A REMARK ON ASYMPTOTIC SUFFICIENCY UP TO HIGHER ORDERS IN MULTI-DIMENSIONAL PARAMETER CASE

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1. Introduction. Suppose that n-dimensional random vector $z_n = (x_1, x_2, \cdots, x_n)$ is distributed according to a probability measure $P_{\theta,n}$ parameterized by $\theta \in \Theta \subset \mathbf{R}^p$, and each component x_i is independently and identically distributed. In Suzuki [3] it was shown that when $p=1$ a statistic $t_n^* = (\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n), \cdots, \Phi_n^{(k)}(z_n, \hat{\theta}_n))$ is asymptotically sufficient up to order $o(n^{-(k-1)/2})$ in the following sense: For each n t_n^* is sufficient for a family $\{Q_{\theta,n}; \theta \in \Theta\}$ of probability measures and that

$$\lim_{n \to \infty} n^{(k-1)/2} ||P_{\theta,n} - Q_{\theta,n}|| = 0$$

uniformly on any compact subset of Θ (where $||\cdot||$ means the total variation norm of a signed measure). Here $\hat{\theta}_n$ is some reasonable estimator of θ and $\Phi_n^{(i)}(z_n, \theta)$ means the i-th logarithmic derivative relative to θ of the density of $P_{\theta,n}$. In this paper we show that the result can be extended to the case where underlying distribution $P_{\theta,n}$ has multi-dimensional parameter θ. Exact form of t_n^* would be found in the statement of the theorem in Section 3. In Michel [2] a similar result was obtained with order of sufficiency $o(n^{-(k-2)/2})$, and hence ours is more accurate one.

2. Notations and assumptions. Let $\Theta(\neq \phi)$ be an open subset of p-dimensional Euclidean space $\mathbf{R}^p$. Suppose that for each $\theta \in \Theta$ there corresponds a probability measure P_θ defined on a measurable space (X, A). For each $n \in N = \{1, 2, \cdots\}$ let $(X^{(n)}, A^{(n)})$ be the cartesian product of n copies of (X, A), and $P_{\theta,n}$ the product measure of n copies of P_θ. For a signed measure $\tilde{\lambda}$ on $(X^{(n)}, A^{(n)})$, $||\tilde{\lambda}||$ means the total variation norm of $\tilde{\lambda}$ over $A^{(n)}$. For a function h and a probability P, $E[h; P]$ stands for the expectation of h under P. In the following it will be assumed that the map: $\theta \to P_\theta$ is one to one, and that for each $\theta \in \Theta$ P_θ has a densiy $f(x, \theta)$ relative to a sigma-finite measure μ on (X, A). We assume that $f(x, \theta) > 0$ for every $x \in X$ and every $\theta \in \Theta$. We denote by μ_n the product measure of n copies of the same com-

ponent μ. We define $\Phi(x, \theta) = \log f(x, \theta)$ for each $x \in X$ and $\theta \in \Theta$, and $\Phi_n(z_n, \theta) = \sum_{\nu=1}^{n} \Phi(x_\nu, \theta)$ for eadch $n \in N$, each $z_n = (x_1, \cdots, x_n) \in X^{(n)}$ and $\theta \in \Theta$. For a vector $u \in R^p$, $||u||$ denotes the usual Eculidean norm of u. For $\varepsilon > 0$ and $a \in R^p$ we define $U(a : \varepsilon) = \{u \in R^p ; ||u-a|| < \varepsilon\}$ and $V(a : \varepsilon) = \{u \in R^p ; ||u-a|| \leq \varepsilon\}$. Let k be a fixed positive integer.

Condition R. (1) For every $x \in X$ $\Phi(x, \theta)$ is $(k+2)$-times continuously differentiable with respect to θ in Θ. For $m \in N$ define $J_m = \{(i_1, \cdots, i_m) ; i_j = 1, \cdots, p(j=1, \cdots, m)\}$. For each $m(1 \leq m \leq k+2)$ and each $(i_1, i_2, \cdots, i_m) \in J_m$ define

$$\Phi^{i_1 \cdots i_m}(x, \theta) = \partial^m \Phi(x, \theta) / \partial \theta_{i_1} \cdots \partial \theta_{i_m},$$

and

$$\Phi_n^{i_1 \cdots i_m}(z_n, \theta) = \sum_{\nu=1}^{n} \Phi^{i_1 \cdots i_m}(x_\nu, \theta).$$

(2) For every $a = (a_1, \cdots, a_p) \in R^p (a \neq 0)$ and every $\theta \in \Theta$ we have

$$P_\theta(\sum_{i=1}^{p} a_i \Phi^i(x, \theta) \neq 0) > 0.$$

(3) For every $\theta \in \Theta$, there exists a positive number ε such that

   a. $\sup_{\tau \in V(\theta : \varepsilon)} E[\sup_{\sigma \in V(\theta : \varepsilon)} \{\Phi^{i_1 \cdots i_{k+2}}(x, \sigma)\}^2 ; P_\tau] < \infty$ for every $(i_1, \cdots, i_{k+2}) \in J_{k+2}$

   b. $\sup_{\tau \in V(\theta : \varepsilon)} E[|\Phi^{i_1 \cdots i_{k+1}}(x, \tau)| \cdot u_\varepsilon^i(x, \tau) ; P_\tau] < \infty$ and $E[u_\varepsilon^i(x, \theta)] < \infty$ for every $(i_1, \cdots, i_{k+1}) \in J_{k+1}$ and every $i(1 \leq i \leq p)$, where $u_\varepsilon^i(x, \tau) = \sup_{\sigma \in V(\tau : \varepsilon)}[|(\partial f(x, \theta) / \partial \theta_i)_{\theta = \sigma}| / f(x, \tau)]$.

   c. $0 < \inf_{\tau \in V(\theta : \varepsilon)} \text{Var}(\Phi^{i_1 \cdots i_{k+1}}(x, \tau) ; P_\tau) \leq \sup_{\tau \in V(\theta : \varepsilon)} \text{Var}(\Phi^{i_1 \cdots i_{k+1}}(x, \tau ; P_\tau) < \infty$

for every $(i_1, \cdots, i_{k+1}) \in J_{k+1}$.
We define for each $\varepsilon' > 0$, $\sigma \in \Theta$ and $(i_1, \cdots, i_{k+1}) \in J_{k+1}$

$$\bar{Z}^{i_1 \cdots i_{k+1}}(x ; \varepsilon', \sigma) = \sup \{\Phi^{i_1 \cdots i_{k+1}}(x, \tau) - E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau) ; P_\tau] ; \tau \in V(\sigma : \varepsilon') \cap \Theta\}$$

and

$$\underline{Z}^{i_1 \cdots i_{k+1}}(x ; \varepsilon', \sigma) = -\inf \{\Phi^{i_1 \cdots i_{k+1}}(x, \tau) - E[(\Phi^{i_1 \cdots i_{k+1}}(x, \tau) ; P_\tau] ; \tau \in V(\sigma : \varepsilon') \cap \Theta\}$$

(4) For each $\theta \in \Theta$ there exist positive numbers η and ρ such that for every $(i_1, \cdots, i_{k+1}) \in J_{k+1}$ and every $(t, \varepsilon') \in (-\rho, \rho) \times (0, \eta]$ the moment generating functions (m.g.f'.s) of $\bar{Z}^{i_1 \cdots i_{k+1}}(x ; \varepsilon', \sigma)$ and $\underline{Z}^{i_1 \cdots i_{k+1}}(x ; \varepsilon', \sigma)$ converge uniformly in $\sigma \in U(\theta : \eta)$.

3. Asymptotic sufficient statistics up to higher orders. An esti-

mator of θ depending on $z_n=(z_1, \cdots, x_n)\in X^{(n)}$ is an $A^{(n)}$-measurable function from $X^{(n)}$ to R^p. Such estimator will be called strict if its range is a subset of Θ. For each $\delta (0<\delta<1/2)$ we denote by $C_k(\delta)$ the class of all sequences of strict estimators $\hat{\theta}_n$ of θ such that for every compact subset K of Θ

$$\sup_{\theta\in K} P_{\theta,n}(n^{1/2}||\hat{\theta}_n(z_n)-\theta||>n^{\delta}) = o(n^{-(k-1)/2}) .$$

Here the notation $o(a_n)$ means that $\lim_{n\to\infty} o(a_n)/a_n=0$. In Pfanzagl [1] it was shown that under Condition R for any δ satisfying $0<\delta<1/2$ $C_k(\delta)$ does not empty. Let $\delta_0=1/[2(k+2)]$ and $C_k=\bigcup_{0<\delta<\delta_0} C_k(\delta)$. We have the following result which is an extension of Tehorem 2 in Suzuki [3] to a multi-dimensional parameter case. Since the proof is much analogous to the one in [3] we shall only sketch the outlines and details will be omitted (see [3] for precise arguments).

Theorem. *Suppose that Condition R is satisfied, and that $\{\hat{\theta}_n\}\in C_k$ then there exists a sequence $\{Q_{\theta,n}; \theta\in\Theta\}$, $n\in N$, of families of probability measures on $(X^{(n)}, A^{(n)})$ with the following properties: (1) For each $n\in N$, the statistic $t_n^*=(\hat{\theta}_n, \Phi_n^i(z_n, \hat{\theta}_n) (i=1, \cdots, p)$, $\Phi_n^{ij}(z_n, \hat{\theta}_n)((i,j)\in J_2), \cdots, \Phi_n^{i_1\cdots i_k}(z_n, \hat{\theta}_n) ((i_1, \cdots, i_k)\in J_k))$ is sufficient for $\{Q_{\theta,n}; \theta\in\Theta\}$. (2) For every compact subset K of Θ*

$$\sup_{\theta\in K} ||P_{\theta,n}-Q_{\theta,n}|| = o(n^{-(k-1)/2})$$

Proof. Suppose that Condition R is satisfied, and that $\{\hat{\theta}_n\}\in C_k(\delta_1)$ where δ_1 satisfies $0<\delta_1<\delta_0$. Let δ and γ be two numbers satisfying $\delta_1<\delta<\delta_0$ and $\delta<\gamma<(1/2)-(k+1)\delta$, and let $\varepsilon_n=n^{\delta-(1/2)}$ $\varepsilon_n'=n^{\gamma-(1/2)}$. Define

$$W_n^1 = \{z_n\in X^{(n)}; ||\theta-\hat{\theta}_n(z_n)||\leqq\varepsilon_n \quad \text{and} \quad [\theta: \hat{\theta}_n]\subset\Theta\}$$
$$W_n^2 = \{z_n\in X^{(n)}; \gamma_n(z_n)\leqq\varepsilon_n'\}$$

where

$$[\theta: \hat{\theta}_n] = \{t\theta+(1-t)\hat{\theta}_n; 0\leqq t\leqq 1\}$$

and

$$\gamma_n(z_n) = \max_{(i_1,\cdots,i_{k+1})\in J_{k+1}} \sup \{|\Phi_n^{i_1\cdots i_{k+1}}(z_n, \tau)/n-E[\Phi^{i_1\cdots i_{k+1}}(x, \tau); P_\tau]|;$$
$$\tau\in V(\hat{\theta}_n: 2\varepsilon_n)\cap\Theta\} .$$

By a Taylor expansion of $\Phi_n(z_n, \theta)$ around $\theta=\hat{\theta}_n$ we have

(3.1) $$\Phi_n(z_n, \theta) = \Phi_n(z_n, \hat{\theta}_n)+\Psi_n(t_n^*, \theta)+R_n(z_n, \theta)$$

where denoting by $\hat{\theta}_{n,i}$ the i-th comonents of $\hat{\theta}_n$

$$\Psi_n(t_n^*, \theta) = \sum_{m=1}^{k}\sum_{i_1=1}^{p}\cdots\sum_{i_m=1}^{p} \prod_{j=1}^{m} (\theta_{i_j}-\hat{\theta}_{n,i_j})\cdot\Phi_n^{i_1\cdots i_m}(z_n, \hat{\theta}_n)/m!+s_n'(\theta_n, \theta),$$

$$s'_n(\hat{\theta}_n, \theta) = n \sum_{i_1=1}^{p} \cdots \sum_{i_{k+1}=1}^{p} \prod_{j=1}^{k+1} (\theta_{i_j} - \hat{\theta}_{n,i_j}) E[\Phi^{i_1 \cdots i_{k+1}}(x, \theta); P_\theta]/(k+1)!$$

$$R_n(Z_n, \theta) = \begin{cases} \Phi_n(z_n, \theta) - \Phi_n(z_n, \hat{\theta}_n) - \Psi_n(t_n^*, \theta) & (\text{if } [\theta: \hat{\theta}_n] \not\subset \Theta), \\ n \sum_{i_1=1}^{p} \cdots \sum_{i_{k+1}=1}^{p} \prod_{j=1}^{k+1} (\theta_{i_j} - \hat{\theta}_{n,i_j}) [\int_0^1 (1-\lambda)^k \{\Phi_n^{i_1 \cdots i_{k+1}}(z_n, \hat{\theta}_n + \lambda(\theta - \hat{\theta}_n))/n \\ \qquad - E[\Phi^{i_1 \cdots i_{k+1}}(x, \theta); P_\theta]/k! \} d\lambda /k! & (\text{if } [\theta: \hat{\theta}_n] \subset \Theta). \end{cases}$$

Define

$$(3.2) \qquad q_n^*(z_n, \theta) = I_{W_n^1}(z_n) \cdot I_{W_n^2}(z_n) \cdot \exp \{\Phi_n(z_n, \hat{\theta}_n) + \Psi_n(t_n^*, \theta)\}$$
$$= r_n^*(t_n^*, \theta) \cdot s_n^*(z_n) \geqq 0,$$

where

$$r_n^*(t_n^*, \theta) = I_{W_n^1}(z_n) \exp \{\Psi_n(t_n^*, \theta)\},$$
$$s_n^*(z_n) = I_{W_n^2}(z_n) \cdot \exp \{\Phi_n(z_n, \hat{\theta}_n)\}$$

and $I_{W_n^i}(z_n)$ mean the indicator functions of W_n^i. The integrability of $q_n^*(\cdot, \theta)$ follows from (3.4). Let $Q_{\theta,n}^*$ be a measure on $(X^{(n)}, A^{(n)})$ defined by

$$Q_{\theta,n}^*(A) = \int_A q_n^*(z_n, \theta) d\mu_n (A \in A^{(n)}).$$

By (3.1) and (3.2) we have

$$(3.3) \qquad \int_{X^{(n)}} |p_n(z_n, \theta) - q_n^*(z_n, \theta)| d\mu_n = T_n^1(\theta) + T_n^2(\theta) + T_n^3(\theta)$$

where

$$p_n(z_n, \theta) = \prod_{\nu=1}^{n} f(x_\nu, \theta),$$

$$T_n^1(\theta) = \int_{W_n^1 \cap W_n^2} |1 - \exp \{-R_n(z_n, \theta)\}| p_n(z_n, \theta) d\mu_n,$$

$$T_n^2(\theta) = P_{\theta,n}((W_n^1)^c) \quad \text{and}$$
$$T_n^3(\theta) = P_{\theta,n}(W_n^1 \cap (W_n^2)^c).$$

Let θ_0 be an arbitrarily fixed point of Θ, and let K be a compact sub-set of Θ. We assume without loss of generality that K contains θ_0. From Condition R it follows that there exist positive numbers ε^*, ρ^* and η^* depending only on K but not depending on θ in K such that

$$M_1 = \sum_{(i_1, \cdots, i_{k+2}) \in J_{k+2}} \sup_{\theta \in K} \sup_{\tau \in V(\theta: \varepsilon^*)} E[\sup_{\sigma \in V(\theta: \varepsilon^*)} \{\Phi^{i_1 \cdots i_{k+2}}(x, \sigma)\}^2; P_\tau] < \infty$$

$$M_2 = \max_{(i_1, \cdots, i_{k+1}) \in J_{k+1}} \sum_{j=1}^{p} \sup_{\theta \in K} E[|\Phi^{i_1 \cdots i_{k+1}}(x, \theta)| \cdot u_{\mathfrak{z}^*}^j(x, \theta); P_\theta] < \infty$$

$$0 < \inf_{\tau \in K} \text{Var}(\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau)$$

$$\leqq \sup_{\tau \in K} \text{Var}(\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau) < \infty \quad (\text{for every } (i_1, \cdots, i_{k+1}) \in J_{k+1})$$

and that for every $\theta \in K$ and every $(t, \varepsilon', \sigma) \in (-\rho^*, \rho^*) \times (0, \eta^*] \times U(\theta: \eta^*)$ the m.g.f.'s of $\bar{Z}^{i_1 \cdots i_{k+1}}(x; \varepsilon', \sigma)$ and $\underline{Z}^{i_1 \cdots i_{k+1}}(z; \varepsilon', \sigma)$ exist and converge uniformly in $\sigma \in U(\theta: \eta^*)$. Hence there exists a number n_1 such that for every $n \geq n_1$, $\theta \in K$, $(i_1, \cdots, i_{k+1}) \in J_{k+1}$ and every $z_n \in W_n^1 \cap W_n^2$ we have

$$\sup_{\tau \in V(\theta: \varepsilon_n)} |\Phi_n^{i_1 \cdots i_{k+1}}(z_n, \tau)/n - E[\Phi^{i_1 \cdots i_{k+1}}(x, \theta); P_\theta]|$$

$$\leq \sup_{\tau \in V(\theta: \varepsilon_n)} |\Phi_n^{i_1 \cdots i_{k+1}}(z_n, \tau)/n - E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau]|$$

$$+ \sup_{\tau \in V(\theta: \varepsilon_n)} |E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau] - E[\Phi^{i_1 \cdots i_{k+1}}(x, \theta); P_\theta]|$$

$$\leq \sup_{\tau \in V(\hat{\theta}_n: 2\varepsilon_n)} |\Phi_n^{i_1 \cdots i_{k+1}}(z_n, \tau)/n - E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau]| + [M_1^{1/2} + M_2] \cdot \varepsilon_n$$

$$\leq \gamma_n(z_n) + \varepsilon_n'$$

$$\leq 2\varepsilon_n'.$$

Thus we have

$$\sup_{\theta \in K} T_n^1(\theta) \leq \sup_{\theta \in \bar{K}} \int_{X^{(n)}} |R_n(z_n, \theta)| \cdot \exp(|R_n|) dP_{\theta, n}$$

$$\leq 4 \cdot n^{-(k-1)/2} n^{(k+1)\delta + \gamma - (1/2)} / (k+1)!$$

for sufficiently large n. Therefore

(3.4) $$\sup_{\theta \in K} T_n^1(\theta) = o(n^{-(k-1)/2}).$$

By the definition of $C_k(\delta_1)$ we have easily

(3.5) $$\sup_{\theta \in \bar{K}} T_n^2(\theta) = o(n^{-(k-1)/2}).$$

Next we evaluate the third term $T_n^3(\theta)$ as follows.

(3.6) $$\sup_{\theta \in K} T_n^3(\theta) = \sup_{\theta \in K} P_{\theta, n}(\|\theta - \hat{\theta}_n(z_n)\| \leq \varepsilon_n, [\theta: \hat{\theta}_n] \subset \Theta, \gamma_n(z_n) > \varepsilon_n')$$

$$\leq \sup_{\theta \in K} P_{\theta, n}(\max_{(i_1, \cdots, i_{k+1}) \in J_{k+1}} \sup_{\tau \in V(\theta: 3\varepsilon_n)} |\Phi_n^{i_1 \cdots i_{k+1}}(z_n, \tau)/n$$

$$- E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau])| > \varepsilon_n')$$

$$\leq \sum_{(i_1 \cdots i_{k+1}) \in J_{k+1}} \sup_{\theta \in K} P_{\theta, n}(\sum_{\nu=1}^n \sup_{\tau \in V(\theta: 3\varepsilon_n)} \{\Phi^{i_1 \cdots i_{k+1}}(x_\nu, \tau)$$

$$- E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau]\} > n\varepsilon_n')$$

$$+ \sum_{(i_1, \cdots, i_{k+1}) \in J_{k+1}} \sup_{\theta \in K} P_{\theta, n}(\sum_{\nu=1}^n \inf_{\tau \in V(\theta: 3\varepsilon_n)} \{\Phi^{i_1 \cdots i_{k+1}}(x_\nu, \tau)$$

$$- E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau]\} < -n\varepsilon_n').$$

Let $a(\varepsilon) = \varepsilon / [4 \cdot M_1^{1/2} + 2M_2]$ and let $Z_\nu(\varepsilon, \theta) = \bar{Z}^{i_1 \cdots i_{k+1}}(x_\nu; a(\varepsilon), \theta)$ $(\nu = 1, \cdots, n)$. According to the lemma in Suzuki [3] there exist constants $\beta > 0$ and $\varepsilon^{**} > 0$ such that

$$\sup_{\theta \in K} P_{\theta,n}(\sum_{\nu=1}^{n} Z_\nu(\mathcal{E}, \theta) \geq n\mathcal{E}) \geq (1-\beta\mathcal{E}^2)^n$$

for every $n \in N$ and every $\mathcal{E}$ satisfying $0 < \mathcal{E} \leq \mathcal{E}^{**}$. Hence we have

$$(3.7) \quad \sup_{\theta \in K} P_{\theta,n}(\sum_{\nu=1}^{n} \sup_{\tau \in V(\theta:3\mathcal{E}_n)} \{\Phi^{i_1 \cdots i_{k+1}}(x_\nu, \tau) - E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau]\} > n\mathcal{E}_n')$$

$$\leq \sup_{\theta \in K} P_{\theta,n}(\sum_{\nu=1}^{n} Z_\nu(\mathcal{E}_n', \theta) \geq n\mathcal{E}_n')$$

$$= o(n^{-(k-1)/2}).$$

Considering the random variable $\underline{Z}^{i_1 \cdots i_{k+1}}(x; a(\mathcal{E}), \theta)$ instead of $\overline{Z}^{i_1 \cdots i_{k+1}}$ $(x; a(\mathcal{E}), \theta)$ we obtain by a similar method to (3.7) that

$$(3.8) \quad \sup_{\theta \in K} P_{\theta,n}(\sum_{\nu=1}^{n} \inf_{\tau \in V(\theta:3\mathcal{E}_n)} \{\Phi^{i_1 \cdots i_{k+1}}(x_\nu, \tau) - E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau]\} < -n\mathcal{E}_n')$$

$$= o(n^{-(k-1)/2}).$$

From (3.6), (3.7) and (3.8) we have

$$(3.9) \qquad\qquad\qquad \sup_{\theta \in K} T_n^3(\theta) = o(n^{-(k-1)/2}).$$

From (3.3), (3.4), (3.5) and (3.9) we have

$$(3.10) \qquad\qquad\qquad \sup_{\theta \in K} \|P_{\theta,n} - Q_{\theta,n}^*\| = o(n^{-(k-1)/2}).$$

Since θ_0 is contained in K it follows from (3.10) that there exists a number n_0^* such that for every n satisfying $n \geq n_0^*$

$$\|P_{\theta_0,n} - Q_{\theta_0,n}^*\| < 1/2.$$

Hence particularly for every $n \geq n_0^*$ we have

$$(3.11) \qquad\qquad\qquad \int_{X^{(n)}} q_n^*(z_n, \theta_0) d\mu_n > 0.$$

Define $\Theta_n = \{\theta \in \Theta; \int_{X^{(n)}} q_n^*(z_n, \theta) d\mu_n > 0\}$, $c_n(\theta) = [\int_{X^{(n)}} q_n^*(z_n, \theta) d\mu_n]^{-1}$ for $\theta \in \Theta_n$ and $c_n(\theta) = 0$ for $\theta \notin \Theta_n$. From (3.11) $n \geq n_0^*$ implies $\theta_0 \in \Theta_n$. Let $d_n(\theta)$ be the indicator function of Θ_n i.e., $d_n(\theta) = 1$ if $\theta \in \Theta_n$ and $d_n(\theta) = 0$ if $\theta \notin \Theta_n$. We define a 'sufficient density' $q_n(z_n, \theta)$ as follows: $q_n(z_n, \theta) = [c_n(\theta) r_n^*(t_n^*, \theta) + c_n(\theta_0)(1-d_n(\theta)) r_n^*(t_n^*, \theta_0)] s_n^*(z_n)$ for each $n \geq n_0^*$, $= p_n(z_n, \theta_0)$ for each n satisfying $1 \leq n \leq n_0^* - 1$ where $z_n \in X^{(n)}$ and $\theta \in \Theta$. It can be easily seen that for every $n \in N$ and every $\theta \in \Theta$

$$\int_{X^{(n)}} q_n(z_n, \theta) d\mu_n = 1.$$

Let $Q_{\theta,n}$ be a probability measure on $(X^{(n)}, A^{(n)})$ defined by

$$Q_{\theta,n}(A) = \int_A q_n(z_n, \theta)d\mu_n \quad (A \in A^{(n)}) .$$

We note that the density $q_n(z_n, \theta)$ has the following form:

$$q_n(z_n, \theta) = r_n(t_n^*, \theta) \cdot s_n(z_n)$$

where

$$\begin{aligned} r_n(t_n^*, \theta) &= c_n(\theta)r_n^*(t_n^*, \theta) + c_n(\theta_0)(1-d_n(\theta))r_n^*(t_n^*, \theta_0) \quad \text{for} \quad n \geq n_0^* \\ &= 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad n \leq n_0^* - 1 \end{aligned}$$

and

$$\begin{aligned} s_n(z_n) &= s_n^*(z_n) \qquad \text{for} \quad n \geq n_0^* \\ &= p_n(z_n, \theta_0) \quad \text{for} \quad n \leq n_0^* - 1 . \end{aligned}$$

Hence according to the factorization theorem t_n^* is sufficient for the family $\{Q_{\theta,n}; \theta \in \Theta\}$ for each $n \in N$.

By (3.10) there exists a number n_1^* such that for every $n \geq n_1^*$ we have

$$\sup_{\theta \in K} ||P_{\theta,n} - Q_{\theta,n}^*|| < 1/2 .$$

Hence $n \geq n_1^*$ implies $K \subset \Theta_n$. Thus if $n \geq n_2^* = \max(n_0^*, n_1^*)$ then for every $\theta \in K$

$$q_n(z_n, \theta) = c_n(\theta) \cdot q_n^*(z_n, \theta) .$$

From this we have for every $n \geq n_2^*$

$$\begin{aligned} 2 \cdot ||Q_{\theta,n}^* - Q_{\theta,n}|| &= |1 - c_n^{-1}(\theta)| = |P_{\theta,n}(X^{(n)}) - Q_{\theta,n}^*(X^{(n)})| \\ &\leq || P_{\theta,n} - Q_{\theta,n}^*|| , \end{aligned}$$

and hence

$$\sup ||P_{\theta,n} - Q_{\theta,n}|| \leq \sup_{\theta \in K} ||P_{\theta,n} - Q_{\theta,n}^*|| + \sup_{\theta \in K} ||Q_{\theta,n}^* - Q_{\theta,n}|| \leq 2 \cdot \sup_{\theta \in K} ||P_{\theta,n} - Q_{\theta,n}^*|| .$$

Thus by (3.10) we obtain

$$\sup_{\theta \in K} ||P_{\theta,n} - Q_{\theta,n}|| = o(n^{-(k-1)/2}) .$$

This completes the proof of the theorem.

References

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