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ON SPECIAL VALUES AT $s=0$ OF PARTIAL ZETA-FUNCTIONS FOR REAL QUADRATIC FIELDS

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1. Introduction

1.1 Let F be a totally real algebraic number field with finite degree, $\mathfrak{a}$ a fractional ideal of F, and F_{ab} the maximal abelian extension of F. We define a map $\xi_{\mathfrak{a}}$ from the quotient space $F/\mathfrak{a}$ to the group $W(F_{ab})$ of roots of unity of F_{ab} using the deep results of Coates-Sinnott [$C-S1$], [$C-S2$] and Deligne-Ribet [$D-R$] on special values of partial zeta functions of F. Under the action of the Galois group $Gal(F_{ab}/F)$ of F_{ab} over F this map behaves formally in a manner similar to Shimura's reciprocity law for elliptic curves with complex multiplication. This reciprocity law for the map $\xi_{\mathfrak{a}}$ is also a direct consequence of those results of Coates-Sinnott and Deligne-Ribet. On the other hand we have studied in [Ar1] a certain Dirichlet series and its relationship with parital zeta functions of real quadratic fields. In particular the special values at $s=0$ of partial zeta functions of real quadratic fields essentially coincide with the residues at the pole $s=0$ of our Dirichlet series. Using those residues, we give another expression for the map $\xi_{\mathfrak{a}}$ in the case of F a real quadratic field. We also show that the expression works in a reasonable manner under the action of the Galois group $Gal(F_{ab}/F)$.

1.2 We summarize our results. For an integral ideal $\mathfrak{c}$ of a totally real algebraic number field F, denote by $H_F(\mathfrak{c})$ the narrow ray class group modulo $\mathfrak{c}$. For each integral ideal $\mathfrak{b}$ prime to $\mathfrak{c}$, we define the partial zeta-function $\zeta_{\mathfrak{c}}(\mathfrak{b}, s)$ to be the sum $\sum_{\mathfrak{a}}(N\mathfrak{a})^{-s}$, $\mathfrak{a}$ running over all integral ideals of the class of $\mathfrak{b}$ in $H_F(\mathfrak{c})$. Let $\mathfrak{a}$ be a fractional ideal of F. For each class $\bar{z}$ of the quotient space $F/\mathfrak{a}$, we take a totally positive representative element $z\in F$ of the class $\bar{z}$, and write

$$z\mathfrak{a}^{-1}=\mathfrak{f}^{-1}\mathfrak{b} \tag{1.1}$$

with coprime integral ideals $\mathfrak{f}, \mathfrak{b}$ of F. Thanks to some results of Coates-Sinnott ([C-S1], [C-S2], [Co]) and Deligne-Ribet ([D-R]), one can define a map $\xi_{\mathfrak{a}}: F/\mathfrak{a}\to W(F_{ab})$ as follows;

$$\xi_{\mathfrak{a}}(\bar{z}) = \exp(2\pi i\zeta_{\mathfrak{f}}(\mathfrak{b}, 0)), \tag{1.2}$$

where the value on the right hand side of the equality depends on the class $\bar{z}$ and not on a representative element z of $\bar{z}$. Denote by $F_A^\times$ the idele group of F and by $F_{A,+}^\times$ the subgroup of $F_A^\times$ consisting of ideles x whose archimedean components x_∞ are totally positive. Each element s of $F_A^\times$ induces a natural isomorphism $s: F/\mathfrak{a} \cong F/s\mathfrak{a}$. We denote by $[s, F]$ the canonical Galois automorphism of the extension F_{ab}/F induced by $s \in F_A^\times$. The following theorem is a reformulation of a part of the results due to Coates-Sinnott and Deligne-Ribet ([C-S1], [C-S2], [D-R]).

Theorem A (Coates-Sinnott, Deligne-Ribet)
Let $s \in F_{A,+}^\times$ and set $\sigma = [s, F]$. Then the following diagram is commutative.

$$\begin{array}{ccc} F/\mathfrak{a} & \xrightarrow{\xi_\mathfrak{a}} & W(F_{ab}) \\ \downarrow{\scriptstyle s^{-1}} & & \downarrow{\scriptstyle \sigma} \\ F/s^{-1}\mathfrak{a} & \xrightarrow{\xi_{s^{-1}\mathfrak{a}}} & W(F_{ab}) \end{array}$$

Namely,

$$\xi_\mathfrak{a}(\bar{z})^\sigma = \xi_{s^{-1}\mathfrak{a}}(\overline{s^{-1}z}),$$

where $\overline{s^{-1}z}$ stands for the image of $\bar{z}$ by the isomorphism $s^{-1}: F/\mathfrak{a} \cong F/s^{-1}\mathfrak{a}$.

In particular if we write, with $\bar{z}$ being specialized at $\bar{0} = 0 \bmod \mathfrak{a}$,

$$\xi(\mathfrak{a}) = \xi_\mathfrak{a}(\bar{0}),$$

then, $\xi(\mathfrak{a})$ is a root of unity contained in the narrow Hilbert class field of F. In this case the Galois action is described in the simple maner:

$$\xi(\mathfrak{a})^{[s,F]} = \xi(s^{-1}\mathfrak{a}) \qquad \text{for any } s \in F_{A,+}^\times .$$

Theorem A will be interpreted as a formal analogy to Shimura's reciprocity law for elliptic curves with complex multiplication (see Theorem 5.4 of [Shm]).

For a real number x, we denote by $\langle x \rangle$ the real number satisfying $x - \langle x \rangle \in \boldsymbol{Z}$ and $0 < \langle x \rangle \leq 1$. Let F be a *real quadratic field* embedded in $\boldsymbol{R}$. We set, for $\alpha \in F - \boldsymbol{Q}$ and $(p, q) \in \boldsymbol{Q}^2$,

$$\eta(\alpha, s, p, q) = \sum_{n=1}^{\infty} n^{s-1} \cdot \frac{\exp(2\pi i n(p\alpha + q))}{1 - \exp(2\pi i n\alpha)} \tag{1.3}$$

and

$$H(\alpha, s, (p, q)) = \eta(\alpha, s, \langle p \rangle, q) + e^{\pi i s} \eta(\alpha, s, \langle -p \rangle, -q) . \tag{1.4}$$

This type of infinite series has been intensively studied by Berndt [Be1], [Be2], if α is a complex number with positive imaginary part. In our case we have proved in [Ar1] that the infinite series $\eta(\alpha, s, p, q)$ is absolutely convergent for $\mathrm{Re}(s)<0$ and moreover that $H(\alpha, s, (p, q))$ can be analytically continued to a meromorphic function of s in the whole s-plane which has a possible simple pole at $s=0$. Let $h_{-1}(\alpha, (p, q))$ denote the residue at the pole $s=0$ of this function $H(\alpha, s, (p, q))$ (see § 3 of this paper). We set

$$\mathfrak{h}(\alpha, (p, q)) = \frac{1}{2}(h_{-1}(\alpha, (p, q)) - h_{-1}(\alpha', (p, q))),$$

where α' denotes the conjugate of α in F. This quantity $\mathfrak{h}(\alpha, (p, q))$ satisfies the transformation law under the action of $SL_2(\boldsymbol{Z})$:

$$\mathfrak{h}(V\alpha, (p, q)) = \mathfrak{h}(\alpha, (p, q)V) \qquad \text{for any } V \in SL_2(\boldsymbol{Z}). \tag{1.5}$$

We denote by $F^\times$ the group of of invertible elements of F. Let $\mathfrak{a}$ be a fractional ideal of F with an oriented basis $\{\alpha_1, \alpha_2\}$ (i.e., $\mathfrak{a}=\boldsymbol{Z}\alpha_1+\boldsymbol{Z}\alpha_2$, $\alpha_1\alpha_2'-\alpha_1'\alpha_2>0$). Denote by q: $F^\times \to GL_2(\boldsymbol{Q})$ the injective homomorphism of $F^\times$ into $GL_2(\boldsymbol{Q})$ defined via the basis $\{\alpha_1, \alpha_2\}$ as follows;

$$\mu\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = q(\mu)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \qquad (\mu \in F^\times). \tag{1.6}$$

This homomorphism q is naturally extended to that of $F_A^\times$ into the adele group $G_A = GL_2(\boldsymbol{Q}_A)$. Denote by $G_{A,+}$ the subgroup of G_A consisting of all elements $x \in G_A$ whose archimedean components x_∞ have positive determinants. By the transformation law (1.5) of $\mathfrak{h}(\alpha, (p, q))$, one can define an action of any $x \in G_{A,+}$ on the coefficient $\mathfrak{h}(\alpha, (p, q))$. This action will be denoted by $\mathfrak{h}^x(\alpha, (p, q))$ (for the precise definition see (3.12)). For an integral ideal $\mathfrak{f}$ of F, we denote by $E_+(\mathfrak{f})$ the group of totally positive units u of F with $u-1 \in \mathfrak{f}$. Another expression for the map $\xi_{\mathfrak{a}}(\bar{z})$ is given by the following theorem.

Theorem B *Let the notation be the same as above. Let $\mathfrak{a}$ be a fractional ideal of a real quadratic field F with the oriented basis $\{\alpha_1, \alpha_2\}$. Choose a representative element $z \in F$, $z \neq 0$ of a class $\bar{z} \in F/\mathfrak{a}$ and determine the ideal $\mathfrak{f}$ by (1.1). Denote by η the generator of the group $E_+(\mathfrak{f})$ with $\eta>1$. Write $z=p\alpha_1+q\alpha_2$ with $(p, q) \in \boldsymbol{Q}^2$ and set $\alpha=\alpha_1/\alpha_2$. Then,*

$$\xi_{\mathfrak{a}}(\bar{z}) = \exp(\log \eta \cdot \mathfrak{h}(\alpha, (p, q))). \tag{1.7}$$

Let $s \in F_{A,+}^\times$. The Galois action on $\xi_{\mathfrak{a}}(\bar{z})$ is given by the equality

$$\xi_{\mathfrak{a}}(\bar{z})^{[s,F]} = \exp(\log \eta \cdot \mathfrak{h}^{q(s)^{-1}}(\alpha, (p, q))). \tag{1.8}$$

In Theorem 3.3 we obtain a stronger result than (1.7); namely, the special value $\zeta_{\mathfrak{f}}(\mathfrak{b}, 0)$ is explicitly given by the value $\mathfrak{h}(\alpha, (p, q))$. We note that, as is essentially known, the value $\xi(\mathfrak{a})=\xi_{\mathfrak{a}}(\bar{0})$ is a twelfth root of unity in the narrow Hilbert class field of F (see the end of §3).

2. Partial zeta-functions for totally real number fields

We recall a part of the results of [C-S1, 2], [Co], and [D-R] concerning special values at non-positive integers of parital zeta-finctions for totally real algebraic number fields.

Let μ_m denote the group of m-th roots of unity. Let L be an algebraic number field. If K is a Galois extension of L, we write $Gal(K/L)$ for the Galois group of K over L. For a positive integer n, we define $w_n(L)$ to be the largest integer m such that the exponent of the group $Gal(L(\mu_m)/L)$ divides n (see 2.2 of [Co]). In particular if $n=1$, $w_1(L)$ is nothing but the number of roots of unity of L. We denote by $W(L)$ the group of roots of unity of L.

Let F be a totally real algebraic number field with finite degree throughout this paragraph. For an integral ideal $\mathfrak{f}$ of F, denote by $H_F(\mathfrak{f})$ the narrow ray class group modulo $\mathfrak{f}$. Namely, $H_F(\mathfrak{f})$ is the quotient group $I_F(\mathfrak{f})/P_+(\mathfrak{f})$, where $I_F(\mathfrak{f})$ is the group of fractional ideals of F prime to $\mathfrak{f}$ and $P_+(\mathfrak{f})$ is the group of principal ideals of F generated by totally positive elements θ of F such that the numerators of $\theta-1$ are divisible by $\mathfrak{f}$. We set, for each class C of $H_F(\mathfrak{f})$,

$$\zeta_{\mathfrak{f}}(C, s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s} \qquad (\mathrm{Re}(s)>1),$$

where $\mathfrak{a}$ runs over all integral ideals of C and $N\mathfrak{a}$ denotes the norm of $\mathfrak{a}$. The partial zetafunction $\zeta_{\mathfrak{f}}(C, s)$ is analytically continued to a meromorphic function in the whole s-plane which is holomorphic at non-positive integers. If $\mathfrak{b}$ is a representative ideal of C, we often write $\zeta_{\mathfrak{f}}(\mathfrak{b}, s)$ in place of $\zeta_{\mathfrak{f}}(C, s)$. Let $K=K_F(\mathfrak{f})$ be the maximal narrow ray class field of F defined modulo $\mathfrak{f}$. We write $[C, K/F]$ for the Artin symbol of the class C of $H_F(\mathfrak{f})$. By the class field theory there exists a canonical isomorphisms of $H_F(\mathfrak{f})$ to the Galois group $Gal(K/F)$ given by the correspondence: $C\rightarrow[C, K/F]$. If $\mathfrak{b}$ is a representative ideal of the class C, we write $[\mathfrak{b}, K/F]$ for $[C, K/F]$. The following theorem is due to Coates-Sinnott [C-S1, 2] in the case of real quadratic fields and to Deligne-Ribet [D-R] in general.

Theorem 2.1. (Coates-Sinnott, Deligne-Ribet) *Let $\mathfrak{f}$ be an integral ideal of F and $\mathfrak{b}$, $\mathfrak{c}$ integral ideals of F which are prime to $\mathfrak{f}$. Set $K=K_F(\mathfrak{f})$. For each non-negative inetger n,*

(i) $w_{n+1}(K)\zeta_{\mathfrak{f}}(\mathfrak{b}, -n)$ *is an integer.*

(ii) *Moreover if $\mathfrak{c}$ is prime to $w_{n+1}(K)$, then the value*

$$(N\mathfrak{c})^{n+1}\zeta_{\mathfrak{f}}(\mathfrak{b}, -n)-\zeta_{\mathfrak{f}}(\mathfrak{bc}, -n)$$

is also an integer.

In the case of $n=0$, we reformulate the above theorem into a slightly different form suitable to our later situation. For that purpose we recall briefly the class field theory in the adelic language (see [C-F]).

Denote by $F_+^\times$ the group of totally positive elements of F. Let $F_A^\times$ denote the idele group of F, $F_\infty^\times$ the archimedean part of $F_A^\times$, and $F_{\infty,+}^\times$ the connected component of the identity of $F_\infty^\times$, respectively. We denote by $F_{A,+}^\times$ the subgroup of $F_A^\times$ consisting of elements $x \in F_A^\times$ whose archimedean component x_∞ are contained in $F_{\infty,+}^\times$. For each element x of $F_A^\times$ and for a finite prime $\mathfrak{p}$ of F, we denote by $x_\mathfrak{p}$ the $\mathfrak{p}$-component of x and define a fractional ideal $il(x)$ of F by putting $il(x)\mathfrak{o}_\mathfrak{p} = x_\mathfrak{p}\mathfrak{o}_\mathfrak{p}$ for all finite $\mathfrak{p}$, where $\mathfrak{o}_\mathfrak{p}$ is the maximal order of the completion $F_\mathfrak{p}$ of F at $\mathfrak{p}$. Set

$$U = \{x \in F_A^\times \mid x_\mathfrak{p} \in \mathfrak{o}_\mathfrak{p}^\times \quad \text{for all finite primes } \mathfrak{p} \text{ of } F\},$$

$\mathfrak{o}_\mathfrak{p}^\times$ being the unit group of $\mathfrak{o}_\mathfrak{p}$. Set, for an integral ideal $\mathfrak{f}$,

$$W_+(\mathfrak{f}) = \{x \in F_A^\times \mid x_\infty \in F_{\infty,+}^\times \text{ and } x_\mathfrak{p} - 1 \in \mathfrak{f}\mathfrak{o}_\mathfrak{p} \text{ for all } \mathfrak{p} \text{ dividing } \mathfrak{f}\},$$
$$U_+(\mathfrak{f}) = U \cap W_+(\mathfrak{f}).$$

By the class field theory there exists a canonical exact sequence

$$\begin{array}{l} 1 \longrightarrow \overline{F^\times F_{\infty,+}^\times} \longrightarrow F_A^\times \longrightarrow Gal(F_{ab}/F) \longrightarrow 1, \\ \qquad\qquad\qquad\qquad\quad s \longrightarrow [s, F] \end{array}$$

where $\overline{F^\times F_{\infty,+}^\times}$ is the closure of $F^\times F_{\infty,+}^\times$ in $F_A^\times$ and where we denote by $[s, F]$ the element of $Gal(F_{ab}/F)$ corresponding to an element s of $F_A^\times$. If we take an element u of $W_+(\mathfrak{f})$, then the Galois automorphism $[u, F]$ coincides with the Artin symbol $[il(u), K_F(\mathfrak{f})/F]$ on the narrow ray class field $K_F(\mathfrak{f})$ over F.

Let $\mathfrak{a}$ be a fractional ideal of F. To define the map $\xi_\mathfrak{a}$ of the quotient space $F/\mathfrak{a}$ to the group $W(F_{ab})$ by the equality (1.2), we have to prove that the right hand side of (1.2) depends only on the class $\bar{z} \in F/\mathfrak{a}$ (not on the choice of a representative element z of $\bar{z}$) and moreover that the image of $\xi_\mathfrak{a}$ is in $W(F_{ab})$. To see this we take another element z_1 of $F_+^\times$ with the condition $z - z_1 \in \mathfrak{a}$. Let $\mathfrak{f}, \mathfrak{b}$ be the same coprime integral ideals of F as in (1.1). Then we have

$$z_1\mathfrak{a}^{-1} = \mathfrak{f}^{-1}\mathfrak{b}_1$$

with some integral ideal $\mathfrak{b}_1$ prime to $\mathfrak{f}$. We see easily that $\mathfrak{b}$ and $\mathfrak{b}_1$ are in the same class of $H_F(\mathfrak{f})$. Therefore,

$$\zeta_\mathfrak{f}(\mathfrak{b}, 0) = \zeta_\mathfrak{f}(\mathfrak{b}_1, 0)$$

By virtue of the assertion (i) of Theorem 2.1 the value

$$\exp(2\pi i \zeta_\mathfrak{f}(\mathfrak{b}, 0))$$

is a root of unity of $K_F(\mathfrak{f})$. Thus the map $\xi_{\mathfrak{a}}$ given by (1.2) defines a map of $F/\mathfrak{a}$ to $W(F_{ab})$.

Any element x of $F_A^\times$ acts naturally on a fractional ideal $\mathfrak{a}$ of F. The ideal $x\mathfrak{a}$ of F is characterized by the property $x\mathfrak{a}=il(x)\mathfrak{a}$. For each element u of F, there exists an element v of F such that

$$v-x_{\mathfrak{p}}u\in x_{\mathfrak{p}}\mathfrak{a}_{\mathfrak{p}} \qquad \text{for all prime ideals } \mathfrak{p} \text{ of } F,$$

where $\mathfrak{a}_{\mathfrak{p}}=\mathfrak{a}\mathfrak{o}_{\mathfrak{p}}$ in $F_{\mathfrak{p}}$. Thus we obtain a natural isomorphism of $F/\mathfrak{a}$ to $F/x\mathfrak{a}$ by the correspondence $u \bmod \mathfrak{a}\to v \bmod x\mathfrak{a}$. We denote this isomorphism by $x\colon F/\mathfrak{a}\to F/x\mathfrak{a}$ and write $xu \bmod x\mathfrak{a}$ for the image of $u \bmod \mathfrak{a}$.

A part of the theorem of Coates-Sinnott and Deligne-Ribet (Theorem 2.1) can be formulated in terms of the adele language as in Theorem A in the introduction. For the completeness we give its proof here.

Proof of Theorem A.

We take a representative element $z\in F_+^\times$ of a class $\bar{z}\in F/\mathfrak{a}$ and write $z\mathfrak{a}^{-1}=\mathfrak{f}^{-1}\mathfrak{b}$ with coprime integral ideals $\mathfrak{f}$, $\mathfrak{b}$ of F as in (1.1). Set $K=K_F(\mathfrak{f})$. For $s\in F_{A,+}^\times$, we decompose $s=au$ with $a\in F_+^\times$, $u\in W_+(\mathfrak{f})$. Moreover we may choose u so that $il(u)$ is an integral ideal prime to $w_1(K)$. Set, for simplicity, $\mathfrak{c}=il(u)$. Since by definition

$$\xi_{\mathfrak{a}}(\bar{z}) = \exp(2\pi i\zeta_{\mathfrak{f}}(\mathfrak{b}, 0))\in W(K),$$

we have, for $\sigma=[s, F]$,

$$\begin{aligned}\xi_{\mathfrak{a}}(\bar{z})^{\sigma} &= \xi_{\mathfrak{a}}(\bar{z})^{[u,F]}\\ &= \exp(2\pi i\zeta_{\mathfrak{f}}(\mathfrak{b}, 0))^{[\mathfrak{c},K/F]}\\ &= \exp(2\pi iN\mathfrak{c}\,\zeta_{\mathfrak{f}}(\mathfrak{b}, 0)).\end{aligned}$$

Therefore Theorem 2.1 implies that

$$\xi_{\mathfrak{a}}(\bar{z})^{\sigma} = \exp(2\pi i\zeta_{\mathfrak{f}}(\mathfrak{c}\mathfrak{b}, 0)). \tag{2.1}$$

On the other hand since $u\in W_+(\mathfrak{f})$ and $u_{\mathfrak{p}}\in\mathfrak{o}_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of F, we see immediately that

$$1-u_{\mathfrak{p}}\in(\mathfrak{f}\mathfrak{b}^{-1})\mathfrak{o}_{\mathfrak{p}} \qquad \text{for all prime ideals } \mathfrak{p} \text{ of } F.$$

Thus for every prime ideal $\mathfrak{p}$ of F,

$$u_{\mathfrak{p}}^{-1}z-z\in z\mathfrak{f}\mathfrak{b}^{-1}u_{\mathfrak{p}}^{-1}\mathfrak{o}_{\mathfrak{p}},$$

which truns out that

$$u^{-1}z\equiv z \bmod u^{-1}\mathfrak{a}.$$

Hence,

$$s^{-1}z \equiv a^{-1}z \bmod s^{-1}\mathfrak{a}\,, \tag{2.2}$$

where we see that

$$a^{-1}z \in F_+^\times \quad \text{and} \quad a^{-1}z(s^{-1}\mathfrak{a})^{-1} = \mathfrak{f}^{-1}\mathfrak{b}\mathfrak{c}\,. \tag{2.3}$$

Therefore,

$$\xi_{s^{-1}\mathfrak{a}}(s^{-1}z \bmod s^{-1}\mathfrak{a}) = \exp(2\pi i \zeta_{\mathfrak{f}}(\mathfrak{b}\mathfrak{c}, 0))\,,$$

which together with (2.1) completes the proof of Theorem A. ■

3. Special values at $s=0$ of partial zeta-functions for real quadratic fields

First we recall some results of [Ar1]. For a real number x, denote by $\{x\}$ (res. $\langle x\rangle$) the real number satisfying

$$0 \leq \{x\} < 1\,, \quad x - \{x\} \in \boldsymbol{Z} \quad (\text{resp. } 0 < \langle x\rangle \leq 1\,, \quad x - \langle x\rangle \in \boldsymbol{Z})\,.$$

We note here that $\{x\} + \langle -x\rangle = 1$. In this paragraph let F be a real quadratic field embedded in $\boldsymbol{R}$ and fix it once and for all. For each α of F, let α' denote the conjugate of α in F. For $\alpha \in F - \boldsymbol{Q}$ and $(p, q) \in \boldsymbol{Q}^2$, we define a Lambert series $\eta(\alpha, s, p, q)$ by the equality (1.3) in the introduction. The infinite series $\eta(\alpha, s, p, q)$ is absolutely convergent for $\mathrm{Re}(s) < 0$ (see Lemma 1 of [Ar1]). We also define the function $H(\alpha, s, (p, q))$ of s by the equality (1.4) in the introduction. We note that $H(\alpha, s, (p, q))$ depends on (p, q) mod $\boldsymbol{Z}^2$. As we have seen in [Ar1], this function $H(\alpha, s, (p, q))$ can be analytically continued to a meromorphic function of s in the whole s-plane and has a Laurent expansion at $s=0$ of the form:

$$H(\alpha, s, (p, q)) = \frac{h_{-1}(\alpha, (p, q))}{s} + h_0(\alpha, (p, q)) + \cdots\,.$$

Moreover the first coefficient $h_{-1}(\alpha, (p, q))$ satisfies under the action of $SL_2(\boldsymbol{Z})$ the following transformation law.

Proposition 3.1. *Let $\alpha \in F - \boldsymbol{Q}$ and $(p, q) \in \boldsymbol{Q}^2$. Then,*

$$h_{-1}(V\alpha, (p, q)) = h_{-1}(\alpha, (p, q)V) \quad \text{for any } V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z})\,, \tag{3.1}$$

where we put $V\alpha = \dfrac{a\alpha + b}{c\alpha + d}$.

Proof. For $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z})$, set $V^* = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ and $(p^*, q^*) = (p, q)V$. If $c > 0$ and $c\alpha + d > 0$, then the identity (3.1) is nothing but the first equality in

Proposition 4 of [Ar1]. Let $c<0$ and $c\alpha+d>0$. In this case since $V^*(-\alpha)=-(V\alpha)$, we get, by Propositions 3, 4 of [Ar1],

$$\begin{aligned} h_{-1}(V\alpha, (p, q)) &= -h_{-1}(-(V\alpha), (-p, q))+2\delta(p, q) \\ &= -h_{-1}(-\alpha, (-p, q)V^*)+2\delta(p, q) \\ &= -h_{-1}(-\alpha, (-p^*, q^*))+2\delta(p^*, q^*) \\ &= h_{-1}(\alpha, (p^*, q^*))\,, \end{aligned}$$

where we put

$$\delta(p, q) = \begin{cases} 1 & \cdots \quad (p, q)\in \boldsymbol{Z}^2 \\ 0 & \cdots \quad \text{otherwise.} \end{cases}$$

If $c=0$, $d=1$, then the assertion easily follows from the definition of $H(\alpha, s, (p, q))$. Finally let $c\alpha+d<0$. Since $V\alpha=(-V)\alpha$, we have

$$h_{-1}(V\alpha, (p, q)) = h_{-1}(\alpha, (-p^*, -q^*))\,.$$

With the help of Lemma 5 of [Ar1], the last term coincides with $h_{-1}(\alpha, (p^*, q^*))$. ■

We set, for positive numbers ω, z,

$$G(z, \omega, t) = \frac{\exp(-zt)}{(1-\exp(-t))(1-\exp(-\omega t))} \qquad (t\in \boldsymbol{C})\,,$$

$$\zeta_2(s, \omega, z) = \sum_{m,n=0}^{\infty} (z+m+n\omega)^{-s} \qquad (\mathrm{Re}(s)>2)\,.$$

The Dirichlet series $\zeta_2(s, \omega, z)$ is absolutely convergent for $\mathrm{Re}(s)>2$. For a sufficiently small positive number ε, let $I_\varepsilon(\infty)$ be the integral path consisting of the oriented half line $(+\infty, \varepsilon)$, the counterclockwise circle of radius ε around the origin, and the oriented half line $(\varepsilon, +\infty)$. Then as is well-known, the zeta-function $\zeta_2(s, \omega, z)$ has the following expression by a contour integral:

$$\zeta_2(s, \omega, z) = \frac{1}{\Gamma(s)(e^{2\pi i s}-1)} \int_{I_\varepsilon(\infty)} t^{s-1} G(z, \omega, t)\,dt\,, \tag{3.2}$$

where $\log t$ is understood to be real valued on the upper half line $(+\infty, \varepsilon)$. This expression (3.2) gives the analytic continuation of $\zeta_2(s, \omega, z)$ to a meromorphic function over the whole s-plane which is holomorphic except at $s=1, 2$. We put, for $r\in \boldsymbol{R}$,

$$\chi(r) = \begin{cases} 1 & \cdots \quad r\in \boldsymbol{Z} \\ 0 & \cdots \quad r\in \boldsymbol{R}-\boldsymbol{Z}. \end{cases}$$

For each $\alpha\in \boldsymbol{F}-\boldsymbol{Q}$ and a pair $(p, q)\in \boldsymbol{Q}^2$, we choose a totally positive unit η of

F and an element $V=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL_2(\boldsymbol{Z})$ which satisfy the following conditions

$$c>0\,, \quad (p, q)V \equiv (p, q) \bmod \boldsymbol{Z}^2\,, \quad \eta\begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha \\ 1 \end{pmatrix}. \tag{3.3}$$

We have obtained in (3.2) of [Ar1] the following expresssion for $h_{-1}(\alpha, (p, q))$ using the data given in (3.3):

$$h_{-1}(\alpha, (p, q)) - \chi(p)\chi(q) = \frac{2\pi i}{\log \eta}\,\chi(p)\left(\frac{1}{2} - \langle -q \rangle\right) - \frac{1}{\log \eta} L(\alpha, 0, (p, q), c, d)\,, \tag{3.4}$$

where $L(\alpha, 0, (p, q), c, d)$ $(s \in \boldsymbol{C})$ is the special value at $s=0$ of the function

$$L(\alpha, s, (p, q), c, d) = -\sum_{j=1}^{c} \int_{I_\varepsilon(\infty)} t^{s-1} G\left(1 - \left\{\frac{jd+\rho}{c}\right\} + \frac{(j-\{p\})\eta}{c}, \eta, t\right) dt$$

with $\rho = \{q\}c - \{p\}d$. Since the above integral on the right hand side of the equality converges absolutely for any $s \in \boldsymbol{C}$, this function $L(\alpha, s, (p, q), c, d)$ of s is holomorphic in the whole complex plane.

Proposition 3.2. *Let $\alpha \in F - \boldsymbol{Q}$ and $(p, q) \in \boldsymbol{Q}^2$. Choose a totally positive unit η of F and $V=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL_2(\boldsymbol{Z})$ as in* (3.3). *Then,*

$$h_{-1}(\alpha, (p, q)) - \chi(p)\chi(q) = \frac{2\pi i}{\log \eta} \sum_{k \bmod c} \zeta_2(0, \eta, x_k + y_k \eta)\,,$$

$$h_{-1}(\alpha', (p, q)) - \chi(p)\chi(q) = -\frac{2\pi i}{\log \eta} \sum_{k \bmod c} \zeta_2(0, \eta', x_k + y_k \eta')\,,$$

where we put, for each integer k,

$$x_k = 1 - \left\{\frac{(k+p)d}{c} - q\right\} \quad \text{and} \quad y_k = \left\{\frac{k+p}{c}\right\}. \tag{3.5}$$

Proof. We know by Lemma 5 of [Ar1] that

$$h_{-1}(\alpha, (-p, -q)) = h_{-1}(\alpha, (p, q))\,.$$

It follows from the identities (3.2) and (3.4) that

$$h_{-1}(\alpha, (-p, -q)) - \chi(p)\chi(q) = \frac{2\pi i}{\log \eta}\chi(p)\left(\frac{1}{2} - \langle q \rangle\right) + \frac{2\pi i}{\log \eta} \sum_{j=1}^{c} \zeta_2\left(0, \eta, 1 - \left\{\frac{jd+\rho^*}{c}\right\} + \frac{(j-\{-p\})\eta}{c}\right), \tag{3.6}$$

where $\rho^*=\{-q\}c-\{-p\}d$. A slight modification of the summation in (3.6) yields

$$(3.7)\quad \sum_{j=1}^{c}\zeta_2\left(0, \eta, 1-\left\{\frac{jd+\rho^*}{c}\right\}+\frac{(j-\{-p\})\eta}{c}\right)-\sum_{k \bmod c}\zeta_2(0, \eta, x_k+y_k\eta)$$
$$=\chi(p)(\zeta_2(0, \eta, 1-\{-q\}+\eta)-\zeta_2(0, \eta, 1-\{-q\}))\,.$$

An easy computation with the use of the identity (3.2) shows that

$$\zeta_2(0, \eta, x+y\eta)=\frac{1}{2}B_2(x)\eta^{-1}+\frac{1}{2}B_2(y)\eta+B_1(x)B_1(y)$$

(see (1.10) of [Sht2]),

where $x, y>0$ and $B_k(x)$ is the k-th Bernoulli polynomial. Thus the right hand side of the equality (3.7) coincides with

$$\chi(p)\left(\langle q\rangle-\frac{1}{2}\right).$$

Therefore the identity (3.6) with the help of (3.7) turns out the first identity in Proposition 3.2. Another identity is similarly verified. ■

Let $a=(a_1, a_2)$ be a pair of positive numbers and $x=(x_1, x_2)$ a pair of non-negative numbers with $x\neq(0, 0)$. Shintani [Sht2] defined the following zeta-function $\zeta(s, a, x)$:

$$\zeta(s, a, x)=\sum_{m,n=0}^{\infty}\prod_{j=1}^{2}(x_1+m+(x_2+n)a_j)^{-s}\,,$$

which is absolutely convergent for $\mathrm{Re}(s)>1$. It has been proved that the zeta-function $\zeta(s, a, x)$ is continued analytically to a meromorphic function of s in the whole complex plane which is holomorphic at $s=0$ and moreover that

$$(3.8)\quad \zeta(0, a, x)=\frac{1}{2}(\zeta_2(0, a_1, x_1+x_2a_1)+\zeta_2(0, a_2, x_1+x_2a_2))$$

(see [Sht 1], (1.11) of [Sht2] and [Eg]).

Let $\mathfrak{f}$ be an integral ideal of F and $E_+(\mathfrak{f})$ the group of totally positive unit u of F with $u-1\in\mathfrak{f}$. We denote by η the generator of the group $E_+(\mathfrak{f})$ with $\eta>1$. For each class C of $H_F(\mathfrak{f})$, take an integral ideal $\mathfrak{b}$ of C and a basis $\{\beta_1, \beta_2\}$ of the ideal $\mathfrak{f}\mathfrak{b}^{-1}$ with the conditions $\beta_1\beta_2'-\beta_1'\beta_2>0$, $\beta_2\beta_2'>0$. We represent the unit η via the basis $\{\beta_1, \beta_2\}$ to get an element V of $SL_2(\boldsymbol{Z})$ such that

$$\eta\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix},\quad V=\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

A pair (p, q) of $\boldsymbol{Q}^2$ is uniquely determined by the relation

$$(3.9)\qquad p\beta_1+q\beta_2=1\,.$$

Since $\eta\in E_+(\mathfrak{f})$, we necessarily have

$$(p, q)V\equiv(p, q) \bmod \boldsymbol{Z}^2\,.$$

Set $\beta=\beta_1/\beta_2$. Then, β, η, V and (p, q) satisfy the conditions in (3.3) with α being replaced by β. We have proved in 4 of [Ar1] that the partial zeta-function $\zeta_{\mathfrak{f}}(\mathfrak{b}, s)$ has the decomposition

$$\begin{aligned}\zeta_{\mathfrak{f}}(\mathfrak{b}, s) &= N(\beta_2\mathfrak{b})^{-s}\sum_{k \bmod c}\sum_{m,n=0}^{\infty} N(x_k+y_k\eta+m+n\eta)^{-s}\\ &= N(\beta_2\mathfrak{b})^{-s}\sum_{k \bmod c}\zeta(s, (\eta, \eta'), (x_k, y_k))\,,\end{aligned}$$

where x_k, y_k are given by (3.5) (see also p. 409, §2 of [Sht1] and [Ar2]). Therefore it is immediate to see from (3.8) that the special value $\zeta_{\mathfrak{f}}(\mathfrak{b}, 0)$ at $s=0$ is given by the identity

$$(3.10)\qquad \zeta_{\mathfrak{f}}(\mathfrak{b}, 0)=\frac{1}{2}\sum_{k \bmod c}(\zeta_2(0, \eta, x_k+y_k\eta)+\zeta_2(0, \eta', x_k+y_k\eta'))\,.$$

The following theorem is immediate from Proposition 3.2 and (3.10).

Theorem 3.3. *Let $\mathfrak{b}, \mathfrak{f}$ be coprime integral ideals of F. Choose a basis $\{\beta_1, \beta_2\}$ of the ideal $\mathfrak{f}\mathfrak{b}^{-1}$ with $\beta_1\beta_2'-\beta_1'\beta_2>0$, $\beta_2\beta_2'>0$. Let η denote the generator of the group $E_+(\mathfrak{f})$ with $\eta>1$. Let $(p, q)\in\boldsymbol{Q}^2$ be the same as in (3.9). Set $\beta=\beta_1/\beta_2$. Then,*

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0)=\frac{\log\eta}{4\pi i}(h_{-1}(\beta, (p, q))-h_{-1}(\beta', (p, q)))\,.$$

Now we descirbe the map $\xi_{\mathfrak{a}}\colon F/\mathfrak{a}\to W(F_{ab})$ in terms of the coefficient $h_{-1}(\alpha, (p, q))$. We set, for $\alpha\in F-\boldsymbol{Q}$ and $(p, q)\in\boldsymbol{Q}^2$,

$$\mathfrak{h}(\alpha, (p, q))=\frac{1}{2}(h_{-1}(\alpha, (p, q))-h_{-1}(\alpha', (p, q)))\,.$$

We denote by G the group GL_2 defined over $\boldsymbol{Q}$. Let $G_A=GL_{2,A}$ be the adelized group of G. For each $x\in G_A$, denote by x_∞ the archimedean component of x. Set

$$\begin{aligned}G_{\infty,+}&=GL_{2,+}(\boldsymbol{R})=\{x\in GL_2(\boldsymbol{R})\,|\,\det x>0\}\,,\\ G_{\boldsymbol{Q},+}&=GL_{2,+}(\boldsymbol{Q})=\{x\in GL_2(\boldsymbol{Q})\,|\,\det x>0\}\,,\\ G_{A,+}&=\{x\in G_A\,|\,\det x_\infty>0\}\,,\end{aligned}$$

and

$$U=\prod_p GL_2(\boldsymbol{Z}_p)\times G_{\infty,+}\,,$$

where $\boldsymbol{Z}_p$ is the ring of p-adic integers. We have the decomposition

$$G_{A,+} = G_{Q,+}U = UG_{Q,+}\,. \tag{3.11}$$

Let L be a $\boldsymbol{Z}$-lattice in $\boldsymbol{Q}^2$. Set $L_p = L \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$. For an element x of G_A, we define Lx to be the $\boldsymbol{Z}$-lattice characterized by $(Lx)_p = L_p x_p$ in $\boldsymbol{Q}_p^2 = L \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$. Moreover any element x of G_A has a natural action on the quotient space $\boldsymbol{Q}^2/L$ by the right multiplication and defines an isomorphism of $\boldsymbol{Q}^2/L$ to $\boldsymbol{Q}^2/Lx$ We denote by rx the image of an element $r \in \boldsymbol{Q}^2/L$ by this isomorphism. For any $x \in G_{A,+}$, we write

$$x = ug \quad \text{with} \quad u \in U, g \in G_{Q,+}.$$

We define the action of x on $\mathfrak{h}(\alpha, (p, q))$ to be

$$\mathfrak{h}^x(\alpha, (p, q)) = \mathfrak{h}(g\alpha, (p, q)u)\,, \tag{3.12}$$

where we note that the element $(p, q)u$ is uniquely determined as an element of $\boldsymbol{Q}^2/\boldsymbol{Z}^2$. Since $G_{Q,+} \cap U = SL_2(\boldsymbol{Z})$, the right hand side of the equality (3.12) is independent of the decomposition $x = ug (u \in U,\ g \in G_{Q,+})$ according to (3.1).

Let $\mathfrak{a}$ be a fractional ideal of F with an oriented basis $\{\alpha_1, \alpha_2\}$ (namely, $\mathfrak{a} = \boldsymbol{Z}\alpha_1 + \boldsymbol{Z}\alpha_2$, $\alpha_1\alpha_2' - \alpha_1'\alpha_2 > 0$). Choose a representative element $z \neq 0$ of the class $\bar{z} \in F/\mathfrak{a}$ and write

$$z\mathfrak{a}^{-1} = \mathfrak{f}^{-1}\mathfrak{b}$$

with coprime integral ideals $\mathfrak{f}$, $\mathfrak{b}$ of F. A pair (p, q) of rational numbers is uniquely determined by

$$z = p\alpha_1 + q\alpha_2\,.$$

Let $q\colon F^\times \to GL_2(\boldsymbol{Q})$ be the homomorphism given by (1.6) in the introduction which is defined via the basis $\{\alpha_1, \alpha_2\}$ of $\mathfrak{a}$. We also use the same symbol q for the natural extension of q to the homomorphism of $F_A^\times$ to G_A. Obviously, $q(F_{A,+}^\times) \subset G_{A,+}$.

A description of the map $\xi_{\mathfrak{a}}\colon F/\mathfrak{a} \to W(F_{ab})$ in this case is formulated in Theorem B in the introduction. Now under the above preparations we can give its proof.

Proof of Theorem B. Let the notation be the same as in the assertion of Theorem B. The expression on the right hand side of (1.7) is independent of the choice of an oriented basis $\{\alpha_1, \alpha_2\}$ of $\mathfrak{a}$ in virtue of Proposition 3.1. Therefore we may assume that

$$\alpha_1\alpha_2' - \alpha_1'\alpha_2 > 0\,, \quad \alpha_2\alpha_2' > 0\,,$$

if necessary, by change of a basis $\{\alpha_1, \alpha_2\}$ of $\mathfrak{a}$. We choose an element z_1 of $F_+^\times$ such that $z - z_1 \in \mathfrak{a}$ and set $z_1 = p_1\alpha_1 + q_1\alpha_2$ with a pair of rational numbers

(p_1, q_1) We can write

$$z_1\mathfrak{a}^{-1} = \mathfrak{f}^{-1}\mathfrak{b}_1$$

with an integral ideal $\mathfrak{b}_1$ of F prime to the same $\mathfrak{f}$. Then,

$$\mathfrak{f}\mathfrak{b}_1^{-1} = z_1^{-1}\boldsymbol{a} = \boldsymbol{Z}(\alpha_1/z_1)+\boldsymbol{Z}(\alpha_2/z_1)\,,$$
$$p_1(\alpha_1/z_1)+q_1(\alpha_2/z_1) = 1\,.$$

Noticing that z_1 is also a representative element of the calss $\bar{z}$, we get, by the definition (1.2) of the map $\xi_{\mathfrak{a}}$,

$$\xi_{\mathfrak{a}}(\bar{z}) = \exp(2\pi i\zeta_{\mathfrak{f}}(\mathfrak{b}_1, 0))\,.$$

By virtue of Theorem 3.3 the special value $\zeta_{\mathfrak{f}}(\mathfrak{b}_1, 0)$ has the expression

$$\zeta_{\mathfrak{f}}(\mathfrak{b}_1, 0) = \frac{\log \eta}{2\pi i}\,\mathfrak{h}(\alpha, (p_1, q_1))\,,$$

where we put $\alpha=\alpha_1/\alpha_2$. Since $(p_1, q_1)\equiv(p, q) \bmod \boldsymbol{Z}^2$, we immediately have the identity (1.7).

Next let $s\in F_{A,+}^{\times}$ and write

$$q(s)^{-1} = ug \quad \text{with} \quad u\in U, g\in G_{\boldsymbol{Q},+}\,.$$

We set

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = g\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}.$$

Obviously,

$$\beta_1\beta_2'-\beta_1'\beta_2>0\,.$$

Then we see easily that

$$\begin{aligned} s^{-1}\mathfrak{a} &= \boldsymbol{Z}^2q(s)^{-1}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \boldsymbol{Z}^2g\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\\ &= \boldsymbol{Z}\beta_1+\boldsymbol{Z}\beta_2\end{aligned}$$

and moreover that

$$s^{-1}z \equiv (p, q)u\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \bmod s^{-1}\mathfrak{a}\,,$$

where $(p, q)u$ stands for an element of $\boldsymbol{Q}^2/\boldsymbol{Z}^2$ and where $s^{-1}z$ is not determined as an element of F but uniquely determined modulo $s^{-1}\mathfrak{a}$. Choose a representative element $\theta(\theta\neq0)$ of the class $\overline{s^{-1}z}=s^{-1}z \bmod s^{-1}\mathfrak{a}$. We see from (2.2), (2.3) in the proof of Theorem A that

$$\theta(s^{-1}\mathfrak{a})^{-1} = \mathfrak{f}^{-1}\mathfrak{b}_2$$

with some integral ideal $\mathfrak{b}_2$ of F prime to $\mathfrak{f}$. Set $\beta=\beta_1/\beta_2$. Thus we have, by the expression (1.7) and the definition (3.12),

$$\begin{aligned}\xi_{s^{-1}\mathfrak{a}}(\overline{s^{-1}z}) &= \exp(\log\eta\cdot\mathfrak{h}(\beta,(p,q)u))\\ &= \exp(\log\eta\cdot\mathfrak{h}(g\alpha,(p,q)u))\\ &= \exp(\log\eta\cdot\mathfrak{h}^{q(s)^{-1}}(\alpha,(p,q)))\,.\end{aligned}$$

Finally thanks to Theorem A in the introductoton we obtain the identity (1.8). ■

We continue the assumption that F is a real quadratic field. For $\alpha\in F-\boldsymbol{Q}$, we define $\xi(s,\alpha)$ to be the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{\cot\pi n\alpha}{n^s}\,.$$

We have proved in [Ar2] that $\xi(s,\alpha)$ is absolutely convergent for $\mathrm{Re}(s)>1$ and that it can be continued analytically to a meromorphic function in the whole s-plane. Moreover, $\xi(s,\alpha)$ has a simple pole at $s=1$. We denote by $c_{-1}(\alpha)$ the residue of $\xi(s,\alpha)$ at the simple pole $s=1$. The function $H(\alpha,s,(0,0))$ given by (1.4) has the following obvious connection with $\xi(s,\alpha)$:

$$H(\alpha,s,(0,0))=\frac{1+e^{\pi is}}{2}\cdot(i\xi(1-s,\alpha)-\zeta(1-s))\,,$$

where $\zeta(s)$ is the Riemann zeta function. Thus we have

$$h_{-1}(\alpha,(0,0))=-ic_{-1}(\alpha)+1\,.$$

Since $c_{-1}(\alpha')=-c_{-1}(\alpha)$ (see Proposition 2.10 of [Ar2]), it follows that

$$\mathfrak{h}(\alpha,(0,0))=-ic_{-1}(\alpha)\,.$$

Let ε be the totally positive fundamental unit of F with $\varepsilon>1$. Choose a basis $\{\alpha_1,\alpha_2\}$ of a fractional ideal $\mathfrak{a}$ of F such that

$$\alpha_1\alpha_2'-\alpha_1'\alpha_2>0\,,\quad \alpha_2\alpha_2'>0\,.$$

We represent ε by the basis $\{\alpha_1,\alpha_2\}$ to get a matrix V of $SL_2(\boldsymbol{Z})$:

$$\varepsilon\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=V\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix},\quad V=\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

We get, by Theorem B,

$$\begin{aligned}\xi_{\mathfrak{a}}(0 \bmod \mathfrak{a}) &= \exp(\log\varepsilon\cdot h((\alpha,(0,0))\\ &= \exp(-i\log\varepsilon\cdot c_{-1}(\alpha))\,,\end{aligned}$$

where we put $\alpha=\alpha_1/\alpha_2$. Taking the facts $V\alpha=\alpha$, $c>0$, $c\alpha+d>0$ into account, we have, with the help of Proposition 2.9, (i) of [Ar2],

$$c_{-1}(\alpha) = -\frac{2\pi}{\log \varepsilon}\left(\frac{a+d}{12c} - s(d, c) - \frac{1}{4}\right),$$

where $s(d, c)$ is the Dedekind sum (for the Dedekind sum we refer the reader to [R-G]). Hence,

$$\xi_{\mathfrak{a}}(0 \bmod \mathfrak{a}) = \exp\left(2\pi i\left(\frac{a+d}{12c} - s(d, c) - \frac{1}{4}\right)\right).$$

It is known that the value $(a+d)/c - 12s(d, c)$ is a rational integer (see Ch. 4 of [R-G] and Remark 3.2 of [Ar2]). Therefore the value $\xi_{\mathfrak{a}}(0 \bmod \mathfrak{a})$ is a twelfth root of unity.

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