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A CHARACTERIZATION OF BOUNDED KRULL PRIME RINGS

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In [9] we defined the concept of non commutative Krull prime rings from the point of view of localizations and we mainly investigated the ideal theory in bounded Krull prime rings (cf. [9], [10]).

The purpose of this paper is to prove the following:

Theorem. *Let R be a prime Goldie ring with two-sided quotient ring Q . Then R is a bounded Krull prime ring if and only if it satisfies the following conditions ;*

- (1) *R is a regular maximal order in Q (in the sense of Asano).*
- (2) *R satisfies the maximum condition for integral right and left v -ideals.*
- (3) *R/P is a prime Goldie ring for any minimal prime ideal P of R .*

As corollary we have

Corollary. *Let R be a noetherian prime ring. If R is a regular maximal order in Q , then it is a bounded Krull prime ring.*

In case R is a commutative domain, the theorem is well known and its proof is easy (cf. [11]). We shall prove the theorem by using properties of one-sided v -ideals and torsion theories.

Throughout this paper let R be a prime Goldie ring, not artinian ring, having identity element 1, and let Q be the two-sided quotient ring of R ; Q is a simple and artinian ring. We say that R is an *order* in Q . If R_1 and R_2 are orders in Q , then they are called *equivalent* (in symbol: $R_1 \sim R_2$) if there exist regular elements a_1, b_1, a_2, b_2 of Q such that $a_1 R_1 b_1 \subseteq R_2, a_2 R_2 b_2 \subseteq R_1$. An order in Q is said to be *maximal* if it is a maximal element in the set of orders which are equivalent to R . A right R -submodule I of Q is called a *right R -ideal* provided I contains a regular element of Q and there is a regular element b of Q such that $bI \subseteq R$. I is called *integral* if $I \subseteq R$. Left R -ideals are defined in a similar way. If I is a right (left) R -ideal of Q , then $O_l(I) = \{x \in Q \mid xI \subseteq I\}$ is an order in Q and is equivalent to R . Similarly $O_r(I) = \{x \in Q \mid Ix \subseteq I\}$ is an order in Q and is equivalent to R . They are called a *left order* and a *right order* of I respectively.

We define the inverse of I to be $I^{-1} = \{q \in Q \mid IqI \subseteq I\}$. Evidently $I^{-1} = \{q \in Q \mid Iq \subseteq O_l(I)\} = \{q \in Q \mid qI \subseteq O_r(I)\}$. Following [2], we define $I^* = (I^{-1})^{-1}$. If $I = I^*$, then it is said to be a *right (left) v-ideal*. If R is a maximal order, then $I^{-1} = I^{-1-1-1}$ and so I^{-1} is a left (right) *v-ideal*, and the concept of right (left) *v-ideals* coincides with one of right (left) *v-ideals* defined in [9]. So the mapping: $I \rightarrow I^*$ of the set of all right (left) R -ideals into the set of all right (left) *v-ideals* is a $*$ -operation in the sense of [9].

Lemma 1. *Let R be a maximal order in Q and let S be any order equivalent to R . Then S is a maximal order if and only if $S = O_l(I)$ for some right *v-ideal* I of Q .*

Proof. If $S = O_l(I)$ for some right *v-ideal* I of Q , then it is a maximal order by Satz 1.3 of [1]. Conversely assume that S is a maximal order, then there are regular elements c, d in R such that $cSd \subseteq R$. So SdR is a right R -ideal and is a left S -module. Hence $(SdR)^{-1}$ is a left R -ideal and is a right S -module. Similarly $I = (SdR)^{-1-1}$ is a right *v-ideal* and is a left S -module so that $O_l(I) \supseteq S$. Hence $S = O_l(I)$.

Lemma 2. *Let R, S be maximal orders in Q such that $R \sim S$, and let $\{I_i\}$, I be right R -ideals. Then*

- (1) *If $\cap_i I_i$ is a right R -ideal, then $\cap_i I_i^* = (\cap_i I_i^*)^*$.*
- (2) *If $\sum_i I_i$ is a right R -ideal, then $(\sum_i I_i)^* = (\sum_i I_i^*)^*$.*
- (3) *If J is a left R and right S -ideal, then $(IJ)^* = (I^*J)^* = (IJ^*)^* = (I^*J^*)^*$.*
- (4) *$(I^{-1}I^*)^* = R$ and $(I^*I^{-1})^* = T$, where $T = O_l(I^*)$.*

Proof. The proofs of (1) and (2) are similar to ones of the corresponding results for commutative rings (cf. Proposition 26.2 of [4]).

To prove (3) assume that $IJ \subseteq cS$, where c is a unit in Q . Then we have $(I^*J) \subseteq cS$ and $(IJ^*) \subseteq cS$, because

$c^{-1}IJ \subseteq S \Rightarrow c^{-1}I \subseteq J^{-1} \Rightarrow c^{-1}IJ^* \subseteq J^{-1}J^* \subseteq S \Rightarrow IJ^* \subseteq cS$, and $c^{-1}I \subseteq J^{-1} \Rightarrow c^{-1}I^* = (c^{-1}I)^* \subseteq J^{-1} \Rightarrow c^{-1}I^*J \subseteq J^{-1}J \subseteq S \Rightarrow I^*J \subseteq cS$. Hence $(IJ)^*$ contains $(IJ^*)^*$ and $(I^*J)^*$ by Proposition 4.1 of [9]. The converse inclusions are clear. Therefore we have $(IJ)^* = (I^*J)^* = (IJ^*)^*$. From these it is clear that $(IJ)^* = (I^*J^*)^*$.

To prove (4), assume that $I^{-1}I^* \subseteq cR$, where c is a unit in Q . Then we have $c^{-1}I^{-1} \subseteq I^{-1}$ so that $c^{-1} \subseteq O_l(I^{-1}) = R$ and thus $R \subseteq cR$. Hence $(I^{-1}I^*)^* \supseteq R$ by Proposition 4.1 of [9]. The converse inclusion is clear. Therefore $(I^{-1}I^*)^* = R$. Similarly $(I^*I^{-1})^* = T$.

Let R be a maximal order in Q . We denote by $F_r^*(R)$ ($F_l^*(R)$) the set of right (left) *v-ideals* and let $F^*(R) = F_r^*(R) \cap F_l^*(R)$. It is clear that $F_r^*(R)$ becomes a lattice by the definition; if $I, J \in F_r^*(R)$, then $I \cup^* J = (I + J)^*$, and the meet “ \cap ” is the set-theoretic intersection. Similarly $F_l^*(R)$ and $F^*(R)$ also become

lattices. For any $I \in F_r^*(R)$ and $L \in F_r^*(R)$, we define the product “ \circ ” of I and L by $I \circ L = (IL)^*$. It is clear that $I \circ L \in F_r^*(S) \cap F_r^*(T)$, where $S = O_l(I)$ and $T = O_r(L)$. In particular, the semi-group $F^*(R)$ becomes an abelian group (cf. Theorem 4.2 of [2]). For convenience we write $F'_r(R)$ for the sublattice of $F_r^*(R)$ consisting of all integral right v -ideals. Similarly we write $F'_l(R)$ and $F'(R)$ for the corresponding sublattices of $F_r^*(R)$ and $F^*(R)$ respectively. Let M and N be subsets of Q . Then we use the following notations: $(M:N)_r = \{x \in R \mid Nx \subseteq M\}$, $(M:N)_l = \{x \in R \mid xN \subseteq M\}$. When N is a single element q of Q , then we denote by $q^{-1}M$ the set $(M:N)_r$.

Lemma 3. *Let R be a maximal order in Q . Then*

- (1) *If $I \in F_r^*(R)$ and $q \in Q$, then $q^{-1}I = (I^{-1}q + R)^{-1}$ and so $q^{-1}I \in F'_r(R)$.*
- (2) *If $I \in F_r^*(R)$ and J is a right R -ideal, then $(I:J)_r \in F'(R)$ or 0.*
- (3) *If $I \in F_r^*(R)$ and $J \in F_r^*(R)$, then $(I \circ J)^{-1} = J^{-1} \circ I^{-1}$.*
- (4) *If $I, J \in F_r^*(R)$ and $L \in F_r^*(R)$, then $(I \cup *J) \circ L = I \circ L \cup *J \circ L$.*

Proof. (1) Since $(I^{-1}q + R)q^{-1}I \subseteq R$, we get $(I^{-1}q + R)^{-1} \supseteq q^{-1}I$. Let x be any element of $(I^{-1}q + R)^{-1}$. Then $(I^{-1}q + R)x \subseteq R$ so that $x \in R$ and $I^{-1}qx \subseteq R$. Let $S = O_l(I)$. Then it is a maximal order equivalent to R by Lemma 1. It is evident that $Sqx + I$ is a left S -ideal and that $II^{-1}(Sqx + I) \subseteq I$. Thus, by Lemma 2, we have

$qx \in S(Sqx + I) \subseteq (II^{-1})^*(Sqx + I)^* = (II^{-1}(Sqx + I))^* \subseteq I$. Hence $x \in q^{-1}I$ and so $q^{-1}I = (I^{-1}q + R)^{-1}$. It is clear that $q^{-1}I \in F'_r(R)$ by Corollary 4.2 of [9].

(2) If $(I:J)_r \neq 0$, then it is an R -ideal of Q and $J(I:J)_r \subseteq I$. So $J((I:J)_r)^* \subseteq (J(I:J)_r)^* \subseteq I$. Hence $((I:J)_r)^* \subseteq (I:J)_r$, so that $((I:J)_r)^* = (I:J)_r$.

(3) It is clear that $O_l(I \circ J) \supseteq O_l(I)$ and so $O_l(I \circ J) = O_l(I)$ by Lemma 1. Since $(I \circ J) \circ (J^{-1} \circ I^{-1}) = S$, where $S = O_l(I)$, we get $(I \circ J)^{-1} \supseteq J^{-1} \circ I^{-1}$. Let x be any element of $(I \circ J)^{-1}$. Then $IJx \subseteq (I \circ J)x \subseteq S$. Let $T = O_r(J)$. Then $Tx + J^{-1}I^{-1}$ is a left T -ideal and $IJ(Tx + J^{-1}I^{-1}) \subseteq S$. Hence $I \circ J \circ (Tx + J^{-1}I^{-1})^* \subseteq S$ by Lemma 2. By multiplying $J^{-1} \circ I^{-1}$ to the both side of the inequality we have $x \in (Tx + J^{-1}I^{-1})^* \subseteq J^{-1} \circ I^{-1}$. Therefore we get $(I \circ J)^{-1} = J^{-1} \circ I^{-1}$.

(4) From Lemma 2, we have: $(I \cup *J) \circ L = [(I + J)^*L]^* = [(I + J)L]^* = (IL + JL)^* = [(IL)^* + (JL)^*]^* = I \circ L \cup *J \circ L$.

Let R be a maximal order. We consider the following condition:

(A): $F'_r(R)$ and $F'_l(R)$ both satisfy the maximum condition.

If R is a maximal order satisfying the condition (A), then $F^*(R)$ is a direct product of infinite cyclic groups with prime v -ideals as their generators by Theorem 4.2 of [2]. It is evident that an element P in $F'(R)$ is a prime element in the lattice if and only if it is a prime ideal of R .

Following [1], R is said to be *regular* if every integral one-sided R -ideal contains a non-zero R -ideal.

Lemma 4. *Let R be a regular maximal order satisfying the condition (A) and let P be a non-zero prime ideal of R . Then P is a minimal prime ideal of R if and only if it is a prime v -ideal.*

Proof. Assume that P is a minimal prime ideal. Let c be any regular element in P . Then since $(cR)^* = cR$ and R is regular, we get $P \supseteq cR \supseteq (P_1^{n_1})^* \circ \dots \circ (P_k^{n_k})^*$, where P_i is a prime v -ideal. Hence $P \supseteq P_i$ for some i and so $P = P_i$. Conversely assume that $P \not\supseteq P_0 \neq 0$, where P_0 is a prime ideal. Then since $P_0^*(P_0^{-1}P_0) = (P_0^*P_0^{-1})P_0 \subseteq RP_0 = P_0$ and $P_0^{-1}P_0 \not\subseteq P_0$, we have $P_0^* \subseteq P_0$ and thus $P_0^* = P_0$. It follows that P_0 is a maximal element in $F'(R)$ by [2, p. 11], a contradiction. Hence P is a minimal prime ideal of R .

REMARK. Let R be a maximal order satisfying the condition (A). Then it is evident from the proof of the lemma that prime v -ideals are minimal prime ideals of R .

Let I be any right ideal of R . Then we denote by \sqrt{I} the set $\bigcup \{(s^{-1}I:R), |s \notin I, s \in R\}$. Following [3], if \sqrt{I} is an ideal of R , then we say that I is *primal* and that \sqrt{I} is the *adjoint ideal* of it. A right ideal I of R is called *primary* if $JA \subseteq I$ and $J \not\subseteq I$ implies that $A^n \subseteq I$ for some positive integer n , where J is a right ideal of R and A is an ideal of R . We shall apply these concepts for integral right v -ideals.

Lemma 5. *Let R be a maximal order satisfying the condition (A) and let I be a meet-irreducible element in $F'_v(R)$. Then I is primal, and \sqrt{I} is a minimal prime ideal of R or 0, and $\sqrt{I} = (x^{-1}I:R)$, for some $x \notin I$.*

Proof. If $\sqrt{I} = 0$, then the assertion is evident. Assume that $\sqrt{I} \neq 0$. By Lemma 3, $(s^{-1}I:R)_r$ is a v -ideal or 0. Hence the set $S = \{(s^{-1}I:R)_r, |s \notin I, s \in R\}$ has a maximal element. Assume that $(s^{-1}I:R)_r$ and $(t^{-1}I:R)_r$ are maximal elements in S . Then $(sR+I)(s^{-1}I:R)_r \subseteq I$ implies that $(sR+I)^*(s^{-1}I:R)_r \subseteq I$ by Lemma 2 and so $(s^{-1}I:R)_r \subseteq (I:(sR+I)^*)_r$. The converse inclusion is clear. Thus we have $(s^{-1}I:R)_r = (I:(sR+I)^*)_r$. Similarly $(t^{-1}I:R)_r = (I:(tR+I)^*)_r$. Since I is irreducible in $F'_v(R)$, we have $I \subseteq (sR+I)^* \cap (tR+I)^* = J$. Let x be any element in J but not in I . Then it follows that $(x^{-1}I:R)_r \supseteq (s^{-1}I:R)_r, (t^{-1}I:R)_r$, so that $\sqrt{I} = (x^{-1}I:R)_r = (s^{-1}I:R)_r$, which is a v -ideal. Hence I is primal. If $AB \subseteq \sqrt{I}$ and $A \not\subseteq \sqrt{I}$, where A and B are ideals of R , then $xAB \subseteq I$ and $xA \not\subseteq I$. Let y be any element in xA but not in I . Then $yB \subseteq I$ and so $B \subseteq (y^{-1}I:R)_r \subseteq \sqrt{I}$. Thus \sqrt{I} is a prime ideal of R . It follows that \sqrt{I} is minimal from the remark to Lemma 4.

A right ideal of R is said to be *bounded* if it contains a non-zero ideal of R .

Lemma 6. *Let R be a maximal order satisfying the condition (A) and let I be an irreducible element in $F'_*(R)$. If I is bounded, then it is primary and $(\sqrt{I})^n \subseteq I$ for some positive integer n .*

Proof. Since $I \in F'_*(R)$ and is bounded, $(I:R)_r$ is non-zero and is a v -ideal. Write $(I:R)_r = (P_1^{n_1})^* \circ \dots \circ (P_k^{n_k})^*$, where P_i are prime v -ideals. For any i ($1 \leq i \leq k$), we let $B_i = (P_1^{n_1})^* \circ \dots \circ (P_{i-1}^{n_{i-1}})^* \circ (P_{i+1}^{n_{i+1}})^* \circ \dots \circ (P_k^{n_k})^*$. Then $B_i \not\subseteq I$ and $B_i P_i^{n_i} \subseteq (I:R)_r \subseteq I$, because $F^*(R)$ is an abelian group. Thus $P_i^{n_i} \subseteq \sqrt{I}$ and so $P_i = \sqrt{I}$ ($1 \leq i \leq k$) by Lemma 5. Therefore $(\sqrt{I})^{n_1 + \dots + n_k} \subseteq I$. It is evident that I is primary.

If A is an ideal of R , then we denote by $C(A)$ those elements of R which are regular mod (A) .

Lemma 7. *Let R be a maximal order satisfying the condition (A). Let P be a prime v -ideal. Then*

- (1) $C(P) = C((P^n)^*)$ for every positive integer n .
- (2) $C(P) \subseteq C(0)$.

Proof. (1) We shall prove by the induction on n (> 1). Assume that $C(P) = C((P^{n-1})^*)$. If $cx \in (P^n)^*$, where $c \in C(P)$ and $x \in R$, then $cx(P^{-1})^{n-1} \subseteq (P^n)^*(P^{-1})^{n-1} \subseteq P$ by Lemma 2. Since $cx \in (P^n)^*$, we get $x \in (P^{n-1})^*$ and so $x(P^{-1})^{n-1} \subseteq R$. Hence $x(P^{-1})^{n-1} \subseteq P$. Then we have $(xR + P^n)(P^{-1})^{n-1} P^{n-1} \subseteq P^n$ so that $x \in (P^n)^*$ by Lemma 2. Conversely suppose that $cx \in P$, $c \in C((P^n)^*)$, $x \in R$. Then $cxP^{n-1} \subseteq (P^n)^*$ and so $xP^{n-1} \subseteq (P^n)^*$. Since $(xP + P^n)P^{n-1}(P^{-1})^{n-1} \subseteq (P^n)^*(P^{-1})^{n-1} \subseteq P$, we get $x \in P$ by Lemma 2. Therefore $C(P) = C((P^n)^*)$.

(2) If $0 \neq \cap_n (P^n)^*$, then it is a v -ideal by Lemma 2. Write $\cap_n (P^n)^* = (P_1^{n_1})^* \circ \dots \circ (P_k^{n_k})^*$, where P_i are prime v -ideals. This is a contradiction, because $F^*(R)$ is an abelian group and P, P_i are minimal prime ideals of R . Hence $0 = \cap_n (P^n)^*$. Therefore (2) follows from (1).

If P is a prime ideal of a ring S , then the family $T_P = \{I: \text{right ideal} \mid s^{-1}I \cap C(P) \neq \emptyset \text{ for any } s \in S\}$ is a right additive topology (cf. Ex. 4 of [12, p. 18]). The following lemma is due to Lambek and Michler if S is right noetherian. However, only trivial modifications to their proof are needed to establish the more general result.

Lemma 8. *Let P be a prime ideal of S and let $\bar{S} = S/P$ be a right prime Goldie ring. Then the torsion theory determined by the S -injective hull $E(\bar{S})$ of \bar{S} coincides with one determined by the right additive topology T_P , that is, a right ideal I of S is an element in T_P if and only if $\text{Hom}_S(S/I, E(\bar{S})) = 0$ (Corollary 3.10 of [8]).*

Lemma 9. *Let R be a maximal order satisfying the condition (A) and let P*

be a prime v -ideal such that $\bar{R}=R/P$ is a prime Goldie ring. If I is any element in $F'_*(R)$ such that $R \supseteq I \supseteq P$, then $I \cap C(P) = \phi$.

Proof. It is enough to prove the lemma when I is a maximal element $F'_*(R)$. Since $I^{-1} \supseteq R$, $P \circ I^{-1} \cap R \supseteq P$. If $P \circ I^{-1} \cap R = P$, then $P^{-1} = (P \circ I^{-1})^{-1} \cup *R$, because the mapping: $J \rightarrow J^{-1}$ is an inverse lattice isomorphism between $F'_*(R)$ and $F^*(R)$. By Lemma 3, $P^{-1} = I \circ P^{-1} \cup *R$. On the other hand $P \subseteq I$ implies that $R \subseteq I \circ P^{-1}$. Hence $P^{-1} = I \circ P^{-1}$ and so $R = I$, a contradiction. Thus we have $P \circ I^{-1} \cap R \supsetneq P$. Let a be any element in $P \circ I^{-1} \cap R$ but not in P . Then $aI \subseteq (P \circ I^{-1})I \subseteq P \circ I^{-1} \circ I = P$ so that $I \subseteq a^{-1}P \subsetneq R$. Since $a^{-1}P$ is a right v -ideal by Lemma 3, we get $I = a^{-1}P$. Then $\text{Hom}(R/I, E(\bar{R})) \neq 0$, because $R/I = R/a^{-1}P \cong (aR + P)/P \subseteq \bar{R}$. Now assume that $I \cap C(P) \neq \phi$ and let c be any element in $I \cap C(P)$. Then $cR + P \in T_P$ by Lemma 3.1 of [6]. Hence $I \in T_P$ and thus $\text{Hom}(R/I, E(\bar{R})) = 0$ by Lemma 8. This is a contradiction and so $I \cap C(P) = \phi$.

For convenience, we write $M(p)$ for the family of minimal prime ideals of R . If R is a regular maximal order satisfying the condition (A), then we know from Lemma 4 that a prime ideal P is an element in $M(p)$ if and only if it is a prime element in $F'(R)$.

Lemma 10. *Let R be a regular maximal order satisfying the condition (A), $P \in M(p)$ and let $I \in F'_*(R)$. If $\bar{R} = R/P$ is a prime Goldie ring, then $I \cup *P = R$ if and only if I contains an ideal B such that $B \not\subseteq P$.*

Proof. Assume that $I \supseteq B$, where B is an ideal not contained in P . Then $I \supseteq B^*$ and $B^* \cup *P = R$, because P is a maximal element in $F'(R)$ (cf. [2, p. 11]). Therefore $I \cup *P = R$. Conversely assume that the family $S = \{I \in F'_*(R) \mid I \cup *P = R, I \neq R \text{ and } I \not\supseteq B \text{ for any ideal } B \text{ not contained in } P\}$ is not empty and let I be a maximal element in S . If I is irreducible in $F'_*(R)$, then there exists P' in $M(p)$ such that $I \supseteq P'^n$ by Lemmas 5 and 6. Since $I \in S$, we have $P = P'$. If $n = 1$, then $R = I \cup *P = I$, a contradiction. We may assume that $I \not\supseteq P^{n-1}$ and $n > 1$. Then $(P^{n-1})^* = (I \cup *P) \circ (P^{n-1})^* = I \circ (P^{n-1})^* \cup *(P^n)^* \subseteq I^* = I$ by Lemmas 2 and 3. This is a contradiction. If I is reducible, then $I = I_1 \cap I_2$, where $I_i \in F'_*(R)$ and $I \subsetneq I_i$ ($i = 1, 2$). There are non zero ideals $B_i (\not\subseteq P)$ such that $I_i \supseteq B_i$. Thus I contains the ideal $B_1 B_2$ not contained in P , a contradiction. Hence $S = \phi$. This implies that if $I \cup *P = R$, then I contains an ideal not contained in P .

Let P be a prime ideal of a ring S . If S satisfies the Ore condition with respect to $C(P)$, then we denote by S_P the quotient ring with respect to $C(P)$.

Lemma 11. *Let R be a regular maximal order satisfying the condition (A) and let P be an element in $M(p)$ such that $\bar{R} = R/P$ is a prime Goldie ring. Then*

- (1) R satisfies the Ore condition with respect to $C(P)$.
- (2) $R_P = \varinjlim B^{-1}$, where B ranges over all non zero ideals not contained in P .
- (3) R_P is a noetherian, local and Asano order.

Proof. (1) It is clear that $T = \varinjlim B^{-1}(B(\nsubseteq P): \text{ideal})$ is an overring of R . Let c be any element in $C(P)$. Then c is regular by Lemma 7 and so $cR \in F'_*(R)$. Since $(cR \cup *P) \cap C(P) \neq \emptyset$, we have $cR \cup *P = R$ by Lemma 9 and so cR contains an ideal not contained in P by Lemma 10. Hence $c^{-1} \in T$. So for any $r \in R$, $c \in C(P)$, there exists an ideal $B(\nsubseteq P)$ such that $c^{-1}rB \subseteq R$. It is evident that $B \cap C(P) \neq \emptyset$. Let d be any element in $B \cap C(P)$. Then we have $c^{-1}rd = s$ for some s in R , that is, $rd = cs$. This implies that R satisfies the right Ore condition with respect to $C(P)$. The other Ore condition is shown to hold by a symmetric proof.

(2) is evident from (1).

(3) We let $P' = PR_P$. Then clearly $P' = R_P P$ and $P = P' \cap R$. So we may assume that $\bar{R} = R/P \subseteq \bar{R}_P = R_P/P'$ as rings. By (1), \bar{R}_P is the quotient ring of \bar{R} . Since \bar{R} is a prime Goldie ring, \bar{R}_P is the simple artinian ring. Hence P' is a maximal ideal of R_P . Let V' be any maximal right ideal of R_P . Suppose that $V' \nsubseteq P'$. Then $V' + P' = R_P$. Write $1 = v + pc^{-1}$, where $v \in V'$, $p \in P$ and $c \in C(P)$. Then $c = vc + p$ and so $vc = c - p \in C(P) \cap V'$. This implies that $V' = R_P$, a contradiction and so $V' \supseteq P'$. Hence P' is the Jacobson radical of R_P . The ideal $P^{-1}P$ properly contains P so that $C(P) \cap P^{-1}P \neq \emptyset$. It follows that $P^{-1}PR_P = R_P$. Similarly $R_P PP^{-1} = R_P$. Hence P' is an invertible ideal of R_P . Therefore R_P/P'^n is an artinian ring for any n , because \bar{R}_P is an artinian ring. Let I' be any essential right ideal of R_P . It is clear that $I' = (I' \cap R)R_P$. Let c be any regular element of $I' \cap R$. Then, since $cR \in F'_*(R)$ and R is regular, cR contains a non zero v -ideal $(P^n)^* \circ (P_1^n)^* \circ \dots \circ (P_k^n)^*$, where $P_i \in M(p)$. So we get $I' \supseteq R_P P^n = P'^n$. Therefore essential right ideals of R_P satisfies the maximum condition. Since R_P is finite dimensional in the sense of Goldie, R_P is right noetherian. Similarly R_P is left noetherian. Hence R_P is a noetherian, local and Asano order by Proposition 1.3 of [7].

After all these preparations we now prove the following theorem which is the purpose of this paper:

Theorem. *A prime Goldie ring R is a bounded Krull prime ring if and only if it satisfies the following conditions:*

- (1) R is a regular maximal order,
- (2) R satisfies the maximum condition for integral right and left v -ideals,
- (3) R/P is a prime Goldie ring for any $P \in M(p)$.

Proof. Assume that $R = \bigcap_i R_i$ ($i \in I$) is a bounded Krull prime ring, where R_i is a noetherian, local and Asano order with unique maximal ideal P_i . (1) is

clear from Corollary 1.4 and Lemma 1.6 of [10]. Let I be any right (left) R -ideal. Then $I^* = \cap IR_i (= \cap R_i I)$ by Proposition 1.10 of [10]. Since R_i is noetherian, (2) follows from the condition (K3) in the definition of Krull rings. Let $P_i = P'_i \cap R$. It follows that $\{P_i | i \in I\} = M(p)$ by Proposition 1.7 of [10] so that (3) is evident from Proposition 1.1 of [9].

It remains to prove that the conditions (1), (2) and (3) are sufficient. Let P be any element in $M(p)$. Then R satisfies the Ore condition with respect to $C(P)$ and R_P is a noetherian, local and Asano order by Lemma 11. Hence R_P is an essential overring of R . It is clear that $R \subseteq T = \cap R_P$, where $P \in M(p)$. To prove the converse inclusion let x be any element of T . Then there is an ideal $B_P (\nsubseteq P)$ such that $xB_P \subseteq R$ by Lemma 11. Let B be the sum of all ideals B_P . If B^* is different from R , then B^* is contained in some P in $M(p)$. But $B^* \nsubseteq P$ so that $B^* = R$. Hence we have $x \in (xR + R) \subseteq (xR + R)^* \circ B^* = (xB + B)^* \subseteq R$. Thus we get $R = \cap R_P$. Let c be any regular element in R . Then cR contains a v -ideal $(P_1^*)^* \circ \dots \circ (P_k^*)^*$, where $P_i \in M(p)$. It follows that $cR_P = R_P$ for every $P \in M(p)$ different to $P_i (1 \leq i \leq k)$ by Lemma 11. Hence R is a bounded Krull prime ring. This completes the proof of the theorem.

Corollary. *Let R be a regular, noetherian and prime ring. If R is a maximal order, then it is a bounded Krull prime ring.*

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References

- [1] K. Asano: *Zur Arithmetik in Schieftringen* I. Osaka Math. J. **1** (1949), 98–134.
- [2] A. Asano and K. Murata: *Arithmetical ideal theory in semigroups*, J. Inst. Polytec. Osaka city Univ. **4** (1953), 9–33.
- [3] W.E. Barnes: *Primal ideals and isolated components in noncommutative rings*, Trans Amer Math. Soc. **82** (1956), 1–16.
- [4] R. Gilmer: *Multiplicative ideal theory II*, Queen's Series in Pure and Applied Math. Queen's Univ. Kingston, Ontario, 1968.
- [5] A.W. Goldie: *The structure of prime rings under ascending chain conditions*, Proc. London Math. Soc. **8** (1958), 589–608.
- [6] A.W. Goldie: *Localization in non-commutative noetherian rings*, J. Algebra **5** (1967), 89–105.
- [7] C.R. Hajarnavis and T.H. Lenagan: *Localisation in Asano orders*, J. Algebra **21** (1972), 441–449.
- [8] J. Lambek and G. Michler: *The torsion theory at a prime ideal of a right noetherian ring*, J. Algebra **25** (1973), 364–389.
- [9] H. Marubayashi: *Non commutative Krull rings*, Osaka J. Math. **12** (1975), 703–714.
- [10] H. Marubayashi: *On bounded Krull prime rings*, **13** (1976), 491–501.
- [11] P. Samuel: *Lectures on unique factorization domains*, Tata Inst. of Fundamental Research, Bombay, 1964.
- [12] B. Stenström: *Rings and modules of quotients*, Lecture notes in Math. vol. 237, Berlin-Heidelberg-New York, Springer, 1971.