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A CHARACTERIZATION OF BOUNDED KRULL PRIME RINGS

HIDETOSHI MARUBAYASHI

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In [9] we defined the concept of non commutative Krull prime rings from the point of view of localizations and we mainly investigated the ideal theory in bounded Krull prime rings (cf. [9], [10]).

The purpose of this paper is to prove the following:

Theorem. Let $R$ be a prime Goldie ring with two-sided quotient ring $Q$. Then $R$ is a bounded Krull prime ring if and only if it satisfies the following conditions:

1. $R$ is a regular maximal order in $Q$ (in the sense of Asano).
2. $R$ satisfies the maximum condition for integral right and left $v$-ideals.
3. $R/P$ is a prime Goldie ring for any minimal prime ideal $P$ of $R$.

As corollary we have

Corollary. Let $R$ be a noetherian prime ring. If $R$ is a regular maximal order in $Q$, then it is a bounded Krull prime ring.

In case $R$ is a commutative domain, the theorem is well known and its proof is easy (cf. [11]). We shall prove the theorem by using properties of one-sided $v$-ideals and torsion theories.

Throughout this paper let $R$ be a prime Goldie ring, not artinian ring, having identity element 1, and let $Q$ be the two-sided quotient ring of $R$; $Q$ is a simple and artinian ring. We say that $R$ is an order in $Q$. If $R_1$ and $R_2$ are orders in $Q$, then they are called equivalent (in symbol: $R_1 \sim R_2$) if there exist regular elements $a_1, b_1, a_2, b_2$ of $Q$ such that $a_1R_1b_1 \subseteq R_2$, $a_2R_2b_2 \subseteq R_1$. An order in $Q$ is said to be maximal if it is a maximal element in the set of orders which are equivalent to $R$. A right $R$-submodule $I$ of $Q$ is called a right $R$-ideal provided $I$ contains a regular element of $Q$ and there is a regular element $b$ of $Q$ such that $bI \subseteq R$. $I$ is called integral if $I \subseteq R$. Left $R$-ideals are defined in a similar way.

If $I$ is a right (left) $R$-ideal of $Q$, then $O_R(I) = \{x \in Q | xI \subseteq I\}$ is an order in $Q$ and is equivalent to $R$. Similarly $O_L(I) = \{x \in Q |Ix \subseteq I\}$ is an order in $Q$ and is equivalent to $R$. They are called a left order and a right order of $I$ respectively.
We define the inverse of \( I \) to be \( I^{-1} = \{ q \in Q \mid Iq = q \subseteq O(I) \} \). Evidently \( I^{-1} = \{ q \in Q \mid Iq \subseteq O(I) \} \). Following [2], we define \( \star(I^{-1})^{-1} \). If \( I = I^* \), then it is said to be a right (left) \( \nu \)-ideal. If \( R \) is a maximal order, then \( I^{-1} = I^{-1} \) and so \( I^{-1} \) is a left (right) \( \nu \)-ideal, and the concept of right (left) \( \nu \)-ideals coincides with one of right (left) \( \nu \)-ideals defined in [9]. So the mapping: \( I \rightarrow I^* \) of the set of all right (left) \( R \)-ideals into the set of all right (left) \( \nu \)-ideals is a \( \star \)-operation in the sense of [9].

**Lemma 1.** Let \( R \) be a maximal order in \( Q \) and let \( S \) be any order equivalent to \( R \). Then \( S \) is a maximal order if and only if \( S = O(I) \) for some right \( \nu \)-ideal \( I \) of \( Q \).

Proof. If \( S = O(I) \) for some right \( \nu \)-ideal \( I \) of \( Q \), then it is a maximal order by Satz 1.3 of [1]. Conversely assume that \( S \) is a maximal order, then there are regular elements \( c, d \) in \( R \) such that \( cdR \subseteq R \). So \( cdR \) is a right \( R \)-ideal and is a left \( S \)-module. Hence \( (cdR)^{-1} \) is a left \( R \)-ideal and is a right \( S \)-module. Similarly \( I = (cdR)^{-1} \) is a right \( \nu \)-ideal and is a left \( S \)-module so that \( O(I) \subseteq S \). Hence \( S = O(I) \).

**Lemma 2.** Let \( R, S \) be maximal orders in \( Q \) such that \( R \sim S \), and let \( \{I_i\}, I \) be right \( R \)-ideals. Then

1. If \( \bigcap_i I_i \) is a right \( R \)-ideal, then \( \bigcap_i I_i^* = (\bigcap_i I_i^*)^* \).
2. If \( \sum I_i \) is a right \( R \)-ideal, then \( (\sum I_i)^* = (\sum I_i^*)^* \).
3. If \( J \) is a left \( R \) and right \( S \)-ideal, then \( (IJ)^* = (J^* I)^* = (I^* J)^* \).
4. \( (I^{-1} I^*)^* = R \) and \( (I^* I^{-1})^* = T \), where \( T = O(I^*) \).

Proof. The proofs of (1) and (2) are similar to ones of the corresponding results for commutative rings (cf. Proposition 26.2 of [4]).

To prove (3) assume that \( I \subseteq cS \), where \( c \) is a unit in \( Q \). Then we have \( (I^* J)^* \subseteq cS \) and \( (IJ)^* \subseteq cS \), because \( c^{-1} I \subseteq S \Rightarrow c^{-1} I \subseteq J^{-1} \Rightarrow c^{-1} I^* \subseteq J^{-1} J^* = (c^{-1} I)^* \subseteq J^{-1} J^* \subseteq S \Rightarrow I^* J \subseteq cS \). Hence \( (IJ)^* \) contains \( (I^* J)^* \) and \( (IJ)^* \) by Proposition 4.1 of [9]. The converse inclusions are clear. Therefore we have \( (IJ)^* = (I^* J)^* = (IJ)^* \). From these it is clear that \( (IJ)^* = (I^* J)^* \).

To prove (4), assume that \( I^{-1} I^* \subseteq cR \), where \( c \) is a unit in \( Q \). Then we have \( c^{-1} I^{-1} \subseteq I \) so that \( c^{-1} \subseteq O(I^{-1}) = R \) and thus \( R \subseteq cR \). Hence \( (I^{-1} I^*)^* = R \) by Proposition 4.1 of [9]. The converse inclusion is clear. Therefore \( (I^{-1} I^*)^* = R \).

Let \( R \) be a maximal order in \( Q \). We denote by \( F^*(R) (F^*(R)) \) the set of right (left) \( \nu \)-ideals and let \( F^*(R) = F^*(R) \cap F^*(R) \). It is clear that \( F^*(R) \) becomes a lattice by the definition; if \( I, J \in F^*(R) \), then \( I \cup J = (I \cup J)^* \), and the meet "\( \cap \)" is the set-theoretic intersection. Similarly \( F^*(R) \) and \( F^*(R) \) also become
lattices. For any \( I \in F^+_*(R) \) and \( L \in F^+_*(R) \), we define the product "\( \circ \)" of \( I \) and \( L \) by \( I \circ L = (IL)^* \). It is clear that \( I \circ L \subseteq F^+_*(S) \cap F^+_*(T) \), where \( S = O_1(I) \) and \( T = O_2(L) \). In particular, the semi-group \( F^*_*(R) \) becomes an abelian group (cf. Theorem 4.2 of [2]). For convenience we write \( F'_*(R) \) for the sublattice of \( F^*_*(R) \) consisting of all integral right \( \triangledown \)-ideals. Similarly we write \( F_{\triangledown}^*(R) \) and \( F'_{\triangledown}(R) \) for the corresponding sublattices of \( F^*(R) \) and \( F^*(R) \) respectively. Let \( M \) and \( N \) be subsets of \( R \). Then we use the following notations: \( (M: N)_\triangledown = \{ x \in R | Nx \subseteq M \} \), \( (M: N)_\triangledown = \{ x \in R | xN \subseteq M \} \). When \( N \) is a single element \( q \) of \( R \), then we denote by \( q^{-1}M \) the set \( (M: N)_\triangledown \).

**Lemma 3.** Let \( R \) be a maximal order in \( Q \). Then

1. If \( I \in F^+_*(R) \) and \( q \in Q \), then \( q^{-1}I = (I^{-1}q + R)^{-1} \) and so \( q^{-1}I \in F'_*(R) \).
2. If \( I \in F^+_*(R) \) and \( J \) is a right \( R \)-ideal, then \( (I:J)_\triangledown \subseteq F'_*(R) \) or \( 0 \).
3. If \( I \in F^+_*(R) \) and \( J \subseteq F^*(R) \), then \( (I \circ J)^{-1} = J^{-1}I^{-1} \).
4. If \( I, J \in F^+_*(R) \) and \( L \subseteq F^*_*(R) \), then \( (I \cup J) \circ L = I \circ L \cup J \circ L \).

Proof. (1) Since \( (I^{-1}q + R)^{-1}q^{-1}I \subseteq R \), we get \( (I^{-1}q + R)^{-1} \supseteq q^{-1}I \). Let \( x \) be any element of \( (I^{-1}q + R)^{-1} \). Then \( (I^{-1}q + R)x \subseteq R \) so that \( x \in R \) and \( I^{-1}q \subseteq R \). Let \( S = O_1(I) \). Then it is a maximal order equivalent to \( R \) by Lemma 1. It is evident that \( Sx + I \) is a left \( S \)-ideal and that \( \triangledown^{-1}(Sx + I) \subseteq I \). Thus, by Lemma 2, we have

\[
q^{-1}I \subseteq (II^{-1})^{-1}S(Sx + I) \subseteq (II^{-1}S(Sx + I))^{-1} \subseteq (II^{-1}(Sx + I))^{-1} \subseteq I.
\]

Hence \( x \in q^{-1}I \) and so \( q^{-1}I \in F'_*(R) \) by Corollary 4.2 of [9].

(2) If \( (I:J)_\triangledown \neq 0 \), then it is an \( R \)-ideal of \( Q \) and \( J(I:J)_\triangledown \subseteq I \). So \( J((I:J)_\triangledown)^* \subseteq (J(I:J)_\triangledown)^* \subseteq I \). Hence \( (I:J)_\triangledown \subseteq (I:J)^* \subseteq ((I:J)_\triangledown)^* \).

(3) It is clear that \( O_1(I \circ J) \supseteq O_1(I) \) and so \( O_1(I \circ J) = O_1(I) \) by Lemma 1. Since \( (I \circ J)^* (J^{-1}I^{-1}) = S \), where \( S = O_1(I) \), we get \( (I \circ J)^{-1} \supseteq J^{-1}I^{-1} \). Let \( x \) be any element of \( (I \circ J)^{-1} \). Then \( Ix \subseteq I \circ J \subseteq S \). Let \( T = O_1(J) \). Then \( Tx + J^{-1}I^{-1} \) is a left \( T \)-ideal and \( I(J(Tx + J^{-1}I^{-1})) \subseteq S \). Hence \( I \circ J^*(Tx + J^{-1}I^{-1}) \subseteq S \) by Lemma 2. By multiplying \( J^{-1}I^{-1} \) to the both side of the inequality we have \( x \subseteq (Tx + J^{-1}I^{-1}) \subseteq J^{-1}I^{-1} \). Therefore we get \( (I \circ J)^{-1} = J^{-1}I^{-1} \).

(4) From Lemma 2, we have: \( (I \cup J)^* \circ L = [(I + J)^* L]^* = [(I + J)L]^* = (IL + JL)^* = [(IL)^* + (JL)^*] = I \circ L \cup J \circ L \).

Let \( R \) be a maximal order. We consider the following condition:

\( (A): F'_*(R) \) and \( F^*(R) \) both satisfy the maximum condition.

If \( R \) is a maximal order satisfying the condition \( (A) \), then \( F^*(R) \) is a direct product of infinite cyclic groups with prime \( \triangledown \)-ideals as their generators by Theorem 4.2 of [2]. It is evident that an element \( P \) in \( F'(R) \) is a prime element in the lattice if and only if it is a prime ideal of \( R \).

Following [1], \( R \) is said to be regular if every integral one-sided \( R \)-ideal contains a non-zero \( R \)-ideal.
Lemma 4. Let $R$ be a regular maximal order satisfying the condition (A) and let $P$ be a non-zero prime ideal of $R$. Then $P$ is a minimal prime ideal of $R$ if and only if it is a prime $v$-ideal.

Proof. Assume that $P$ is a minimal prime ideal. Let $c$ be any regular element in $P$. Then since $(cR)^* = cR$ and $R$ is regular, we get $P \supseteq cR \supseteq (P_i)^* \circ \ldots \circ (P_0)^*$, where $P_i$ is a prime $v$-ideal. Hence $P \supseteq P_i$ for some $i$ and so $P = P_i$. Conversely assume that $P \supseteq P_0 \neq 0$, where $P_0$ is a prime ideal. Then since $P_0^*(P_0^*P_0) = (P_0^*P_0)R_0 = P_0$ and $P_0 = P_0^*P_0$, we have $P_0^* \subseteq P_0$ and thus $P_0^* = P_0$. It follows that $P_0$ is a maximal element in $\mathcal{F}'(R)$ by [2, p. 11], a contradiction. Hence $P$ is a minimal prime ideal of $R$.

Remark. Let $R$ be a maximal order satisfying the condition (A). Then it is evident from the proof of the lemma that prime $v$-ideals are minimal prime ideals of $R$.

Let $I$ be any right ideal of $R$. Then we denote by $\sqrt{I}$ the set $\cup \{ (s^{-1}I:R), |s \in I, s \in R \}$. Following [3], if $\sqrt{I}$ is an ideal of $R$, then we say that $I$ is primal and that $\sqrt{I}$ is the adjoint ideal of it. A right ideal $I$ of $R$ is called primary if $JA \subseteq I$ and $J \not\subseteq I$ implies that $A^* \subseteq I$ for some positive integer $n$, where $J$ is a right ideal of $R$ and $A$ is an ideal of $R$. We shall apply these concepts for integral right $v$-ideals.

Lemma 5. Let $R$ be a maximal order satisfying the condition (A) and let $I$ be a meet-irreducible element in $\mathcal{F}'(R)$. Then $I$ is primal, and $\sqrt{I}$ is a minimal prime ideal of $R$ or 0, and $\sqrt{I} = (x^{-1}I:R)$, for some $x \in I$.

Proof. If $\sqrt{I} = 0$, then the assertion is evident. Assume that $\sqrt{I} \neq 0$. By Lemma 3, $(s^{-1}I:R)$ is a $v$-ideal or 0. Hence the set $S = \{ (s^{-1}I:R), |s \in I, s \in R \}$ has a maximal element. Assume that $(s^{-1}I:R)$, and $(t^{-1}I:R)$, are maximal elements in $S$. Then $(sR + I)(s^{-1}I:R) \subseteq I$ implies that $(sR + I)^*(s^{-1}I:R) \subseteq I$ by Lemma 2 and so $(s^{-1}I:R) \subseteq (I:(sR + I)^*)$. The converse inclusion is clear. Thus we have $(s^{-1}I:R) = (I:(sR + I)^*)$. Similarly $(t^{-1}I:R) = (I:(tR + I)^*)$. Since $I$ is irreducible in $\mathcal{F}'(R)$, we have $I \subseteq (sR + I)^* \cap (tR + I)^* = J$. Let $x$ be any element in $J$ but not in $I$. Then it follows that $(x^{-1}I:R), \supseteq (s^{-1}I:R), (t^{-1}I:R)$, so that $\sqrt{I} = (x^{-1}I:R) = (s^{-1}I:R)^*$, which is a $v$-ideal. Hence $I$ is primal. If $AB \subseteq \sqrt{I}$ and $A \subseteq \sqrt{I}$, where $A$ and $B$ are ideals of $R$, then $xAB \subseteq I$ and $xA \subseteq I$. Let $y$ be any element in $xA$ but not in $I$. Then $yB \subseteq I$ and so $B \subseteq (y^{-1}I:R) \subseteq \sqrt{I}$. Thus $\sqrt{I}$ is a prime ideal of $R$. It follows that $\sqrt{I}$ is minimal from the remark to Lemma 4.

A right ideal of $R$ is said to be bounded if it contains a non-zero ideal of $R$. 
Lemma 6. Let $R$ be a maximal order satisfying the condition $(A)$ and let $I$ be an irreducible element in $F'(R)$. If $I$ is bounded, then it is primary and $(\sqrt{I})^n \subseteq I$ for some positive integer $n$.

Proof. Since $I \in F'(R)$ and is bounded, $(I:R)_r$ is non-zero and is a $v$-ideal. Write $(I:R)_r = (P_1^n)^* \cdot \cdots \cdot (P_k^n)^*$, where $P_i$ are prime $v$-ideals. For any $i$ ($1 \leq i \leq k$), we let $B_i = (P_i^n)^* \cdot \cdots \cdot (P_{i-1}^n)^* \cdot (P_i^{n+1})^* \cdot \cdots \cdot (P_k^n)^*$. Then $B_i \notin I$ and $B_i P_i \subseteq (I:R)_r \subseteq I$, because $F^*(R)$ is an abelian group. Thus $P_i^n \subseteq \sqrt{I}$ and so $P_i = \sqrt{I}$ ($1 \leq i \leq k$) by Lemma 5. Therefore $(\sqrt{I})^{n+\cdots+n} \subseteq I$. It is evident that $I$ is primary.

If $A$ is an ideal of $R$, then we denote by $C(A)$ those elements of $R$ which are regular mod $(A)$.

Lemma 7. Let $R$ be a maximal order satisfying the condition $(A)$. Let $P$ be a prime $v$-ideal. Then

1. $C(P) = C((P^n)^*)$ for every positive integer $n$.
2. $C(P) \subseteq C(0)$.

Proof. (1) We shall prove by the induction on $n$ ($>1$). Assume that $C(P) = C((P^{n-1})^*)$. If $cx \in (P^n)^*$, where $c \in C(P)$ and $x \in R$, then $cx(P^{n-1}) \subseteq (P^n)^*(P^{n-1})^* \subseteq P$ by Lemma 2. Since $cx \in (P^{n-1})^*$, we get $x \in (P^{n-1})^*$ and so $x(P^{n-1})^* \subseteq P$. Hence $x(P^{n-1})^* \subseteq P$. Then we have $(xR + P^*) (P^{n-1})^* \subseteq P^*$ so that $x \in (P^n)^*$ by Lemma 2. Conversely suppose that $cx \in P$, $c \in C((P^n)^*)$, $x \in R$. Then $cxP^{n-1} \subseteq (P^n)^*$ and so $xP^{n-1} \subseteq (P^n)^*$. Since $(xP + P^n)P^{n-1} \subseteq (P^n)^*(P^{n-1})^* \subseteq P^*$, we get $x \in P$ by Lemma 2. Therefore $C(P) = C((P^n)^*)$.

(2) If $0 \neq \cap_n (P^n)^*$, then it is a $v$-ideal by Lemma 2. Write $\cap_n (P^n)^* = (P_1^n)^* \cdot \cdots \cdot (P_k^n)^*$, where $P_i$ are prime $v$-ideals. This is a contradiction, because $F^*(R)$ is an abelian group and $P_i$ are minimal prime ideals of $R$. Hence $0 = \cap_n (P^n)^*$. Therefore (2) follows from (1).

If $P$ is a prime ideal of a ring $S$, then the family $T_P = \{I: \text{right ideal } s^{-1}I \cap C(P) \neq \emptyset \text{ for any } s \in S\}$ is a right additive topology (cf. Ex. 4 of [12, p. 18]). The following lemma is due to Lambek and Michler if $S$ is right noetherian. However, only trivial modifications to their proof are needed to establish the more general result.

Lemma 8. Let $P$ be a prime ideal of $S$ and let $\bar{S} = S/P$ be a right prime Goldie ring. Then the torsion theory determined by the $S$-injective hull $E(\bar{S})$ of $\bar{S}$ coincides with one determined by the right additive topology $T_P$, that is, a right ideal $I$ of $S$ is an element in $T_P$ if and only if $\text{Hom}_S(S/I, E(\bar{S})) = 0$ (Corollary 3.10 of [8]).

Lemma 9. Let $R$ be a maximal order satisfying the condition $(A)$ and let $P$
be a prime $v$-ideal such that $\overline{R}=R/P$ is a prime Goldie ring. If $I$ is any element in $F' \cap C(R)$ such that $R \cong I \supseteq P$, then $I \cap C(P) = \emptyset$.

Proof. It is enough to prove the lemma when $I$ is a maximal element $F' \cap C(R)$. Since $I^{-1} \cong R$, $P \circ I^{-1} \cap R \supseteq P$. If $P \circ I^{-1} \cap R = P$, then $P^{-1} = (P \circ I^{-1})^{-1} \cup *R$, because the mapping: $J \rightarrow J^{-1}$ is an inverse lattice isomorphism between $F'_\alpha(R)$ and $F'_\beta(R)$. By Lemma 3, $P^{-1} = I \circ P^{-1}$ and so $R = I$, a contradiction. Thus we have $P \circ I^{-1} \cap R \supsetneq P$. Let $a$ be any element in $P \circ I^{-1} \cap R$ but not in $P$. Then $a \subseteq (P \circ I^{-1}) \subseteq P \circ I^{-1} \cap I = P$ so that $I \subseteq a^{-1}P \subseteq R$. Since $a^{-1}P$ is a right $v$-ideal by Lemma 3, we get $I = a^{-1}P$. Then $\text{Hom}(R/I, E(\overline{R})) \neq 0$, because $R/I = (aR + P)/P \subseteq \overline{R}$. Now assume that $I \cap C(P) \neq \emptyset$ and let $c$ be any element in $I \cap C(P)$. Then $cR + P \subseteq T$, by Lemma 3.1 of [6]. Hence $I \subseteq T$ and thus $\text{Hom}(R/I, E(\overline{R})) = 0$ by Lemma 8. This is a contradiction and so $I \cap C(P) = \emptyset$.

For convenience, we write $M(p)$ for the family of minimal prime ideals of $R$. If $R$ is a regular maximal order satisfying the condition (A), then we know from Lemma 4 that a prime ideal $P$ is an element in $M(p)$ if and only if it is a prime element in $F'(R)$.

**Lemma 10.** Let $R$ be a regular maximal order satisfying the condition (A), $P \in M(p)$ and let $I \in F'(R)$. If $\overline{R}=R/P$ is a prime Goldie ring, then $I \cup *P = R$ if and only if $I$ contains an ideal $B$ such that $B \supseteq P$.

Proof. Assume that $I \supseteq B$, where $B$ is an ideal not contained in $P$. Then $I \supseteq B^*$ and $B^* \cup *P = R$, because $P$ is a maximal element in $F'(R)$ (cf. [2, p. 11]). Therefore $I \cup *P = R$. Conversely assume that the family $S = \{ I \in F'(R) | I \cup *P = R, I \neq R \}$ is not empty and let $I$ be a maximal element in $S$. If $I$ is irreducible in $F'(R)$, then there exists $P'$ in $M(p)$ such that $I \supseteq P'$ by Lemmas 5 and 6. Since $I \subseteq S$, we have $P = P'$. If $n=1$, then $R = I \cup *P = I$, a contradiction. We may assume that $I \supseteq P^{n-1}$ and $n > 1$. Then $(P^{n-1})^* = (I \cup *P) \circ (P^{n-1})^* = I \circ (P^{n-1})^* \cup *(P^*)^* \subseteq I^* = I$ by Lemmas 2 and 3. This is a contradiction. If $I$ is reducible, then $I = I_1 \cap I_2$, where $I_i \in F'(R)$ and $I \supseteq I_i$ ($i=1, 2$). There are non zero ideals $B_i \supseteq P$ such that $I_i \supseteq B_i$. Thus $I$ contains the ideal $B_1B_2$ not contained in $P$, a contradiction. Hence $S = \emptyset$. This implies that if $I \cup *P = R$, then $I$ contains an ideal not contained in $P$.

Let $P$ be a prime ideal of a ring $S$. If $S$ satisfies the Ore condition with respect to $C(P)$, then we denote by $S_P$ the quotient ring with respect to $C(P)$.

**Lemma 11.** Let $R$ be a regular maximal order satisfying the condition (A) and let $P$ be an element in $M(p)$ such that $\overline{R}=R/P$ is a prime Goldie ring. Then
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(1) R satisfies the Ore condition with respect to C(P).
(2) \( R_p = \lim_{\rightarrow} B^{-1}, \) where B ranges over all non zero ideals not contained in P.
(3) \( R_p \) is a noetherian, local and Asano order.

Proof. (1) It is clear that \( T = \lim_{\rightarrow} B^{-1}(B(\mathcal{P}); \text{ ideal}) \) is an overring of \( R. \)
Let \( c \) be any element in \( C(P). \) Then \( c \) is regular by Lemma 7 and so \( c R^F = \mathbb{R}. \)
Since \( (cR \cup \mathcal{P}) \cap C(P) \neq \emptyset, \) we have \( c R \cup \mathcal{P} = R \) by Lemma 9 and so \( c R \) contains an ideal not contained in \( P \) by Lemma 10. Hence \( c^{-1} \in T. \) So for any \( r \in R, \)
\( c \in C(P), \) there exists an ideal \( (cR^F) \) such that \( c^{-1} \in R. \) It is evident that
\( B \cap C(P) \neq \emptyset. \) Let \( d \) be any element in \( B \cap C(P). \) Then we have \( c^{-1}rd = s \)
for some \( s \in R, \) that is, \( rd = cs. \) This implies that \( R \) satisfies the right Ore condition
with respect to \( C(P). \) The other Ore condition is shown to hold by a symmetric proof.

(2) is evident from (1).

(3) We let \( P' = R_pP. \) Then clearly \( P' = R_pP \) and \( P = P' \cap R. \) So we may assume that \( \tilde{R} = R/P \cong \tilde{R}_p = R/P' \) as rings. By (1), \( \tilde{R}_p \) is the quotient ring of \( \tilde{R}. \)
Since \( P' \) is a maximal ideal of \( R_p, \) \( \tilde{R}_p \) is the simple artinian ring. Hence \( P' \) is a maximal ideal of \( R_p. \) Let \( V' \) be any maximal right ideal of \( R_p. \) Suppose
that \( V' \not\subseteq P'. \) Then \( V' + P' = R_p. \) Write \( 1 = v + pc^{-1}, \) where \( v \in V', \)
\( p \in P \) and \( c \in C(P). \) Then \( c = vc + p \) and so \( vc = c - p \in C(P) \cap V'. \) This implies that
\( V' = R_p, \) a contradiction and so \( V' \subseteq P'. \) Hence \( P' \) is the Jacobson radical of \( R_p. \)
The ideal \( P^{-1}P \) properly contains \( P \) so that \( C(P) \cap P^{-1}P \neq \emptyset. \) It follows that
\( P^{-1}PR_p = R_p. \) Similarly \( R_pPP^{-1} = R_p. \) Hence \( P' \) is an invertible ideal of \( R_p. \)
Therefore \( R_p/P'^n \) is an artinian ring for any \( n, \) because \( \tilde{R}_p \) is an artinian ring.
Let \( L' \) be any essential right ideal of \( R_p. \) It is clear that \( L' = (L' \cap R)R_p. \)
Let \( c \) be any regular element of \( L' \cap R. \) Then, since \( c R \subseteq F'(R) \) and \( R \) is regular,
\( c R \) contains a non zero \( v \)-ideal \( (P'^* \cap \mathcal{P}) \). Therefore \( L' = R_pP^n = P^n. \) Now we get \( L' \subsetneq R_pP^n = R_p. \) Hence \( R_p \) is left noetherian, local and Asano order by Proposition 1.3 of [7].

After all these preparations we now prove the following theorem which is
the purpose of this paper:

Theorem. A prime Goldie ring \( R \) is a bounded Krull prime ring if and only
if it satisfies the following conditions:
(1) \( R \) is a regular maximal order,
(2) \( R \) satisfies the maximum condition for integral right and left \( v \)-ideals,
(3) \( R/P \) is a prime Goldie ring for any \( P \in M(p). \)

Proof. Assume that \( R = \cap_i R_i \) is a bounded Krull prime ring, where \( R_i \) is a noetherian, local and Asano order with unique maximal ideal \( P'_i. \) (1) is
clear from Corollary 1.4 and Lemma 1.6 of [10]. Let \( I \) be any right (left) \( R \)-ideal. Then \( I^* = \cap IR_i (= \cap R_i I) \) by Proposition 1.10 of [10]. Since \( R_i \) is noetherian, (2) follows from the condition (K3) in the definition of Krull rings. Let \( P_i = P_i \cap R \). It follows that \( \{ P_i | i \in I \} = \mathcal{M}(p) \) by Proposition 1.7 of [10] so that (3) is evident from Proposition 1.1 of [9].

It remains to prove that the conditions (1), (2) and (3) are sufficient. Let \( P \) be any element in \( \mathcal{M}(p) \). Then \( R \) satisfies the Ore condition with respect to \( C(P) \) and \( R_P \) is a noetherian, local and Asano order by Lemma 11. Hence \( R_P \) is an essential overring of \( R \). It is clear that \( R \subseteq T = \cap R_p \), where \( P \in \mathcal{M}(p) \). To prove the converse inclusion let \( x \) be any element of \( T \). Then there is an ideal \( B_P (\subseteq P) \) such that \( xB_P \subseteq R \) by Lemma 11. Let \( B \) be the sum of all ideals \( B_P \). If \( B^* \) is different from \( R \), then \( B^* \) is contained in some \( P \) in \( \mathcal{M}(p) \). But \( B^* \subseteq P \) so that \( B^* = R \). Hence we have \( x \in (xR + R) \subseteq (xR + R)^* \cdot B^* = (xB + B)^* \subseteq R \). Thus we get \( R = \cap R_p \). Let \( c \) be any regular element in \( R \). Then \( cR \) contains a \( \mathfrak{u} \)-ideal \( (P_1^*) \cdot \cdots \cdot (P_k^*) \), where \( P_i \in \mathcal{M}(p) \). It follows that \( cR_P = R_P \) for every \( P \in \mathcal{M}(p) \) different to \( P_1 (1 \leq i \leq k) \) by Lemma 11. Hence \( R \) is a bounded Krull prime ring. This completes the proof of the theorem.

**Corollary.** Let \( R \) be a regular, noetherian and prime ring. If \( R \) is a maximal order, then it is a bounded Krull prime ring.

**References**


