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ON SPECIAL VALUES AT \( s=0 \) OF PARTIAL ZETA-FUNCTIONS FOR REAL QUADRATIC FIELDS

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1. Introduction

1.1 Let \( F \) be a totally real algebraic number field with finite degree, \( \alpha \) a fractional ideal of \( F \), and \( F_{ab} \) the maximal abelian extension of \( F \). We define a map \( \xi_\alpha \) from the quotient space \( F/\alpha \) to the group \( W(F_{ab}) \) of roots of unity of \( F_{ab} \) using the deep results of Coates-Sinnott \([C-S1], [C-S2]\) and Deligne-Ribet \([D-R]\) on special values of partial zeta functions of \( F \). Under the action of the Galois group \( \text{Gal}(F_{ab}/F) \) this map behaves formally in a manner similar to Shimura's reciprocity law for elliptic curves with complex multiplication. This reciprocity law for the map \( \xi_\alpha \) is also a direct consequence of those results of Coates-Sinnott and Deligne-Ribet. On the other hand we have studied in \([Ar1]\) a certain Dirichlet series and its relationship with partial zeta functions of real quadratic fields. In particular the special values at \( s=0 \) of partial zeta functions of real quadratic fields essentially coincide with the residues at the pole \( s=0 \) of our Dirichlet series. Using those residues, we give another expression for the map \( \xi_\alpha \) in the case of \( F \) a real quadratic field. We also show that the expression works in a reasonable manner under the action of the Galois group \( \text{Gal}(F_{ab}/F) \).

1.2 We summarize our results. For an integral ideal \( c \) of a totally real algebraic number field \( F \), denote by \( H_F(c) \) the narrow ray class group modulo \( c \). For each integral ideal \( b \) prime to \( c \), we define the partial zeta-function \( \zeta_c(b, s) \) to be the sum \( \sum_{\alpha} (N\alpha)^{-s} \), \( \alpha \) running over all integral ideals of the class of \( b \) in \( H_F(c) \). Let \( \alpha \) be a fractional ideal of \( F \). For each class \( \bar{z} \) of the quotient space \( F/\alpha \), we take a totally positive representative element \( z \in F \) of the class \( \bar{z} \), and write

\[
\bar{z} \alpha^{-1} = \bar{f}^{-1} \bar{b}
\]

with coprime integral ideals \( \bar{f}, \bar{b} \) of \( F \). Thanks to some results of Coates-Sinnott \([C-S1], [C-S2], [Co]\) and Deligne-Ribet \([D-R]\), one can define a map \( \xi_\alpha : F/\alpha \to W(F_{ab}) \) as follows;

\[
\xi_\alpha(\bar{z}) = \exp(2\pi i \zeta_{\bar{f}}(b, 0)),
\]
where the value on the right hand side of the equality depends on the class $\bar{z}$ and not on a representative element $z$ of $\bar{z}$. Denote by $F_A^\times$ the idele group of $F$ and by $F_A^\times,+\,$ the subgroup of $F_A^\times\,$ consisting of ideles $x$ whose archimedean components $x_\infty$ are totally positive. Each element $s$ of $F_A^\times\,$ induces a natural isomorphism $s: F|a \approx F|a$. We denote by $[s, F]$ the canonical Galois automorphism of the extension $F_{ab}/F$ induced by $s\in F_A^\times\,$. The following theorem is a reformulation of a part of the results due to Coates-Sinnott and Deligne-Ribet ([C-S1], [C-S2], [D-R]).

**Theorem A** (Coates-Sinnott, Deligne-Ribet)

Let $s\in F_A^\times,+\,$ and set $\sigma=[s, F]$. Then the following diagram is commutative.

\[
\begin{array}{ccc}
F|a & \xrightarrow{s\bar{a}} & W(F_{ab}) \\
\downarrow s^{-1} & & \downarrow \sigma \\
F|s^{-1}a & \xrightarrow{s^{-1}\bar{a}} & W(F_{ab})
\end{array}
\]

Namely,

\[\xi_a(\bar{z})^\sigma = \xi_{s^{-1}a}(s^{-1}z),\]

where $s^{-1}z$ stands for the image of $\bar{z}$ by the isomorphism $s^{-1}: F|a \approx F|a$.

In particular if we write, with $\bar{z}$ being specialized at $0=0 \mod a$,

\[\xi(a) = \xi_a(0),\]

then, $\xi(a)$ is a root of unity contained in the narrow Hilbert class field of $F$. In this case the Galois action is described in the simple manner:

\[\xi(a)^{[r,F]} = \xi(s^{-1}a) \quad \text{for any } s\in F_A^\times,+\,.
\]

Theorem A will be interpreted as a formal analogy to Shimura's reciprocity law for elliptic curves with complex multiplication (see Theorem 5.4 of [Shm]).

For a real number $x$, we denote by $\langle x \rangle$ the real number satisfying $x-\langle x \rangle \in \mathbb{Z}$ and $0 < \langle x \rangle \leq 1$. Let $F$ be a real quadratic field embedded in $\mathbb{R}$. We set, for $\alpha \in F-\mathbb{Q}$ and $(p, q) \in \mathbb{Q}^2$,

\[\eta(\alpha, s, p, q) = \sum_{n=1}^{\infty} \frac{\exp(2\pi in(p\alpha+q))}{1-\exp(2\pi in\alpha)}\]

and

\[H(\alpha, s, (p, q)) = \eta(\alpha, s, \langle p \rangle, q) + e^{\pi i s} \eta(\alpha, s, \langle -p \rangle, -q) .\]

This type of infinite series has been intensively studied by Berndt [Be1], [Be2],
if $\alpha$ is a complex number with positive imaginary part. In our case we have proved in [Ar1] that the infinite series $\eta(\alpha, s, p, q)$ is absolutely convergent for $\Re(s)<0$ and moreover that $H(\alpha, s, (p, q))$ can be analytically continued to a meromorphic function of $s$ in the whole $s$-plane which has a possible simple pole at $s=0$. Let $h_{-1}(\alpha, (p, q))$ denote the residue at the pole $s=0$ of this function $H(\alpha, s, (p, q))$ (see §3 of this paper). We set

$$h(\alpha, (p, q)) = \frac{1}{2}(h_{-1}(\alpha, (p, q)) - h_{-1}(\alpha', (p, q))),$$

where $\alpha'$ denotes the conjugate of $\alpha$ in $F$. This quantity $h(\alpha, (p, q))$ satisfies the transformation law under the action of $SL_2(\mathbb{Z})$:

$$(1.5) \quad h(V\alpha, (p, q)) = h(\alpha, (p, q)V) \quad \text{for any } V \in SL_2(\mathbb{Z}).$$

We denote by $F^\times$ the group of of invertible elements of $F$. Let $\alpha$ be a fractional ideal of $F$ with an oriented basis $\{a_1, a_2\}$ (i.e., $\alpha = \mathcal{O}_F + a_1\mathcal{O}_F$, $a_1a_2 - a_1a_2 > 0$). Denote by $q : F^\times \rightarrow GL_2(\mathbb{Q})$ the injective homomorphism of $F^\times$ into $GL_2(\mathbb{Q})$ defined via the basis $\{a_1, a_2\}$ as follows;

$$(1.6) \quad \mu^{(a_1, a_2)} = q(\mu)(\frac{a_1}{a_2}) \quad (\mu \in F^\times).$$

This homomorphism $q$ is naturally extended to that of $F^\times_\mathfrak{p}$ into the adele group $G_A = GL_2(\mathbb{Q}_A)$. Denote by $G_{A,+}$ the subgroup of $G_A$ consisting of all elements $x \in G_A$ whose archimedean components $x_\mathfrak{p}$ have positive determinants. By the transformation law (1.5) of $h(\alpha, (p, q))$, one can define an action of any $x \in G_{A,+}$ on the coefficient $h(\alpha, (p, q))$. This action will be denoted by $h^q(\alpha, (p, q))$ (for the precise definition see (3.12)). For an integral ideal $\mathfrak{f}$ of $F$, we denote by $E_+(\mathfrak{f})$ the group of totally positive units $u$ of $F$ with $u - 1 \in \mathfrak{f}$. Another expression for the map $\xi_\alpha(z)$ is given by the following theorem.

**Theorem B** Let the notation be the same as above. Let $\alpha$ be a fractional ideal of a real quadratic field $F$ with the oriented basis $\{a_1, a_2\}$. Choose a representative element $z \in F$, $z \neq 0$ of a class $\bar{z} \in F'/\alpha$ and determine the ideal $\mathfrak{f}$ by (1.1). Denote by $\eta$ the generator of the group $E_+(\mathfrak{f})$ with $\eta > 1$. Write $z = p\alpha_1 + q\alpha_2$ with $(p, q) \in \mathbb{Q}^2$ and set $\alpha = \alpha_1\alpha_2$. Then, 

$$(1.7) \quad \xi_\alpha(\bar{z}) = \exp(\log \eta \cdot h(\alpha, (p, q))).$$

Let $s \in F_{A,+}^\times$. The Galois action on $\xi_\alpha(\bar{z})$ is given by the equality 

$$(1.8) \quad (\xi_\alpha(\bar{z}))^{\mathfrak{f}^{-1}} = \exp(\log \eta \cdot h^{\mathfrak{f}^{-1}}(\alpha, (p, q))).$$

In Theorem 3.3 we obtain a stronger result than (1.7); namely, the special value $\zeta(\mathfrak{f}, 0)$ is explicitly given by the value $h(\alpha, (p, q))$. We note that, as
is essentially known, the value $\xi(\alpha) = \xi_{a}(\bar{0})$ is a twelfth root of unity in the narrow Hilbert class field of $F$ (see the end of §3).

2. Partial zeta-functions for totally real number fields

We recall a part of the results of [C-Sl, 2], [Co], and [D-R] concerning special values at non-positive integers of parital zeta-functions for totally real algebraic number fields.

Let $\mu_m$ denote the group of $m$-th roots of unity. Let $L$ be an algebraic number field. If $K$ is a Galois extension of $L$, we write $Gal(K/L)$ for the Galois group of $K$ over $L$. For a positive integer $n$, we define $w_n(L)$ to be the largest integer $m$ such that the exponent of the group $Gal(L(\mu_m)/L)$ divides $n$ (see 2.2 of [Co]). In particular if $n=1$, $w_1(L)$ is nothing but the number of roots of unity of $L$. We denote by $W(L)$ the group of roots of unity of $L$.

Let $F$ be a totally real algebraic number field with finite degree throughout this paragraph. For an integral ideal $\mathfrak{f}$ of $F$, denote by $H_F(\mathfrak{f})$ the narrow ray class group modulo $\mathfrak{f}$. Namely, $H_F(\mathfrak{f})$ is the quotient group $I_F(\mathfrak{f})/P_+(\mathfrak{f})$, where $I_F(\mathfrak{f})$ is the group of fractional ideals of $F$ prime to $\mathfrak{f}$ and $P_+(\mathfrak{f})$ is the group of principal ideals of $F$ generated by totally positive elements $\theta$ of $F$ such that the numerators of $\theta-1$ are divisible by $\mathfrak{f}$. We set, for each class $C$ of $H_F(\mathfrak{f})$,

$$
\zeta_f(C, s) = \sum_{\alpha} (Na)^{-s} \quad (\text{Re}(s) > 1),
$$

where $\alpha$ runs over all integral ideals of $C$ and $Na$ denotes the norm of $\alpha$. The partial zetafunction $\zeta_f(C, s)$ is analytically continued to a meromorphic function in the whole $s$-plane which is holomorphic at non-positive integers. If $b$ is a representative ideal of $C$, we often write $\zeta_f(b, s)$ in place of $\zeta_f(C, s)$. Let $K=K_F(\mathfrak{f})$ be the maximal narrow ray class field of $F$ defined modulo $\mathfrak{f}$. We write $[C, K/F]$ for the Artin symbol of the class $C$ of $H_F(\mathfrak{f})$. By the class field theory there exists a canonical isomorphism of $H_F(\mathfrak{f})$ to the Galois group $Gal(K/F)$ given by the correspondence: $C \rightarrow [C, K/F]$. If $b$ is a representative ideal of the class $C$, we write $[b, K/F]$ for $[C, K/F]$. The following theorem is due to Coates-Sinnott [C-S1, 2] in the case of real quadratic fields and to Deligne-Ribet [D-R] in general.

**Theorem 2.1.** (Coates-Sinnott, Deligne-Ribet) Let $\mathfrak{f}$ be an integral ideal of $F$ and $b, c$ integral ideals of $F$ which are prime to $\mathfrak{f}$. Set $K=K_F(\mathfrak{f})$. For each non-negative integer $n$,

(i) $w_{n+1}(K)\zeta_f(b, -n)$ is an integer.

(ii) Moreover if $c$ is prime to $w_{n+1}(K)$, then the value

$$
(Nc)^{n+1} \zeta_f(b, -n) - \zeta_f(bc, -n)
$$

is also an integer.
In the case of \( n=0 \), we reformulate the above theorem into a slightly different form suitable to our later situation. For that purpose we recall briefly the class field theory in the adelic language (see [C-F]).

Denote by \( F_+^\times \) the group of totally positive elements of \( F \). Let \( F_+^\times_A \) denote the idele group of \( F \), \( F_+^\times \) the archimedean part of \( F_+^\times_A \), and \( F_+^\times_{a,+} \) the connected component of the identity of \( F_+^\times \), respectively. We denote by \( F_+^\times_A,+ \) the subgroup of \( F_+^\times_A \) consisting of elements \( x \in F_+^\times_A \) whose archimedean component \( x_\infty \) are contained in \( F_+^\times_{a,+} \). For each element \( x \) of \( F_+^\times_A \) and for a finite prime \( p \) of \( F \), we denote by \( x_p \) the \( p \)-component of \( x \) and define a fractional ideal \( \mathcal{I}(x) \) of \( F \) by putting \( \mathcal{I}(x)p = x_p\mathcal{O}_p \) for all finite \( p \), where \( \mathcal{O}_p \) is the maximal order of the completion \( F_p \) of \( F \) at \( p \).

Set
\[
U = \{ x \in F_+^\times_A | x_p \in \mathcal{O}_p^\times \text{ for all finite primes } p \text{ of } F \},
\]
\( \mathcal{O}_p^\times \) being the unit group of \( \mathcal{O}_p \). Set, for an integral ideal \( \mathfrak{f} \),
\[
W_+(\mathfrak{f}) = \{ x \in F_+^\times_A | x_\infty \in F_+^\times_{a,+} \text{ and } x_p - 1 \in \mathcal{O}_p \text{ for all } p \text{ dividing } \mathfrak{f} \},
\]
\[
U_+(\mathfrak{f}) = U \cap W_+(\mathfrak{f}).
\]

By the class field theory there exists a canonical exact sequence
\[
1 \longrightarrow \overline{F_+^\times F_+^\times_{a,+}} \longrightarrow F_+^\times_A \longrightarrow Gal(F_{ab}/F) \longrightarrow 1,
\]
s \mapsto \([s, F]\)

where \( \overline{F_+^\times F_+^\times_{a,+}} \) is the closure of \( F_+^\times F_+^\times_{a,+} \) in \( F_+^\times_A \) and where we denote by \([s, F]\) the element of \( Gal(F_{ab}/F) \) corresponding to an element \( s \) of \( F_+^\times_A \). If we take an element \( u \) of \( W_+(\mathfrak{f}) \), then the Galois automorphism \([u, F]\) coincides with the Artin symbol \([\mathcal{I}(u), K_F(\mathfrak{f})/F]\) on the narrow ray class field \( K_F(\mathfrak{f}) \) over \( F \).

Let \( \mathfrak{a} \) be a fractional ideal of \( F \). To define the map \( \xi_\mathfrak{a} \) of the quotient space \( F/\mathfrak{a} \) to the group \( W(F_{ab}) \) by the equality (1.2), we have to prove that the right hand side of (1.2) depends only on the class \( \bar{z} \in F/\mathfrak{a} \) (not on the choice of a representative element \( z \) of \( \bar{z} \)) and moreover that the image of \( \xi_\mathfrak{a} \) is in \( W(F_{ab}) \). To see this we take another element \( z_1 \) of \( F_+^\times \) with the condition \( z - z_1 \subseteq \mathfrak{a} \). Let \( \mathfrak{f}, \mathfrak{b} \) be the same coprime integral ideals of \( F \) as in (1.1). Then we have
\[
z_1 \mathfrak{a}^{-1} = \mathfrak{f}^{-1} \mathfrak{b}_1
\]
with some integral ideal \( \mathfrak{b}_1 \) prime to \( \mathfrak{f} \). We see easily that \( \mathfrak{b} \) and \( \mathfrak{b}_1 \) are in the same class of \( H_F(\mathfrak{f}) \). Therefore,
\[
\xi_{\mathfrak{f}}(\mathfrak{b}, 0) = \xi_{\mathfrak{f}}(\mathfrak{b}_1, 0)
\]
By virtue of the assertion (i) of Theorem 2.1 the value
\[
\exp(2\pi i \xi_{\mathfrak{f}}(\mathfrak{b}, 0))
\]
is a root of unity of $K_F(f)$. Thus the map $\xi_\alpha$ given by (1.2) defines a map of $F/\alpha$ to $W(F_\alpha)$.

Any element $x$ of $F_\alpha$ acts naturally on a fractional ideal $\alpha$ of $F$. The ideal $x\alpha$ of $F$ is characterized by the property $x\alpha = \iota(l(x)\alpha$. For each element $u$ of $F$, there exists an element $v$ of $F$ such that

$$ v = x_p u \in x_p \alpha_p \quad \text{for all prime ideals } p \text{ of } F, $$

where $\alpha_p = \alpha_p$ in $F_p$. Thus we obtain a natural isomorphism of $F/\alpha$ to $F/x\alpha$ by the correspondence $u \mod \alpha \leftrightarrow v \mod x\alpha$. We denote this isomorphism by $x: F/\alpha \to F/x\alpha$ and write $xu \mod x\alpha$ for the image of $u \mod \alpha$.

A part of the theorem of Coates-Sinnott and Deligne-Ribet (Theorem 2.1) can be formulated in terms of the adele language as in Theorem A in the introduction. For the completeness we give its proof here.

**Proof of Theorem A.**

We take a representative element $z \in F_\alpha$ of a class $z \in F/\alpha$ and write $z\alpha^{-1} = f^{-1}b$ with coprime integral ideals $f, b$ of $F$ as in (1.1). Set $K = K_F(f)$. For $s \in F_\alpha$, we decompose $s = au$ with $a \in F_\alpha$, $u \in W_+(f)$. Moreover we may choose $u$ so that $il(u)$ is an integral ideal prime to $\omega(K)$. Set, for simplicity, $c = il(u)$. Since by definition

$$ \xi_\alpha(z) = \exp(2\pi i \xi_f(b, 0)) \in W(K), $$

we have, for $\sigma = [s, F]$,

$$ \xi_\alpha(z)^\sigma = \xi_\alpha(z)^{[\sigma, F]} = \exp(2\pi i \xi_f(b, 0))^{[\sigma, F]} = \exp(2\pi i Nc \xi_f(b, 0)). $$

Therefore Theorem 2.1 implies that

$$ (2.1) \quad \xi_\alpha(z)^\sigma = \exp(2\pi i \xi_f(ch, 0)). $$

On the other hand since $u \in W_+(f)$ and $u_p \in \mathcal{O}_p$ for all prime ideals $p$ of $F$, we see immediately that

$$ 1 - u_p \in (f^{-1}b) \mathcal{O}_p \quad \text{for all prime ideals } p \text{ of } F. $$

Thus for every prime ideal $p$ of $F$,

$$ u_p^{-1}x - x \in x f^{-1} u_p^{-1} \mathcal{O}_p, $$

which turns out that

$$ u^{-1}x \equiv x \mod u^{-1} \alpha. $$

Hence,
(2.2) \[ s^{-1}z = a^{-1}z \mod s^{-1}a, \]
where we see that
(2.3) \[ a^{-1}z \in F^\times_+ \quad \text{and} \quad a^{-1}z(s^{-1}a)^{-1} = f^{-1}b. \]

Therefore,
\[ \xi_{s^{-1}a}(s^{-1}z \mod s^{-1}a) = \exp(2\pi i \xi_f(b, 0)), \]
which together with (2.1) completes the proof of Theorem A.

3. Special values at \( s=0 \) of partial zeta-functions for real quadratic fields

First we recall some results of \([Ar1]\). For a real number \( x \), denote by \( \{x\} \) (resp. \( <x> \)) the real number satisfying \( x - \{x\} \in \mathbb{Z} \) (resp. \( 0 < <x> \leq 1, \quad x - <x> \in \mathbb{Z} \)).

We note here that \( \{x\} + <-x> = 1 \). In this paragraph let \( F \) be a real quadratic field embedded in \( \mathbb{R} \) and fix it once and for all. For each \( \alpha \) of \( F \), let \( \alpha' \) denote the conjugate of \( \alpha \) in \( F \). For \( \alpha \in F - \mathbb{Q} \) and \( (p, q) \in \mathbb{Q}^2 \), we define a Lambert series \( \eta(\alpha, s, p, q) \) by the equality (1.3) in the introduction. The infinite series \( \eta(\alpha, s, p, q) \) is absolutely convergent for \( \Re(s) < 0 \) (see Lemma 1 of \([Ar1]\)). We also define the function \( H(\alpha, s, (p, q)) \) of \( s \) by the equality (1.4) in the introduction. We note that \( H(\alpha, s, (p, q)) \) depends on \( (p, q) \mod \mathbb{Z}^2 \). As we have seen in \([Ar1]\), this function \( H(\alpha, s, (p, q)) \) can be analytically continued to a meromorphic function of \( s \) in the whole \( s \)-plane and has a Laurent expansion at \( s=0 \) of the form:
\[ H(\alpha, s, (p, q)) = \frac{h_{-1}(\alpha, (p, q))}{s} + h_0(\alpha, (p, q)) + \cdots. \]

Moreover the first coefficient \( h_{-1}(\alpha, (p, q)) \) satisfies under the action of \( SL_2(\mathbb{Z}) \) the following transformation law.

**Proposition 3.1.** Let \( \alpha \in F - \mathbb{Q} \) and \( (p, q) \in \mathbb{Q}^2 \). Then,
(3.1) \[ h_{-1}(V\alpha, (p, q)) = h_{-1}(\alpha, (p, q)V) \quad \text{for any} \quad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \]
where we put \( V\alpha = \frac{a\alpha+b}{c\alpha+d} \).

**Proof.** For \( V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \), set \( V^* = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \) and \( (p, q)^* = (p, q)V \).

If \( c > 0 \) and \( c\alpha + d > 0 \), then the identity (3.1) is nothing but the first equality in
Proposition 4 of [Ar1]. Let \( c<0 \) and \( ca+d>0 \). In this case since \( V^*(-\alpha)=-(V\alpha) \), we get, by Propositions 3, 4 of [Ar1],

\[
h_{-1}(V\alpha, (p, q)) = -h_{-1}(-(V\alpha), (-p, q)) + 2\delta(p, q)
\]

\[
= -h_{-1}(-\alpha, (-p, q)V^*) + 2\delta(p, q)
\]

\[
= -h_{-1}(-\alpha, (-p^*, q^*)) + 2\delta(p^*, q^*)
\]

\[
= h_{-1}(\alpha, (p^*, q^*)�)
\]

where we put

\[
\delta(p, q) = \begin{cases} 1 & \text{if } (p, q) \in \mathbb{Z}^2 \\ 0 & \text{otherwise} \end{cases}
\]

If \( c=0, d=1 \), then the assertion easily follows from the definition of \( H(\alpha, s, (p, q)) \).

Finally let \( ca+d<0 \). Since \( V\alpha=-(V)\alpha \), we have

\[
h_{-1}(V\alpha, (p, q)) = h_{-1}(\alpha, (-p^*, -q^*)�)
\]

With the help of Lemma 5 of [Ar1], the last term coincides with \( h_{-1}(\alpha, (p^*, q^*)�) \).

We set, for positive numbers \( \omega, z \),

\[
G(z, \omega, t) = \frac{\exp(-zt)}{(1-\exp(-t))(1-\exp(-\omega t))} \quad (t \in \mathbb{C})
\]

\[
\zeta_2(s, \omega, z) = \sum_{m,n=0}^{\infty} (z+m+n\omega)^{-s} \quad (\text{Re}(s)>2).
\]

The Dirichlet series \( \zeta_2(s, \omega, z) \) is absolutely convergent for \( \text{Re}(s)>2 \). For a sufficiently small positive number \( \varepsilon \), let \( I_\varepsilon(\infty) \) be the integral path consisting of the oriented half line \((+\infty, \varepsilon)\), the counterclockwise circle of radius \( \varepsilon \) around the origin, and the oriented half line \((\varepsilon, +\infty)\). Then as is well-known, the zeta-function \( \zeta_2(s, \omega, z) \) has the following expression by a contour integral:

\[
\zeta_2(s, \omega, z) = \frac{1}{\Gamma(s)(e^{\pi is}-1)} \int_{I_\varepsilon(\infty)} t^{s-1}G(z, \omega, t)dt,
\]

where \( \log t \) is understood to be real valued on the upper half line \((+\infty, \varepsilon)\). This expression (3.2) gives the analytic continuation of \( \zeta_2(s, \omega, z) \) to a meromorphic function over the whole \( s \)-plane which is holomorphic except at \( s=1, 2 \). We put, for \( r \in \mathbb{R} \),

\[
\chi(r) = \begin{cases} 1 & \text{if } r \in \mathbb{Z} \\ 0 & \text{if } r \in \mathbb{R}-\mathbb{Z} \end{cases}
\]

For each \( \alpha \in F-Q \) and a pair \( (p, q) \in Q^2 \), we choose a totally positive unit \( \eta \) of
F and an element $V=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL_2(\mathbb{Z})$ which satisfy the following conditions

$$c>0, \quad (p, q)V \equiv (p, q) \mod \mathbb{Z}^2, \quad \eta(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix}.$$ (3.3)

We have obtained in (3.2) of [Ar1] the following expression for $h_{-1}(\alpha, (p, q))$ using the data given in (3.3):

$$h_{-1}(\alpha, (p, q)) - \chi(p)\chi(q) = \frac{2\pi i}{\log \eta} \chi(p) \left( \frac{1}{2} - \langle -q \rangle \right)$$

$$- \frac{1}{\log \eta} L(\alpha, 0, (p, q), c, d) ,$$ (3.4)

where $L(\alpha, 0, (p, q), c, d) (s \in \mathbb{C})$ is the special value at $s=0$ of the function

$$L(\alpha, s, (p, q), c, d) = -\sum_{j=1}^{c} \int_{t \in \mathbb{R}} t^{s-1} G' \left( 1 - \left\{ \frac{jd+\rho}{c} \right\} + \left\{ \frac{j - \{p\} \eta}{c} \right\}, \eta, t \right) dt$$

with $\rho = \{q\}c - \{p\}d$. Since the above integral on the right hand side of the equality converges absolutely for any $s \in \mathbb{C}$, this function $L(\alpha, s, (p, q), c, d)$ of $s$ is holomorphic in the whole complex plane.

**Proposition 3.2.** Let $\alpha \in F - \mathbb{Q}$ and $(p, q) \in \mathbb{Q}^2$. Choose a totally positive unit $\eta$ of $F$ and $V=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL_2(\mathbb{Z})$ as in (3.3). Then,

$$h_{-1}(\alpha, (p, q)) - \chi(p)\chi(q) = \frac{2\pi i}{\log \eta} \sum_{k \mod \nu} \xi(0, \eta, x_k + y_k \eta) ,$$

$$h_{-1}(\alpha', (p, q)) - \chi(p)\chi(q) = -\frac{2\pi i}{\log \eta} \sum_{k \mod \nu} \xi(0, \eta', x_k + y_k \eta') ,$$

where we put, for each integer $k$,

$$x_k = 1 - \left\{ \frac{(k+p)d}{c} - q \right\} \quad \text{and} \quad y_k = \left\{ \frac{k+p}{c} \right\} .$$ (3.5)

**Proof.** We know by Lemma 5 of [Ar1] that

$$h_{-1}(\alpha, (-p, -q)) = h_{-1}(\alpha, (p, q)) .$$

It follows from the identities (3.2) and (3.4) that

$$h_{-1}(\alpha, (-p, -q)) - \chi(p)\chi(q) = \frac{2\pi i}{\log \eta} \chi(p) \left( \frac{1}{2} - \langle q \rangle \right)$$

$$+ \frac{2\pi i}{\log \eta} \sum_{j=1}^{c} \xi(0, \eta, 1 - \left\{ \frac{jd+p}{c} \right\} + \left\{ \frac{j - \{p\} \eta}{c} \right\} ,$$ (3.6)
where \( \rho^* = \{-q\} c - \{-p\} d \). A slight modification of the summation in (3.6) yields

\[
\sum_{j=1}^{l} \zeta(0, \eta, 1 - \left\{ \frac{j + \rho^*}{c} \right\} + \left\{ \frac{j - \{-p\}}{c} \right\}) = \sum_{k \mod \ell} \zeta(0, \eta, x_k + y_k \eta)
\]

\[
= \chi(p)(\zeta(0, \eta, 1 - \{-q\} + \eta) - \zeta(0, \eta, 1 - \{-q\})).
\]

An easy computation with the use of the identity (3.2) shows that

\[
\zeta(0, \eta, x + y \eta) = \frac{1}{2} B_2(x) \eta + \frac{1}{2} B_2(y) \eta + B_1(x) B_1(y)
\]

(see (1.10) of [Sht2]),

where \( x, y > 0 \) and \( B_k(x) \) is the \( k \)-th Bernoulli polynomial. Thus the right hand side of the equality (3.7) coincides with

\[
\chi(p) \left( \left\{ q \right\} - \frac{1}{2} \right).
\]

Therefore the identity (3.6) with the help of (3.7) turns out the first identity in Proposition 3.2. Another identity is similarly verified.

Let \( a = (a_1, a_2) \) be a pair of positive numbers and \( x = (x_1, x_2) \) a pair of non-negative numbers with \( x + (0, 0) \). Shintani [Sht2] defined the following zeta-function \( \zeta(s, a, x) \):

\[
\zeta(s, a, x) = \sum_{m, n = 0}^{\infty} \prod_{j=1}^{2} (x_1 + m + (x_2 + n) a_j)^{-s},
\]

which is absolutely convergent for \( \Re(s) > 1 \). It has been proved that the zeta-function \( \zeta(s, a, x) \) is continued analytically to a meromorphic function of \( s \) in the whole complex plane which is holomorphic at \( s = 0 \) and moreover that

\[
\zeta(0, a, x) = \frac{1}{2} (\zeta_2(0, a_1, x_1 + x_2 a_1) + \zeta_2(0, a_2, x_1 + x_2 a_2))
\]

(see [Sht1], (1.11) of [Sht2] and [Eg]).

Let \( \bar{f} \) be an integral ideal of \( F \) and \( E_s(\bar{f}) \) the group of totally positive unit \( u \) of \( F \) with \( u - 1 \in \bar{f} \). We denote by \( \eta \) the generator of the group \( E_s(\bar{f}) \) with \( \eta > 1 \). For each class \( C \) of \( H_s(\bar{f}) \), take an integral ideal \( b \) of \( C \) and a basis \( \{\beta_1, \beta_2\} \) of the ideal \( b^{-1} \) with the conditions \( \beta_1 \beta_2 - \beta_1^2 > 0, \beta_1 \beta_2 > 0 \). We represent the unit \( \eta \) via the basis \( \{\beta_1, \beta_2\} \) to get an element \( V \) of \( SL_2(\mathbb{Z}) \) such that

\[
\eta \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

A pair \((p, q)\) of \( \mathbb{Q}^* \) is uniquely determined by the relation
(3.9) \[ p\beta_1 + q\beta_2 = 1. \]

Since \( \eta \in E_+(f) \), we necessarily have

\[(p, q)V \equiv (p, q) \mod \mathbb{Z}^2.\]

Set \( \beta = \beta_1 / \beta_2 \). Then, \( \beta, \eta, V \) and \( (p, q) \) satisfy the conditions in (3.3) with \( \alpha \) being replaced by \( \beta \). We have proved in 4 of [Ar1] that the partial zeta-function \( \zeta_f(b, s) \) has the decomposition

\[
\zeta_f(b, s) = N(bb')^{-s} \sum_{k \mod c} \sum_{m,n=0} \zeta(s, \eta, \eta'), (x_k, y_k)) ,
\]

where \( x_k, y_k \) are given by (3.5) (see also p.409, §2 of [Sht1] and [Ar2]). Therefore it is immediate to see from (3.8) that the special value \( \zeta_f(b, 0) \) at \( s=0 \) is given by the identity

(3.10) \[
\zeta_f(b, 0) = \frac{1}{2} \sum_{k \mod c} (\zeta_2(0, \eta, x_k + y_k \eta) + \zeta_2(0, \eta', x_k + y_k \eta')) .
\]

The following theorem is immediate from Proposition 3.2 and (3.10).

**Theorem 3.3.** Let \( b, f \) be coprime integral ideals of \( F \). Choose a basis \( \{ \beta_1, \beta_2 \} \) of the ideal \( \mathfrak{f} \mathfrak{b}^{-1} \) with \( \beta_1 \beta_2 = \beta_1 \beta_2 > 0, \beta_2 \beta_2 > 0 \). Let \( \eta \) denote the generator of the group \( E_+(\mathfrak{f}) \) with \( \eta > 1 \). Let \( (p, q) \in \mathbb{Q}^2 \) be the same as in (3.9). Set \( \beta = \beta_1 / \beta_2 \). Then,

\[
\zeta_f(b, 0) = \frac{\log \eta}{4\pi i} (h_-(\beta, (p, q)) - h_-(\beta', (p, q))) .
\]

Now we describe the map \( \xi_a : F/a \to W(F_{\mathfrak{a}}) \) in terms of the coefficient \( h_-(\alpha, (p, q)) \). We set, for \( \alpha \in F - \mathbb{Q} \) and \( (p, q) \in \mathbb{Q}^2 \),

\[
b(\alpha, (p, q)) = \frac{1}{2} (h_-(\alpha, (p, q)) - h_-(\alpha', (p, q))) .
\]

We denote by \( G \) the group \( GL_2 \) defined over \( \mathbb{Q} \). Let \( G_A = GL_{2,A} \) be the adelized group of \( G \). For each \( x \in G_A \), denote by \( x_\infty \) the archimedean component of \( x \). Set

\[
G_{\mathfrak{m}, +} = GL_{2, +}(\mathbb{R}) = \{ x \in GL_2(\mathbb{R}) \mid \det x > 0 \} ,
\]

\[
G_{\mathfrak{q}, +} = GL_{2, +}(\mathbb{Q}) = \{ x \in GL_2(\mathbb{Q}) \mid \det x > 0 \} ,
\]

\[
G_{A, +} = \{ x \in G_A \mid \det x_\infty > 0 \} ,
\]

and

\[
U = \prod_p GL_2(\mathbb{Z}_p) \times G_{\mathfrak{m}, +} ,
\]
where \( \mathbb{Z}_p \) is the ring of \( p \)-adic integers. We have the decomposition

\[
G_{A,+} = G_{Q,+} U = U G_{Q,+}.
\]

Let \( L \) be a \( \mathbb{Z} \)-lattice in \( \mathbb{Q}^2 \). Set \( L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p \). For an element \( x \) of \( G_A \), we define \( L x \) to be the \( \mathbb{Z} \)-lattice characterized by \( (Lx)_p = L_p x_p \) in \( \mathbb{Q}_p^2 = L \otimes_{\mathbb{Q}} \mathbb{Q}_p \). Moreover any element \( x \) of \( G_A \) has a natural action on the quotient space \( \mathbb{Q}_p^2 / L \) by the right multiplication and defines an isomorphism of \( \mathbb{Q}_p^2 / L \) to \( \mathbb{Q}_p^2 / L x \). We denote by \( rx \) the image of an element \( r \in \mathbb{Q}_p^2 / L \) by this isomorphism. For any \( x \in G_{A,+} \), we write

\[
x = u g \quad \text{with} \quad u \in U, \ g \in G_{Q,+}.
\]

We define the action of \( x \) on \( \delta(\alpha, (p, q)) \) to be

\[
\delta'(\alpha, (p, q)) = g \delta(\alpha, (p, q)) u,
\]

where we note that the element \( (p, q) u \) is uniquely determined as an element of \( \mathbb{Q}_p^2 / \mathbb{Z}_p^2 \). Since \( G_{Q,+} \cap U = SL_2(\mathbb{Z}) \), the right hand side of the equality (3.12) is independent of the decomposition \( x = u g (u \in U, \ g \in G_{Q,+}) \) according to (3.1).

Let \( \alpha \) be a fractional ideal of \( F \) with an oriented basis \( \{a_1, a_2\} \) (namely, \( a = Z a_1 + Z a_2 \)). Choose a representative element \( z \in F/\alpha \) and write

\[
z = p a_1 + q a_2.
\]

with coprime integral ideals \( f, b \) of \( F \). A pair \( (p, q) \) of rational numbers is uniquely determined by

\[
z = p a_1 + q a_2.
\]

Let \( q : F^* \rightarrow GL_2(\mathbb{Q}) \) be the homomorphism given by (1.6) in the introduction which is defined via the basis \( \{\alpha_1, \alpha_2\} \) of \( \alpha \). We also use the same symbol \( q \) for the natural extension of \( q \) to the homomorphism of \( F_{\alpha}^* \) to \( G_A \). Obviously, \( q(F_{\alpha}^*) \subset G_{A,+} \).

A description of the map \( \xi_\alpha : F/\alpha \rightarrow W(F_{\alpha}) \) in this case is formulated in Theorem \( B \) in the introduction. Now under the above preparations we can give its proof.

Proof of Theorem \( B \). Let the notation be the same as in the assertion of Theorem \( B \). The expression on the right hand side of (1.7) is independent of the choice of an oriented basis \( \{\alpha_1, \alpha_2\} \) of \( \alpha \) in virtue of Proposition 3.1. Therefore we may assume that

\[
\alpha_1 \alpha_2' - \alpha_1' \alpha_2 > 0, \ \alpha_2 \alpha_1' > 0,
\]

if necessary, by change of a basis \( \{\alpha_1, \alpha_2\} \) of \( \alpha \). We choose an element \( z_1 \) of \( F_{\alpha}^* \) such that \( z - z_1 \in \alpha \) and set \( z_1 = p_1 \alpha_1 + q_1 \alpha_2 \) with a pair of rational numbers
(p₁, q₁) We can write
\[ x₁a^{-1} = f^{-1}b₁ \]
with an integral ideal \( b₁ \) of \( F \) prime to the same \( f \). Then,
\[ f₁b₁^{-1} = z₁^a = Z(\alpha₁/z₁)+Z(\alpha₂/z₁), \]
\[ p₁(\alpha₁/z₁)+q₁(\alpha₂/z₁) = 1. \]
Noticing that \( z₁ \) is also a representative element of the class \( z \), we get, by the
definition (1.2) of the map \( ξ_α \),
\[ ξ_α(z) = \exp(2πiξ₁(b₁, 0)). \]
By virtue of Theorem 3.3 the special value \( ζ₁(b₁, 0) \) has the expression
\[ ζ₁(b₁, 0) = \log \frac{η}{2πi} b(α, (p₁, q₁)), \]
where we put \( α=α₁/α₂ \). Since \( (p₁, q₁)≡(p, q) \mod Z² \), we immediately have
the identity (1.7).
Next let \( s∈F₊ \) and write
\[ q(s)^{-1} = uσ \quad \text{with} \quad u∈U, g∈G₊. \]
We set
\[ \left( \begin{array}{c} \beta₁ \\ \beta₂ \end{array} \right) = g\left( \begin{array}{c} α₁ \\ α₂ \end{array} \right). \]
Obviously,
\[ β₁β₂ - β₂β₁ > 0. \]
Then we see easily that
\[ s^{-1}a = Z²q(s)^{-1}\left( \begin{array}{c} α₁ \\ α₂ \end{array} \right) = Z²g\left( \begin{array}{c} α₁ \\ α₂ \end{array} \right) \]
\[ = Zβ₁ + Zβ₂ \]
and moreover that
\[ s^{-1}z = (p, q)u\left( \begin{array}{c} β₁ \\ β₂ \end{array} \right) \mod s^{-1}a, \]
where \( (p, q)u \) stands for an element of \( Q'/Z² \) and where \( s^{-1}z \) is not determined
as an element of \( F \) but uniquely determined modulo \( s^{-1}a \). Choose a representative element \( θ(θ±0) \) of the class \( s^{-1}z = s^{-1}z \mod s^{-1}a \). We see from (2.2),
(2.3) in the proof of Theorem A that
\[ θ(s^{-1}a)^{-1} = f^{-1}b₂ \]
with some integral ideal \( b₂ \) of \( F \) prime to \( f \). Set \( β=β₁/β₂ \). Thus we have,
by the expression (1.7) and the definition (3.12),
\[ \xi_{s-10}(t^{-1}z) = \exp(\log \eta \cdot h((p, q), u)) \]
\[ = \exp(\log \eta \cdot h((\alpha, q), u)) \]
\[ = \exp(\log \eta \cdot h((\alpha, q), (p, q))). \]

Finally thanks to Theorem A in the introduction we obtain the identity (1.8).

We continue the assumption that \( F \) is a real quadratic field. For \( F = \mathbb{Q} \), we define \( \xi(s, \alpha) \) to be the Dirichlet series
\[ \sum_{n=1}^{\infty} \frac{\cot \pi n\alpha}{n}. \]

We have proved in [Ar2] that \( \xi(s, \alpha) \) is absolutely convergent for \( \Re(s) > 1 \) and that it can be continued analytically to a meromorphic function in the whole \( s \)-plane. Moreover, \( \xi(s, \alpha) \) has a simple pole at \( s = 1 \). We denote by \( c_{-1}(\alpha) \) the residue of \( \xi(s, \alpha) \) at the simple pole \( s = 1 \). The function \( H(\alpha, s, (0, 0)) \) given by (1.4) has the following obvious connection with \( \xi(s, \alpha) \):
\[ H(\alpha, s, (0, 0)) = \frac{1 + e^{s\pi i}}{2} \cdot (i\xi(1-s, \alpha) - \xi(1-s)), \]
where \( \zeta(s) \) is the Riemann zeta function. Thus we have
\[ h_{-1}(\alpha, (0, 0)) = -ic_{-1}(\alpha) + 1. \]

Since \( c_{-1}(\alpha') = -c_{-1}(\alpha) \) (see Proposition 2.10 of [Ar2]), it follows that
\[ h(\alpha, (0, 0)) = -ic_{-1}(\alpha). \]

Let \( \varepsilon \) be the totally positive fundamental unit of \( F \) with \( \varepsilon > 1 \). Choose a basis \( \{\alpha_1, \alpha_2\} \) of a fractional ideal \( \alpha \) of \( F \) such that
\[ \alpha_1\alpha_2 - \alpha_1\alpha_2 > 0, \quad \alpha_2\alpha_2 > 0. \]

We represent \( \varepsilon \) by the basis \( \{\alpha_1, \alpha_2\} \) to get a matrix \( V \) of \( SL_2(\mathbb{Z}) \):
\[ \varepsilon(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}) = V(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}), \quad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]

We get, by Theorem B,
\[ \xi_\alpha(0 \mod \alpha) = \exp(\log \varepsilon \cdot h((\alpha, (0, 0))) \]
\[ = \exp(-i \log \varepsilon \cdot c_{-1}(\alpha)), \]
where we put \( \alpha = \alpha_1/\alpha_2 \). Taking the facts \( V\alpha = \alpha, \ v > 0, \ e\alpha + d > 0 \) into account, we have, with the help of Proposition 2.9, (i) of [Ar2],
where \( s(d, c) \) is the Dedekind sum (for the Dedekind sum we refer the reader to [R-G]). Hence,

\[
\xi_{\alpha}(0 \mod \alpha) = \exp \left( 2\pi i \left( \frac{a + d}{12c} - s(d, c) - \frac{1}{4} \right) \right).
\]

It is known that the value \((a + d)/c - 12s(d, c)\) is a rational integer (see Ch. 4 of [R-G] and Remark 3.2 of [Ar2]). Therefore the value \(\xi_{\alpha}(0 \mod \alpha)\) is a twelfth root of unity.

References


[Ar2] T. Arakawa: Dirichlet series \( \sum_{n=1}^{\infty} \frac{\text{Col} \, \frac{a + d}{n}}{n^s} \), Dedekind sums, and Hecke L-functions for real quadratic fields, Commentarii Math. Universitatis Sancti Pauli 37 (1988), 209–235.


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