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Osaka University
THE RIEMANN-ROCH THEOREM FOR COMPLEX V-MANIFOLDS

TETSURO KAWASAKI

(Received January 31, 1978)

Introduction and statement of theorem. This note is the sequel to our work [10]. We shall apply our method to the $\bar{\partial}$-operators over complex $V$-manifolds. Our result is a generalization of the Hirzebruch-Riemann-Roch Theorem (see Atiyah-Singer [4] and Hirzebruch [8]) to the case of complex $V$-manifolds and holomorphic vector $V$-bundles.

Let $M$ be a compact complex manifold with a holomorphic action of a finite group $G$ and let $E \rightarrow M$ be a $G$-equivariant holomorphic vector bundle. We denote by $\mathcal{O}(E)$ the sheaf of local holomorphic sections of $E$. Then Atiyah-Singer [4] proved: For each $g \in G$,

\[ \chi(g, M; \mathcal{O}(E)) = \sum T(-1)^t \text{trace} \{ g|H^t(M; \mathcal{O}(E)) \} = \langle \mathcal{D}^g(M; E), [M^e] \rangle. \]

We shall generalize this formula to the case of complex $V$-manifolds. The notion of $V$-manifold was introduced by Satake [11]. In [10] we have stated the precise definitions concerning $V$-manifold structures. So, here we put a brief description of complex $V$-manifolds and holomorphic vector $V$-bundles. Let $X$ be an analytic space admitting only quotient singularities. A complex $V$-manifold structure $\mathcal{C}^V$ over $X$ is the following: For each sufficiently small connected open set $U$ in $X$, $\mathcal{C}^V(U) = (G, \bar{U}) \rightarrow U$ is a ramified covering $\bar{U} \rightarrow U$ such that $\bar{U}$ is a connected complex manifold with an effective 1.) From April 1, 1979, the author will move to: Gakushuin University, Faculty of Science, Tokyo.
holomorphic action of a finite group $G$ and the projection $\bar{U}\to U$ gives an identification $U\simeq \bar{U}|G_U$ of analytic spaces. For a connected open subset $V\subset U$, we assume also, that there is a biholomorphic open embedding $\varphi: \bar{V}\to \bar{U}$ that covers the inclusion $V\subset U$. Then the choice of $\varphi$ is unique up to the action of $G$ and each $\varphi$ defines an injective group homomorphism $\lambda_\varphi: G_U\to G_U$ that makes $\varphi$ be $\lambda_\varphi$-equivariant. Let $p: E\to X$ be a holomorphic map between analytic spaces. A holomorphic vector $V$-bundle structure $\mathcal{B}$ on "$E\to X$" is the following: For small $U\subset X$, $\mathcal{B}(U) = (G_U, \tilde{\mathcal{E}}_U\to \bar{U})$ is a $G_U$-equivariant holomorphic vector bundle with an identification "$p|p^{-1}(U): p^{-1}(U)\to U" \Rightarrow \tilde{\mathcal{E}}_U|G_U\to \bar{U}|G_U".$ For $V\subset U$, we assume that there is a holomorphic bundle map $\Phi: E\to \bar{U}$ over some open embedding $\Phi: V\to U$ that covers the inclusions $p^{-1}(V)\subset p^{-1}(U)$ and $V\subset U$. Then $\Phi$ becomes a $\lambda_\varphi$-equivariant bundle map. (In the terminology of [10], $(E, \mathcal{B})$ is a "proper" holomorphic vector $V$-bundle).

Now let $X$ be a compact complex $V$-manifold and let $E\to X$ be a holomorphic vector $V$-bundle. The local $G_U$-invariant holomorphic sections of $\tilde{\mathcal{E}}_U\to \bar{U}$ define a coherent analytic sheaf $\mathcal{O}_V(E)$ over an analytic space $X$. Then we have the arithmetic genus $\chi(X; \mathcal{O}_V(E)) = \sum (-1)^i \dim \mathcal{H}^i(X; \mathcal{O}_V(E))$. We can choose invariant smooth linear connections on complex vector bundles $\tilde{\mathcal{E}}_U\to \bar{U}$, complex tangent bundles $T\bar{U}\to \bar{U}$ and complex normal bundles $\nu(U^g\subset \bar{U})\to \bar{U}^g$ for all $U$ and for all $g\in G_U$, such that they are compatible with open embeddings $\Phi$'s and $\varphi$'s. Then, by the Weil homomorphism, we have the equivariant Todd form $\tau_g^X(\bar{U}; \tilde{\mathcal{E}}_U)$ for each $\bar{U}^g$. Then we can state our theorem in the following form. Let $\{f_\varphi\}$ be a (smooth or continuous) partition of unity on $X$, then,

$$\chi(X; \mathcal{O}_V(E)) = \sum \frac{1}{|G_U|} \sum_{g\in G_U} \int_{\bar{U}^g} \tau_g^X(\bar{U}; \tilde{\mathcal{E}}_U).$$

For each local coordinate $(G_U, \bar{U})$ and for each $g\in G_U$, we consider $\bar{U}^g$ as a complex manifold on which the centralizer $Z_{G_U}(g)$ acts. For $V\subset U$, the open embedding $\varphi: \bar{V}\to \bar{U}$ defines a natural open embedding $\bar{V}^g|Z_{G_U}(h)\to \bar{U}^g|Z_{G_U}(g)$ of analytic spaces, where $g=\lambda_\varphi(h)$. This embedding is unique for a fixed pair $(g, h)$. We patch all $\bar{U}^g|Z_{G_U}(g)$'s together by these identifications. Then we get a disjoint union of complex $V$-manifolds of various dimensions:

$$X \sqcup \sum X = \bigcup_{(g, h)\in G_U} \bar{U}^g|Z_{G_U}(g),$$

($X$ corresponds to the portion defined by $g=1$).

We have a canonical map $\sum X\to X$ covered locally by the inclusion $\bar{U}^g\subset \bar{U}$. For each $x\in X$, we can choose a coordinate neighbourhood $(G_x, \bar{U}_x)$ such that $x\in \bar{U}_x$ is a fixed point of $G_x$. $G_x$ is unique up to isomorphisms. Then the number of pieces of $\sum X$ over $x$ is equal to the number of the conjugacy classes of $G_x$ other
than the identity class.

Let \( \Xi_1, \Xi_2, \ldots, \Xi_\varepsilon \) be all the connected components of \( \Xi X \). To each \( \Xi_i \), we assign a number \( m_i \), defined by:

\[
m_i = \frac{1}{|\text{kernel}[Z_{\text{sp}}(g) \to \text{Aut}(\mathcal{U})]|},
\]

Now the formal sum \( \sum_{\xi \in \mathcal{V}} \frac{\xi^*(\alpha)}{\xi} \) defines a "differential form" on \( X \perp \Xi X \).

It represents a cohomology class \( \mathcal{I}(X; E) + \mathcal{I}(\Xi X; E) \) in \( H^*(X \perp \Xi X; \mathbb{C}) \).

This class is independent of the choice of the connections. Then we get the following theorem:

**Theorem.** Let \( X \) be a compact complex \( \mathcal{V} \)-manifold and let \( E \to X \) be a holomorphic vector \( \mathcal{V} \)-bundle. Then:

\[
\chi(X; \mathcal{O}_v(E)) = \langle \mathcal{I}(X; E), [X] \rangle + \sum_{i=1}^{\varepsilon} m_i \langle \mathcal{I}(\Xi_i X), [\Xi_i X] \rangle.
\]

**Remark 1.** Since the class \( \mathcal{I}(X; E) \) is defined over rationals, the term \( \langle \mathcal{I}(X; E), [X] \rangle \) is a rational number.

**Remark 2.** For the case when \( X = \Gamma \backslash X \), where \( X \) is a complex manifold and \( \Gamma \) is a properly discontinuous group acting holomorphically on \( \bar{X} \), the number \( \langle \mathcal{I}(X; E), [X] \rangle \) is just the \( \Gamma \)-index \( \text{ind}_\Gamma((\bar{\partial} + \bar{\partial}^*)_E) \) defined by Atiyah [1]. (Though \( \Gamma \) acts freely in [1], the similar argument holds for the case when \( \Gamma \) has finite isotropies, see III) below).

The proof of our theorem is a combination of our work [10] and Gilkey's result [7] on the Lefschetz fixed point formula for the Dolbeault complexes. Here we shall place a complete proof.

**Proof of Theorem.** In this proof, we use the "heat kernel-zeta function" method. We review the results briefly. (See Seeley [12], Atiyah-Bott-Patodi [2], Gilkey [6, 7], Donnelly-Patodi [5] and Kawasaki [10]).

Let \( U \) be a germ of a Riemannian manifold and let \( E_U \to \bar{U} \) be a smooth complex vector bundle with a smooth Hermitian fibre metric. Let \( g: E_U \to E_U \) be an isometry of the pair \((U, E_U)\). Let \( A: \mathcal{C}^\infty(U; E_U) \to \mathcal{C}^\infty(U; E_U) \) be a \( g \)-invariant, formally self-adjoint, positive semi-definite, elliptic differential operator. Then we have a smooth measure \( Z^g_A \) on the fixed point set \( U^g \). \( Z^g_A \) is a local invariant of the action of \( g \) and of the operator \( A \). It is given by a universal expression in \( g \) and \( A \). The explicit form of \( Z^g_A \) is given in [10]. \( Z^g_A \) has the following properties:
I) Let $M$ be a compact Riemannian manifold and let $g: M \to M$ be an isometry. Let $E, F$ be two $g$-equivariant smooth complex vector bundles over $M$ with $g$-invariant Hermitian fibre metrics. Let $D: \mathcal{C}^\infty(M; E) \to \mathcal{C}^\infty(M; F)$ be a $g$-invariant elliptic differential operator. Then we have the adjoint operator $D^*: \mathcal{C}^\infty(M; F) \to \mathcal{C}^\infty(M; E)$ and two $g$-invariant, self-adjoint, positive semi-definite, elliptic differential operators $D^*D$ and $DD^*$. Put $\mu^g_{D} = Z_{D^*D} - Z_{DD^*}$. Then the equivariant index $\text{ind}(g, D)$ is given by:

$$\text{ind}(g, D) = \int_{M^g} d\mu^g_{D}. \tag{1}$$

II) (Kawasaki [10]). Let $X$ be a compact Riemannian $V$-manifold and let $E, F$ be two "proper" differentiable complex vector $V$-bundles over $X$. Let $D: \mathcal{C}^\infty(X; E) \to \mathcal{C}^\infty(X; F)$ be an elliptic differential operator, that is, a family $\{D_U: \mathcal{C}^\infty(\bar{U}; E_U) \to \mathcal{C}^\infty(\bar{U}; F_U)\}_{(\bar{G}_U, \bar{V})}$ of invariant elliptic differential operators that are compatible with attaching maps $\{\Phi\}: E_U \to E_V$ and $\{\Psi\}: F_U \to F_V$. Then $D$ operates on the differentiable $V$-sections and the kernel and the cokernel of the operator $D$ are finite dimensional. We define the $V$-index $\text{ind}_V(D)$ of the operator $D$ by:

$$\text{ind}_V(D) = \dim_C \ker[D: \mathcal{C}^\infty(X; E) \to \mathcal{C}^\infty(X; F)] - \dim_C \text{coker}[D: \mathcal{C}^\infty(X; E) \to \mathcal{C}^\infty(X; F)]. \tag{2}$$

For each coordinate neighbourhood $(G_U, \bar{U})$, we have a formal sum of measures:

$$\sum_{g \in G_U} \mu^g_{D_U} = \sum_{g \in G_U} (Z^g_{D_U} - Z^g_{D_U}). \tag{3}$$

These formal sums define a measure $\mu_D + \mu_{\tilde{D}}$ over $X \amalg \bar{\Sigma}X$. Then the $V$-index $\text{ind}_V(D)$ is given by:

$$\text{ind}_V(D) = \int_X d\mu_D + \sum_{i=1}^\infty \int_{\Sigma X_i} d\mu^\vee_{\tilde{D}}. \tag{4}$$

III) (See Atiyah [1]). Let $\bar{X}$ be a (non-compact) Riemannian manifold and let $\Gamma$ be a properly discontinuous group acting on $\bar{X}$ as isometries. We assume that the orbit $V$-manifold $X = \Gamma \backslash \bar{X}$ is compact. Let $\bar{E}, \bar{F}$ be two $\Gamma$-equivariant complex vector bundles over $\bar{X}$ with $\Gamma^*$-invariant Hermitian fibre metrics. Let $\bar{D}: \mathcal{C}^\infty(\bar{X}; \bar{E}) \to \mathcal{C}^\infty(\bar{X}; \bar{F})$ be a $\Gamma$-invariant elliptic differential operator. Then we consider the completions $\mathcal{L}^2(\bar{X}; \bar{E}), \mathcal{L}^2(\bar{X}; \bar{F})$ and the unbounded operators $\bar{D}: \mathcal{L}^2(\bar{X}; \bar{E}) \to \mathcal{L}^2(\bar{X}; \bar{F}), \bar{D}^*: \mathcal{L}^2(\bar{X}; \bar{F}) \to \mathcal{L}^2(\bar{X}; \bar{E})$. (In this case the formal adjoint coincides with the Hilbert space adjoint). We put:

$$\mathcal{A}_0 = \{ f \in \mathcal{L}^2(\bar{X}; \bar{E}) | \bar{D} f = 0 \} \subset \mathcal{L}^2(\bar{X}; \bar{E}),$$

$$\mathcal{A}_1 = \{ g \in \mathcal{L}^2(\bar{X}; \bar{F}) | \bar{D}^* g = 0 \} \subset \mathcal{L}^2(\bar{X}; \bar{F}).$$
Then $\mathcal{H}_i$ becomes a $\Gamma$-invariant closed subspace ($i=0, 1$). Let $H_i$ be the orthogonal projection onto $\mathcal{H}_i$. Then $H_i$ has a smooth kernel $H_i(\mathfrak{F}, \mathfrak{F})$ and we get a smooth measure $\text{trace}_c[H_i(\mathfrak{F}, \mathfrak{F})]$ over $\mathfrak{F}$. Since the operator $H_i$ is $\Gamma$-invariant, we may consider $\text{trace}_c[H_i(\mathfrak{F}, \mathfrak{F})]$ as a measure over $\mathfrak{F}=\Gamma \backslash \mathfrak{F}$. Then the $\Gamma$-index of the operator $\bar{D}$ is defined by:

$$\text{ind}_\Gamma(\bar{D}) = \int \mathfrak{F} \left( \text{trace}_c[H_0(\mathfrak{F}, \mathfrak{F})] - \text{trace}_c[H_0(\mathfrak{F}, \mathfrak{F})] \right).$$

Now the elliptic differential operator $\bar{D}$ over $\mathfrak{F}$ defines an elliptic differential operator $D: \mathcal{C}^\infty(\mathfrak{F}; E) \to \mathcal{C}^\infty(\mathfrak{F}; F)$ over a $V$-manifold $\mathfrak{F}$ and we have a measure $\mu_D$ over $\mathfrak{F}$. Then $\text{ind}_\Gamma(\bar{D})$ is given by:

$$\text{ind}_\Gamma(\bar{D}) = \int \mathfrak{F} \mu_D.$$

Now we return to our problem: Let $\mathfrak{F}$ be a compact complex $V$-manifold and let $E \to \mathfrak{F}$ be a holomorphic vector $V$-bundle. We denote by $\mathcal{T}$ the holomorphic part of the complexified cotangent vector $V$-bundle. Consider the sheaf $\mathcal{A}_p(E) = \mathcal{C}^\infty(\Lambda^\mathfrak{F} \mathcal{T} \otimes \Lambda^\mathfrak{F} \mathcal{T} \otimes E)$ of germs of $E$-valued $(p, q)$-forms over $\mathfrak{F}$. Then we have the $\bar{\partial}$-operators $\bar{\partial}: \mathcal{A}_p(E) \to \mathcal{A}_{q+1}(E)$ and a soft resolution:

$$0 \to \mathcal{O}_\mathfrak{F}(\Lambda^\mathfrak{F} \mathcal{T} \otimes E) \to \mathcal{A}_{q}(E) \to \mathcal{A}_{q+1}(E) \to \cdots \to \mathcal{A}_{n}(E) \to 0.$$

Put $A_{q}^\mathfrak{F}(X; E) = \Gamma(X; \mathcal{A}_q(E))$, then we have a complex:

$$0 \to A_{q}^\mathfrak{F}(X; E) \to A_{q-1}^\mathfrak{F}(X; E) \to \cdots \to A_{0}^\mathfrak{F}(X; E) \to 0,$$

whose $i$-th cohomology group is $H^i(X; \mathcal{O}_\mathfrak{F}(\Lambda^\mathfrak{F} \mathcal{T} \otimes E))$. Choose a Hermitian metric $h$ on $\mathfrak{F}$ and a Hermitian fibre metric $h_E$ on $E$. Then we have the adjoint operator $\bar{\partial}^\ast: A_{q}^\mathfrak{F}(X; E) \to A_{q-1}^\mathfrak{F}(X; E)$ of $\bar{\partial}$. Consider a differential operator:

$$(\bar{\partial} + \bar{\partial}^\ast)^{0, \mathfrak{F}}_{E}v = \bar{\partial} + \bar{\partial}^\ast |_{A_{q}^\mathfrak{F}}: A_{q}^\mathfrak{F}_{\mathfrak{F}}(X; E) \to A_{q-1}^\mathfrak{F}_{\mathfrak{F}}(X; E),$$

$$A_{q}^\mathfrak{F}_{\mathfrak{F}}(X; E) = \bigoplus_{q} A_{q}^\mathfrak{F}_{\mathfrak{F}}(X; E).$$

Then $(\bar{\partial} + \bar{\partial}^\ast)^{0, \mathfrak{F}}_{E}$ is an elliptic operator and:

$$\text{ind}_\Gamma((\bar{\partial} + \bar{\partial}^\ast)^{0, \mathfrak{F}}_{E}) = \chi(\mathfrak{F}; \mathcal{O}_\mathfrak{F}(E)).$$

Thus we can express the arithmetic genus as the $V$-index of an elliptic operator $(\bar{\partial} + \bar{\partial}^\ast)^{0, \mathfrak{F}}_{E}$. Then, by II) above, we have a measure $\mu_{(\bar{\partial} + \bar{\partial}^\ast)^{0, \mathfrak{F}}_{E}}$ over $X \sqcup \Sigma X$ that gives the arithmetic genus. But this measure is not equal to the Todd class in general. So we use the $Spin^c$ Dirac operator instead, which gives the arithmetic genus for complex $V$-manifolds and is defined over more general $V$-manifolds.
Now let \((X, h)\) and \((E, h_E)\) be as before. Consider the almost complex structure \((TX, J)\). \((TX, J)\) is a holomorphic vector \(V\)-bundle. The Hermitian metric \(h\) define a reduction \(U(n)\) of the principal tangent \(V\)-bundle. We consider \(U(n)\) as a subgroup of \(\text{Spin}'(2n)=\text{Spin}(2n)\times\mathbb{Z}_2\). (See Atiyah-Bott-Shapiro [2]). Let \(\text{Spin}'(2n)\) be the associated \(\text{Spin}'(2n)\)-principal tangent \(V\)-bundle. We construct a connection \(\nabla^c\) on \(\text{Spin}'(2n)\) as follows: We have a Riemannian connection \(\nabla_{SO}\) on \(SO(2n)\) and a Hermitian connection \(\nabla_L\) on \(L=\Lambda^*(TX, J)\). Then \(\nabla^c\) is a unique lift of \(\nabla_{SO}\times\nabla_L\) on \((SO(2n)\times U(1))(TX)\) by the double covering \(\text{Spin}'(2n)\to SO(2n)\times U(1)\). Let \(\Delta^{\pm,c}\) be the half \(\text{Spin}'\)-representations. Then we have two complex vector \(V\)-bundles:

\[
\Delta^{\pm,c}(TX) = \text{Spin}'(2n)\times_{\text{Spin}(2n)} \Delta^{\pm,c},
\]

with induced connections \(\nabla^{\pm,c}\). The Clifford module structures on \(\Delta^{\pm,c}\) define the Clifford multiplications:

\[
m: TX\otimes_R \Delta^{\pm,c}(TX) \to \Delta^{\mp,c}(TX).
\]

On \((E, h_E)\) we have the Hermitian connection \(\nabla_E\). Then the \(\text{Spin}'\) Dirac operator \(d^{\pm,c}_E\) is defined by:

\[
d^{\pm,c}_E: C^\infty(V(X; \Delta^{\pm,c}(TX)\otimes cE)\nabla^{\pm,c}\otimes 1 + 1\otimes \nabla_E C^\infty(V(X; T^*X\otimes_R \Delta^{\pm,c}(TX)\otimes cE))
\]

\[
m \to C^\infty(V(X; \Delta^{\pm,c}(TX)\otimes cE)).
\]

Here we identify \(TX=T^*X\) by the real Hermitian metric \(R_e h\). Since \(\text{Spin}'(2n)\) has a reduction \(U(n)\), we have:

\[
\Delta^{\pm,c}(TX) \cong \Lambda^{\varepsilon\varepsilon}_c(TX, J).
\]

The Hermitian metric \(h\) defines a \(V\)-bundle isometry \(\psi: (TX, J)\cong \tilde{T}\). So we have a \(V\)-bundle isomorphism:

\[
\psi^\pm: \Delta^{\pm,c}(TX)\otimes E \cong \Lambda^{\varepsilon\varepsilon}_c \tilde{T}\otimes E.
\]

By a standard computation (see Hitchin [9]), we have:

**Proposition.** The two operators \((\bar{\partial} + \bar{\partial}^*)^\pm_E\) and \(d^{\pm,c}_E\) have the same principal symbol (via \(\psi^\pm\) up to a constant factor.

As a corollary, we have:

\[
\chi(X; \mathcal{O}_V(E)) = \text{ind}_V(d^{\pm,c}_E) = \int_X d\mu_{d^{\pm,c}_E} + \sum_{i=1}^n \frac{1}{m_i} \int_{\mathbb{F}X_i} d\mu_{\tilde{d}^{\pm,c}_E}.
\]
Now the operator $d_{L}^*\omega$ does not depend on the complex structure on $X$. It depends only on the $\text{Spin}^c$-structure $\text{Spin}^c(2n)(TX)$, the metric connection $\nabla_L$ and the Hermitian $V$-bundle $(E, h_E, \nabla_E)$. Its index $\text{ind}_V(d_{L}^*\omega)$ does not depend on the choices of metrics $h$ and $h_E$, nor the choices of connections $\nabla_L$ and $\nabla_E$. So we can change metrics and connections.

We consider over a coordinate neighbourhood $(G_U, \tilde{U})\to U$. Choose a metric $h$ on $\tilde{U}$ so that, for each $g \in G_U$, on a neighbourhood of $U^\varepsilon$ in $\tilde{U}$, $h$ is equal to the Riemannian metric on the total space $N_g$ of the normal bundle $\nu_g = \nu(\tilde{U}^\varepsilon \subset \tilde{U})$ induced from a $g$-invariant Hermitian structure $(\nu_g, h_{\nu_g}, \nabla_{\nu_g})$. We identify $N_g$ with a neighbourhood of $U^\varepsilon$ in $\tilde{U}$. Then, over $N_g$, the principal bundle $\text{Spin}^c(2n)(T\tilde{U})$ reduces equivariantly to $\pi^*(\text{Spin}^c(2n_0)(T\tilde{U}^\varepsilon) \times_{\mathbb{Z}_2} U(n-n_0)(\nu_g))$, where $\pi: N_g \to \tilde{U}^\varepsilon$ is the projection of $\nu_g$ and $2n_0=\dim_{\mathbb{R}}U^\varepsilon$. The associated line bundle $L$ splits into a tensor product $\pi^*(L_{0} \otimes \Lambda^{\varepsilon-n_0} \nu_g)$, where $L_0$ is the associated line bundle of $\text{Spin}^c(2n_0)(T\tilde{U}^\varepsilon)$.

The actions of $g$ on the first factors $\text{Spin}^c(2n_0)(T\tilde{U}^\varepsilon)$ and $L_0$ are trivial. On $L_0$, we have the induced metric $h_{L_0}$. Choose a metric connection $\nabla_{L_0}$ on $(L_0, h_{L_0})$. Then we choose a metric connection $\nabla_L$ so that, over $N_g$, $\nabla_L$ is equal to the induced connection $\pi^*(\nabla_{L_0} \otimes \Lambda^{\varepsilon-n_0} \nabla_{\nu_g})$. Also, we choose a Hermitian structure $(E, h_E, \nabla_E)$ so that, over $N_g$, it is equal to the induced structure $(\pi^*(E | \tilde{U}^\varepsilon)$, $\pi^*(h_E | \tilde{U}^\varepsilon)$, $\pi^*(\nabla_E | \tilde{U}^\varepsilon)$).

Then, over a neighbourhood $N_g$ of $U^\varepsilon$ in $\tilde{U}$, the operator $d_{L}^*\omega$ is completely determined by the data over $\tilde{U}^\varepsilon$, that is, the $\text{Spin}^c$-structure $\text{Spin}^c(2n_0)(T\tilde{U}^\varepsilon)$, the metric connection $\nabla_{L_0}$ and the $g$-equivariant Hermitian bundles $(g; \nu_g, h_{\nu_g}, \nabla_{\nu_g})$ and $(g; E | U^\varepsilon, h_E | U^\varepsilon, \nabla_E | U^\varepsilon)$.

We remark here that we can choose a metric $h$, a metric connection $\nabla_L$ and a hermitian structure $(E, h_E, \nabla_E)$ over a $V$-manifold $X$ so that the above conditions are satisfied for all coordinate neighbourhood $(G_U, \tilde{U})\to U$ and for all $g \in G_U$ at the same time.

Now we consider differently: Let $(U_0, h_0)$ be a germ of $(2n_0)$-dimensional Riemannian manifold with trivial $g$-action and assume that we are given a Hermitian line bundle $(L_0, h_{L_0}, \nabla_{L_0})$ with trivial $g$-action and two $g$-equivariant Hermitian bundles $(g; \nu, h_\nu, \nabla_\nu) (\dim_{\mathbb{C}} \nu = n-n_0)$ and $(g; E, h_E, \nabla_E)$ over $U_0$. So $g$ acts on each fibre of $\nu$ and $E$. We assume that the fixed points in $\nu$ are all in the zero section. We may assume that $U_0$ is contractible. Then an orientation $o$, the Riemannian metric $h_0$ and the Hermitian line bundle $(L_0, h_{L_0}, \nabla_{L_0})$ define a unique $\text{Spin}^c$-structure $\text{Spin}^c(2n_0)(TU_0)$ up to $\text{Spin}^c$-isomorphisms. (There are two canonical isomorphisms). The Riemannian metric $h_0$ and the metric connection $\nabla_{L_0}$ define a connection $\nabla_0$ on $\text{Spin}^c(2n_0)(TU_0)$. Consider the total space $N$ of $\nu$. The Hermitian structure $(\nu, h_\nu, \nabla_\nu)$ define a $\text{Spin}^c(2n_0) \times U(n-n_0)$-structure over $N$. Also we have the action of $g$ that preserve the above structure. Then we have the associated $\text{Spin}^c(2n)$-structure with
$g$-action over $N$. Its associated line bundle is $\pi^*(L_0 \otimes \Lambda^*\nu)$ and the metric connection $\nabla_{L_0} \otimes \Lambda^*\nu \nabla_\nu$ defines a connection $\nabla^c$ on $\text{Spin}^c(2n)$ ($TN$). Also we have an induced $g$-equivariant Hermitian bundle $(g; \pi^*E, \pi^*h, \pi^*\Delta_\nu)$ over $N$.

Then the $\text{Spin}^c$-structure $\text{Spin}^c(2n)$ ($TN$) with connection $\nabla^c$ and the Hermitian bundle $(\pi^*E, \pi^*h, \pi^*\nabla_\nu)$ define the $\text{Spin}^c$ Dirac operator $d_{\pi^*E}^c$. The operator $d_{\pi^*E}^c$ and the action of $g$ define a measure $\mu_{d_{\pi^*E}^c}$ over $U_0$. The only ambiguity of this construction comes from the choice of the orientation $o$ over $U_0$. If we change the orientation, then the measure $\mu_{d_{\pi^*E}^c}$ changes its sign. So the measure $\mu_{d_{\pi^*E}^c}$ defines a $2n_0$-form $d\mu_{d_{\pi^*E}^c}$ with no ambiguity.

Thus we have shown that the $2n_0$-form $d\mu_{d_{\pi^*E}^c}$ is a local invariant of a Riemannian structure $(U_0, h_0)$ and Hermitian bundles $(L_0, h_{L_0}, \nabla_{L_0}), (g; \nu, h, \nabla,)$ and $(g; E, h_E, \nabla_E)$. In [10], we have an explicit form of $\mu_{d_{\pi^*E}^c}$. Then we can see that the $2n_0$-form $d\mu_{d_{\pi^*E}^c}$ is a homogeneous regular local invariant of weight 0, in the terminology of Atiyah-Bott-Patodi [2]. Then, by Gilkey’s Theorem (see [2]), we can conclude:

**Proposition.** $d\mu_{d_{\pi^*E}^c}$ is expressed by a universal polynomial in the Pontrjagin forms of $(U_0, h_0)$, the first Chern form of $(L_0, h_{L_0}, \nabla_{L_0})$, the equivariant Chern forms of $(g; \nu, h, \nabla,)$ and the equivariant Chern forms of $(g; E, h_E, \nabla_E)$.

We restrict ourselves to the case when $TU_0$ has an almost complex structure $J_0$ and $L_0 = \Lambda^*\nu(TU_0, J_0)$. Let $M$ be a compact complex manifold and let $E \to M$ be a holomorphic vector bundle. Let $g$ be an automorphism of the pair $(M, E)$ that generates a compact transformation group. Then by Atiyah-Singer [4] we know:

$$\int_M d\mu_{d_{\pi^*E}^c} = \sum_i (-1)^i \text{trace}_g[H'(M; \mathcal{O}(E))]$$

$$= \langle \mathcal{D}(M; E), [M^g] \rangle.$$

The computations over the products of complex projective spaces with linear actions show that the expression of $d\mu_{d_{\pi^*E}^c}$ in the characteristic classes must be unique. This shows:

$$d\mu_{d_{\pi^*E}^c} = \mathcal{D}(M; E).$$

Now we return to the original situation. Over a coordinate neighbourhood $(G, U) \to U$, we have:

$$\mu_{d_E^c} + \mu_{\pi^*E}^c = \sum_{g \in G} \mu_{d_E^c}^g.$$ 

Then, by choosing suitable metrics and connections, we have:

$$d\mu_{d_E^c} + d\mu_{\pi^*E}^c = \mathcal{D}(X; E) + \mathcal{D}(X; E).$$

Hence we have:
\[ \chi(X; \mathcal{O}_V(E)) = \int_X d\mu_{dE} + \sum_{i=1}^n \frac{1}{m_i} \int_{C_i} d\mu_{dE} \]
\[ = \langle \mathbb{D}(X; E), [X] \rangle + \sum_{i=1}^n \frac{1}{m_i} \langle \mathbb{D}_i(X; E), \mathbb{C} X_i \rangle. \]

The both sides are independent of the metrics and connections.

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**References**


