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Author(s)	Marubayashi, Hidetoshi
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A CHARACTERIZATION OF BOUNDED KRULL PRIME RINGS

HIDETOSHI MARUBAYASHI

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In [9] we defined the concept of non commutative Krull prime rings from the point of view of localizations and we mainly investigated the ideal theory in bounded Krull prime rings (cf. [9], [10]).

The purpose of this paper is to prove the following:

Theorem. Let R be a prime Goldie ring with two-sided quotient ring Q. Then R is a bounded Krull prime ring if and only if it satisfies the following conditions;

- (1) R is a regular maximal order in Q (in the sense of Asano).
- (2) R satisfies the maximum condition for integral right and left v-ideals.
- (3) R/P is a prime Goldie ring for any minimal prime ideal P of R.

As corollary we have

Corollary. Let R be a noetherian prime ring. If R is a regular maximal order in Q, then it is a bounded Krull prime ring.

In case R is a commutative domain, the theorem is well known and its proof is easy (cf. [11]). We shall prove the theorem by using properties of one-sided v-ideals and torsion theories.

Throughout this paper let R be a prime Goldie ring, not artinian ring, having identity element 1, and let Q be the two-sided quotient ring of R; Q is a simple and artinian ring. We say that R is an order in Q. If R_1 and R_2 are orders in Q, then they are called *equivalent* (in symbol: $R_1 \sim R_2$) if there exist regular elements a_1, b_1, a_2, b_2 of Q such that $a_1R_1b_1 \subseteq R_2, a_2R_2b_2 \subseteq R_1$. An order in Q is said to be *maximal* if it is a maximal element in the set of orders which are equivalent to R. A right R-submodule I of Q is called a *right* R-*ideal* provided I contains a regular element of Q and there is a regular element b of Q such that $bI \subseteq R$. I is called *integral* if $I \subseteq R$. Left R-ideals are defined in a similar way. If I is a right (left) R-ideal of Q, then $O_i(I) = \{x \in Q | xI \subseteq I\}$ is an order in Qand is equivalent to R. They are called a *left order* and a *right order* of I respectively. H. MARUBAYASHI

We define the inverse of I to be $I^{-1} = \{q \in Q \mid IqI \subseteq I\}$. Evidently $I^{-1} = \{q \in Q \mid Iq \subseteq O_i(I)\} = \{q \in Q \mid qI \subseteq O_r(I)\}$. Following [2], we define $I^* = (I^{-1})^{-1}$. If $I = I^*$, then it is said to be a *right (left) v-ideal*. If R is a maximal order, then $I^{-1} = I^{-1-1-1}$ and so I^{-1} is a left (right) *v*-ideal, and the concept of right (left) *v*-ideals coincides with one of right (left) *v*-ideals defined in [9]. So the mapping: $I \rightarrow I^*$ of the set of all right (left) R-ideals into the set of all right (left) *v*-ideals is a *-operation in the sense of [9].

Lemma 1. Let R be a maximal order in Q and let S be any order equivalent to R. Then S is a maximal order if and only if $S=O_i(I)$ for some right v-ideal I of Q.

Proof. If $S=O_l(I)$ for some right v-ideal I of Q, then it is a maximal order by Satz 1.3 of [1]. Conversely assume that S is a maximal order, then there are regular elements c, d in R such that $cSd \subseteq R$. So SdR is a right R-ideal and is a left S-module. Hence $(SdR)^{-1}$ is a left R-ideal and is a right S-module. Similarly $I=(SdR)^{-1-1}$ is a right v-ideal and is a left S-module so that $O_l(I) \supseteq S$. Hence $S=O_l(I)$.

Lemma 2. Let R, S be maximal orders in Q such that $R \sim S$, and let $\{I_i\}$, I be right R-ideals. Then

- (1) If $\cap_i I_i$ is a right R-ideal, then $\cap_i I_i^* = (\cap_i I_i^*)^*$.
- (2) If $\sum_{i} I_{i}$ is a right R-ideal, then $(\sum I_{i})^{*} = (\sum I_{i}^{*})^{*}$.
- (3) If J is a left R and right S-ideal, then $(IJ)^* = (I^*J)^* = (IJ^*)^* = (I^*J^*)^*$.
- (4) $(I^{-1}I^*)^* = R$ and $(I^*I^{-1})^* = T$, where $T = O_i(I^*)$.

Proof. The proofs of (1) and (2) are similar to ones of the corresponding results for commutative rings (cf. Proposition 26.2 of [4]).

To prove (3) assume that $IJ \subseteq cS$, where c is a unit in Q. Then we have $(I^*J) \subseteq cS$ and $(IJ^*) \subseteq cS$, because

 $c^{-1}IJ \subseteq S \Rightarrow c^{-1}I \subseteq J^{-1} \Rightarrow c^{-1}IJ^* \subseteq J^{-1}J^* \subseteq S \Rightarrow IJ^* \subseteq cS$, and $c^{-1}I \subseteq J^{-1} \Rightarrow c^{-1}I^*$ = $(c^{-1}I)^* \subseteq J^{-1} \Rightarrow c^{-1}I^*J \subseteq J^{-1}J \subseteq S \Rightarrow I^*J \subseteq cS$. Hence $(IJ)^*$ contains $(IJ^*)^*$ and $(I^*J)^*$ by Proposition 4.1 of [9]. The converse inclusions are clear. Therefore we have $(IJ)^* = (I^*J)^* = (IJ^*)^*$. From these it is clear that $(IJ)^* = (I^*J^*)^*$.

To prove (4), assume that $I^{-1}I^* \subseteq cR$, where *c* is a unit in *Q*. Then we have $c^{-1}I^{-1} \subseteq I^{-1}$ so that $c^{-1} \subseteq O_l(I^{-1}) = R$ and thus $R \subseteq cR$. Hence $(I^{-1}I^*)^* \supseteq R$ by Proposition 4.1 of [9]. The converse inclusion is clear. Therefore $(I^{-1}I^*)^* = R$. Similarly $(I^*I^{-1})^* = T$.

Let R be a maximal order in Q. We denote by $F_r^*(R)$ $(F_r^*(R))$ the set of right (left) v-ideals and let $F^*(R) = F_r^*(R) \cap F_r^*(R)$. It is clear that $F_r^*(R)$ becomes a lattice by the definition; if $I, J \in F_r^*(R)$, then $I \cup *J = (I+J)^*$, and the meet " \cap " is the set-theoretic intersection. Similarly $F_r^*(R)$ and $F^*(R)$ also become

lattices. For any $I \in F_r^*(R)$ and $L \in F_l^*(R)$, we define the product " \circ " of I and L by $I \circ L = (IL)^*$. It is clear that $I \circ L \in F_r^*(S) \cap F_r^*(T)$, where $S = O_i(I)$ and $T = O_r(L)$. In particular, the semi-group $F^*(R)$ becomes an abelian group (cf. Theorem 4.2 of [2]). For convenience we write $F_r'(R)$ for the sublattice of $F_r^*(R)$ consisting of all integral right v-ideals. Similarly we write $F_i'(R)$ and F'(R) for the corresponding sublattices of $F_l^*(R)$ and $F^*(R)$ respectively. Let M and N be subsets of Q. Then we use the following notations: $(M:N)_r = \{x \in R \mid Nx \subseteq M\}, (M:N)_i = \{x \in R \mid xN \subseteq M\}$. When N is a single element q of Q, then we denote by $q^{-1}M$ the set $(M:N)_r$.

Lemma 3. Let R be a maximal order in Q. Then

- (1) If $I \in F_r^*(R)$ and $q \in Q$, then $q^{-1}I = (I^{-1}q + R)^{-1}$ and so $q^{-1}I \in F_r'(R)$.
- (2) If $I \in F_r^*(R)$ and J is a right R-ideal, then $(I:J)_r \in F'(R)$ or 0.
- (3) If $I \in F_r^*(R)$ and $J \in F_l^*(R)$, then $(I \circ J)^{-1} = J^{-1} \circ I^{-1}$.
- (4) If $I, J \in F_r^*(R)$ and $L \in F_l^*(R)$, then $(I \cup *J) \circ L = I \circ L \cup *J \circ L$.

Proof. (1) Since $(I^{-1}q+R)q^{-1}I \subseteq R$, we get $(I^{-1}q+R)^{-1} \supseteq q^{-1}I$. Let x be any element of $(I^{-1}q+R)^{-1}$. Then $(I^{-1}q+R)x \subseteq R$ so that $x \in R$ and $I^{-1}qx \subseteq R$. Let $S=O_l(I)$. Then it is a maximal order equivalent to R by Lemma 1. It is evident that Sqx+I is a left S-ideal and that $II^{-1}(Sqx+I) \subseteq I$. Thus, by Lemma 2, we have

 $qx \in S(Sqx+I) \subseteq (II^{-1})^* \circ (Sqx+I)^* = (II^{-1}(Sqx+I))^* \subseteq I$. Hence $x \in q^{-1}I$ and so $q^{-1}I = (I^{-1}q+R)^{-1}$. It is clear that $q^{-1}I \in F'_{*}(R)$ by Corollary 4.2 of [9]. (2) If $(I:J)_r \neq 0$, then it is an *R*-ideal of *Q* and $J(I:J)_r \subseteq I$. So $J((I:J)_r)^*$

 $\subseteq (J(I:J)_r)^* \subseteq I. \quad \text{Hence } ((I:J)_r)^* \subseteq (I:J)_r \text{ so that } ((I:J)_r)^* = (I:J)_r.$

(3) It is clear that $O_l(I \circ J) \supseteq O_l(I)$ and so $O_l(I \circ J) = O_l(I)$ by Lemma 1. Since $(I \circ J) \circ (J^{-1} \circ I^{-1}) = S$, where $S = O_l(I)$, we get $(I \circ J)^{-1} \supseteq J^{-1} \circ I^{-1}$. Let x be any element of $(I \circ J)^{-1}$. Then $IJx \subseteq (I \circ J)x \subseteq S$. Let $T = O_r(J)$. Then $Tx + J^{-1}I^{-1}$ is a left T-ideal and $IJ(Tx + J^{-1}I^{-1}) \subseteq S$. Hence $I \circ J \circ (Tx + J^{-1}I^{-1})^*$ $\subseteq S$ by Lemma 2. By multiplying $J^{-1} \circ I^{-1}$ to the both side of the inequality we have $x \in (Tx + J^{-1}I^{-1})^* \subseteq J^{-1} \circ I^{-1}$. Therefore we get $(I \circ J)^{-1} = J^{-1} \circ I^{-1}$.

(4) From Lemma 2, we have: $(I \cup *J) \circ L = [(I+J)*L]* = [(I+J)L]* = (IL+JL)* = [(IL)*+(JL)*]* = I \circ L \cup *J \circ L.$

Let R be a maximal order. We consider the following condition:

 $(A):F'_{r}(R)$ and $F'_{l}(R)$ both satisfy the maximum condition.

If R is a maximal order satisfying the condition (A), then $F^*(R)$ is a direct product of infinite cyclic groups with prime v-ideals as their generators by Theorem 4.2 of [2]. It is evident that an element P in F'(R) is a prime element in the lattice if and only if it is a prime ideal of R.

Following [1], R is said to be *regular* if every integral one-sided R-ideal contains a non-zero R-ideal. **Lemma 4.** Let R be a regular maximal order satisfying the condition (A) and let P be a non-zero prime ideal of R. Then P is a minimal prime ideal of R if and only if it is a prime v-ideal.

Proof. Assume that P is a minimal prime ideal. Let c be any regular element in P. Then since $(cR)^* = cR$ and R is regular, we get $P \supseteq cR \supseteq (P_1^{n_1})^* \circ \cdots \circ (P_k^{n_k})^*$, where P_i is a prime v-ideal. Hence $P \supseteq P_i$ for some i and so $P = P_i$. Conversely assume that $P \supseteq P_0 = 0$, where P_0 is a prime ideal. Then since $P_0^*(P_0^{-1}P_0) = (P_0^*P_0^{-1})P_0 \subseteq RP_0 = P_0$ and $P_0^{-1}P_0 \subseteq P_0$, we have $P_0^* \subseteq P_0$ and thus $P_0^* = P_0$. It follows that P_0 is a maximal element in F'(R) by [2, p. 11], a contradiction. Hence P is a minimal prime ideal of R.

REMARK. Let R be a maximal order satisfying the condition (A). Then it is evident from the proof of the lemma that prime v-ideals are minimal prime ideals of R.

Let I be any right ideal of R. Then we denote by \sqrt{I} the set $\bigcup \{(s^{-1}I:R), |s \notin I, s \in R\}$. Following [3], if \sqrt{I} is an ideal of R, then we say that I is primal and that \sqrt{I} is the *adjoint ideal* of it. A right ideal I of R is called primary if $JA \subseteq I$ and $J \notin I$ implies that $A^n \subseteq I$ for some positive integer n, where J is a right ideal of R and A is an ideal of R. We shall apply these concepts for integral right v-ideals.

Lemma 5. Let R be a maximal order satisfying the condition (A) and let I be a meet-irreducible element in $F'_r(R)$. Then I is primal, and \sqrt{I} is a minimal prime ideal of R or 0, and $\sqrt{I} = (x^{-1}I: R)$, for some $x \notin I$.

Proof. If $\sqrt{I}=0$, then the assertion is evident. Assume that $\sqrt{I}\pm 0$. By Lemma 3, $(s^{-1}I:R)$, is a v-ideal or 0. Hence the set $S = \{(s^{-1}I:R), |s \notin I, s \in R\}$ has a maximal element. Assume that $(s^{-1}I:R)$, and $(t^{-1}I:R)$, are maximal elements in S. Then $(sR+I)(s^{-1}I:R)$, $\subseteq I$ implies that $(sR+I)^*(s^{-1}I:R)$, $\subseteq I$ by Lemma 2 and so $(s^{-1}I:R)$, $\subseteq (I:(sR+I)^*)$,. The converse inclusion is clear. Thus we have $(s^{-1}I:R)$, $=(I:(sR+I)^*)$,. Similarly $(t^{-1}I:R)$, $=(I:(tR+I)^*)$,. Since I is irreducible in $F'_i(R)$, we have $I \subseteq (sR+I)^* \cap (tR+I)^* = J$. Let x be any element in J but not in I. Then it follows that $(x^{-1}I:R)$, $\supseteq(s^{-1}I:R)$, $(t^{-1}I:R)$, so that $\sqrt{I}=(x^{-1}I:R)_r=(s^{-1}I:R)_r$, which is a v-ideal. Hence I is primal. If $AB \subseteq \sqrt{I}$ and $A \notin \sqrt{I}$, where A and B are ideals of R, then $xAB \subseteq I$ and $xA \notin I$. Let y be any element in xA but not in I. Then $yB \subseteq I$ and so $B \subseteq (y^{-1}I:R)_r \subseteq \sqrt{I}$. Thus \sqrt{I} is a prime ideal of R. It follows that \sqrt{I} is minimal from the remark to Lemma 4.

A right ideal of R is said to be *bounded* if it contains a non-zero ideal of R.

Lemma 6. Let R be a maximal order satisfying the condition (A) and let I be an irreducible element in $F'_{r}(R)$. If I is bounded, then it is primary and $(\sqrt{I})^{n} \subseteq I$ for some positive integer n.

Proof. Since $I \in F'_i(R)$ and is bounded, $(I:R)_r$ is non-zero and is a v-ideal. Write $(I:R)_r = (P_1^{n_1})^* \circ \cdots \circ (P_k^{n_k})^*$, where P_i are prime v-ideals. For any i $(1 \le i \le k)$, we let $B_i = (P_1^{n_1})^* \circ \cdots \circ (P_{i-1}^{n_{i-1}})^* \circ (P_{i+1}^{n_{i+1}})^* \circ \cdots \circ (P_k^{n_k})^*$. Then $B_i \subseteq I$ and $B_i P_i^{n_i} \subseteq (I:R)_r \subseteq I$, because $F^*(R)$ is an abelian group. Thus $P_i^{n_i} \subseteq \sqrt{I}$ and so $P_i = \sqrt{I} (1 \le i \le k)$ by Lemma 5. Therefore $(\sqrt{I})^{n_1 + \cdots + n_k} \subseteq I$. It is evident that I is primary.

If A is an ideal of R, then we denote by C(A) those elements of R which are regular mod (A).

Lemma 7. Let R be a maximal order satisfying the condition (A). Let P be a prime v-ideal. Then

- (1) $C(P) = C((P^n)^*)$ for every positive integer n.
- (2) $C(P) \subseteq C(0)$.

Proof. (1) We shall prove by the induction on n (>1). Assume that $C(P) = C((P^{n-1})^*)$. If $cx \in (P^n)^*$, where $c \in C(P)$ and $x \in R$, then $cx(P^{-1})^{n-1} \subseteq (P^n)^*(P^{-1})^{n-1} \subseteq P$ by Lemma 2. Since $cx \in (P^{n-1})^*$, we get $x \in (P^{n-1})^*$ and so $x(P^{-1})^{n-1} \subseteq R$. Hence $x(P^{-1})^{n-1} \subseteq P$. Then we have $(xR+P^n) (P^{-1})^{n-1}P^{n-1} \subseteq P^n$ so that $x \in (P^n)^*$ by Lemma 2. Conversely suppose that $cx \in P$, $c \in C((P^n)^*)$, $x \in R$. Then $cxP^{n-1} \subseteq (P^n)^*$ and so $xP^{n-1} \subseteq (P^n)^*$. Since $(xP+P^n)P^{n-1}(P^{-1})^{n-1} \subseteq P^n$ is $(P^n)^*(P^{-1})^{n-1} \subseteq P$, we get $x \in P$ by Lemma 2. Therefore $C(P) = C((P^n)^*)$.

(2) If $0 \neq \bigcap_{n}(P^{n})^{*}$, then it is a v-ideal by Lemma 2. Write $\bigcap_{n}(P^{n})^{*} = (P_{1}^{n_{1}})^{*} \circ \cdots \circ (P_{k}^{n_{k}})^{*}$, where P_{i} are prime v-ideals. This is a contradiction, because $F^{*}(R)$ is an abelian group and P, P_{i} are minimal prime ideals of R. Hence $0 = \bigcap_{n}(P^{n})^{*}$. Therefore (2) follows from (1).

If P is a prime ideal of a ring S, then the family $T_P = \{I: \text{right ideal } | s^{-1}I \cap C(P) \neq \phi \text{ for any } s \in S\}$ is a right additive topology (cf. Ex. 4 of [12, p. 18]). The following lemma is due to Lambek and Michler if S is right noetherian. However, only trivial modifications to their proof are needed to establish the more general result.

Lemma 8. Let P be a prime ideal of S and let $\overline{S}=S|P$ be a right prime Goldie ring. Then the torsion theory determined by the S-injective hull $E(\overline{S})$ of \overline{S} coincides with one determined by the right additive topology T_P , that is, a right ideal I of S is an element in T_P if and only if $Hom_S(S|I,E(\overline{S}))=0$ (Corollary 3.10 of [8]).

Lemma 9. Let R be a maximal order satisfying the condition (A) and let P

be a prime v-ideal such that $\overline{R} = R/P$ is a prime Goldie ring. If I is any element in $F'_r(R)$ such that $R \supseteq I \supseteq P$, then $I \cap C(P) = \phi$.

Proof. It is enough to prove the lemma when I is a maximal element $F'_{r}(R)$. Since $I^{-1} \supseteq R$, $P \circ I^{-1} \cap R \supseteq P$. If $P \circ I^{-1} \cap R = P$, then $P^{-1} = (P \circ I^{-1})^{-1} \cup *R$, because the mapping: $J \to J^{-1}$ is an inverse lattice isomorphism between $F'_{r}(R)$ and $F^{*}(R)$. By Lemma 3, $P^{-1} = I \circ P^{-1} \cup *R$. On the other hand $P \subseteq I$ implies that $R \subseteq I \circ P^{-1}$. Hence $P^{-1} = I \circ P^{-1} \cup *R$. On the other hand $P \subseteq I$ implies that $R \subseteq I \circ P^{-1}$. Hence $P^{-1} = I \circ P^{-1}$ and so R = I, a contradiction. Thus we have $P \circ I^{-1} \cap R \supseteq P$. Let a be any element in $P \circ I^{-1} \cap R$ but not in P. Then $aI \subseteq (P \circ I^{-1})I \subseteq P \circ I^{-1} \circ I = P$ so that $I \subseteq a^{-1}P \subseteq R$. Since $a^{-1}P$ is a right v-ideal by Lemma 3, we get $I = a^{-1}P$. Then $\operatorname{Hom}(R/I, E(\overline{R})) = 0$, because $R/I = R/a^{-1}P \cong (aR+P)/P \subseteq \overline{R}$. Now assume that $I \cap C(P) \neq \phi$ and let c be any element in $I \cap C(P)$. Then $cR + P \in T_P$ by Lemma 3.1 of [6]. Hence $I \in T_P$ and thus $\operatorname{Hom}(R/I, E(\overline{R})) = 0$ by Lemma 8. This is a contradiction and so $I \cap C(P) = \phi$.

For convenience, we write M(p) for the family of minimal prime ideals of R. If R is a regular maximal order satisfying the condition (A), then we know from Lemma 4 that a prime ideal P is an element in M(p) if and only if it is a prime element in F'(R).

Lemma 10. Let R be a regular maximal order satisfying the condition (A), $P \in M(p)$ and let $I \in F'_{r}(R)$. If $\overline{R} = R/P$ is a prime Goldie ring, then $I \cup *P = R$ if and only if I contains an ideal B such that $B \subseteq P$.

Proof. Assume that $I \supseteq B$, where *B* is an ideal not contained in *P*. Then $I \supseteq B^*$ and $B^* \cup *P = R$, because *P* is a maximal element in F'(R) (cf. [2, p. 11]). Therefore $I \cup *P = R$. Conversely assume that the family $S = \{I \in F'_r(R) | I \cup *P = R, I \neq R \text{ and } I \not\supseteq B \text{ for any ideal } B \text{ not contained in } P\}$ is not empty and let *I* be a maximal element in *S*. If *I* is irreducible in $F'_r(R)$, then there exists P' in M(p) such that $I \supseteq P'^n$ by Lemmas 5 and 6. Since $I \in S$, we have P = P'. If n = 1, then $R = I \cup *P = I$, a contradiction. We may assume that $I \not\supseteq P^{n-1}$ and n > 1. Then $(P^{n-1})^* = (I \cup *P) \circ (P^{n-1})^* = I \circ (P^{n-1})^* \cup *(P^n)^* \subseteq I^* = I$ by Lemmas 2 and 3. This is a contradiction. If *I* is reducible, then $I = I_1 \cap I_2$, where $I_i \in F'_r(R)$ and $I \subseteq I_i$ (i = 1, 2). There are non zero ideals $B_i (\subseteq P)$ such that $I_i \supseteq B_i$. Thus *I* contains the ideal B_1B_2 not contained in *P*, a contradiction. Hence $S = \phi$. This implies that if $I \cup *P = R$, then *I* contains an ideal not contained in *P*.

Let P be a prime ideal of a ring S. If S satisfies the Ore condition with respect to C(P), then we denote by S_P the quotient ring with respect to C(P).

Lemma 11. Let R be a regular maximal order satisfying the condition (A) and let P be an element in M(p) such that $\overline{R} = R/P$ is a prime Goldie ring. Then

- (1) R satisfies the Ore condition with respect to C(P).
- (2) $R_P = \lim B^{-1}$, where B ranges over all non zero ideals not contained in P.
- (3) R_P is a noetherian, local and Asano order.

Proof. (1) It is clear that $T = \lim_{\to} B^{-1}(B(\nsubseteq P))$: ideal) is an overring of R. Let c be any element in C(P). Then c is regular by Lemma 7 and so $cR \in F'_r(R)$. Since $(cR \cup *P) \cap C(P) \neq \phi$, we have $cR \cup *P = R$ by Lemma 9 and so cR contains an ideal not contained in P by Lemma 10. Hence $c^{-1} \in T$. So for any $r \in R$, $c \in C(P)$, there exists an ideal $B(\oiint P)$ such that $c^{-1}rB \subseteq R$. It is evident that $B \cap C(P) \neq \phi$. Let d be any element in $B \cap C(P)$. Then we have $c^{-1}rd = s$ for some s in R, that is, rd = cs. This implies that R satisfies the right Ore condition with respect to C(P). The other Ore condition is shown to hold by a symmetric proof.

(2) is evident from (1).

(3) We let $P'=PR_P$. Then clearly $P'=R_PP$ and $P=P'\cap R$. So we may assume that $\overline{R} = R/P \subseteq \overline{R}_P = R_P/P'$ as rings. By (1), \overline{R}_P is the quotient ring of \overline{R} . Since \overline{R} is a prime Goldie ring, \overline{R}_{P} is the simple artinian ring. Hence P' is a maximal ideal of R_p . Let V' be any maximal right ideal of R_p . Suppose that $V' \supseteq P'$. Then $V' + P' = R_p$. Write $1 = v + pc^{-1}$, where $v \in V'$, $p \in P$ and $c \in C(P)$. Then c = vc + p and so $vc = c - p \in C(P) \cap V'$. This implies that $V'=R_P$, a contradiction and so $V'\supseteq P'$. Hence P' is the Jacobson radical of R_{P} . The ideal $P^{-1}P$ properly contains P so that $C(P) \cap P^{-1}P \neq \phi$. It follows that $P^{-1}PR_P = R_P$. Similarly $R_P P P^{-1} = R_P$. Hence P' is an invertible ideal of Therefore R_P/P'^n is an artinian ring for any *n*, because \overline{R}_P is an artinian R_{P} . ring. Let I' be any essential right ideal of R_P . It is clear that $I' = (I' \cap R)R_P$. Let c be any regular element of $I' \cap R$. Then, since $cR \in F'_r(R)$ and R is regular, cR contains a non zero v-ideal $(P^n)^* \circ (P_{1^1}^n)^* \circ \cdots \circ (P_{k^k}^n)^*$, where $P_i \in M(p)$. So we get $I' \supseteq R_P P^n = P'^n$. Therefore essential right ideals of R_P satisfies the maximum condition. Since R_P is finite dimensional in the sense of Goldie, R_P is right noetherian. Similarly R_P is left noetherian. Hence R_P is a noetherian, local and Asano order by Proposition 1.3 of [7].

After all these preparations we now prove the following theorem which is the purpose of this paper:

Theorem. A prime Goldie ring R is a bounded Krull prime ring if and only if it satisfies the following conditions:

- (1) R is a regular maximal order,
- (2) R satisfies the maximum condition for integral right and left v-ideals,
- (3) R/P is a prime Goldie ring for any $P \in M(p)$.

Proof. Assume that $R = \bigcap_{i} R_i$ $(i \in I)$ is a bounded Krull prime ring, where R_i is a noetherian, local and Asano order with unique maximal ideal P'_i . (1) is

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clear from Corollary 1.4 and Lemma 1.6 of [10]. Let I be any right (left) Rideal. Then $I^* = \cap IR_i (= \cap R_i I)$ by Proposition 1.10 of [10]. Since R_i is noetherian, (2) follows from the condition (K3) in the definition of Krull rings. Let $P_i = P'_i \cap R$. It follows that $\{P_i | i \in I\} = M(p)$ by Proposition 1.7 of [10] so that (3) is evident from Proposition 1.1 of [9].

It remains to prove that the conditions (1), (2) and (3) are sufficient. Let P be any element in M(p). Then R satisfies the Ore condition with respect to C(P) and R_P is a noetherian, local and Asano order by Lemma 11. Hence R_P is an essential overring of R. It is clear that $R \subseteq T = \bigcap R_P$, where $P \in M(p)$. To prove the converse inclusion let x be any element of T. Then there is an ideal $B_P (\subseteq P)$ such that $xB_P \subseteq R$ by Lemma 11. Let B be the sum of all ideals B_P . If B^* is different from R, then B^* is contained in some P in M(p). But $B^* \subseteq P$ so that $B^* = R$. Hence we have $x \in (xR+R) \subseteq (xR+R)^* \circ B^* = (xB+B)^* \subseteq R$. Thus we get $R = \bigcap R_P$. Let c be any regular element in R. Then cR contains a v-ideal $(P_1^{n_1})^* \circ \cdots \circ (P_k^{n_k})^*$, where $P_i \in M(p)$. It follows that $cR_P = R_P$ for every $P \in M(p)$ different to $P_i(1 \le i \le k)$ by Lemma 11. Hence R is a bounded Krull prime ring. This completes the proof of the theorem.

Corollary. Let R be a regular, noetherian and prime ring. If R is a maximal order, then it is a bounded Krull prime ring.

OSAKA UNIVERSITY

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