times r(\bar{Y}))_u(\bar{Y} \times {}^t r(X))_{\bar{v}} - 4(X \times s(X))_u(\bar{Y} \times \bar{s}(\bar{Y}))_{\bar{v}}]\}.\end{aligned}$$

Proof. By $\bar{X} = Y = 0$ in (3.7) $\sim$ (3.10), we have $\hat{g}_{hi}\tilde{R}^i_{jkl}\tilde{X}^h\tilde{X}^k\tilde{Y}^j\tilde{Y}^l = 0$ and

$$[\tilde{\nabla}_k(\hat{g}_{hi}W^i_{lj}) - \tilde{\nabla}_l(\hat{g}_{hi}W^i_{kj})]\tilde{X}^h\tilde{X}^k\tilde{Y}^j\tilde{Y}^l = 0,$$
$$(\hat{g}_{jp}W^p_{kh}\tilde{X}^k\tilde{X}^h) = 2t(X \times s(X), 0),$$
$$(\hat{g}_{jp}W^p_{lh}\tilde{X}^h\tilde{Y}^l) = t(X \times r(\bar{Y}), \bar{Y} \times {}^t r(X)),$$
$$(\hat{g}_{jp}W^p_{kh}\tilde{Y}^k\tilde{Y}^h) = 2t(0, \bar{Y} \times \bar{s}(\bar{Y})).$$

Substituting these into (3.5), proof is completed. □

The sectional curvature $\hat{K}(\tilde{X}, \tilde{Y})$ for an orthonormal pair $\{\tilde{X}, \tilde{Y}\}$ with respect to $\tilde{g}$ at a point of $(S^3 \times S^3, \hat{g}(t))$ is given by

$$\hat{K}(\tilde{X}, \tilde{Y}) = \hat{g}(\hat{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X})/\hat{D}(\tilde{X}, \tilde{Y}), \tag{3.11}$$

where $\hat{D}(\tilde{X}, \tilde{Y}) = \hat{g}(\tilde{X}, \tilde{X})\hat{g}(\tilde{Y}, \tilde{Y}) - \hat{g}(\tilde{X}, \tilde{Y})^2$. As far as we are concerned with the sign of sectional curvatures, it suffices to consider $\hat{g}(\hat{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X})$.

4. The case where $s = \bar{s} = 0$

In this section we assume $s = \bar{s} = 0$ in (3.2), i.e.

$$\hat{g} = \tilde{g} + t\, r_{u\bar{v}}(\eta^u \otimes \eta^{\bar{v}} + \eta^{\bar{v}} \otimes \eta^u). \tag{4.1}$$

The restriction of $\hat{g}$ to each factor S^3 is identical with the canonical metric g on S^3. By Lemma 3.1 and (3.7) $\sim$ (3.10), we obtain

Proposition 4.1. *For the metric* (4.1) *on* $S^3 \times S^3$, $\hat{g}(\hat{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X})$ *for an orthonormal pair* $\{\tilde{X}, \tilde{Y}\}$ *with respect to* $\tilde{g}$ *is given by*

$$\hat{g}(\hat{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X}) = \|X \times Y\|^2 + \|\bar{X} \times \bar{Y}\|^2 + G_1 t + G_2 t^2, \tag{4.2}$$

where we have put G_1 *and* $G_2 = G_{21} + G_{22}$ *as*

$$G_1 = 2[r(X, \bar{X}) + r(Y, \bar{Y}) - 3r(X \times Y, \bar{X} \times \bar{Y})], \tag{4.3}$$

$$\begin{aligned} G_{21} = &-4\hat{g}^{uv}(X \times r(\bar{X}))_u(Y \times r(\bar{Y}))_v \\ &-4\hat{g}^{u\bar{v}}[(X \times r(\bar{X}))_u(\bar{Y} \times {}^t r(Y))_{\bar{v}} + (Y \times r(\bar{Y}))_u(\bar{X} \times {}^t r(X))_{\bar{v}}] \\ &-4\hat{g}^{\bar{u}\bar{v}}(\bar{X} \times {}^t r(X))_{\bar{u}}(\bar{Y} \times {}^t r(Y))_{\bar{v}}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} G_{22} = &\ \hat{g}^{uv}(X \times r(\bar{Y}) + Y \times r(\bar{X}))_u(X \times r(\bar{Y}) + Y \times r(\bar{X}))_v \\ &+2\hat{g}^{u\bar{v}}(X \times r(\bar{Y}) + Y \times r(\bar{X}))_u(\bar{X} \times {}^t r(Y) + \bar{Y} \times {}^t r(X))_{\bar{v}} \\ &+\hat{g}^{\bar{u}\bar{v}}(\bar{X} \times {}^t r(Y) + \bar{Y} \times {}^t r(X))_{\bar{u}}(\bar{X} \times {}^t r(Y) + \bar{Y} \times {}^t r(X))_{\bar{v}}. \end{aligned} \tag{4.5}$$

The inverse matrix of $\hat{g} = (\hat{g}_{ij})$ is given by

$$\text{(4.6)}\qquad \hat{g}^{ij} = \tilde{g}^{ij} + t^2 \sum_{l=1}^{\infty} t^{2(l-1)} \sum_{z,w=1}^{3} [((r \cdot {}^t r)^l)_{zw} \xi_z^i \xi_w^j + (({}^t r \cdot r)^l)_{\bar{z}\bar{w}} \xi_{\bar{z}}^i \xi_{\bar{w}}^j]$$
$$- t \sum_{l=1}^{\infty} t^{2(l-1)} \sum_{z,w=1}^{3} (r({}^t r \cdot r)^{l-1})_{z\bar{w}} (\xi_z^i \xi_{\bar{w}}^j + \xi_z^j \xi_{\bar{w}}^i),$$

where $r \cdot {}^t r$ means $(r \cdot {}^t r)_{uw} = \sum_{\bar{v}} r_{u\bar{v}} r_{\bar{v}w} = \sum_{\bar{v}} r_{u\bar{v}} r_{w\bar{v}}$ and ${}^t r \cdot r$ means $({}^t r \cdot r)_{\bar{u}\bar{v}} = \sum r_{w\bar{u}} r_{w\bar{v}}$. So we have $({}^t r \cdot r \cdot {}^t r)_{\bar{v}z} = (r \cdot {}^t r \cdot r)_{z\bar{v}}$, etc. Thus, we obtain the following:

Lemma 4.2. (i) *If r is an orthogonal matrix, then we have*

$$\text{(4.7)}\qquad \hat{g}^{-1} = [1/(1-t^2)]\tilde{g}^{-1} - [t/(1-t^2)] \sum_{z,w=1}^{3} r_{z\bar{w}} (\xi_z \otimes \xi_{\bar{w}} + \xi_{\bar{w}} \otimes \xi_z).$$

(ii) *If r is diagonal, i.e., $r = (\lambda_u \delta_{uv})$, then*

$$\text{(4.8)}\qquad \hat{g}^{uv} = \hat{g}^{\bar{u}\bar{v}} = [1/(1-\lambda_u^2 t^2)]\delta^{uv}, \quad \hat{g}^{u\bar{v}} = -[\lambda_u t/(1-\lambda_u^2 t^2)]\delta^{uv}.$$

Proposition 4.3. *If $r \in O(3)$, then*

$$\text{(4.9)}\quad (1-(\det r)t)\hat{g}(\hat{R}(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X}) = (1-(\det r)t)(\|X \times Y\|^2 + \|\bar{X} \times \bar{Y}\|^2)$$
$$+ 2t(1-(\det r)t)[r(X,\bar{X}) + r(Y,\bar{Y}) - 3r(X \times Y, \bar{X} \times \bar{Y})]$$
$$+ 2t^2[\|X \times r(\bar{Y}) - Y \times r(\bar{X})\|^2 - 4\langle X \times Y, r(\bar{X}) \times r(\bar{Y})\rangle].$$

Proof. We apply (4.7) to (4.4) and (4.5). In the calculation one may notice that $r \in O(3)$ satisfies $r({}^t r(X) \times \bar{X}) = (\det r) X \times r(\bar{X})$, etc. □

Proposition 4.4. *Let $\{\tilde{X},\tilde{Y}\}$ be an orthonormal pair with respect to $\tilde{g}$ such that $\tilde{X} = (X,0)$ and $\tilde{Y} = (0,\bar{Y})$. Then the sectional curvature $\hat{K}(\tilde{X},\tilde{Y})$ is non-negative.*

$\hat{K}(\tilde{X},\tilde{Y})$ vanishes with respect to $\hat{g}(t)$ for each $t \in (-t_o,t_o)$, if and only if $r(\bar{Y})$ is proportional to X and ${}^t r(X)$ is proportional to $\bar{Y}$. So, let $\bar{Y}$ be a unit eigenvector of the symmetric matrix ${}^t r \cdot r$ corresponding to a non-zero eigenvalue. We define X by $X = r(\bar{Y})/\|r(\bar{Y})\|$. Then the sectional curvature $\hat{K}(\tilde{X},\tilde{Y}) = 0$ for $\tilde{X} = (X,0)$ and $\tilde{Y} = (0,\bar{Y})$.

Proof. The first part is verified by Proposition 3.4 and the fact that $\hat{g}(t)^{-1}$ is also positive definite. The second part follows from the expression of $\hat{g}(\hat{R}(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X})$. □

Corollary 4.5. *We assume that ${}^t r \cdot r$ has three different non-zero eigenvalues. Then for each point of $(S^3 \times S^3, \hat{g}(t))$, there are only three sections of the form $\{\tilde{X}, \tilde{Y}\}$ with $\tilde{X} = (X, 0)$ and $\tilde{Y} = (0, Y)$ and with vanishing sectional curvature with respect to each $\hat{g}(t)$, $t \in (-t_o, t_o)$.*

REMARK 1. If one expands (4.2) with respect to t up to $[t^3]$, then one obtains

$$\begin{aligned}(4.10)\quad \hat{g}(\hat{R}(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X}) &= \|X \times Y\|^2 + \|\bar{X} \times \bar{Y}\|^2 \\ &+ 2t[r(X,\bar{X}) + r(Y,\bar{Y}) - 3r(X \times Y, \bar{X} \times \bar{Y})] \\ &+ t^2\{\|X \times r(\bar{Y}) - Y \times r(\bar{X})\|^2 + \|{}^t r(X) \times \bar{Y} - {}^t r(Y) \times \bar{X}\|^2 \\ &- 4[\langle X \times Y, r(\bar{X}) \times r(\bar{Y})\rangle + \langle \bar{X} \times \bar{Y}, {}^t r(X) \times {}^t r(Y)\rangle]\} + [t^3].\end{aligned}$$

5. Proof of Theorem A

Let $r = (-\delta_{uv})$ and let $\{\tilde{X}, \tilde{Y}\}$ be an orthonormal pair with respect to $\tilde{g}$. We can assume $\langle X, Y\rangle = \langle \bar{X}, \bar{Y}\rangle = 0$ by Lemma 3.3. By Proposition 4.1 and Lemma 4.2 we see that $F(t, \tilde{X}, \tilde{Y}) = (1+t)\hat{g}(\hat{R}(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X})$ is expressed as

$$\begin{aligned}(5.1)\quad F(t,\tilde{X},\tilde{Y}) &= \|X\|^2\|Y\|^2 + \|\bar{X}\|^2\|\bar{Y}\|^2 \\ &+ t[\|X\|^2\|Y\|^2 + \|\bar{X}\|^2\|\bar{Y}\|^2 - 2\langle X,\bar{X}\rangle - 2\langle Y,\bar{Y}\rangle \\ &\quad + 6\langle X,\bar{X}\rangle\langle Y,\bar{Y}\rangle - 6\langle X,\bar{Y}\rangle\langle \bar{X},Y\rangle] \\ &+ 2t^2[\|X\|^2\|\bar{Y}\|^2 + \|\bar{X}\|^2\|Y\|^2 - \langle X,\bar{X}\rangle - \langle Y,\bar{Y}\rangle \\ &\quad + \langle X,\bar{X}\rangle\langle Y,\bar{Y}\rangle + \langle X,\bar{Y}\rangle\langle \bar{X},Y\rangle - \langle X,\bar{Y}\rangle^2 - \langle \bar{X},Y\rangle^2].\end{aligned}$$

We put $\varepsilon_0 = 1/100\sqrt{2}$. If we have

$$\|X\|^2\|Y\|^2 + \|\bar{X}\|^2\|\bar{Y}\|^2 \geq \varepsilon_0^2,$$

then (5.1) shows that we have some real number t_3 such that $F(t,\tilde{X},\tilde{Y}) > 0$ holds for any $t \in (-t_3, t_3)$ (where t_3 is independent of the choice of orthonormal pairs $\{\tilde{X},\tilde{Y}\}$). So, in the following we suppose

$$(5.2)\qquad \|X\|^2\|Y\|^2 + \|\bar{X}\|^2\|\bar{Y}\|^2 < \varepsilon_0^2.$$

We can assume $\|\bar{X}\| \leq \|X\|$. Then $\|Y\| \leq \|\bar{Y}\|$ follows from (5.2). Also we have $\|\bar{X}\|\|\bar{Y}\| < \varepsilon_0$. By $\|\bar{Y}\| \geq 1/\sqrt{2}$, we obtain $\|\bar{X}\| < \sqrt{2}\varepsilon_0$. Similarly we obtain $\|Y\| < \sqrt{2}\varepsilon_0$. Therefore we get $\|X\|^2 > 1 - 2\varepsilon_0^2$ and $\|\bar{Y}\|^2 > 1 - 2\varepsilon_0^2$.

If $\bar{X} = Y = 0$, then Proposition 4.4 shows that $\hat{g}(\hat{R}(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X})$ is non-negative. So, in the following in this section we assume $\bar{X} \neq 0$ or $Y \neq 0$. By symmetry we assume $Y \neq 0$.

Now for any orthonormal pair $\{\tilde{X},\tilde{Y}\}$ we can change the frames $\{\xi_u,\xi_{\bar{u}}\} \to \{\xi'_u,\xi'_{\bar{u}}\}$ by an orthogonal 3×3 matrix A (i.e., $\xi'_u = A^v_u\xi_v$, $\xi'_{\bar{u}} = A^v_u\xi_{\bar{v}}$) so that

$$\tilde{X} = (\sqrt{1-\varepsilon_1^2},0,0;\bar{X}_1,\bar{X}_2,\bar{X}_3), \quad \tilde{Y} = (0,\varepsilon_2,0;\bar{Y}_1,\bar{Y}_2,\bar{Y}_3) \tag{5.3}$$

with the property; $X_1 = \|X\| = \sqrt{1-\varepsilon_1^2}$, $Y_2 = \|Y\| = \varepsilon_2 > 0$ and

$$\begin{aligned} &\bar{X}_1^2+\bar{X}_2^2+\bar{X}_3^2 = \varepsilon_1^2, \qquad \bar{Y}_1^2+\bar{Y}_2^2+\bar{Y}_3^2 = 1-\varepsilon_2^2,\\ &\bar{X}_1\bar{Y}_1+\bar{X}_2\bar{Y}_2+\bar{X}_3\bar{Y}_3 = 0, \end{aligned} \tag{5.4}$$

where $\varepsilon_1 = \|\bar{X}\| < \sqrt{2}\varepsilon_0 = 1/100$ and $\varepsilon_2 < 1/100$.
Notice that the expression of $\hat{g}(\hat{R}(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X})$ is unchanged. By (5.1) we obtain

$$F(t,\tilde{X},\tilde{Y}) = F_0 + F_1 t + F_2 t^2, \tag{5.5}$$

where we put F_0, $F_1 = F_1(t,\tilde{X},\tilde{Y})$ and $F_2 = F_2(t,\tilde{X},\tilde{Y})$ as

$$\begin{aligned} F_0 &= \varepsilon_1^2+\varepsilon_2^2-2\varepsilon_1^2\varepsilon_2^2,\\ F_1 &= -2X_1\bar{X}_1 - 2\varepsilon_2\bar{Y}_2 + 6\varepsilon_2X_1(\bar{X}_1\bar{Y}_2-\bar{X}_2\bar{Y}_1) + \varepsilon_1^2+\varepsilon_2^2-2\varepsilon_1^2\varepsilon_2^2,\\ F_2 &= 2[\varepsilon_2X_1(\bar{X}_1\bar{Y}_2+\bar{X}_2\bar{Y}_1)+\varepsilon_2^2(\bar{X}_1^2+\bar{X}_3^2)+X_1^2(\bar{Y}_2^2+\bar{Y}_3^2)-X_1\bar{X}_1-\varepsilon_2\bar{Y}_2]. \end{aligned}$$

We consider t in the range $0<t<1/100$.

First we assume $\varepsilon_1 = 0$, i.e., $\bar{X}_1=\bar{X}_2=\bar{X}_3=0$ with respect to the expression (5.3). Putting $\varepsilon=\varepsilon_2$, we obtain

$$F(t,\tilde{X},\tilde{Y}) = \varepsilon^2 + (\varepsilon^2-2\varepsilon\bar{Y}_2)t + 2(\bar{Y}_2^2+\bar{Y}_3^2-\varepsilon\bar{Y}_2)t^2.$$

By using an inequality $-2\varepsilon\bar{Y}_2t^2 \geq -(\varepsilon^2+\bar{Y}_2^2)t^2$, we get

$$F(t,\tilde{X},\tilde{Y}) \geq (\varepsilon-\bar{Y}_2t)^2 + 2\bar{Y}_3^2t^2 + \varepsilon^2t(1-t) > 0.$$

Therefore, sectional curvatures are positive in this case. So, in the following in this section we assume $\varepsilon_1>0$ and $\varepsilon_2>0$.

Lemma 5.1. *For fixed* t, ε_1 *and* ε_2, *if* $F(t,\tilde{X},\tilde{Y}) = F(t,\varepsilon_1,\varepsilon_2,\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{Y}_1,\bar{Y}_2,\bar{Y}_3)$ *attains its minimum at* $(t,\tilde{X}^*,\tilde{Y}^*) = (t,\varepsilon_1,\varepsilon_2,\bar{X}_1^*,\bar{X}_2^*,\bar{X}_3^*,\bar{Y}_1^*,\bar{Y}_2^*,\bar{Y}_3^*)$, *then* $\bar{X}_3^*=\bar{Y}_3^*=0$.

Proof. First we consider the following deformation;

$$\begin{aligned} &\bar{X}_1(\theta) = \cos\theta\,\bar{X}_1^* - \sin\theta\bar{X}_3^*, \qquad \bar{X}_3(\theta) = \sin\theta\,\bar{X}_1^* + \cos\theta\bar{X}_3^*,\\ &\bar{Y}_1(\theta) = \cos\theta\,\bar{Y}_1^* - \sin\theta\bar{Y}_3^*, \qquad \bar{Y}_3(\theta) = \sin\theta\,\bar{Y}_1^* + \cos\theta\bar{Y}_3^*, \end{aligned}$$

and $\bar{X}_2(\theta) = \bar{X}_2^*$, $\bar{Y}_2(\theta) = \bar{Y}_2^*$ for $\theta \in (-\delta, \delta)$. Calculating $(dF(t, \tilde{X}(\theta), \tilde{Y}(\theta))/d\theta)(0) = 0$ and noticing $X_1 > 0$, we obtain

$$\bar{X}_3^* + 3\varepsilon_2(\bar{X}_2^*\bar{Y}_3^* - \bar{X}_3^*\bar{Y}_2^*) + [\bar{X}_3^* + 2X_1\bar{Y}_1^*\bar{Y}_3^* - \varepsilon_2(\bar{X}_2^*\bar{Y}_3^* + \bar{X}_3^*\bar{Y}_2^*)]t = 0.$$

Therefore we get

$$[1 - 3\varepsilon_2\bar{Y}_2^* + (1 - \varepsilon_2\bar{Y}_2^*)t]\bar{X}_3^* = [-3\varepsilon_2\bar{X}_2^* + (\varepsilon_2\bar{X}_2^* - 2X_1\bar{Y}_1^*)t]\bar{Y}_3^*,$$

and hence $(1 - 3\varepsilon_2)|\bar{X}_3^*| \leq [3\varepsilon_1\varepsilon_2 + (2 + \varepsilon_1\varepsilon_2)t]|\bar{Y}_3^*|$. Consequently, we obtain $(3/4)|\bar{X}_3^*| \leq (3/100)|\bar{Y}_3^*|$, and $|\bar{X}_3^*| \leq (1/25)|\bar{Y}_3^*|$.

Next, we consider the following deformation;

$$\begin{aligned} \bar{X}_2(\tau) &= \cos\tau\,\bar{X}_2^* - \sin\tau\bar{X}_3^*, & \bar{X}_3(\tau) &= \sin\tau\,\bar{X}_2^* + \cos\tau\bar{X}_3^*, \\ \bar{Y}_2(\tau) &= \cos\tau\,\bar{Y}_2^* - \sin\tau\bar{Y}_3^*, & \bar{Y}_3(\tau) &= \sin\tau\,\bar{Y}_2^* + \cos\tau\bar{Y}_3^*, \end{aligned}$$

and $\bar{X}_1(\tau) = \bar{X}_1^*, \bar{Y}_1(\tau) = \bar{Y}_1^*$ for $\tau \in (-\delta, \delta)$. Calculating $(dF(t, \tilde{X}(\tau), \tilde{Y}(\tau))/d\tau)(0) = 0$ and noticing $\varepsilon_2 > 0$, we obtain

$$\bar{Y}_3^* - 3X_1(\bar{X}_1^*\bar{Y}_3^* - \bar{X}_3^*\bar{Y}_1^*) + [\bar{Y}_3^* + 2\varepsilon_2\bar{X}_2^*\bar{X}_3^* - X_1(\bar{X}_3^*\bar{Y}_1^* + \bar{X}_1^*\bar{Y}_3^*)]t = 0.$$

If $\bar{Y}_3^* > 0$ (< 0, resp.), we can show

$$\begin{aligned} &\bar{Y}_3^* - 3X_1(\bar{X}_1^*\bar{Y}_3^* - \bar{X}_3^*\bar{Y}_1^*) > 0, \quad (< 0, \text{ resp.}) \\ &\bar{Y}_3^* + 2\varepsilon_2\bar{X}_2^*\bar{X}_3^* - X_1(\bar{X}_3^*\bar{Y}_1^* + \bar{X}_1^*\bar{Y}_3^*) > 0, \quad (< 0, \text{ resp.}) \end{aligned}$$

using the inequality $|\bar{X}_3^*| \leq (1/25)|\bar{Y}_3^*|$. This is a contradiction. So we have $\bar{Y}_3^* = 0$ and $\bar{X}_3^* = 0$. □

In the following we consider $\tilde{X}$ and $\tilde{Y}$ of the form;

$$\bar{X} = (\bar{X}_1, \bar{X}_2, 0), \qquad \bar{Y} = (\bar{Y}_1, \bar{Y}_2, 0) \tag{5.6}$$

and we put $\rho = |\bar{Y}_2|$. Then we have

$$\bar{X}_1^2 = \rho^2\varepsilon_1^2/(1 - \varepsilon_2^2), \qquad \bar{X}_2^2 = (1 - \varepsilon_2^2 - \rho^2)\varepsilon_1^2/(1 - \varepsilon_2^2), \qquad \bar{Y}_1^2 = 1 - \varepsilon_2^2 - \rho^2.$$

We consider the following two cases (i) and (ii).

(i) The case where $\rho \leq 4\max\{\varepsilon_1, \varepsilon_2\}$.

Lemma 5.2. *There is a positive number t_4 such that $F(t, \tilde{X}, \tilde{Y}) > 0$ holds for any $t \in (0, t_4)$.*

Proof. We put $\hat{\varepsilon} = \max\{\varepsilon_1, \varepsilon_2\}$. For example we have

$$|X_1 \bar{X}_1| < |\bar{X}_1| < 2\rho\varepsilon_1 \leq 8\hat{\varepsilon}\varepsilon_1 \leq 4(\hat{\varepsilon}^2 + \varepsilon_1^2).$$

Therefore, we see that $|F_1| < a(\varepsilon_1^2 + \varepsilon_2^2)$ holds for some positive number a. Similarly, we see that $|F_2| < a'(\varepsilon_1^2 + \varepsilon_2^2)$ holds for some positive number a'. Then (5.5) shows

$$F(t, \tilde{X}, \tilde{Y}) > (\varepsilon_1^2 + \varepsilon_2^2)(1 - at - a't^2) - 2\varepsilon_1^2\varepsilon_2^2,$$

where a and a' are universal constant. So, we have some t_4 so that $1 - at - a't^2 > 1/2$ for $t \in (0, t_4)$. Since $-2\varepsilon_1^2\varepsilon_2^2 > -\varepsilon_1\varepsilon_2$, we have $F(t, \tilde{X}, \tilde{Y}) > 0$ for any $t \in (0, t_4)$. □

(ii) The case where $\rho \geq 4\max\{\varepsilon_1, \varepsilon_2\}$.

Lemma 5.3. *For fixed t, ε_1 and ε_2, if $F(t, \tilde{X}, \tilde{Y}) = F(t, \varepsilon_1, \varepsilon_2, \bar{X}_1, \bar{X}_2, 0, \bar{Y}_1, \bar{Y}_2, 0)$ attains its minimum at $(t, \tilde{X}^*, \tilde{Y}^*) = (t, \varepsilon_1, \varepsilon_2, \bar{X}_1^*, \bar{X}_2^*, 0, \bar{Y}_1^*, \bar{Y}_2^*, 0)$, then we have $\bar{X}_1^* > 0$ and $\bar{Y}_2^* > 0$.*

Proof. We compare $\bar{X}^* = (\bar{X}_1^*, \bar{X}_2^*, 0)$ and $\bar{Y}^* = (\bar{Y}_1^*, \bar{Y}_2^*, 0)$ with

$$\bar{X} = (-\bar{X}_1^*, \bar{X}_2^*, 0), \qquad \bar{Y} = (-\bar{Y}_1^*, \bar{Y}_2^*, 0).$$

By (5.5), $F(t, \tilde{X}, \tilde{Y}) \geq F(t, \tilde{X}^*, \tilde{Y}^*)$ is expressed as

$$\bar{X}_1^* - 3\varepsilon_2(\bar{X}_1^*\bar{Y}_2^* - \bar{X}_2^*\bar{Y}_1^*) - [\varepsilon_2(\bar{X}_1^*\bar{Y}_2^* + \bar{X}_2^*\bar{Y}_1^*) - \bar{X}_1^*]t \geq 0,$$

which is equivalent to

$$[1 - 3\varepsilon_2\bar{Y}_2^* + (1 - \varepsilon_2\bar{Y}_2^*)t]\bar{X}_1^* \geq (t - 3)\varepsilon_2\bar{X}_2^*\bar{Y}_1^*.$$

If $\bar{X}_1^* \leq 0$, then we have $(1 - 3\varepsilon_2)|\bar{X}_1^*| \leq 3\varepsilon_1\varepsilon_2$. By $|\bar{X}_1^*| = \rho\varepsilon_1/\sqrt{1 - \varepsilon_2^2}$, we obtain

$$\rho \leq 3\varepsilon_2\sqrt{1 - \varepsilon_2^2}\Big/(1 - 3\varepsilon_2) < 3\varepsilon_2/(1 - 3\varepsilon_2).$$

This contradicts $\rho \geq 4\max\{\varepsilon_1, \varepsilon_2\}$ and we have $\bar{X}_1^* > 0$.
Next we compare $\bar{X}^* = (\bar{X}_1^*, \bar{X}_2^*, 0)$ and $\bar{Y}^* = (\bar{Y}_1^*, \bar{Y}_2^*, 0)$ with

$$\bar{X} = (\bar{X}_1^*, -\bar{X}_2^*, 0), \qquad \bar{Y} = (\bar{Y}_1^*, -\bar{Y}_2^*, 0).$$

By (5.5), $F(t, \tilde{X}, \tilde{Y}) \geq F(t, \tilde{X}^*, \tilde{Y}^*)$ is expressed as

$$\bar{Y}_2^* - 3X_1(\bar{X}_1^*\bar{Y}_2^* - \bar{X}_2^*\bar{Y}_1^*) - [X_1(\bar{X}_1^*\bar{Y}_2^* + \bar{X}_2^*\bar{Y}_1^*) - \bar{Y}_2^*]t \geq 0,$$

which is equivalent to

$$[1-3X_1\bar{X}_1^*+(1-X_1\bar{X}_1^*)t]\bar{Y}_2^*\geq(t-3)X_1\bar{Y}_1^*\bar{X}_2^*.$$

If $\bar{Y}_2^*\leq 0$, then we have $(1-3\varepsilon_1)|\bar{Y}_2^*|\leq 3\varepsilon_1$. This contradicts $\rho=|\bar{Y}_2^*|\geq 4\max\{\varepsilon_1,\varepsilon_2\}$ and we have $\bar{Y}_2^*>0$. □

In the following we consider $\tilde{X}$ and $\tilde{Y}$ of the form;

$$\bar{X}_1=\rho\varepsilon_1\Big/\sqrt{1-\varepsilon_2^2},\qquad \bar{X}_2=\beta\varepsilon_1\sqrt{1-\varepsilon_2^2-\rho^2}\Big/\sqrt{1-\varepsilon_2^2},$$
$$\bar{Y}_1=-\beta\sqrt{1-\varepsilon_2^2-\rho^2},\qquad \bar{Y}_2=\rho,$$

where $\beta=\pm1$. Now F_1 and F_2 in (5.5) are expressed as

$$\begin{aligned} F_1 &= -2\rho\varepsilon_1\sqrt{1-\varepsilon_1^2}\Big/\sqrt{1-\varepsilon_2^2}-2\rho\varepsilon_2+6\varepsilon_1\varepsilon_2\sqrt{1-\varepsilon_1^2}\sqrt{1-\varepsilon_2^2}\\ &\quad+\varepsilon_1^2+\varepsilon_2^2-2\varepsilon_1^2\varepsilon_2^2,\end{aligned}$$
$$\begin{aligned} F_2/2 &= -\varepsilon_1\varepsilon_2\sqrt{1-\varepsilon_1^2}\sqrt{1-\varepsilon_2^2}+\rho\varepsilon_1(2\rho\varepsilon_2-1)\sqrt{1-\varepsilon_1^2}\Big/\sqrt{1-\varepsilon_2^2}\\ &\quad+\rho^2\varepsilon_1^2\varepsilon_2^2/(1-\varepsilon_2^2)+(1-\varepsilon_1^2)\rho^2-\rho\varepsilon_2.\end{aligned}\tag{5.7}$$

Lemma 5.4. *We have $F_2>0$.*

Proof. We neglect some positive terms of the right hand side of (5.7) and use an inequality $1/\sqrt{1-\varepsilon_2^2}<1+\varepsilon_2^2$. Then we obtain

$$\begin{aligned} F_2/2 &> -\varepsilon_1\varepsilon_2-\rho\varepsilon_1(1+\varepsilon_2^2)+(1-\varepsilon_1^2)\rho^2-\rho\varepsilon_2\\ &= (\rho^2/4-\varepsilon_1\varepsilon_2)+\rho[(1/4-\varepsilon_1^2)\rho-\varepsilon_1\varepsilon_2^2]+\rho(\rho/2-\varepsilon_1-\varepsilon_2)>0.\end{aligned}$$

Therefore we have $F_2>0$. □

Lemma 5.5. *For fixed ρ, ε_1 and ε_2, if $F(t,\tilde{X},\tilde{Y})=F_2t^2+F_1t+F_0$ takes its minimum at $\hat{t}$, then we have $\hat{t}>(\varepsilon_1+\varepsilon_2)/16$.*

Proof. We estimate $\hat{t}=-F_1/2F_2$. Since $\sqrt{1-\mu}=1-\mu/2-\mu^2/8+[\mu^3]$ and $1/\sqrt{1-\mu}=1+\mu/2+3\mu^2/8+[\mu^3]$, we see that F_1 and F_2 are expressed as

$$\begin{aligned} F_1 &= -2\rho(\varepsilon_1+\varepsilon_2)+\varepsilon_1^2+\varepsilon_2^2+6\varepsilon_1\varepsilon_2+\rho\varepsilon_1(\varepsilon_1^2-\varepsilon_2^2)\\ &\quad-\varepsilon_1\varepsilon_2(3\varepsilon_1^2+2\varepsilon_1\varepsilon_2+3\varepsilon_2^2)+(\rho\varepsilon_1/4)(\varepsilon_1^2-\varepsilon_2^2)(\varepsilon_1^2+3\varepsilon_2^2)+[*],\end{aligned}$$
$$F_2/2 = \rho^2-\rho(\varepsilon_1+\varepsilon_2)-\varepsilon_1\varepsilon_2+\rho^2\varepsilon_1(2\varepsilon_2-\varepsilon_1)+(\rho\varepsilon_1/2)(\varepsilon_1^2-\varepsilon_2^2)\tag{5.7'}$$

$$+(\varepsilon_1\varepsilon_2/2)(\varepsilon_1^2+\varepsilon_2^2)+\rho^2\varepsilon_1\varepsilon_2(\varepsilon_1\varepsilon_2-\varepsilon_1^2+\varepsilon_2^2)$$
$$+(\rho\varepsilon_1/8)(\varepsilon_1^2-\varepsilon_2^2)(\varepsilon_1^2+3\varepsilon_2^2)+[*],$$

where $[*]$ denotes terms of higher order $\varepsilon_1^a\varepsilon_2^b$ with $a+b\geq 6$. First we see that the terms of higher order $\varepsilon_1^a\varepsilon_2^b$ with $a+b\geq 3$ in F_1 are covered by $2(\varepsilon_1^2+\varepsilon_2^2)$. So we have

$$\begin{aligned}-F_1 &> 2\rho(\varepsilon_1+\varepsilon_2)-3\varepsilon_1^2-3\varepsilon_2^2-6\varepsilon_1\varepsilon_2\\ &= 2\rho(\varepsilon_1+\varepsilon_2)-3(\varepsilon_1+\varepsilon_2)^2\\ &= (\rho/2)(\varepsilon_1+\varepsilon_2)+3(\varepsilon_1+\varepsilon_2)(\rho/2-\varepsilon_1-\varepsilon_2)\\ &\geq (\rho/2)(\varepsilon_1+\varepsilon_2).\end{aligned}$$

Next neglecting the negative terms in (5.7′) and putting $\hat{\varepsilon}=\max\{\varepsilon_1,\varepsilon_2\}$, we obtain

$$\begin{aligned}F_2/2 &< \rho^2+2\rho^2\varepsilon_1\varepsilon_2+(\rho/2)\varepsilon_1^3+16\hat{\varepsilon}^4\\ &< \rho^2+2\rho^2\varepsilon_1\varepsilon_2+(\rho^2/8)\varepsilon_1^2+\rho^2\hat{\varepsilon}^2<2\rho^2<2\rho.\end{aligned}$$

Therefore we get $-F_1/2F_2>(\varepsilon_1+\varepsilon_2)/16$. $\square$

Finally we show $F(t,\tilde{X},\tilde{Y})>0$ for $t\in(0,1/100)$. We rewrite $F(t,\tilde{X},\tilde{Y})$ as $F(t,\tilde{X},\tilde{Y})=J_2\rho^2+J_1\rho+J_0$, where we have put

$$\begin{aligned}J_0 &= (\varepsilon_1^2+\varepsilon_2^2-2\varepsilon_1^2\varepsilon_2^2)(1+t)+2\varepsilon_1\varepsilon_2t(3-t)\sqrt{1-\varepsilon_1^2}\sqrt{1-\varepsilon_2^2},\\ J_1 &= -2t(1+t)\left(\varepsilon_2+\varepsilon_1\sqrt{1-\varepsilon_1^2}\Big/\sqrt{1-\varepsilon_2^2}\right),\\ J_2 &= 2t^2\left[1-\varepsilon_1^2+2\varepsilon_1\varepsilon_2\sqrt{1-\varepsilon_1^2}\Big/\sqrt{1-\varepsilon_2^2}+\varepsilon_1^2\varepsilon_2^2/(1-\varepsilon_2^2)\right].\end{aligned}$$

Clearly we have $J_2>0$. To show $F(t,\tilde{X},\tilde{Y})>0$, it suffices to show that the discriminant $D=J_1^2-4J_0J_2$ is negative. After some calculation we obtain

$$\begin{aligned}D/4t^2 = &-(\varepsilon_1-\varepsilon_2)^2(1-\varepsilon_1^2+3\varepsilon_1\varepsilon_2)-4t\varepsilon_1\varepsilon_2(2-3\varepsilon_1^2+4\varepsilon_1\varepsilon_2-\varepsilon_2^2)\\ &+t^2[(\varepsilon_1-\varepsilon_2)^2+8\varepsilon_1\varepsilon_2-(\varepsilon_1^4+7\varepsilon_1^3\varepsilon_2-9\varepsilon_1^2\varepsilon_2^2+\varepsilon_1\varepsilon_2^3)]+[*],\end{aligned}$$

where $[*]$ denotes terms of higher order $\varepsilon_1^a\varepsilon_2^b$ with $a+b\geq 6$. We see that $\hat{\varepsilon}^5>[*]$ holds. Neglecting some negative terms we obtain

$$\begin{aligned}D/4t^2 &< -(\varepsilon_1-\varepsilon_2)^2(1-\varepsilon_1^2)-4t\varepsilon_1\varepsilon_2(2-3\varepsilon_1^2-\varepsilon_2^2)\\ &\quad+t^2[(\varepsilon_1-\varepsilon_2)^2+8\varepsilon_1\varepsilon_2+9\varepsilon_1^2\varepsilon_2^2]+\hat{\varepsilon}^5\\ &< -(9/10)[(\varepsilon_1-\varepsilon_2)^2+8t\varepsilon_1\varepsilon_2]+\hat{\varepsilon}^5. \qquad (5.8)\end{aligned}$$

By Lemma 5.5, it suffices to show $D < 0$ for $t = (\varepsilon_1 + \varepsilon_2)/16$. By symmetry of ε_1 and ε_2 in (5.8) we can assume $\hat{\varepsilon} = \varepsilon_2 \geq \varepsilon_1$. Then the inequality

$$-(9/20)[2(\varepsilon_1 - \varepsilon_2)^2 + \varepsilon_1\varepsilon_2(\varepsilon_1 + \varepsilon_2)] + \hat{\varepsilon}^5 < 0$$

is verified by considering two cases; $\varepsilon_1 \leq \hat{\varepsilon}/2$ and $\varepsilon_1 \geq \hat{\varepsilon}/2$.

Proof of Theorem A. We define t_* by $t_* = \min\{t_3, t_4, 1/100\}$. Then sectional curvatures are non-negative. Furthermore, by Proposition 4.4 and the above discussion, we see that the sections $\{\tilde{X}, \tilde{Y}\}$ with zero sectional curvature are of the form $\tilde{X} = (X, 0)$ and $\tilde{Y} = (0, X)$ for $t \in (0, t_*)$. □

REMARK 1. For $\varepsilon_1 = \varepsilon_2 = \varepsilon$, we consider $\tilde{X} = (\rho, 0, 0; \varepsilon, 0, 0)$ and $\tilde{Y} = (0, \varepsilon, 0; 0, \rho, 0)$ where $\rho = \sqrt{1 - \varepsilon^2}$. Then F_1 and F_2 are expressed as

$$F_1 = -4\rho\varepsilon + 8\varepsilon^2(1 - \varepsilon^2), \qquad F_2 = 2 - 4\rho\varepsilon - 2\varepsilon^2 + 2\varepsilon^4.$$

Therefore, $\hat{t} = -F_1/2F_2 = \varepsilon + \varepsilon^3/2 + [\varepsilon^4]$ and for $t = \varepsilon + \varepsilon^3/2$, we obtain

$$F(t, \tilde{X}, \tilde{Y}) = 4\varepsilon^3 - 2\varepsilon^4 + [\varepsilon^5].$$

6. Proof of Theorem B

Suppose $r = (\lambda_u \delta_{uv})$ with $1 = \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. (i) follows from Proposition 4.4 and Corollary 4.5. To prove (ii) we define $\{\tilde{X}, \tilde{Y}\}$ by

$$\begin{aligned} \tilde{X} &= (X_1, 0, 0; -t, 0, 0), \qquad & X_1 &= \sqrt{1 - t^2}, \\ \tilde{Y} &= (0, -\lambda_2 t, 0; 0, \bar{Y}_2, 0), \qquad & \bar{Y}_2 &= \sqrt{1 - \lambda_2^2 t^2}. \end{aligned}$$

By Proposition 4.1 and Lemma 4.2, we have the following:

$$\begin{aligned} &\|X\|^2\|Y\|^2 + \|\bar{X}\|^2\|\bar{Y}\|^2 = t^2 + \lambda_2^2 t^2 - 2\lambda_2^2 t^4, \\ &G_1 = 2(-X_1 t - \lambda_2^2 \bar{Y}_2 t - 3\lambda_2\lambda_3 X_1 \bar{Y}_2 t^2), \\ &G_2 = [1/(1 - \lambda_3^2 t^2)]\{\lambda_2^2(t^2 - X_1\bar{Y}_2)^2 + (\lambda_2^2 t^2 - X_1\bar{Y}_2)^2 \\ &\qquad +2\lambda_2\lambda_3(t^2 - X_1\bar{Y}_2)(\lambda_2^2 t^2 - X_1\bar{Y}_2)t\}. \end{aligned}$$

Therefore, using $X_1 = 1 - t^2/2 + [t^4]$ and $\bar{Y}_2 = 1 - \lambda_2^2 t^2/2 + [t^4]$, we get

$$(6.1) \quad (1 - \lambda_3^2 t^2)\hat{g}(\hat{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X}) = -4\lambda_2\lambda_3 t^3 - (8\lambda_2^2 - \lambda_2^2\lambda_3^2 - \lambda_3^2)t^4 + [t^5],$$

where $[t^5]$ denotes the term of higher order. So, for a sufficiently small t, we obtain $\hat{g}(\hat{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X}) < 0$ for $\hat{g}(t)$ and $\{\tilde{X}, \tilde{Y}\}$. This proves Proposition B.

REMARK 1. By (6.1) we see that (ii) of Theorem B works for the cases;

$$(\lambda_1, \lambda_2, \lambda_3) = (+, +, 0), (+, -, 0), (+, -, -).$$

REMARK 2. Hopf problem asks whether $S^2 \times S^2$ admits a Riemannian metric of positive sectional curvature. One of the related problems is whether $S^3 \times S^3$ admits a Riemannian metric of positive sectional curvature. On the other hand, Hopf conjecture says that the Euler-Poincaré characteristic of a compact oriented $2n$-dimensional Riemannian manifold is > 0 ($\geq 0, \leq 0, < 0$ for $n = 2r + 1$; $\geq 0, \geq 0, > 0$ for $n = 2r$, respectively), if and only if the sectional curvature is > 0 ($\geq 0, \leq 0, < 0$, respectively). If $2n = 4$, the Hopf conjecture is true. However, for $2n \geq 6$ this conjecture is open, and some people focus their study on 6-dimensional or 8-dimensional case (cf. Klembeck [2], etc.). $S^3 \times S^3$ lies at a point of intersection of the above two problems.

Let $\hat{g}(t)$ be one defined by (1.1). Then, $(SU(2) \times SU(2), \hat{g}(t))$ admits Killing vector fields which are right invariant vector fields on $SU(2) \times SU(2)$. Since the Euler-Poincaré characteristic of $S^3 \times S^3$ is zero, $(SU(2) \times SU(2), \hat{g}(t))$ can not be of positive sectional curvature (cf. Weinstein [4]). Therefore, we have one question if it is possible to deform $\hat{g}(t)$ in Theorem A to a Riemannian metric which is not left invariant and has positive sectional curvature.

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