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A CHARACTERIZATION OF BOUNDED KRULL
PRIME RINGS

HIDETOSHI MARUBAYASHI

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In [9] we defined the concept of non commutative Krull prime rings from the point of view of localizations and we mainly investigated the ideal theory in bounded Krull prime rings (cf. [9], [10]).

The purpose of this paper is to prove the following:

**Theorem.** Let R be a prime Goldie ring with two-sided quotient ring Q. Then R is a bounded Krull prime ring if and only if it satisfies the following conditions:

1. R is a regular maximal order in Q (in the sense of Asano).
2. R satisfies the maximum condition for integral right and left \( v \)-ideals.
3. \( R/P \) is a prime Goldie ring for any minimal prime ideal P of R.

As corollary we have

**Corollary.** Let R be a noetherian prime ring. If R is a regular maximal order in Q, then it is a bounded Krull prime ring.

In case R is a commutative domain, the theorem is well known and its proof is easy (cf. [11]). We shall prove the theorem by using properties of one-sided \( v \)-ideals and torsion theories.

Throughout this paper let R be a prime Goldie ring, not artinian ring, having identity element 1, and let Q be the two-sided quotient ring of R; Q is a simple and artinian ring. We say that R is an order in Q. If \( R_1 \) and \( R_2 \) are orders in Q, then they are called equivalent (in symbol: \( R_1 \sim R_2 \)) if there exist regular elements \( a_1, b_1, a_2, b_2 \) of Q such that \( a_1 R b_1 \subseteq R_2, a_2 R b_2 \subseteq R_1 \). An order in Q is said to be maximal if it is a maximal element in the set of orders which are equivalent to R. A right R-submodule I of Q is called a right R-ideal provided I contains a regular element of Q and there is a regular element b of Q such that \( bi \subseteq R \). I is called integral if \( I \subseteq R \). Left R-ideals are defined in a similar way. If I is a right (left) R-ideal of Q, then \( O(I) = \{ x \in Q | xI \subseteq I \} \) is an order in Q and is equivalent to R. Similarly \( O(I) = \{ x \in Q |Ix \subseteq I \} \) is an order in Q and is equivalent to R. They are called a left order and a right order of I respectively.
We define the inverse of $I$ to be $I^{-1} = \{ q \in \mathbb{Q} | Iq \subseteq O_{\mathbb{Q}}(I) \} = \{ q \in \mathbb{Q} | qI \subseteq O_{\mathbb{Q}}(I) \}$. Evidently $I^{-1} = (I^{-1})^{-1}$. If $I = I^*$, then it is said to be a right (left) $\nu$-ideal. If $R$ is a maximal order, then $I^{-1} = I^{-1}_{\text{right}}$ and so $I^{-1}$ is a left (right) $\nu$-ideal, and the concept of right (left) $\nu$-ideals coincides with one of right (left) $\nu$-ideals defined in [9]. So the mapping: $I \mapsto I^*$ of the set of all right (left) $R$-ideals into the set of all right (left) $\nu$-ideals is an $*$-operation in the sense of [9].

**Lemma 1.** Let $R$ be a maximal order in $Q$ and let $S$ be any order equivalent to $R$. Then $S$ is a maximal order if and only if $S = O_{\mathbb{Q}}(I)$ for some right $\nu$-ideal $I$ of $Q$.

Proof. If $S = O_{\mathbb{Q}}(I)$ for some right $\nu$-ideal $I$ of $Q$, then it is a maximal order by Satz 1.3 of [1]. Conversely assume that $S$ is a maximal order, then there are regular elements $c$, $d$ in $R$ such that $cSd \subseteq R$. So $SdR$ is a right $R$-ideal and is a left $S$-module. Hence $(SdR)^{-1}$ is a left $R$-ideal and is a right $S$-module. Similarly $I = (SdR)^{-1}$ is a right $\nu$-ideal and is a left $S$-module so that $O_{\mathbb{Q}}(I) \subseteq S$. Hence $S = O_{\mathbb{Q}}(I)$.

**Lemma 2.** Let $R$, $S$ be maximal orders in $Q$ such that $R \sim S$, and let $\{I_i\}$, $I$ be right $R$-ideals. Then

1. If $\bigcap_i I_i$ is a right $R$-ideal, then $\bigcap_i I_i^* = (\bigcap_i I_i)^*$. 
2. If $\bigcup_i I_i$ is a right $R$-ideal, then $(\bigcup_i I_i)^* = (\bigcup_i I_i^*)^*$. 
3. If $J$ is a left $R$ and right $S$-ideal, then $(IJ)^* = (I^* J)^* = (IJ^*)^* = (I^* J^*)^*$. 
4. $(I^{-1} I)^* = R$ and $(I^* I^{-1})^* = T$, where $T = O_{\mathbb{Q}}(I^*)$.

Proof. The proofs of (1) and (2) are similar to ones of the corresponding results for commutative rings (cf. Proposition 26.2 of [4]).

To prove (3) assume that $IJ \subseteq cS$, where $c$ is a unit in $Q$. Then we have $(I^* J) \subseteq cS$ and $(IJ^*) \subseteq cS$, because $c^{-1} I J \subseteq S \Rightarrow c^{-1} I \subseteq J^{-1} \Rightarrow c^{-1} I \subseteq J^{-1} \Rightarrow (IJ^*) \subseteq S \Rightarrow (IJ^*) \subseteq cS$, and $c^{-1} I \subseteq J^{-1} \Rightarrow c^{-1} I \subseteq J^{-1} \Rightarrow (IJ^*) \subseteq S \Rightarrow (IJ^*) \subseteq cS$. Hence $(IJ)^*$ contains $(IJ^*)^*$ and $(IJ^*)^*$ by Proposition 4.1 of [9]. The converse inclusions are clear. Therefore we have $(IJ)^* = (I^* J)^* = (IJ^*)^*$. From these it is clear that $(IJ)^* = (I^* J)^*$. 

To prove (4), assume that $I^{-1} I^* \subseteq cR$, where $c$ is a unit in $Q$. Then we have $c^{-1} I^{-1} \subseteq I^{-1}$ so that $c^{-1} \subseteq O_{\mathbb{Q}}(I^{-1}) = R$ and thus $R \subseteq cR$. Hence $(I^{-1} I^*)^* \subseteq R$ by Proposition 4.1 of [9]. The converse inclusion is clear. Therefore $(I^{-1} I^*)^* = R$. Similarly $(I^* I^{-1})^* = T$.

Let $R$ be a maximal order in $Q$. We denote by $F^*_R(R)$ ($F^*_R(R)$) the set of all right (left) $\nu$-ideals and let $F^*_R(R) = F^*_R(R) \cap F^*_R(R)$. It is clear that $F^*_R(R)$ becomes a lattice by the definition; if $I, J \in F^*_R(R)$, then $I \cap J = (I \cap J)^*$, and the meet "$\cap$" is the set-theoretic intersection. Similarly $F^*_R(R)$ and $F^*_R(R)$ also become...
lattices. For any \( I \in F^*(R) \) and \( L \in F^*(R) \), we define the product "\( \circ \)" of \( I \) and \( L \) by \( I \circ L = (IL)^* \). It is clear that \( I \circ L \subseteq F^*(S) \cap F^*(T) \), where \( S = O_L(I) \) and \( T = O_L(L) \). In particular, the semi-group \( F^*(R) \) becomes an abelian group (cf. Theorem 4.2 of [2]). For convenience we write \( F'(R) \) for the sublattice of \( F^*(R) \) consisting of all integral right \( \nu \)-ideals. Similarly we write \( F'_{\sig}(R) \) and \( F^*(R) \) for the corresponding sublattices of \( F^*(R) \) and \( F^*(R) \) respectively. Let \( M \) and \( N \) be subsets of \( Q \). Then we use the following notations: \((M: N) = \{x \in R \mid Nx \subseteq M\} \), \((M: N)_I = \{x \in R \mid xN \subseteq M\} \). When \( N \) is a single element \( q \) of \( Q \), then we denote by \( q^{-1}M \) the set \((M: N)_I \).

**Lemma 3.** Let \( R \) be a maximal order in \( Q \). Then

1. If \( I \in F^*(R) \) and \( q \in Q \), then \( q^{-1}I = (I^{-1}q + R)^{-1} \) and so \( q^{-1}I \in F'(R) \).
2. If \( I \in F^*(R) \) and \( J \) is a right \( R \)-ideal, then \((I: J) \subseteq F'(R) \) or \( 0 \).
3. If \( I \in F^*(R) \) and \( J \in F^*(R) \), then \((I \circ J)^{-1} = J^{-1} \circ I^{-1} \).
4. If \( I, J \in F^*(R) \) and \( L \in F^*(R) \), then \((I \cup J) \circ L = I \circ L \cup J \circ L \).

Proof. (1) Since \((I^{-1}q + R)^{-1} \subseteq R \), we get \((I^{-1}q + R)^{-1} \subseteq q^{-1}I \). Let \( x \) be any element of \((I^{-1}q + R)^{-1} \). Then \((I^{-1}q + R)x \subseteq R \) so that \( x \in R \) and \( I^{-1}q \subseteq R \). Let \( S = O_L(I) \). Then it is a maximal order equivalent to \( R \) by Lemma 1. It is evident that \( Sx + I \) is a left \( S \)-ideal and so \((I^{-1}q + R)^{-1} \subseteq \). Hence \( x \in q^{-1}I \) and so \( q^{-1}I = (I^{-1}q + R)^{-1} \). It is clear that \( q^{-1}I \subseteq F'(R) \) by Corollary 4.2 of [9].

(2) If \((I: J) \subseteq Q \), then it is an \( \Lambda \)-ideal of \( Q \) and \( J \subseteq J \subseteq I \). So \((J(I: J)) \subseteq (I: J) \subseteq I \). Hence \((I: J) \subseteq I \), so that \((I: J) = (I: J) \).

(3) It is clear that \( O_L(J) = O_L(I) \) and so \( O_L(J) = O_L(I) \) by Lemma 1. Since \((I: J) \subseteq (J^{-1}I^{-1}) = S \), where \( S = O_L(I) \), we get \((J^{-1}I^{-1}) \subseteq J^{-1}I^{-1} \). Let \( x \) be any element of \((J^{-1}I^{-1}) \). Then \( Jx \subseteq (IJ)x \subseteq S \). Let \( T = O_L(J) \). Then \( Jx + T \subseteq I \) is a left \( T \)-ideal and \( IJ(Tx + J^{-1}I^{-1}) \subseteq S \). Hence \((I: J) \subseteq (I: J) \) by Lemma 2. By multiplying \((J^{-1}I^{-1}) \) to the both side of the inequality we have \( x \in (Tx + J^{-1}I^{-1}) \subseteq J^{-1}I^{-1} \). Therefore we get \((J^{-1}I^{-1}) \subseteq J^{-1}I^{-1} \).

(4) From Lemma 2, we have: \((I \cup J) \circ L = [(I + J)L]^* = [(I + J)L]^* = (IL + JL)^* = [(IL)^* \cup (JL)^*]^* = I \circ L \cup J \circ L \).

Let \( R \) be a maximal order. We consider the following condition:

\((A)\): \( F'(R) \) and \( F^*(R) \) both satisfy the maximum condition.

If \( R \) is a maximal order satisfying the condition \((A) \), then \( F^*(R) \) is a direct product of infinite cyclic groups with prime \( \nu \)-ideals as their generators by Theorem 4.2 of [2]. It is evident that an element \( P \) in \( F^*(R) \) is a prime element in the lattice if and only if it is a prime ideal of \( R \).

Following [1], \( R \) is said to be regular if every integral one-sided \( R \)-ideal contains a non-zero \( R \)-ideal.
Lemma 4. Let $R$ be a regular maximal order satisfying the condition (A) and let $P$ be a non-zero prime ideal of $R$. Then $P$ is a minimal prime ideal of $R$ if and only if it is a prime $v$-ideal.

Proof. Assume that $P$ is a minimal prime ideal. Let $c$ be any regular element in $P$. Then since $(cR)^* = cR$ and $R$ is regular, we get $P \supseteq P_0 \supseteq (P_i^*)^* \supseteq (P_i^*)^*$, where $P_i$ is a prime $v$-ideal. Hence $P \supseteq P_i$ for some $i$ and so $P = P_i$. Conversely assume that $P \supseteq P_0 \neq 0$, where $P_0$ is a prime ideal. Then since $P_0^* (P_0^* P_0) = P_0^* P_0 R = P_0$ and $P_0^* P_0 \subseteq P_0$, we have $P_0^* \subseteq P_0$ and thus $P_0^* = P_0$. It follows that $P_0$ is a maximal element in $F'(R)$ by [2, p. 11], a contradiction. Hence $P$ is a minimal prime ideal of $R$.

Remark. Let $R$ be a maximal order satisfying the condition (A). Then it is evident from the proof of the lemma that prime $v$-ideals are minimal prime ideals of $R$.

Let $I$ be any right ideal of $R$. Then we denote by $\sqrt{I}$ the set $\cup \{ (s^{-1}I:R), s \in I, s \in R \}$. Following [3], if $\sqrt{I}$ is an ideal of $R$, then we say that $I$ is primal and that $\sqrt{I}$ is the adjoint ideal of it. A right ideal $I$ of $R$ is called primary if $JA \subseteq I$ and $J \not\subseteq I$ implies that $A^* \subseteq I$ for some positive integer $n$, where $J$ is a right ideal of $R$ and $A$ is an ideal of $R$. We shall apply these concepts for integral right $v$-ideals.

Lemma 5. Let $R$ be a maximal order satisfying the condition (A) and let $I$ be a meet-irreducible element in $F'(R)$. Then $I$ is primal, and $\sqrt{I}$ is a minimal prime ideal of $R$ or $0$, and $\sqrt{I} = (x^{-1}I : R)$, for some $x \in I$.

Proof. If $\sqrt{I} = 0$, then the assertion is evident. Assume that $\sqrt{I} \neq 0$. By Lemma 3, $(s^{-1}I : R)$ is a $v$-ideal or 0. Hence the set $S = \{(s^{-1}I : R), s \in I, s \in R \}$ has a maximal element. Assume that $(s^{-1}I : R)$, and $(t^{-1}I : R)$, are maximal elements in $S$. Then $(sR+I)(s^{-1}I : R), I$ implies that $(sR+I)(s^{-1}I : R) \subseteq I$ by Lemma 2 and so $(s^{-1}I : R), \subseteq (I : (sR+I)^*)$. The converse inclusion is clear. Thus we have $(s^{-1}I : R), = (I : (sR+I)^*)$. Similarly $(t^{-1}I : R), = (I : (tR+I)^*)$. Since $I$ is irreducible in $F'(R)$, we have $I \subseteq (sR+I)^* \cap (tR+I)^* = I$. Let $x$ be any element in $J$ but not in $I$. Then it follows that $(x^{-1}I : R), \subseteq (s^{-1}I : R), (t^{-1}I : R), \sigma$, so that $\sqrt{I} = (x^{-1}I : R), = (s^{-1}I : R),$, which is a $v$-ideal. Hence $I$ is primal. If $AB \subseteq \sqrt{I}$ and $A \not\subseteq \sqrt{I}$, where $A$ and $B$ are ideals of $R$, then $xA \subseteq I$ and $xA \not\subseteq I$. Let $y$ be any element in $xA$ but not in $I$. Then $yB \subseteq I$ and so $B \subseteq (y^{-1}I : R), \subseteq \sqrt{I}$. Thus $\sqrt{I}$ is a prime ideal of $R$. It follows that $\sqrt{I}$ is minimal from the remark to Lemma 4.

A right ideal of $R$ is said to be bounded if it contains a non-zero ideal of $R$. 
Lemma 6. Let $R$ be a maximal order satisfying the condition (A) and let $I$ be an irreducible element in $F(R)$. If $I$ is bounded, then it is primary and $(\sqrt{I})^n \subseteq I$ for some positive integer $n$.

Proof. Since $I \in F(R)$ and is bounded, $(I:R)$ is non-zero and is a $v$-ideal. Write $(I:R) = \left( (P_1^*)^\cdot \cdot \cdot (P_k^*)^* \right)^*$, where $P_i$ are prime $v$-ideals. For any $i$ ($1 \leq i \leq k$), we let $B_i = (P_1^*)^\cdot \cdot \cdot (P_{i-1}^*)^* o (P_{i+1}^*)^* \cdot \cdot \cdot o (P_k^*)^*$. Then $B_i \subseteq I$ and $B_i P_i \subseteq (I:R)$, because $F^*(R)$ is an abelian group. Thus $P_i \subseteq \sqrt{I}$ and so $\sqrt{I} = \sqrt{I} (1 \leq i \leq k)$ by Lemma 5. Therefore $(\sqrt{I})^n \subseteq I$. It is evident that $I$ is primary.

If $A$ is an ideal of $R$, then we denote by $C(A)$ those elements of $R$ which are regular mod $(A)$.

Lemma 7. Let $R$ be a maximal order satisfying the condition (A). Let $P$ be a prime $v$-ideal. Then

(1) $C(P) = C((P^n)^*)$ for every positive integer $n$.

(2) $C(P) \subseteq C(0)$.

Proof. (1) We shall prove by the induction on $n (>1)$. Assume that $C(P) = C((P^n)^*)$. If $cx \in (P^n)^*$, where $c \in C(P)$ and $x \in R$, then $cx (P^{-1})^{n-1} \subseteq (P^n)^*(P^{-1})^{n-1} \subseteq P$ by Lemma 2. Since $cx \in (P^n)^*$, we get $x \in (P^{-1})^{n-1}$ and so $x(P^{-1})^{n-1} \subseteq P$. Hence $x(P^{-1})^{n-1} \subseteq P$. Then we have $(xR + P^n)(P^{-1})^{n-1}P^{-1} \subseteq P$ so that $x \in (P^n)^*$ by Lemma 2. Conversely suppose that $cx \in P$, $c \in C((P^n)^*)$, $x \in R$. Then $cx P^{n-1} \subseteq (P^n)^*$ and so $cx P^{n-1} \subseteq (P^n)^*$. Since $(xP + P^n) P^{n-1}(P^{-1})^{n-1} \subseteq (P^n)^*(P^{-1})^{n-1} \subseteq P$, we get $x \in P$ by Lemma 2. Therefore $C(P) = C((P^n)^*)$.

(2) If $0 \neq \cap_n (P^n)^*$, then it is a $v$-ideal by Lemma 2. Write $\cap_n (P^n)^* = (P_1^*)^\cdot \cdot \cdot o (P_k^*)^*$, where $P_i$ are prime $v$-ideals. This is a contradiction, because $F^*(R)$ is an abelian group and $P_i$ are minimal prime ideals of $R$. Hence $0 = \cap_n (P^n)^*$. Therefore (2) follows from (1).

If $P$ is a prime ideal of a ring $S$, then the family $T_P = \{ I : \text{right ideal } | s^{-1}I \cap C(P) \neq 0 \}$ for any $s \in S$ is a right additive topology (cf. Ex. 4 of [12, p. 18]). The following lemma is due to Lambek and Michler if $S$ is right noetherian. However, only trivial modifications to their proof are needed to establish the more general result.

Lemma 8. Let $P$ be a prime ideal of $S$ and let $\overline{S} = S/P$ be a right prime Goldie ring. Then the torsion theory determined by the $S$-injective hull $E(\overline{S})$ of $\overline{S}$ coincides with one determined by the right additive topology $T_P$, that is, a right ideal $I$ of $S$ is an element in $T_P$ if and only if $\text{Hom}_S(S/I, E(\overline{S})) = 0$ (Corollary 3.10 of [8]).

Lemma 9. Let $R$ be a maximal order satisfying the condition (A) and let $P$
be a prime \( v \)-ideal such that \( \overline{R} = R/P \) is a prime Goldie ring. If \( I \) is any element in \( F'_i(R) \) such that \( R \ni I \supseteq P \), then \( I \cap C(P) = \phi \).

Proof. It is enough to prove the lemma when \( I \) is a maximal element \( F'_i(R) \). Since \( I^{-1} \ni R, P \ni I^{-1} \cap R \supseteq P \). If \( P \ni I^{-1} \cap R = P \), then \( P^{-1} = (P \ni I^{-1})^{-1} \cup *R \), because the mapping: \( J \rightarrow J^{-1} \) is an inverse lattice isomorphism between \( F'_i(R) \) and \( F'_i(R) \). By Lemma 3, \( P^{-1} = I \ni P^{-1} \cup *R \). On the other hand \( P \subseteq I \) implies that \( R \subseteq I \ni P^{-1} \). Hence \( P^{-1} = I \ni P^{-1} \) and so \( R = I \), a contradiction. Thus we have \( P \ni I^{-1} \cap R \supseteq P \). Let \( a \) be any element in \( P \ni I^{-1} \cap R \) but not in \( P \). Then \( a \subseteq \ni (P \ni I^{-1}) \ni \subseteq P \ni I^{-1} \ni I = P \) so that \( I \ni a^{-1}P \supseteq R \). Since \( a^{-1}P \) is a right \( v \)-ideal by Lemma 3, we get \( I = a^{-1}P \). Then \( \text{Hom}(R/I, E(R)) \neq 0 \), because \( R/I = a^{-1}P \ni (aR + P)/P \ni R \). Now assume that \( I \cap C(P) = \phi \) and let \( e \) be any element in \( I \cap C(P) \). Then \( eR + P \in T \), by Lemma 3.1 of [6]. Hence \( I \subseteq T \), and thus \( \text{Hom}(R/I, E(R)) = 0 \) by Lemma 8. This is a contradiction and so \( I \cap C(P) = \phi \).

For convenience, we write \( M(p) \) for the family of minimal prime ideals of \( R \). If \( R \) is a regular maximal order satisfying the condition (A), then we know from Lemma 4 that a prime ideal \( P \) is an element in \( M(p) \) if and only if it is a prime element in \( F'(R) \).

**Lemma 10.** Let \( R \) be a regular maximal order satisfying the condition (A), \( P \subseteq M(p) \) and let \( I \in F'_i(R) \). If \( \overline{R} = R/P \) is a prime Goldie ring, then \( I \ni *P = R \) if and only if \( I \ni B \) such that \( B \subseteq P \).

Proof. Assume that \( I \ni B \), where \( B \) is an ideal not contained in \( P \). Then \( I \ni B * \) and \( *P \ni B * = R \), because \( P \) is a maximal element in \( F'(R) \) (cf. [2, p. 11]). Therefore \( I \ni *P = R \). Conversely assume that the family \( S = \{ I \in F'_i(R) \mid I \ni *P = R, I \ni P \text{ and } I \ni B \text{ for any ideal } B \text{ not contained in } P \} \) is not empty and let \( I \) be a maximal element in \( S \). If \( I \) is irreducible in \( F'_i(R) \), then there exists \( P' \) in \( M(p) \) such that \( I \ni P' \) by Lemmas 5 and 6. Since \( I \subseteq S \), we have \( P = P' \). If \( n = 1 \), then \( R = I \ni *P = I \), a contradiction. We may assume that \( I \ni P \ni 1 \) and \( n > 1 \). Then \( (P \ni 1)^* = (I \ni *P \ni (P \ni 1)^*) = I \ni (P \ni 1)^* \ni (P \ni *) \ni I \approx I \) by Lemmas 2 and 3. This is a contradiction. If \( I \) is reducible, then \( I = I_1 \cap I_2 \), where \( I \ni F'_i(R) \) and \( I \ni I_i (i = 1, 2) \). There are non zero ideals \( B_i (\ni P) \) such that \( I_i \ni B_i \). Thus \( I \) contains the ideal \( B_1B_2 \) not contained in \( P \), a contradiction. Hence \( S = \phi \). This implies that if \( I \ni *P = R \), then \( I \) contains an ideal not contained in \( P \).

Let \( P \) be a prime ideal of a ring \( S \). If \( S \) satisfies the Ore condition with respect to \( C(P) \), then we denote by \( S_P \) the quotient ring with respect to \( C(P) \).

**Lemma 11.** Let \( R \) be a regular maximal order satisfying the condition (A) and let \( P \) be an element in \( M(p) \) such that \( \overline{R} = R/P \) is a prime Goldie ring. Then
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(1) $R$ satisfies the Ore condition with respect to $C(P)$.
(2) $R_p = \lim_{\rightarrow} B^{-1}$, where $B$ ranges over all non zero ideals not contained in $P$.
(3) $R_p$ is a noetherian, local and Asano order.

Proof. (1) It is clear that $T = \lim_{\rightarrow} B^{-1}(B(\not\subseteq P); \text{ideal})$ is an overring of $R$.
Let $c$ be any element in $C(P)$. Then $c$ is regular by Lemma 7 and so $cR \subseteq P$ by Lemma 9 and so $cR$ contains an ideal not contained in $P$ by Lemma 10. Hence $c^{-1} \subseteq T$. So for any $r \in R$, $c \in C(P)$, there exists an ideal $B$ of $R$ such that $c^{-1}rB \subseteq R$. It is evident that $B \cap C(P) \not\subseteq P$. Then we have $c^{-1}rB = s$ for some $s \in R$, that is, $rd = cs$. This implies that $R$ satisfies the right Ore condition with respect to $C(P)$. The other Ore condition is shown to hold by a symmetric proof.

(2) is evident from (1).

(3) We let $P' = PR_p$. Then clearly $P' = R_p$ and $P \subseteq P' \cap R$. So we may assume that $R/R' \subseteq R_p = R_p$ as rings. By (1), $R_p$ is the quotient ring of $R$. Since $P'$ is a maximal ideal of $R_p$. Let $V'$ be any maximal right ideal of $R_p$. Suppose that $V' \not\subseteq P'$. Then $V' + P' = R_p$. Write $1 = v + pc^{-1}$, where $v \in V'$, $p \in P$ and $c \in C(P)$. Then $c = vc + p$ and so $vc = c - p \in C(P) \cap V'$. This implies that $V' = R_p$, a contradiction and so $V' \subseteq P'$. Hence $P'$ is the Jacobson radical of $R_p$. The ideal $P^{-1}P$ properly contains $P$ so that $C(P) \cap P^{-1}P \not\subseteq P$. It follows that $P^{-1}PR_p = R_p$. Similarly $R_pPP^{-1} = R_p$. Hence $P'$ is an invertible ideal of $R_p$. Therefore $R_p/P'^n$ is an artinian ring for any $n$, because $R_p$ is an artinian ring. Let $I'$ be any essential right ideal of $R_p$. It is clear that $I' = (I' \cap R)R_p$. Let $c$ be any regular element of $I' \cap R$. Then, since $cR \subseteq F'(R)$ and $R$ is regular, $cR$ contains a non zero $v$-ideal $(P_1^n) \cdots (P_k^n)$, where $P_i \subseteq M(p)$. So we get $I' \subseteq R_pP^n = P^n$. Therefore essential right ideals of $R_p$ satisfies the maximum condition. Since $R_p$ is finite dimensional in the sense of Goldie, $R_p$ is right noetherian. Similarly $R_p$ is left noetherian. Hence $R_p$ is a noetherian, local and Asano order by Proposition 1.3 of [7].

After all these preparations we now prove the following theorem which is the purpose of this paper:

Theorem. A prime Goldie ring $R$ is a bounded Krull prime ring if and only if it satisfies the following conditions:

(1) $R$ is a regular maximal order,
(2) $R$ satisfies the maximum condition for integral right and left $v$-ideals,
(3) $R/P$ is a prime Goldie ring for any $P \subseteq M(p)$.

Proof. Assume that $R = \bigcap_{i \in I} R_i$ is a bounded Krull prime ring, where $R_i$ is a noetherian, local and Asano order with unique maximal ideal $P_i$. (1) is
clear from Corollary 1.4 and Lemma 1.6 of [10]. Let $I$ be any right (left) $R$-ideal. Then $I^* = \cap IR_i (= \cap R_i I)$ by Proposition 1.10 of [10]. Since $R_i$ is noetherian, (2) follows from the condition (K3) in the definition of Krull rings. Let $P_i = P \cap R$. It follows that $\{P_i | i \in I\} = M(p)$ by Proposition 1.7 of [10] so that (3) is evident from Proposition 1.1 of [9].

It remains to prove that the conditions (1), (2) and (3) are sufficient. Let $P$ be any element in $M(p)$. Then $R$ satisfies the Ore condition with respect to $C(P)$ and $R_P$ is a noetherian, local and Asano order by Lemma 11. Hence $R_P$ is an essential overring of $R$. It is clear that $R \subseteq T = \cap R_p$, where $P \in M(p)$. To prove the converse inclusion let $x$ be any element of $T$. Then there is an ideal $B_x (\subseteq P)$ such that $xB_x \subseteq R$ by Lemma 11. Let $B$ be the sum of all ideals $B_x$. If $B^*$ is different from $R$, then $B^*$ is contained in some $P$ in $M(p)$. But $B^* \subseteq P$ so that $B^* = R$. Hence we have $x \in (xR + R) \subseteq (xR + R)^* \subseteq B^* = (xB + B)^* \subseteq R$. Thus we get $R = \cap R_p$. Let $c$ be any regular element in $R$. Then $cR$ contains a v-ideal $(P_1)^* \cdots (P_k)^*$, where $P_i \in M(p)$. It follows that $cR_p = R_p$ for every $P \in M(p)$ different to $P_i (1 \leq i \leq k)$ by Lemma 11. Hence $R$ is a bounded Krull prime ring. This completes the proof of the theorem.

**Corollary.** Let $R$ be a regular, noetherian and prime ring. If $R$ is a maximal order, then it is a bounded Krull prime ring.

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**References**


