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THE RIEMANN-ROCH THEOREM FOR COMPLEX V-MANIFOLDS

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Introduction and statement of theorem. This note is the sequel to our work [10]. We shall apply our method to the $\bar{\partial}$ -operators over complex V -manifolds. Our result is a generalization of the Hirzebruch-Riemann-Roch Theorem (see Atiyah-Singer [4] and Hirzebruch [8]) to the case of complex V -manifolds and holomorphic vector V -bundles.

Let M be a compact complex manifold with a holomorphic action of a finite group G and let $E \rightarrow M$ be a G -equivariant holomorphic vector bundle. We denote by $\mathcal{O}(E)$ the sheaf of local holomorphic sections of E . Then Atiyah-Singer [4] proved: For each $g \in G$,

$$(I) \quad \chi(g, M; \mathcal{O}(E)) = \sum_i (-1)^i \text{trace}_c [g | H^i(M; \mathcal{O}(E))] \\ = \langle \mathcal{T}^g(M; E), [M^g] \rangle.$$

Here $\mathcal{T}^g(M; E)$ is the equivariant Todd class.

Now the orbit space M/G has a structure of an analytic space and the local G -invariant holomorphic sections of E define a coherent analytic sheaf $\mathcal{O}_V(E/G)$ over M/G . Then, by averaging (I) for all $g \in G$, we have:

$$(II) \quad \chi(M/G; \mathcal{O}_V(E/G)) = \sum_i (-1)^i \dim_c H^i(M/G; \mathcal{O}_V(E/G)) \\ = \frac{1}{|G|} \sum_{g \in G} \langle \mathcal{T}^g(M; E), [M^g] \rangle.$$

We shall generalize this formula to the case of complex V -manifolds. The notion of V -manifold was introduced by Satake [11]. In [10] we have stated the precise definitions concerning V -manifold structures. So, here we put a brief description of complex V -manifolds and holomorphic vector V -bundles. Let X be an analytic space admitting only quotient singularities. A complex V -manifold structure \mathcal{V}^c over X is the following: For each sufficiently small connected open set U in X , $\mathcal{V}^c(U) = "(G_U, \tilde{U}) \rightarrow U"$ is a ramified covering $\tilde{U} \rightarrow U$ such that \tilde{U} is a connected complex manifold with an effective

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holomorphic action of a finite group G_U and the projection $\tilde{U} \rightarrow U$ gives an identification $U \simeq \tilde{U}/G_U$ of analytic spaces. For a connected open subset $V \subset U$, we assume also, that there is a biholomorphic open embedding $\varphi: \tilde{V} \rightarrow \tilde{U}$ that covers the inclusion $V \subset U$. Then the choice of φ is unique upto the action of G_U and each φ defines an injective group homomorphism $\lambda_\varphi: G_V \rightarrow G_U$ that makes φ be λ_φ -equivariant. Let $p: E \rightarrow X$ be a holomorphic map between analytic spaces. A holomorphic vector V -bundle structure \mathcal{B} on “ $E \rightarrow X$ ” is the following: For small $U \subset X$, $\mathcal{B}(U) = (G_U, \tilde{p}_U: \tilde{E}_U \rightarrow \tilde{U})$ is a G_U -equivariant holomorphic vector bundle with an identification “ $p|_{p^{-1}(U)}: p^{-1}(U) \rightarrow U \simeq \tilde{p}_U/G_U: \tilde{E}_U/G_U \rightarrow \tilde{U}/G_U$ ”. For $V \subset U$, we assume that there is a holomorphic bundle map $\Phi: \tilde{E}_V \rightarrow \tilde{E}_U$ over some open embedding $\varphi: \tilde{V} \rightarrow \tilde{U}$ that covers the inclusions $p^{-1}(V) \subset p^{-1}(U)$ and $V \subset U$. Then Φ becomes a λ_φ -equivariant bundle map. (In the terminology of [10], (E, \mathcal{B}) is a “proper” holomorphic vector V -bundle).

Now let X be a compact complex V -manifold and let $E \rightarrow X$ be a holomorphic vector V -bundle. The local G_U -invariant holomorphic sections of $\tilde{E}_U \rightarrow \tilde{U}$ define a coherent analytic sheaf $\mathcal{O}_V(E)$ over an analytic space X . Then we have the arithmetic genus $\chi(X; \mathcal{O}_V(E)) = \sum_i (-1)^i \dim_{\mathbb{C}} H^i(X; \mathcal{O}_V(E))$. We can choose invariant smooth linear connections on complex vector bundles $\tilde{E}_U \rightarrow \tilde{U}$, complex tangent bundles $T\tilde{U} \rightarrow \tilde{U}$ and complex normal bundles $\nu(\tilde{U}^g \subset \tilde{U}) \rightarrow \tilde{U}^g$ for all U and for all $g \in G_U$, such that they are compatible with open embeddings Φ 's and φ 's. Then, by the Weil homomorphism, we have the equivariant Todd form $\mathcal{T}^g(\tilde{U}; \tilde{E}_U)$ for each \tilde{U}^g . Then we can state our theorem in the following form. Let $\{f_U\}$ be a (smooth or continuous) partition of unity on X , then,

$$(III) \quad \chi(X; \mathcal{O}_V(E)) = \sum_U \frac{1}{|G_U|} \sum_{g \in G_U} \int_{\tilde{U}^g} f_U \mathcal{T}^g(\tilde{U}; \tilde{E}_U).$$

For each local coordinate (G_U, \tilde{U}) and for each $g \in G_U$, we consider \tilde{U}^g as a complex manifold on which the centralizer $Z_{G_U}(g)$ acts. For $V \subset U$, the open embedding $\varphi: \tilde{V} \rightarrow \tilde{U}$ defines a natural open embedding $\tilde{V}^h/Z_{G_V}(h) \rightarrow \tilde{U}^g/Z_{G_U}(g)$ of analytic spaces, where $g = \lambda_\varphi(h)$. This embedding is unique for a fixed pair (g, h) . We patch all $\tilde{U}^g/Z_{G_U}(g)$'s together by these identifications. Then we get a disjoint union of complex V -manifolds of various dimensions:

$$X \amalg \tilde{\Sigma}X = \bigcup_{(G_U, \tilde{U}), g \in G_U} \tilde{U}^g/Z_{G_U}(g),$$

(X corresponds to the portion defined by $g=1$).

We have a canonical map $\tilde{\Sigma}X \rightarrow X$ covered locally by the inclusion $\tilde{U}^g \subset \tilde{U}$. For each $x \in X$, we can choose a coordinate neighbourhood (G_x, \tilde{U}_x) such that $x \in \tilde{U}_x$ is a fixed point of G_x . G_x is unique upto isomorphisms. Then the number of pieces of $\tilde{\Sigma}X$ over x is equal to the number of the conjugacy classes of G_x other

than the identity class.

Let $\widetilde{\Sigma}X_1, \widetilde{\Sigma}X_2, \dots, \widetilde{\Sigma}X_c$ be all the connected components of $\widetilde{\Sigma}X$. To each $\widetilde{\Sigma}X_i$, we assign a number m_i , defined by:

$$m_i = |\text{kernel}[Z_{G_U}(g) \rightarrow \text{Aut}(\widetilde{U}^g)]| ,$$

$$(\widetilde{U}^g/Z_{G_U}(g) \subset \widetilde{\Sigma}X_i).$$

Now the formal sum $\sum_{g \in G_U} \mathcal{Q}^g(U; E_U)$ defines a “differential form” on $X \amalg \widetilde{\Sigma}X$.

It represents a cohomology class $\mathcal{Q}(X; E) + \mathcal{Q}^Z(X; E)$ in $H^*(X \amalg \widetilde{\Sigma}X; \mathbf{C})$. This class is independent of the choice of the connections. Then we get the following theorem:

Theorem. *Let X be a compact complex V -manifold and let $E \rightarrow X$ be a holomorphic vector V -bundle. Then:*

$$(IV) \quad \chi(X; \mathcal{O}_V(E)) = \langle \mathcal{Q}(X; E), [X] \rangle$$

$$+ \sum_{i=1}^c \frac{1}{m_i} \langle \mathcal{Q}^Z(X; E), [\widetilde{\Sigma}X_i] \rangle .$$

REMARK 1. Since the class $\mathcal{Q}(X; E)$ is defined over rationals, the term $\langle \mathcal{Q}(X; E), [X] \rangle$ is a rational number.

REMARK 2. For the case when $X = \Gamma \backslash \widetilde{X}$, where \widetilde{X} is a complex manifold and Γ is a properly discontinuous group acting holomorphically on \widetilde{X} , the number $\langle \mathcal{Q}(X; E), [X] \rangle$ is just the Γ -index $\text{ind}_\Gamma((\bar{\partial} + \bar{\partial}^*)_{E}^{0, ev})$ defined by Atiyah [1]. (Though Γ acts freely in [1], the similar argument holds for the case when Γ has finite isotropies, see III) below).

The proof of our theorem is a combination of our work [10] and Gilkey’s result [7] on the Lefschetz fixed point formula for the Dolbeault complexes. Here we shall place a complete proof.

Proof of Theorem. In this proof, we use the “heat kernel-zeta function” method. We reivew the results briefly. (See Seeley [12], Atiyah-Bott-Patodi [2], Gilkey [6], [7], Donnelly-Patodi [5] and Kawasaki [10]).

Let U be a germ of a Riemannian manifold and let $E_U \rightarrow U$ be a smooth complex vector bundle with a smooth Hermitian fibre metric. Let $g: E_U \rightarrow E_U$ be an isometry of the pair (U, E_U) . Let $A: C^\infty(U; E_U) \rightarrow C^\infty(U; E_U)$ be a g -invariant, formally self-adjoint, positive semi-definite, elliptic differential operator. Then we have a smooth measure Z_A^g on the fixed point set U^g . Z_A^g is a local invariant of the action of g and of the operator A . It is given by a universal expression in g and A . The explicit form of Z_A^g is given in [10]. Z_A^g has the following properties:

I) Let M be a compact Riemannian manifold and let $g: M \rightarrow M$ be an isometry. Let E, F be two g -equivariant smooth complex vector bundles over M with g -invariant Hermitian fibre metrics. Let $D: C^\infty(M; E) \rightarrow C^\infty(M; F)$ be a g -invariant elliptic differential operator. Then we have the adjoint operator $D^*: C^\infty(M; F) \rightarrow C^\infty(M; E)$ and two g -invariant, self-adjoint, positive semi-definite, elliptic differential operators D^*D and DD^* . Put $\mu_D^g = Z_{D^*D}^g - Z_{DD^*}^g$. Then the equivariant index $\text{ind}(g, D)$ is given by:

$$\text{ind}(g, D) = \int_{M^g} d\mu_D^g.$$

II) (Kawasaki [10]). Let X be a compact Riemannian V -manifold and let E, F be two "proper" differentiable complex vector V -bundles over X . Let $D: C_V^\infty(X; E) \rightarrow C_V^\infty(X; F)$ be an elliptic differential operator, that is, a family $\{\tilde{D}_U: C^\infty(\tilde{U}; \tilde{E}_U) \rightarrow C^\infty(\tilde{U}; \tilde{F}_U)\}_{(G_U, \tilde{U})}$ of invariant elliptic differential operators that are compatible with attaching maps $\{\Phi\}: \tilde{E}_V \rightarrow \tilde{E}_U$ and $\{\Psi\}: \tilde{F}_V \rightarrow \tilde{F}_U$. Then D operates on the differentiable V -sections and the kernel and the cokernel of the operator D are finite dimensional. We define the V -index $\text{ind}_V(D)$ of the operator D by:

$$\begin{aligned} \text{ind}_V(D) &= \dim_C \text{kernel}[D: C_V^\infty(X; E) \rightarrow C_V^\infty(X; F)] \\ &\quad - \dim_C \text{cokernel}[D: C_V^\infty(X; E) \rightarrow C_V^\infty(X; F)]. \end{aligned}$$

For each coordinate neighbourhood (G_U, \tilde{U}) , we have a formal sum of measures:

$$\sum_{g \in G_U} \mu_{\tilde{D}_U}^g = \sum_{g \in G_U} (Z_{\tilde{D}_U^* \tilde{D}_U}^g - Z_{\tilde{D}_U \tilde{D}_U^*}^g).$$

These formal sums define a measure $\mu_D + \mu_{\tilde{D}}$ over $X \amalg \tilde{\Sigma}X$. Then the V -index $\text{ind}_V(D)$ is given by:

$$\text{ind}_V(D) = \int_X d\mu_D + \sum_{i=1}^c \frac{1}{m_i} \int_{\tilde{\Sigma}X_i} d\mu_{\tilde{D}}.$$

III) (See Aityah [1]). Let \tilde{X} be a (non-compact) Riemannian manifold and let Γ be a properly discontinuous group acting on \tilde{X} as isometries. We assume that the orbit V -manifold $X = \Gamma \backslash \tilde{X}$ is compact. Let \tilde{E}, \tilde{F} be two Γ -equivariant complex vector bundles over \tilde{X} with Γ -invariant Hermitian fibre metrics. Let $\tilde{D}: C^\infty(\tilde{X}; \tilde{E}) \rightarrow C^\infty(\tilde{X}; \tilde{F})$ be a Γ -invariant elliptic differential operator. Then we consider the completions $\mathcal{L}^2(\tilde{X}; \tilde{E}), \mathcal{L}^2(\tilde{X}; \tilde{F})$ and the unbounded operators $\tilde{D}: \mathcal{L}^2(\tilde{X}; \tilde{E}) \rightarrow \mathcal{L}^2(\tilde{X}; \tilde{F}), \tilde{D}^*: \mathcal{L}^2(\tilde{X}; \tilde{F}) \rightarrow \mathcal{L}^2(\tilde{X}; \tilde{E})$. (In this case the formal adjoint coincides with the Hilbert space adjoint). We put:

$$\begin{aligned} \mathcal{A}_0 &= \{f \in \mathcal{L}^2(\tilde{X}; \tilde{E}) \mid \tilde{D}f = 0\} \subset \mathcal{L}^2(\tilde{X}; \tilde{E}), \\ \mathcal{A}_1 &= \{g \in \mathcal{L}^2(\tilde{X}; \tilde{F}) \mid \tilde{D}^*g = 0\} \subset \mathcal{L}^2(\tilde{X}; \tilde{F}). \end{aligned}$$

Then \mathcal{H}_i becomes a Γ -invariant closed subspace ($i=0, 1$). Let H_i be the orthogonal projection onto \mathcal{H}_i . Then H_i has a smooth kernel $H_i(\tilde{x}, \tilde{y})$ and we get a smooth measure $\text{trace}_c[H_i(\tilde{x}, \tilde{x})]$ over \tilde{X} . Since the operator H_i is Γ -invariant, we may consider $\text{trace}_c[H_i(\tilde{x}, \tilde{x})]$ as a measure over $X=\Gamma\backslash\tilde{X}$. Then the Γ -index of the operator \tilde{D} is defined by:

$$\text{ind}_\Gamma(\tilde{D}) = \int_X d(\text{trace}_c[H_0(\tilde{x}, \tilde{x})] - \text{trace}_c[H_1(\tilde{x}, \tilde{x})]).$$

Now the elliptic differential operator \tilde{D} over \tilde{X} defines an elliptic differential operator $D: C^\infty_V(X; E) \rightarrow C^\infty_V(X; F)$ over a V -manifold X and we have a measure μ_D over X . Then $\text{ind}_\Gamma(\tilde{D})$ is given by:

$$\text{ind}_\Gamma(\tilde{D}) = \int_X d\mu_D.$$

Now we return to our problem: Let X be a compact complex V -manifold and let $E \rightarrow X$ be a holomorphic vector V -bundle. We denote by T the holomorphic part of the complexified cotangent vector V -bundle. Consider the sheaf $\mathcal{A}^{p,q}_V(E) = C^\infty_V(\Lambda^p T \otimes \Lambda^q \bar{T} \otimes E)$ of germs of E -valued (p, q) -forms over X . Then we have the $\bar{\partial}$ -operators $\bar{\partial}: \mathcal{A}^{p,q}_V(E) \rightarrow \mathcal{A}^{p,q+1}_V(E)$ and a soft resolution:

$$0 \rightarrow \mathcal{O}_V(\Lambda^p T \otimes E) \hookrightarrow \mathcal{A}^{p,0}_V(E) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}_V(E) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{A}^{p,n}_V(E) \rightarrow 0.$$

Put $A^{p,q}_V(X; E) = \Gamma(X; \mathcal{A}^{p,q}_V(E))$, then we have a complex:

$$0 \rightarrow A^{p,0}_V(X; E) \xrightarrow{\bar{\partial}} A^{p,1}_V(X; E) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} A^{p,n}_V(X; E) \rightarrow 0,$$

whose i -th cohomology group is $H^i(X; \mathcal{O}_V(\Lambda^p T \otimes E))$. Choose a Hermitian metric h on X and a Hermitian fibre metric h_E on E . Then we have the adjoint operator $\bar{\partial}^*: A^{p,q}_V(X; E) \rightarrow A^{p,q-1}_V(X; E)$ of $\bar{\partial}$. Consider a differential operator:

$$(\bar{\partial} + \bar{\partial}^*)_E^{0, ev} = \bar{\partial} + \bar{\partial}^*|_{A^{0, ev}_V}: A^{0, ev}_V(X; E) \rightarrow A^{0, od}_V(X; E),$$

$$(A^{0, od}_V(X; E) = \bigoplus_{q: \text{odd}} A^{0,q}_V(X; E)).$$

Then $(\bar{\partial} + \bar{\partial}^*)_E^{0, ev}$ is an elliptic operator and:

$$\text{ind}_V((\bar{\partial} + \bar{\partial}^*)_E^{0, ev}) = \chi(X; \mathcal{O}_V(E)).$$

Thus we can express the arithmetic genus as the V -index of an elliptic operator $(\bar{\partial} + \bar{\partial}^*)_E^{0, ev}$. Then, by II) above, we have a measure $\mu_{(\bar{\partial} + \bar{\partial}^*)_E^{0, ev}} + \mu_{(\bar{\partial} + \bar{\partial}^*)_E^{0, ev}}$ over $X \amalg \bar{X}$ that gives the arithmetic genus. But this measure is not equal to the Todd class in general. So we use the $Spin^c$ Dirac operator instead, which gives the arithmetic genus for complex V -manifolds and is defined over more general V -manifolds.

Now let (X, h) and (E, h_E) be as before. Consider the almost complex structure (TX, J) . (TX, J) is a holomorphic vector V -bundle. The Hermitian metric h define a reduction $U(n)$ (TX) of the principal tangent V -bundle. We consider $U(n)$ as a subgroup of $Spin^c(2n) = Spin(2n) \times_{\mathbb{Z}_2} U(1)$. (See Atiyah-Bott-Shapiro [2]). Let $Spin^c(2n)(TX)$ be the associated $Spin^c(2n)$ -principal tangent V -bundle. We construct a connection ∇^c on $Spin^c(2n)(TX)$ as follows: We have a Riemannian connection ∇_{SO} on $SO(2n)(TX)$ and a Hermitian connection ∇_L on $L = \Lambda^n((TX, J))$. Then ∇^c is a unique lift of $\nabla_{SO} \times \nabla_L$ on $(SO(2n) \times U(1))(TX)$ by the double covering $Spin^c(2n) \rightarrow SO(2n) \times U(1)$. Let $\Delta^{\pm, c}$ be the half $Spin^c$ -representations. Then we have two complex vector V -bundles:

$$\Delta^{\pm, c}(TX) = Spin^c(2n)(TX) \times_{Spin^c(2n)} \Delta^{\pm, c},$$

with induced connections $\nabla^{\pm, c}$. The Clifford module structures on $\Delta^{\pm, c}$ define the Clifford multiplications:

$$m: TX \otimes_{\mathbb{R}} \Delta^{\pm, c}(TX) \rightarrow \Delta^{\mp, c}(TX).$$

On (E, h_E) we have the Hermitian connection ∇_E . Then the $Spin^c$ Dirac operator $d_E^{\pm, c}$ is defined by:

$$\begin{aligned} d_E^{\pm, c}: C_V^\infty(X; \Delta^{\pm, c}(TX) \otimes_{\mathbb{C}} E) \\ \xrightarrow{\nabla^{\pm, c} \otimes 1 + 1 \otimes \nabla_E} C_V^\infty(X; T^*X \otimes_{\mathbb{R}} \Delta^{\pm, c}(TX) \otimes_{\mathbb{C}} E) \\ \xrightarrow{m} C_V^\infty(X; \Delta^{\mp, c}(TX) \otimes_{\mathbb{C}} E). \end{aligned}$$

Here we identify $TX = T^*X$ by the real Hermitian metric $\mathcal{R}_e h$.

Since $Spin^c(2n)(TX)$ has a reduction $U(n)(TX)$, we have:

$$\Delta^{\pm, c}(TX) \cong \Lambda^{od, ev}(TX, J).$$

The Hermitian metric h defines a V -bundle isometry $\psi: (TX, J) \cong \bar{T}$. So we have a V -bundle isomorphism:

$$\psi^\pm: \Delta^{\pm, c}(TX) \otimes_{\mathbb{C}} E \cong \Lambda^{od, ev} \bar{T} \otimes_{\mathbb{C}} E.$$

By a standard computation (see Hitchin [9]), we have:

Proposition. *The two operators $(\bar{\partial} + \bar{\partial}^*)_E^{0, ev}$ and $d_E^{\pm, c}$ have the same principal symbol (via ψ^\pm) upto a constant factor.*

As a corollary, we have:

$$\begin{aligned} \chi(X; \mathcal{O}_V(E)) &= \text{ind}_V(d_E^{\pm, c}) \\ &= \int_X d\mu_{d_E^{\pm, c}} + \sum_{i=1}^c \frac{1}{m_i} \int_{\bar{\Sigma}_X} d\mu_{\bar{d}_k^{\pm, c}}. \end{aligned}$$

Now the operator $d_E^{\pm,c}$ does not depend on the complex structure on X . It depends only on the $Spin^c$ -structure $Spin^c(2n)(TX)$, the metric connection ∇_L and the Hermitian V -bundle (E, h_E, ∇_E) . Its index $\text{ind}_V(d_E^{\pm,c})$ does not depend on the choices of metrics h and h_E , nor the choices of connections ∇_L and ∇_E . So we can change metrics and connections.

We consider over a coordinate neighbourhood $(G_U, \tilde{U}) \rightarrow U$. Choose a metric h on \tilde{U} so that, for each $g \in G_U$, on a neighbourhood of \tilde{U}^g in \tilde{U} , h is equal to the Riemannian metric over the total space N_g of the normal bundle $\nu_g = \nu(\tilde{U}^g \subset \tilde{U})$ induced from a g -invariant Hermitian structure $(\nu_g, h_{\nu_g}, \nabla_{\nu_g})$. We identify N_g with a neighbourhood of \tilde{U}^g in \tilde{U} . Then, over N_g , the principal bundle $Spin^c(2n)(T\tilde{U})$ reduces equivariantly to $\pi^*(Spin^c(2n_0)(T\tilde{U}^g) \times_{\tilde{\nu}_g} U(n-n_0)(\nu_g))$, where $\pi: N_g \rightarrow \tilde{U}^g$ is the projection of ν_g and $2n_0 = \dim_R U^g$. The associated line bundle L splits into a tensor product $\pi^*(L_0 \otimes \Lambda^{n-n_0} \nu_g)$, where L_0 is the associated line bundle of $Spin^c(2n_0)(T\tilde{U}^g)$.

The actions of g on the first factors $Spin^c(2n_0)(T\tilde{U}^g)$ and L_0 are trivial. On L_0 , we have the induced metric h_{L_0} . Choose a metric connection ∇_{L_0} on (L_0, h_{L_0}) . Then we choose a metric connection ∇_L so that, over N_g , ∇_L is equal to the induced connection $\pi^*(\nabla_{L_0} \otimes \Lambda^{n-n_0} \nabla_{\nu_g})$. Also, we choose a Hermitian structure (E, h_E, ∇_E) so that, over N_g , it is equal to the induced structure $(\pi^*(E|_{\tilde{U}^g}), \pi^*(h_E|_{\tilde{U}^g}), \pi^*(\nabla_E|_{\tilde{U}^g}))$.

Then, over a neighbourhood N_g of \tilde{U}^g in \tilde{U} , the operator $d_E^{\pm,c}$ is completely determined by the data over \tilde{U}^g , that is, the $Spin^c$ -structure $Spin^c(2n_0)(T\tilde{U}^g)$, the metric connection ∇_{L_0} and the g -equivariant Hermitian bundles $(g; \nu_g, h_{\nu_g}, \nabla_{\nu_g})$ and $(g; E|_{U^g}, h_E|_{U^g}, \nabla_E|_{U^g})$.

We remark here that we can choose a metric h , a metric connection ∇_L and a hermitian structure (E, h_E, ∇_E) over a V -manifold X so that the above conditions are satisfied for all coordinate neighbourhood $(G_U, \tilde{U}) \rightarrow U$ and for all $g \in G_U$ at the same time.

Now we consider differently: Let (U_0, h_0) be a germ of $(2n_0)$ -dimensional Riemannian manifold with trivial g -action and assume that we are given a Hermitian line bundle $(L_0, h_{L_0}, \nabla_{L_0})$ with trivial g -action and two g -equivariant Hermitian bundles $(g; \nu, h_\nu, \nabla_\nu)$ ($\dim_C \nu = n - n_0$) and $(g; E, h_E, \nabla_E)$ over U_0 . So g acts on each fibre of ν and E . We assume that the fixed points in ν are all in the zero section. We may assume that U_0 is contractible. Then an orientation σ , the Riemannian metric h_0 and the Hermitian line bundle $(L_0, h_{L_0}, \nabla_{L_0})$ define a unique $Spin^c$ -structure $Spin^c(2n_0)(TU_0)$ upto $Spin^c$ -isomorphisms. (There are two canonical isomorphisms). The Riemannian metric h_0 and the metric connection ∇_{L_0} define a connection ∇_0^σ on $Spin^c(2n_0)(TU_0)$. Consider the total space N of ν . The Hermitian structure (ν, h_ν, ∇_ν) define a $Spin^c(2n_0) \times U(n-n_0)$ -structure over N . Also we have the action of g that preserve the above structure. Then we have the associated $Spin^c(2n)$ -structure with

g -action over N . Its associated line bundle is $\pi^*(L_0 \otimes \Lambda^{n-n_0}\nu)$ and the metric connection $\nabla_{L_0} \otimes \Lambda^{n-n_0}\nabla_\nu$ defines a connection ∇^c on $Spin^c(2n) (TN)$. Also we have an induced g -equivariant Hermitian bundle $(g; \pi^*E, \pi^*h_E, \pi^*\Delta_E)$ over N .

Then the $Spin^c$ -structure $Spin^c(2n) (TN)$ with connection ∇^c and the Hermitian bundle $(\pi^*E, \pi^*h_E, \pi^*\nabla_E)$ define the $Spin^c$ Dirac operator $d_{\pi^*E}^{+,c}$. The operator $d_{\pi^*E}^{+,c}$ and the action of g define a measure $\mu_{\pi^*E}^{g,+c}$ over U_0 . The only ambiguity of this construction comes from the choice of the orientation o over U_0 . If we change the orientation, then the measure $\mu_{\pi^*E}^{g,+c}$ changes its sign. So the measure $\mu_{\pi^*E}^{g,+c}$ defines a $2n_0$ -form $d\mu_{\pi^*E}^{g,+c}$ with no ambiguity.

Thus we have shown that the $2n_0$ -form $d\mu_{\pi^*E}^{g,+c}$ is a local invariant of a Riemannian structure (U_0, h_0) and Hermitian bundles $(L_0, h_{L_0}, \nabla_{L_0}), (g; \nu, h_\nu, \nabla_\nu)$ and $(g; E, h_E, \nabla_E)$. In [10], we have an explicit form of μ_b^g . Then we can see that the $2n_0$ -form $d\mu_{\pi^*E}^{g,+c}$ is a homogeneous regular local invariant of weight 0, in the terminology of Atiyah-Bott-Patodi [2]. Then, by Gilkey's Theorem (see [2]), we can conclude:

Proposition. $d\mu_{\pi^*E}^{g,+c}$ is expressed by a universal polynomial in the Pontrjagin forms of (U_0, h_0) , the first Chern form of $(L_0, h_{L_0}, \nabla_{L_0})$, the equivariant Chern forms of $(g; \nu, h_\nu, \nabla_\nu)$ and the equivariant Chern forms of $(g; E, h_E, \nabla_E)$.

We restrict ourselves to the case when TU_0 has an almost complex structure J_0 and $L_0 = \Lambda^{n_0}(TU_0, J_0)$. Let M be a compact complex manifold and let $E \rightarrow M$ be a holomorphic vector bundle. Let g be an automorphism of the pair (M, E) that generates a compact transformation group. Then by Atiyah-Singer [4] we know:

$$\begin{aligned} \int_{M^g} d\mu_E^{g,+c} &= \sum_i (-1)^i \text{trace}_c [g | H^i(M; \mathcal{O}(E))] \\ &= \langle \mathcal{I}^g(M; E), [M^g] \rangle. \end{aligned}$$

The computations over the products of complex projective spaces with linear actions show that the expression of $d\mu_{\pi^*E}^{g,+c}$ in the characteristic classes must be unique. This shows:

$$d\mu_E^{g,+c} = \mathcal{I}^g(M; E).$$

Now we return to the original situation. Over a coordinate neighbourhood $(G_U, \tilde{U}) \rightarrow U$, we have:

$$\mu_{d_E^{+,c}} + \mu_{d_E^{\nu,+c}} = \sum_{g \in G_U} \mu_{d_E^{g,+c}}.$$

Then, by choosing suitable metrics and connections, we have:

$$d\mu_{d_E^{+,c}} + d\mu_{d_E^{\nu,+c}} = \mathcal{I}(X; E) + \mathcal{I}^\nu(X; E).$$

Hence we have:

$$\begin{aligned}
\chi(X; \mathcal{O}_V(E)) &= \int_X d\mu_{d_E^{1,c}} + \sum_{i=1}^c \frac{1}{m_i} \int_{\tilde{\Sigma} X_i} d\mu_{d_E^{1,c}} \\
&= \langle \mathcal{Q}(X; E), [X] \rangle \\
&\quad + \sum_{i=1}^c \frac{1}{m_i} \langle \mathcal{Q}^Y(X; E), [\tilde{\Sigma} X_i] \rangle.
\end{aligned}$$

The both sides are independent of the metrics and connections.

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References

- [1] M.F. Atiyah: *Elliptic operators, discrete groups and von Neumann algebras*, Société Mathématique de France, Astérisque **32–33** (1976), 43–72.
- [2] M.F. Atiyah, R. Bott and V.K. Patodi: *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 379–330.
- [3] M.A. Atiyah, R. Bott and A. Shapiro: *Clifford modules*, Topology **3** (1964), Suppl. 1, 3–38.
- [4] M.F. Atiyah and I.M. Singer: *The index of elliptic operators: III*, Ann. of Math. **87** (1968), 546–604.
- [5] H. Donnelly and V.K. Patodi: *Spectrum and the fixed point sets of isometries - II*, Topology **16** (1977), 1–11.
- [6] P.B. Gilkey: *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Advances in Math. **10** (1973), 344–382.
- [7] P.B. Gilkey: *Spectral geometry and the Lefschetz formulas for a holomorphic isometry of an almost complex manifold*, to appear.
- [8] F. Hirzebruch: *Topological methods in algebraic geometry*, Springer-Verlag, 1966.
- [9] N. Hitchin: *Harmonic spinors*, Advances in Math. **14** (1974), 1–55.
- [10] T. Kawasaki: *The signature theorem for V-manifolds*, Topology **17** (1978) 75–83.
- [11] I. Satake: *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan **9** (1957), 464–492.
- [12] R.T. Seeley: *Complex powers of an elliptic operator*, Proc. Sympos. Pure Math. Math. 10, Amer. Math. Soc. (1967), 288–307.

