



Title	Homeomorphisms of 3-manifolds and topological entropy
Author(s)	Kobayashi, Tsuyoshi
Citation	Osaka Journal of Mathematics. 1985, 22(2), p. 289-297
Version Type	VoR
URL	https://doi.org/10.18910/8703
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

Kobayashi, T.
Osaka J. Math.
22 (1985), 289-297

HOMEOMORPHISMS OF 3-MANIFOLDS AND TOPOLOGICAL ENTROPY

Dedicated to Professor Itiro Tamura on his 60th birthday

TSUYOSHI KOBAYASHI

(Received June 21, 1984)

1. Introduction

The *topological entropy* $h(f)$ of a self-map f of a metric space is a measure of its dynamical complexity (for the definition of topological entropy see section 2 below). In [T₁] Thurston has shown that any homeomorphism f of a compact hyperbolic surface is isotopic to φ which is either periodic, pseudo-Anosov, or reducible (see also [F-L-P], [H-T], [M]). We call φ *Thurston's canonical form* of f . In section 2 we show that $h(\varphi) \leq h(f)$ i.e. φ attains the minimal entropy in its isotopy class. Hence from the dynamical viewpoint Thurston's canonical form plays an important role ([H], [K], [Smi]).

In this paper, we find a similar canonical form of a homeomorphism of a class of geometric 3-manifolds (for the definition and fundamental properties of geometric 3-manifolds see [Sc]). We note that every self-homeomorphism of a hyperbolic 3-manifold is homotopic to a periodic one ([Mo]). In the following, we consider homeomorphisms of an $H^2 \times R$, $\widetilde{SL_2(R)}$, or Nil 3-manifold M .

Then our main result is:

Theorem 2. *Let f be a homeomorphism of an $H^2 \times R$, $\widetilde{SL_2(R)}$, or Nil 3-manifold M . Then f is homotopic to φ such that either:*

- (i) φ is of type periodic,
- (ii) φ is of type pseudo-Anosov, or
- (iii) there is a system \mathcal{I} of tori in M such that φ is reducible by \mathcal{I} . There is a φ -invariant regular neighborhood $\eta(\mathcal{I})$ of \mathcal{I} such that each φ -component of $M - \text{Int } \eta(\mathcal{I})$ satisfies (i) or (ii). Each component $\eta(T_i)$ of $\eta(\mathcal{I})$ is mapped to itself by some positive iterate φ^{m_i} of φ and $\varphi^{m_i}|_{\eta(T_i)}$ is a twist homeomorphism.

For the definitions of the terms which appear in Theorem 2, see section 4 below. We note that if M is sufficiently large, then φ is isotopic to f ([Wa]).

In section 5 we show that the above φ attains the minimal entropy in its homotopy class, and $h(\varphi)$ is positive if and only if φ contains a component of type pseudo-Anosov.

2. Preliminaries

Let $f: X \rightarrow X$ be a continuous map of a metric space (X, d) . In this section we recall the definition of the topological entropy $h(f)$ of f in [Bo], and show that Thurston's canonical form of a surface homeomorphism attains the minimal entropy in its homotopy class.

Let $K \subset X$ be compact, $\varepsilon > 0$ a positive number, and n a positive integer. We say that $E \subset K$ is (n, ε) -separated, if $x, y \in E$, $x \neq y$, then there is $0 \leq i < n$ such that $d(f^i(x), f^i(y)) \geq \varepsilon$. Let $s_K(n, \varepsilon)$ be the maximal cardinality of an (n, ε) -separated set in K . We say that $E \subset K$ is (n, ε) -spanning for K , if $x \in K$, then there is a $y \in E$ such that $d(f^i(x), f^i(y)) < \varepsilon$ for all i with $0 \leq i < n$. Let $r_K(n, \varepsilon)$ be the minimal cardinality of an (n, ε) -spanning set in K . Let $\bar{s}_K(\varepsilon) = \limsup_{n \rightarrow \infty} 1/n \cdot \log(s_K(n, \varepsilon))$, and $\bar{r}_K(\varepsilon) = \limsup_{n \rightarrow \infty} 1/n \cdot \log(r_K(n, \varepsilon))$. Then it can be shown that $\lim_{\varepsilon \rightarrow 0} \bar{s}_K(\varepsilon) = \lim_{\varepsilon \rightarrow 0} \bar{r}_K(\varepsilon)$, and we denote this value by $h_K(f)$. Finally, we put $h(f) = \sup \{h_K(f) \mid K: \text{compact} \subset X\}$.

For the proof of the following theorem see [Bo].

Theorem A. *Let X, Y be compact metric spaces, $p: X \rightarrow Y$, $f: X \rightarrow X$, $g: Y \rightarrow Y$ continuous maps such that $f \circ p = p \circ g$. Then $h(g) \leq h(f) \leq h(g) + \sup \{h_{p^{-1}(y)}(f) \mid y \in Y\}$.*

Let F be a compact hyperbolic surface. A *measured foliation* (\mathcal{F}, μ) on F is a pair of a singular foliation on F and a transverse invariant measure μ of \mathcal{F} .

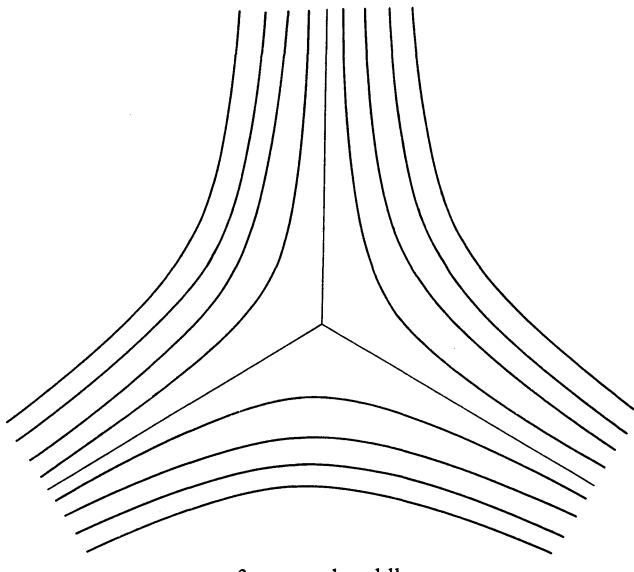


Fig. 1

([F-L-P], [T₁]). \mathcal{F} may have a finite number of singularities a_1, \dots, a_l , where a_i is an r_i -pronged saddle with $r_i \geq 3$. If M has boundary, then each boundary component is a leaf of the foliation and has at least one singularity. A self-homeomorphism $f: F \rightarrow F$ is *pseudo-Anosov* if there is a pair of mutually transverse measured foliations $(\mathcal{F}^s, \mu^s), (\mathcal{F}^u, \mu^u)$ and a number $\lambda > 1$ such that f preserves two foliations $\mathcal{F}^s, \mathcal{F}^u$ and $f_*(\mu^s) = 1/\lambda \cdot \mu^s, f_*(\mu^u) = \lambda \cdot \mu^u$. Then λ is called the *expanding factor* of f . f is *reducible* by Γ if Γ is a system of mutually disjoint and non-parallel loops, each of which is non-contractible, non-peripheral and $f(\Gamma) = \Gamma$. f is *periodic* if there is a positive integer n such that $f^n = id_F$. Let A be an annulus, $g: A \rightarrow A$ a homeomorphism, $\tilde{A} = [-1, 1] \times R$ the universal cover of A where the covering translations are generated by $(x, y) \rightarrow (x, y+1)$. g is a *twist homeomorphism* if there is a lift $\tilde{g}: \tilde{A} \rightarrow \tilde{A}$ of g such that $\tilde{g}(x, y) = (\pm x, h(x, y))$ for some map h .

Then the precise statement of Thurston's result is:

Theorem B (Thurston [T₁]). *If f is a self-homeomorphism of a compact hyperbolic surface F then f is isotopic to φ such that either:*

- (i) φ is periodic,
- (ii) φ is pseudo-Anosov, or

(iii) there is a system of simple loops Γ on F such that φ is reducible by Γ . There is a φ -invariant regular neighborhood $\eta(\Gamma)$. Each φ -component of $F - \text{Int } \eta(\Gamma)$ satisfies (i) or (ii). Each component, A_i , of $\eta(\Gamma)$ is mapped to itself by some positive iterate φ^{m_i} of φ , and $\varphi^{m_i}|_{A_i}$ is a twist homeomorphism.

Then we can show:

Proposition 2.1. *Thurston's canonical form φ attains the minimal entropy in its homotopy class. Moreover $h(\varphi) > 0$ if and only if φ contains a pseudo-Anosov component.*

Proof. If φ is periodic then $h(\varphi) = 0$ for $h(\varphi^n) = n \cdot h(\varphi)$. If φ is pseudo-Anosov then by [F-L-P] Exposé 10, $h(\varphi) > 0$ and it attains the minimal entropy in its homotopy class. Suppose that φ is reducible. Let A_1, \dots, A_m ($m \geq 1$) be the components of $\eta(\Gamma)$ and F_1, \dots, F_n ($n \geq 1$) be the closures of the components of $F - \cup A_i$. Let l be a positive integer such that $\varphi^l(A_i) = A_i$ ($1 \leq i \leq m$) and $\varphi^l(F_j) = F_j$ ($1 \leq j \leq n$). By the definition of topological entropy we see that $h(\varphi^l) = \max_{i,j} \{h(\varphi^l|_{A_i}), h(\varphi^l|_{F_j})\}$. It is easy to show that for each homeomorphism g of a circle we have $h(g) = 0$. By Theorem A we see that $h(\varphi^l|_{A_i}) = 0$ for each i . Hence $h(\varphi^l) = \max_j \{h(\varphi^l|_{F_j})\}$.

If each $\varphi^l|_{F_j}$ is periodic, then $h(\varphi^l) = 0$. Hence $h(\varphi) = 0$.

If φ^l contains a pseudo-Anosov component, then $h(\varphi^l) = \log \lambda_i$, where λ_i is the expanding factor of some pseudo-Anosov component $\varphi^l|_{F_i}$. Since $\chi(F_i)$

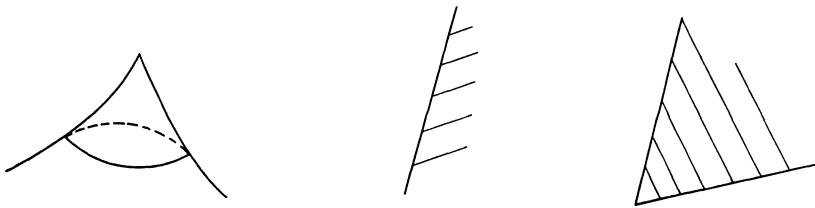
<0 , there is an essential, non-peripheral (not necessarily simple) loop α on F_i . Then by [F-L-P] $\lim_{n \rightarrow \infty} 1/n \cdot \log(L(\varphi^n(\alpha))) = \lambda_i$, where $L(\alpha)$ denotes the infimum of the length of loops which are homotopic to α . By [F-L-P] if g is homotopic to φ^l then $h(g) \geq \log \lambda_i$, i.e. φ^l attains the minimal entropy in its homotopy class. Since $h(\varphi^l) = l \cdot h(\varphi)$ we see that φ attains the minimal entropy in its homotopy class.

This completes the proof of Proposition 2.1.

3. Homeomorphisms of 2-dimensional orbifolds

In this section, we give a classification theorem for homeomorphisms of 2-dimensional orbifolds. We assume that the reader is familiar with [T₂, §13] or [Sc, §2].

By [Sc], [T₂] every singularity on a 2-dimensional orbifold is either cone, reflector line, or corner reflector. Throughout this paper we consider orbifolds



Singualrities of 2-orbifolds

Fig. 2

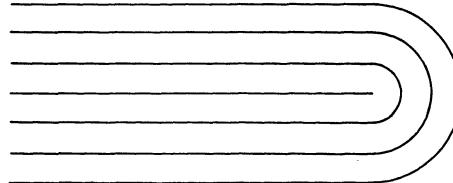
whose singularities are cones. Let O (O' resp.) be a 2-dimensional orbifold with singularities x_1, \dots, x_n ($n \geq 1$) ($x'_1, \dots, x'_{n'}$ ($n' \geq 1$) resp.), where the cone angle of x_i (x'_i resp.) is $2\pi/p_i$ ($2\pi/p'_i$ resp.).

Let f be a map from O to O' . f is called O -homeomorphism if f satisfies:

- (i) f is a topological homeomorphism,
- (ii) $f(\text{sing}(O)) = \text{sing}(O')$, where $\text{sing}(O)$ denotes the set of the singularities of O ,
- (iii) if $f(x_i) = x'_j$ then $p_i = p'_j$.

Let $f, f': O \rightarrow O'$ be O -homeomorphisms. f and f' are O -isotopic if there is a topological isotopy $F_t: O \rightarrow O'$ ($0 \leq t \leq 1$) such that each F_t is an O -homeomorphism and $F_0 = f$, $F_1 = f'$.

The definition of a measured foliation (\mathcal{F}, μ) on O is the same as the definition for surfaces in section 2 except the fact that if x is a singularity of O then \mathcal{F} may have 1-pronged saddle at x . A self- O -homeomorphism $f: O \rightarrow O$ is pseudo-Anosov if there are a pair of mutually transverse measured foliations (\mathcal{F}^s, μ^s) , (\mathcal{F}^u, μ^u) and a number $\lambda > 1$ such that f preserves the two foliations, and $f_*(\mu^s) = 1/\lambda \cdot \mu^s$, $f_*(\mu^u) = \lambda \cdot \mu^u$.



1-pronged saddle

Fig. 3

Let $a_1, a_2 \subset O$ be simple loops which do not meet singularities of O . a_1 and a_2 are *parallel* if $a_1 \cup a_2$ bounds an annulus which does not contain singularities. a_1 is *peripheral* if a_1 is parallel to a component of ∂O . a_1 is *essential* if a_1 is not peripheral and a_1 does not bound a disk on O which contains at most one singular point.

A self- O -homeomorphism $f: O \rightarrow O$ is *reducible* by Γ if Γ is a system of simple loops on O each of which does not meet a singular point and is essential, which are mutually disjoint and non-parallel, and $f(\Gamma) = \Gamma$.

Then we have:

Theorem 1. *Let O be a compact 2-dimensional orbifold whose (possibly empty) singular points are all cone type and f a self- O -homeomorphism of O . Then f is O -isotopic to φ such that either:*

(i) φ is periodic,

(ii) φ is pseudo-Anosov, or

(iii) there is a system Γ of simple loops on O such that φ is reducible by Γ .

There is a φ -invariant regular neighborhood $\eta(\Gamma)$ of Γ such that each φ -component of $O - \text{Int } \eta(\Gamma)$ satisfies (i) or (ii). Each component, A_j , of $\eta(\Gamma)$ is mapped to itself by some positive iterate φ^{m_j} of φ and $\varphi^{m_j}|_{A_j}$ is a twist homeomorphism.

Proof. First, suppose that O contains no singularities i.e. O is a surface. In case of $\chi(O) < 0$ Theorem 1 is just Theorem B in section 2. There are four distinct compact surfaces with Euler characteristic zero, say annulus, Möbius band, Klein bottle, and torus [Sc]. It is easy to see that every homeomorphism of an annulus or a Möbius band is homotopic to a periodic one, and then is isotopic to a periodic one [E]. By Lickorish [Li] every homeomorphism of a Klein bottle is isotopic to a periodic one. Let O be a torus, A a 2×2 matrix representing $f_*: (O)_{\pi_1} \rightarrow \pi_1(O)$ for a fixed basis of $\pi_1(O)$. Then f is isotopic to a reducible, periodic, or (pseudo-)Anosov map according to whether A is conjugate to $\begin{pmatrix} \varepsilon & n \\ 0 & \varepsilon \end{pmatrix}$ ($\varepsilon = \pm 1, n \neq 0$), to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $|\lambda_1| = |\lambda_2| = 1$, or to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $|\lambda_1| \neq |\lambda_2|$ [F-L-P, Exposé 1]. There are three distinct compact surfaces with positive Euler characteristic say sphere, disk, and projective plane. By Smale [Sma], [F-L-P, Exposé 2] every homeomorphism of them is isotopic to a

periodic one.

We suppose that O has singularities x_1, \dots, x_n ($n \geq 1$) which are cone type. Let S be a surface obtained from $O - \{x_1, \dots, x_n\}$ by adding a circle to each non-compact end. By moving f by an O -isotopy we may suppose that $f|_{O - \{x_1, \dots, x_n\}}$ extends to $\tilde{f}: S \rightarrow S$.

If $\chi(S) \geq 0$, then by the above \tilde{f} is isotopic to a periodic map $\bar{\varphi}$. Let $\varphi: O \rightarrow O$ be the projection of $\bar{\varphi}$. Then φ is O -isotopic to f , and periodic.

If $\chi(S) < 0$, then by Theorem B \tilde{f} is isotopic to Thurston's canonical form $\bar{\varphi}$. Let $\varphi: O \rightarrow O$ be the projection of $\bar{\varphi}$. Then φ is O -isotopic to f , and we easily see that φ satisfies the conclusion (i), (ii), or (iii) of Theorem 1 according to $\bar{\varphi}$ is periodic, pseudo-Anosov, or reducible.

This completes the proof of Theorem 1.

4. Homeomorphisms of $H^2 \times R$, $\widetilde{SL_2(R)}$ and Nil-manifolds

Throughout this section let M be a compact, orientable 3-manifold with an $H^2 \times R$, $\widetilde{SL_2(R)}$, or Nil structure. In this section we prove Theorem 2 and investigate some properties of homeomorphisms of type periodic and pseudo-Anosov. By [Sc] M admits a Seifert fibration $p: M \rightarrow O$ where O is a good 2-dimensional orbifold whose (possibly empty) singularities are all cones and by moving f by a homotopy we may suppose that f is fiber preserving. We note that this deformation can be realized by an isotopy if M is sufficiently large [Wa]. Then we have an O -homeomorphism $\psi: O \rightarrow O$ which satisfies:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ p \downarrow & & \downarrow p \\ O & \xrightarrow{\psi} & O \end{array}$$

A fiber preserving self-homeomorphism f is of *type periodic* if ψ is periodic. f is of *type pseudo-Anosov* if ψ is pseudo-Anosov. Let F be a 2-sided surface properly embedded in a 3-manifold M' . F is *incompressible* if $i_*: \pi_1(F) \rightarrow \pi_1(M')$ is injective. Let E be a subset of a Seifert fibered manifold S . E is *saturated* if E is a union of fibers of S . $f: M \rightarrow M$ is *reducible* by \mathcal{D} if \mathcal{D} is a system of mutually disjoint, non-parallel incompressible tori, and $f(\mathcal{D}) = \mathcal{D}$. Let $q: N \rightarrow A$ be a circle bundle over an annulus A , $g: N \rightarrow N$ a fiber preserving homeomorphism, $\varphi: A \rightarrow A$ a map induced from g . g is a *twist homeomorphism* if φ is a twist homeomorphism.

Lemma 4.1. *Let M be an $H^2 \times R$, $\widetilde{SL_2(R)}$, or Nil-manifold with a Seifert fibration, T an incompressible torus in M . Then T is isotopic to a saturated torus.*

Proof. Asumme that T is not isotopic to a saturated torus. By [Sc],

[Wa] Seifert fibrations on M are unique up to isotopy. Then by [J] Theorem VI. 34 we have either:

- (i) M is a torus bundle over a circle and T is a fiber of the bundle, or
- (ii) $M = M_1 \cup M_2$ where $M_1 \cap M_2 = \partial M_1 = \partial M_2 = T$, and M_i ($i=1, 2$) is a twisted I -bundle over the Klein bottle.

Assume that (i) holds. The monodromy of M is represented by a 2×2 matrix $A \in SL_2(\mathbb{Z})$. By [Sc] a torus bundle over the circle admits either an E^3 , Nil, or Sol structure, and M admits a Nil structure if and only if A is conjugate to $\begin{pmatrix} \varepsilon & n \\ 0 & \varepsilon \end{pmatrix}$ ($\varepsilon = \pm 1$, $n \neq 0$). Then we easily see that there is a Seifert fibration on M such that T is saturated with respect to the fibration. This contradicts the assumption. Assume that (ii) holds. By [Sc] M admits an E^3 or Sol structure and does not admit an $H^2 \times R$, $\widetilde{SL_2(R)}$, or Nil structure, a contradiction. Hence T is isotopic to a saturated torus.

Proof of Theorem 2. If T is a non-peripheral, saturated, incompressible torus in M , then $p(T)$ is an essential loop on O . Conversely, if a is an essential loop on O , then $p^{-1}(a)$ is a non-peripheral, saturated, incompressible torus in M . Moreover, if a_1, a_2 are mutually non-parallel, essential loops on O , then $p^{-1}(a_1), p^{-1}(a_2)$ are mutually non-parallel incompressible tori. From these facts the proof of Theorem 2 follows immediately from Theorem 1.

Now, we investigate homeomorphisms of type periodic on M . Suppose that f is of type periodic. Let G be a subgroup of $\text{Out } \pi_1(M)$ generated by f_* . Let $\psi: \text{Out } \pi_1(M) \rightarrow \text{Out } \pi_1^{\text{orb}}(O)$ be a canonical homomorphism, where $\pi_1^{\text{orb}}(O)$ denotes the fundamental group of O as an orbifold ([T₂ § 13]). Then we have an exact sequence:

$$1 \rightarrow \text{Ker } \psi|_G \rightarrow G \xrightarrow{\psi} \psi(G) \rightarrow 1.$$

$\psi(G)$ is a finite cyclic group. If O is non-orientable, then by Kojima [Ko] $\text{Ker } \psi|_G$ is a finite group and G itself a finite cyclic group. This fact together with [Zi] implies:

Proposition 4.2. *If O is non-orientable and $f: M \rightarrow M$ is a homeomorphism of type periodic, then f is homotopic to a periodic one.*

5. Topological entropy of homeomorphisms of $H^2 \times R$, $\widetilde{SL_2(R)}$, or Nil-manifolds

In this section we see that the map φ obtained in Theorem 2 attains the minimal entropy in its homotopy class. Throughout this section, M denotes a compact, orientable $H^2 \times R$, $\widetilde{SL_2(R)}$, or Nil manifold with Seifert fibration $M \rightarrow O$.

Lemma 5.1. *Let $g: M \rightarrow M$ be a fiber preserving homeomorphism. Then $h(g) = h(\psi)$, where $\psi: O \rightarrow O$ is the homeomorphism induced from g .*

Proof. By the argument in the proof of Lemma 3.1 of [S-S] we see that $h_c(g) = 0$ for each fiber C of M . Hence, by Theorem A we have $h(g) = h(\psi)$.

By Lemma 5.1 we have:

Proposition 5.2. *If $f: M \rightarrow M$ is a homeomorphism of type periodic, then $h(f) = 0$.*

For the homeomorphisms of type pseudo-Anosov, we have:

Proposition 5.3. *If $f: M \rightarrow M$ is a homeomorphism of type pseudo-Anosov, then $h(f) > 0$. Moreover, it attains the minimal entropy in its homotopy class.*

Proof. By [Sc] M admits a finite covering $p: \bar{M} \rightarrow M$ such that the Seifert bundle structure on M lifts to a circle bundle structure $q: \bar{M} \rightarrow S$. We may suppose that some power of f , f^n , lifts to a homeomorphism $\bar{f}: \bar{M} \rightarrow \bar{M}$. Let $\bar{\psi}: S \rightarrow S$ be a homeomorphism induced from \bar{f} . Then we have $h(f) = 1/n \cdot h(f^n)$. By Lemma 5.1 $h(f^n) = h(\bar{f}) = h(\bar{\psi})$.

Then we note that $\bar{\psi}$ is a pseudo-Anosov homeomorphism. Let $\lambda (> 1)$ be the expanding factor of f . Then λ^n is the expanding factor of $\bar{\psi}$. By [F-L-P] $h(\bar{\psi}) = n \cdot \log \lambda > 0$. Hence, we have $h(f) = \log \lambda$. Since $\chi(S) < 0$, there is a loop l in \bar{M} such that $q(l) (\subset S)$ is an essential loop of S . We note that $q(\bar{f}(l)) = \bar{\psi}(q(l)) (\subset S)$. Then by [Sc] we see that $L(l') \geq L(q(l'))$ for each loop l' on \bar{M} , where $L(l')$ denotes the infimum of the length of loops which are homotopic to l' . Since $\lim_{m \rightarrow \infty} 1/m \cdot \log L(\bar{\psi}^m(q(l))) = \lambda^n$, we see that $\lim_{m \rightarrow \infty} 1/m \cdot \log L(\bar{f}^m(l)) \geq \lambda^n$. Hence if f' is homotopic to f then $h(f') \geq h(f) = n \cdot \log \lambda$. From this we see that f attains the minimal entropy in its homotopy class.

By using the argument as in the proof of Proposition 2.1 and by Lemma 5.1 we easily have:

Proposition 5.4. *Let $\varphi: M \rightarrow M$ be as in Theorem 2. Then φ attains the minimal entropy in its homotopy class and $h(\varphi)$ is positive if and only if φ contains a component of type pseudo-Anosov.*

References

- [Bo] Bowen, R.: *Entropy for group endomorphism and homogeneous spaces*, Trans. Amer. Math. Soc. **233** (1976), 401–414.
- [E] Epstein, D.B.A.: *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966), 83–107.

- [F-L-P] Fathi, A., Laudenbach, F., Poénaru, V.: *Travaux de Thurston sur les surfaces*, Astérisque **66–67** (1979).
- [H] Handel, M.: *The entropy of orientation reversing homeomorphisms of surfaces*, Topology **21** (1982), 291–296.
- [H-T] Handel, M., Thurston, W.: *New proofs of some results of Nielsen*, preprint.
- [J] Jaco, W.: *Lectures on three manifold topology*, CBMS Regional conference series in Math. No. 43 (1980).
- [K] Kobayashi, T.: *Links of homeomorphisms of a disk and topological entropy*, Invent. Math. (to appear).
- [Ko] Kojima, S.: *Bounding finite groups acting on 3-manifolds*, preliminary report.
- [Li] Lickorish, W.B.R.: *Homeomorphisms of non-orientable two-manifolds*, Proc. Cambridge Philos. Soc. **59** (1963), 307–317.
- [M] Miller, R.: *Geodesic laminations from Nielsen's viewpoint*, Advances in Math. **45** (1982), 189–212.
- [Mo] Mostow, G.: Strong rigidity of locally symmetric spaces, Ann. of Math. Studies **78**, Princeton Univ. Press, 1973.
- [Sc] Scott, P.: *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.
- [Sma] Smale, S.: *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621–626.
- [Smi] Smillie, J.: *Periodic points of surface homeomorphisms with zero entropy*, Ergodic Theory Dynamical Systems **3** (1983), 315–334.
- [S-S] Sacksteder R., Shub M.: *Entropy on sphere bundles*, Advances in Math. **28** (1978), 174–177.
- [T₁] Thurston, W.: *On the geometry and dynamics of diffeomorphisms of surfaces I*, preprint.
- [T₂] Thurston, W.: The geometry and topology of 3-manifolds, Lecture note, Princeton University, 1978.
- [Wa] Waldhausen, F.: *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. **87** (1968), 56–88.
- [Zi] Zimmermann, B.: *Periodische Homöomorphismen Seifertscher Faserräume*, Math. Z. **166** (1979), 289–297.

Department of Mathematics
 Osaka University
 Toyonaka Osaka, 560
 Japan

