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AN INFINITE PRESENTATION FOR THE MAPPING CLASS GROUP OF A NON-ORIENTABLE SURFACE WITH BOUNDARY

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Abstract

We give an infinite presentation for the mapping class group of a non-orientable surface with boundary components. The presentation is a generalization of the presentation given by the second author [16]. We also give a finite presentation for the mapping class group to obtain the infinite presentation.

1. Introduction

Let $\Sigma_{g,n}$ be a compact connected oriented surface of genus $g \geq 0$ with $n \geq 0$ boundary components. The *mapping class group* $\mathcal{M}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is the group of isotopy classes of orientation preserving self-diffeomorphisms on $\Sigma_{g,n}$ fixing the boundary pointwise. Hatcher-Thurston [8] were the first to give a finite presentation for $\mathcal{M}(\Sigma_{g,0})$ in general genus cases. By applying their method in [8] to non-empty boundary cases, Harer [7] gave a finite presentation for $\mathcal{M}(\Sigma_{g,0})$ for $n \geq 1$. Wajnryb [22] simplified their presentation for $n \in \{0, 1\}$. Furthermore, for $n \geq 0$, a finite presentation for $\mathcal{M}(\Sigma_{g,n})$ was given by Gervais [6] and Labruère-Paris [12]. Gervais [5] gave an infinite presentation for $\mathcal{M}(\Sigma_{g,n})$ for arbitrary $g \geq 0$ and $n \geq 0$ by using Harer's and Wajnryb's finite presentations for $\mathcal{M}(\Sigma_{g,n})$ ([7, 22]). To prove this, Gervais constructed explicit isomorphisms between the group obtained from his infinite presentations and the group obtained from Harer's and Wajnryb's finite presentations for $\mathcal{M}(\Sigma_{g,n})$. This Gervais' presentation has infinitely many generators and relations, however, the relations are simple. Luo [15] reduced relations in Gervais' presentation into a simpler infinite presentation (see Theorem 2.5).

Let $N_{g,n}$ be a compact connected non-orientable surface of genus $g \geq 1$ with $n \geq 0$ boundary components. The surface $N_{g,0}$ is a connected sum of g real projective planes. The mapping class group $\mathcal{M}(N_{g,n})$ of $N_{g,n}$ is the group of isotopy classes of self-diffeomorphisms on $N_{g,n}$ fixing the boundary pointwise. For $n \in \{0, 1\}$, Paris-Szepietowski [17] were the first to give a finite presentation for $\mathcal{M}(N_{g,n})$ in general genus cases. Stukow [20] rewrote Paris-Szepietowski's presentation into a finite presentation with Dehn twists and one "Y-homeomorphism" as generators (see Theorem 3.1). In low genus cases for $n \in \{0, 1\}$, finite presentations for $\mathcal{M}(N_{g,n})$ were given by Lickorish [13], Birman-Chillingworth [2], and Stukow [18]. A finite presentation for $\mathcal{M}(N_{g,n})$ for $n \geq 2$ was not known.

In this paper, we give a simple infinite presentation for $\mathcal{M}(N_{g,n})$ for $g \geq 1$ and $n \geq 2$

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(Theorem 4.1). The generating set consists of all Dehn twists and all "crosscap pushing maps" along simple loops. We review the crosscap pushing map in Section 2. In the case of $n \in \{0, 1\}$, an infinite presentation for $\mathcal{M}(N_{g,n})$ was given by the second author [16]. To prove Theorem 4.1, we construct an explicit finite presentation for $\mathcal{M}(N_{g,n})$ for $g \geq 1$ and $n \geq 2$ (Proposition 3.2), and apply Gervais' argument to the finite presentation in Proposition 3.2. We prove Proposition 3.2 by inductively applying the forgetful exact sequence to the group obtained from known finite presentation for $\mathcal{M}(N_{g,1})$.

Contents of this paper are as follows. In Section 2, we prepare some elements of $\mathcal{M}(N_{g,n})$ and some relations among their elements in $\mathcal{M}(N_{g,n})$, and review the infinite presentation for $\mathcal{M}(\Sigma_{g,n})$ (Theorem 2.5) which is an improvement by Luo [15] of Gervais' presentation in [5]. In Section 3, we review Stukow's finite presentation for $\mathcal{M}(N_{g,n})$ when $n \in \{0, 1\}$ (Theorem 3.1) and give a finite presentation for $\mathcal{M}(N_{g,n})$ when $n \geq 2$ (Proposition 3.2). In the proof of the main theorem in Section 4, we use their finite presentations for $\mathcal{M}(N_{g,n})$. In Section 4, we give the main theorem (Theorem 4.1) in this paper and a proof of the main theorem. Finally, in Section 5, we give a proof of Proposition 3.2.

2. Preliminaries

In this section, we recall the definitions of Dehn twists and crosscap pushing maps, and their important relations from Section 2 in [16].

2.1. Relations among Dehn twists and Gervais' presentation. Let S be either $N_{g,n}$ or $\Sigma_{g,n}$. We denote by $\mathcal{N}_S(A)$ a regular neighborhood of a subset A in S. We assume that every simple closed curve on S is oriented throughout this paper, and for simple closed curves c_1, c_2 on S, $c_1 = c_2$ means c_1 is isotopic to c_2 in consideration of their orientations. Denote by c^{-1} the inverse curve of a simple closed curve c on S. Note that $(c^{-1})^{-1} = c$. For a two-sided simple closed curve c on S, we can take two orientations $+_c$ and $-_c$ of $\mathcal{N}_S(c)$. When S is orientable, we take $+_c$ as the orientation of $\mathcal{N}_S(c)$ which is induced by the orientation of S. For a two-sided simple closed curve c on S and an orientation $\theta \in \{+_c, -_c\}$ of $\mathcal{N}_S(c)$, denote by $t_{c;\theta}$ the right-handed Dehn twist along c on S with respect to θ. Note that $t_{c;+_c} = t_{c^{-1};+_c} = t_{c;-_c}^{-1}$. For some convenience, we write $t_c = t_{c;+_c}$ for a two-sided simple closed curve c, where the orientation of $\mathcal{N}_S(c)$ is given explicitly (for instance, S is an oriented surface). In particular, for a given explicit two-sided simple closed curve, an arrow on a side of the simple closed curve indicates the direction of the Dehn twist (see Figure 1). For elements $f = [\varphi]$, $h = [\psi] \in \mathcal{M}(S)$, we define $fh := [\varphi \circ \psi] \in \mathcal{M}(S)$.

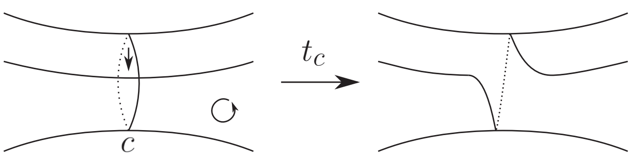


Fig. 1. The right-handed Dehn twist $t_c = t_{c;\theta}$ along a two-sided simple closed curve c on S with respect to the orientation $\theta \in \{+_c, -_c\}$ of $\mathcal{N}_S(c)$ as in the figure.

Recall the following relations in $\mathcal{M}(S)$ among Dehn twists along two-sided simple closed curves on S.

Lemma 2.1. *Let c be a two-sided simple closed curve on S and $\theta \in \{+_c, -_c\}$ an orientation of $\mathcal{N}_S(c)$. If c bounds a disk or a Möbius band in S, then we have $t_{c;\theta} = 1$ in $\mathcal{M}(S)$.*

For a two-sided simple closed curve c on S and $f \in \mathcal{M}(S)$, we have a bijection $f_* = (f|_{\mathcal{N}_S(c)})_* : \{+_c, -_c\} \to \{+_{f(c)}, -_{f(c)}\}$.

Lemma 2.2 (The braid relation (i))**.** *For a two-sided simple closed curve c on S and $f \in \mathcal{M}(S)$, we have*

$$f t_{c;\theta} f^{-1} \quad = \quad t_{f(c);f_*(\theta)}.$$

When f is a Dehn twist $t_{d;\theta'}$ along a two-sided simple closed curve d and the geometric intersection number $|c \cap d|$ of c and d is m, we denote by T_m the braid relation.

Let $c_1, c_2, \ldots, c_k$ be two-sided simple closed curves on S. The sequence $c_1, c_2, \ldots, c_k$ is a *k-chain on S* if $c_1, c_2, \ldots, c_k$ satisfy $|c_i \cap c_{i+1}| = 1$ for each $i = 1, 2, \ldots, k-1$ and $|c_i \cap c_j| = 0$ for $|j - i| > 1$.

Lemma 2.3 (The k-chain relation)**.** *Let $c_1, c_2, \ldots, c_k$ be a k-chain on S and let δ, δ' (resp. δ) be distinct boundary components (resp. the boundary component) of $\mathcal{N}_S(c_1 \cup c_2 \cup \cdots \cup c_k)$ when k is odd (resp. even). We give an orientation of $\mathcal{N}_S(c_1 \cup c_2 \cup \cdots \cup c_k)$, and it induces orientations θ_i $(i = 1, 2, \ldots, k)$, θ, and θ' of $\mathcal{N}_S(c_i)$ $(i = 1, 2, \ldots, k)$, $\mathcal{N}_S(\delta)$, and $\mathcal{N}_S(\delta')$, respectively. Then we have*

$$(t_{c_1;\theta_1} t_{c_2;\theta_2} \cdots t_{c_k;\theta_k})^{k+1} \quad = \quad t_{\delta;\theta} t_{\delta';\theta'} \quad \text{when } k \text{ is odd},$$
$$(t_{c_1;\theta_1} t_{c_2;\theta_2} \cdots t_{c_k;\theta_k})^{2k+2} \quad = \quad t_{\delta;\theta} \quad \text{when } k \text{ is even}.$$

Lemma 2.4 (The lantern relation)**.** *Let Σ be a subsurface of S which is diffeomorphic to $\Sigma_{0,4}$ and let δ_{12}, δ_{23}, δ_{13}, δ_1, δ_2, δ_3 and δ_4 be simple closed curves on Σ as in Figure 2. We give an orientation of Σ, and it induces orientations θ_i $(i = 1, 2, 3, 4)$, and θ_{ij} $((i, j) = (1, 2), (2, 3), (1, 3))$ of $\mathcal{N}_S(\delta_i)$ $(i = 1, 2, 3, 4)$, and $\mathcal{N}_S(\delta_{ij})$ $((i, j) = (1, 2), (2, 3), (1, 3))$, respectively. Then we have*

$$t_{\delta_{12};\theta_{12}} t_{\delta_{23};\theta_{23}} t_{\delta_{13};\theta_{13}} = t_{\delta_1;\theta_1} t_{\delta_2;\theta_2} t_{\delta_3;\theta_3} t_{\delta_4;\theta_4}.$$

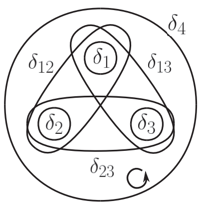


Fig. 2. The simple closed curves δ_{12}, δ_{23}, δ_{13}, δ_1, δ_2, δ_3 and δ_4 on Σ.

Luo's presentation for $\mathcal{M}(\Sigma_{g,n})$, which is an improvement of Gervais' one, is as follows.

Theorem 2.5 ([5], [15])**.** *For $g \geq 0$ and $n \geq 0$, $\mathcal{M}(\Sigma_{g,n})$ has the following presentation:*
generators: $\{t_c \mid c :$ s.c.c. on $\Sigma_{g,n}\}$, where s.c.c. means simple closed curve.
relations:

(0′) $t_c = 1$ *when c bounds a disk in $\Sigma_{g,n}$,*

(I′) *All the braid relations T_0 and T_1,*
(II) *All the 2-chain relations,*
(III) *All the lantern relations.*

2.2. Relations among the crosscap pushing maps and Dehn twists. Let μ be a one-sided simple closed curve on $N_{g,n}$ and let α be a simple closed curve on $N_{g,n}$ such that μ and α intersect transversely at one point. Recall that α is oriented. For these simple closed curves μ and α, we denote by $Y_{\mu,\alpha}$ a self-diffeomorphism on $N_{g,n}$ which is described as the result of pushing the Möbius band $\mathcal{N}_{N_{g,n}}(\mu)$ once along α. We call $Y_{\mu,\alpha}$ a *crosscap pushing map*. In particular, if α is two-sided, we call $Y_{\mu,\alpha}$ a *Y-homeomorphism* (or a *crosscap slide*), where a *crosscap* means a Möbius band in the interior of a surface. Note that $Y_{\mu,\alpha} = Y^{-1}_{\mu,\alpha^{-1}} = Y_{\mu^{-1},\alpha}$. The Y-homeomorphism was originally defined by Lickorish [13]. We have the following fundamental relation in $\mathcal{M}(N_{g,n})$ and we also call the relation the *braid relation*.

Lemma 2.6 (The braid relation (ii)). *Let μ be a one-sided simple closed curve on $N_{g,n}$ and let α be simple closed curve on $N_{g,n}$ such that μ and α intersect transversely at one point. For $f \in \mathcal{M}(N_{g,n})$, we have*

$$fY_{\mu,\alpha}f^{-1} = Y_{f(\mu),f(\alpha)}.$$

We describe crosscap pushing maps from a different point of view. Let $e : D' \hookrightarrow \mathrm{int}S$ be a smooth embedding of the unit disk $D' \subset \mathbb{C}$. Put $D := e(D')$. Let S' be the surface obtained from $S - \mathrm{int}D$ by the identification of antipodal points of ∂D. We call the manipulation that gives S' from S the *blowup of S on D*. Note that the image $M \subset S'$ of $\mathcal{N}_{S-\mathrm{int}D}(\partial D) \subset S - \mathrm{int}D$ with respect to the blowup of S on D is a crosscap. Conversely, the *blowdown of S' on M* is the following manipulation that gives S from S'. We paste a disk on the boundary obtained by cutting S along the center line μ of M. The blowdown of S' on M is the inverse manipulation of the blowup of S on D.

Let μ be a one-sided simple closed curve on $N_{g,n}$ and let S be the surface which is obtained from $N_{g,n}$ by the blowdown of $N_{g,n}$ on $\mathcal{N}_{N_{g,n}}(\mu)$. Note that S is diffeomorphic to $N_{g-1,n}$ or $\Sigma_{h,n}$ for $g = 2h+1$. Denote by x_μ the center point of a disk D_μ that is pasted on the boundary obtained by cutting S along μ. Let $e : D' \hookrightarrow D_\mu \subset S$ be a smooth embedding of the unit disk $D' \subset \mathbb{C}$ to S such that $D_\mu = e(D')$ and $e(0) = x_\mu$. Let $\mathcal{M}(S, x_\mu)$ be the group of isotopy classes of self-diffeomorphisms on S fixing the boundary ∂S and the point x_μ, where isotopies also fix the boundary ∂S and x_μ. Then we have the *blowup homomorphism*

$$\varphi_\mu : \mathcal{M}(S, x_\mu) \to \mathcal{M}(N_{g,n})$$

that is defined as follows. For $h \in \mathcal{M}(S, x_\mu)$, we take a representative diffeomorphism $\omega \in h$ of the mapping class h which satisfies either of the following conditions: (a) $\omega|_{D_\mu}$ is the identity map on D_μ, (b) $\omega(x) = e(\overline{e^{-1}(x)})$ for $x \in D_\mu$, where $\overline{e^{-1}(x)}$ is the complex conjugate of $e^{-1}(x) \in \mathbb{C}$. Such ω is compatible with the blowup of S on D_μ, thus $\varphi_\mu(h) \in \mathcal{M}(N_{g,n})$ is induced and well-defined (c.f. [21, Subsection 2.3]).

The *point pushing map*

$$j_{x_\mu} : \pi_1(S, x_\mu) \to \mathcal{M}(S, x_\mu)$$

is a homomorphism that is defined as follows. For $\gamma \in \pi_1(S, x_\mu)$, $j_{x_\mu}(\gamma) \in \mathcal{M}(S, x_\mu)$ is described as the result of pushing the point x_μ once along γ. The point pushing map comes from the Birman exact sequence. Note that for $\gamma_1, \gamma_2 \in \pi_1(S, x_\mu)$, $\gamma_1\gamma_2$ means $\gamma_1\gamma_2(t) = \gamma_2(2t)$ for $0 \le t \le \frac{1}{2}$ and $\gamma_1\gamma_2(t) = \gamma_1(2t-1)$ for $\frac{1}{2} \le t \le 1$.

Following Szepietowski [21] we define the composition of the homomorphisms:

$$\psi_{x_\mu} := \varphi_\mu \circ j_{x_\mu} : \pi_1(S, x_\mu) \to \mathcal{M}(N_{g,n}).$$

For each closed curve α on $N_{g,n}$ which transversely intersects with μ at one point, we take a loop $\overline{\alpha}$ on S based at x_μ such that $\overline{\alpha}$ has no self-intersection points on D_μ and α is the image of $\overline{\alpha}$ with respect to the blowup of S on D_μ. If α is simple, we take $\overline{\alpha}$ as a simple loop. The next two lemmas follow from the description of the point pushing map (see [11, Lemma 2.2, Lemma 2.3]).

Lemma 2.7. *For a simple closed curve α on $N_{g,n}$ which transversely intersects with a one-sided simple closed curve μ on $N_{g,n}$ at one point, we have*

$$\psi_{x_\mu}(\overline{\alpha}) = Y_{\mu,\alpha}.$$

Lemma 2.8. *For a one-sided simple closed curve α on $N_{g,n}$ which transversely intersects with a one-sided simple closed curve μ on $N_{g,n}$ at one point, we take $\mathcal{N}_S(\overline{\alpha})$ such that the interior of $\mathcal{N}_S(\overline{\alpha})$ contains D_μ and an orientation $\theta_{\overline{\alpha}} \in \{+_{\overline{\alpha}}, -_{\overline{\alpha}}\}$ of $\mathcal{N}_S(\overline{\alpha})$. Denote by $\overline{\delta_1}$ (resp. $\overline{\delta_2}$) the boundary component of $\mathcal{N}_S(\overline{\alpha})$ on the right (resp. left) side of $\overline{\alpha}$, and by δ_i $(i = 1, 2)$ the two-sided simple closed curve on $N_{g,n}$ which is the image of $\overline{\delta_i}$ with respect to the blowup of S on D_μ. Let $\overline{\theta_i} \in \{+_{\overline{\delta_i}}, -_{\overline{\delta_i}}\}$ $(i = 1, 2)$ be the orientation of $\mathcal{N}_S(\overline{\delta_i})$ which is induced by $\theta_{\overline{\alpha}}$ and $\theta_i \in \{+_{\delta_i}, -_{\delta_i}\}$ $(i = 1, 2)$ the orientation of $\mathcal{N}_{N_{g,n}}(\delta_i)$ which is induced by $\overline{\theta_i}$ (see Figure 3). Then we have*

$$Y_{\mu,\alpha} = t_{\delta_1;\theta_1} t_{\delta_2;\theta_2}^{-1}.$$

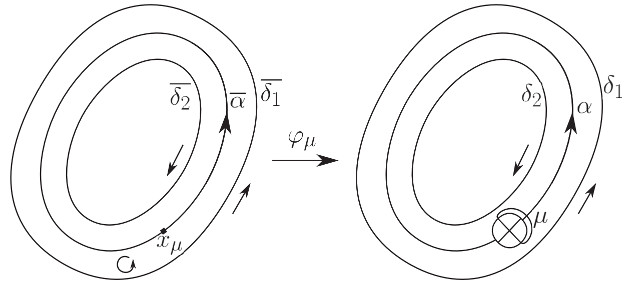


Fig.3. Simple closed curves $\overline{\delta_1}$, $\overline{\delta_2}$, δ_1 and δ_2, and orientations $\overline{\theta_1}$, $\overline{\theta_2}$, θ_1 and θ_2 of their regular neighborhoods. The x-mark means that antipodal points of ∂D_μ are identified.

By the definition of the homomorphism ψ_{x_μ} and Lemma 2.7, we have the following lemma.

Lemma 2.9. *Let α and β be simple closed curves on $N_{g,n}$ each of which transversely intersect with a one-sided simple closed curve μ on $N_{g,n}$ at one point. Suppose that the product $\overline{\alpha}\overline{\beta}$ of $\overline{\alpha}$ and $\overline{\beta}$ in $\pi_1(S, x_\mu)$ is represented by a simple loop on S, and $\alpha\beta$ is a simple closed curve on $N_{g,n}$ which is the image of the representative of $\overline{\alpha\beta}$ with respect to the blow-up of S on D_μ. Then we have*

$$Y_{\mu,\alpha\beta} = Y_{\mu,\alpha} Y_{\mu,\beta}.$$

3. Finite presentation for $\mathcal{M}(N_{g,n})$

In this section, we review Stukow's finite presentation for $\mathcal{M}(N_{g,n})$ when $n \in \{0, 1\}$ and give a finite presentation for $\mathcal{M}(N_{g,n})$ when $n \geq 2$. We use their finite presentations for $\mathcal{M}(N_{g,n})$ in the proof of the main theorem in Section 4.

Let $e_i : D' \hookrightarrow \mathrm{int}\Sigma_{0,1}$ for $i = 1, 2, \ldots, g+n-1$ be smooth embeddings of the unit disk $D' \subset \mathbb{C}$ to a disk $\Sigma_{0,1}$ such that $D_i := e_i(D')$ and D_j are disjoint for distinct $1 \leq i, j \leq g+n-1$. For $n \geq 1$, we take a model of $N_{g,n}$ as the surface obtained from $\Sigma_{0,1} - (\mathrm{int}D_{g+1} \sqcup \cdots \sqcup \mathrm{int}D_{n-1})$ by the blowups on $D_1, \ldots, D_g$ and we describe the identification of ∂D_i by the x-mark as in Figures 4. We denote by $\delta_1, \ldots, \delta_{n-1}$ and δ boundary components of $N_{g,n}$ as in Figure 4 which are obtained from $\partial D_{g+1}, \ldots, \partial D_{g+n-1}$ and $\partial\Sigma_{0,1}$, respectively. Let $\alpha_1, \ldots, \alpha_{g-1}, \beta$ and μ_1 be simple closed curves on $N_{g,n}$ as in Figure 4 and let $\alpha_{i;j}$ for $1 \leq i \leq g-1$ and $1 \leq j \leq n-1$, $\rho_{i;j}$ for $1 \leq i \leq g$ and $1 \leq j \leq n-1$ and $\sigma_{i,j}$, $\bar{\sigma}_{i,j}$ for $1 \leq i < j \leq n-1$ be simple closed curves on $N_{g,n}$ as in Figure 5. We give orientations of regular neighborhoods of their simple closed curves as in Figure 4 and 5. Then we define the mapping classes

$$\begin{aligned}
a_i &:= t_{\alpha_i} && \text{for } 1 \leq i \leq g-1,\\
b &:= t_{\beta}, &&\\
y &:= Y_{\mu_1,\alpha_1}, &&\\
d_i &:= t_{\delta_i} && \text{for } 1 \leq i \leq n-1,\\
a_{i;j} &:= t_{\alpha_{i;j}} && \text{for } 1 \leq i \leq g-1 \text{ and } 1 \leq j \leq n-1,\\
r_{i;j} &:= t_{\rho_{i;j}} && \text{for } 1 \leq i \leq g \text{ and } 1 \leq j \leq n-1,\\
s_{i,j} &:= t_{\sigma_{i,j}} && \text{for } 1 \leq i < j \leq n-1,\\
\bar{s}_{i,j} &:= t_{\bar{\sigma}_{i,j}} && \text{for } 1 \leq i < j \leq n-1,\\
\bar{s}_{j,k;i} &:= \{(a_{1;k}a_1^{-1})^{-1} r_{2;k} \cdots (a_{i-1;k}a_{i-1}^{-1})^{-1} r_{i;k}\}^{-1} \bar{s}_{j,k} &&\\
&\quad\ \{(a_{1;k}a_1^{-1})^{-1} r_{2;k} \cdots (a_{i-1;k}a_{i-1}^{-1})^{-1} r_{i;k}\} &&\\
&\quad\ \text{for } 2 \leq i \leq g \text{ and } 1 \leq j < k \leq n-1.
\end{aligned}$$

Remark that, for $2 \leq i \leq g$ and $1 \leq j < k \leq n-1$, $\bar{s}_{j,k;i}$ is the Dehn twist along the simple closed curve $\bar{\sigma}_{j,k;i}$ on $N_{g,n}$ as in Figure 6.

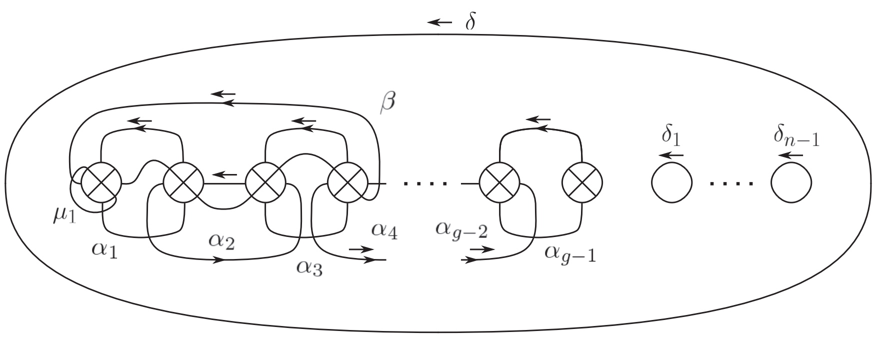


Fig. 4. A model of $N_{g,n}$ and simple closed curves $\alpha_1, \ldots, \alpha_{g-1}, \beta$ and μ_1 on $N_{g,n}$.

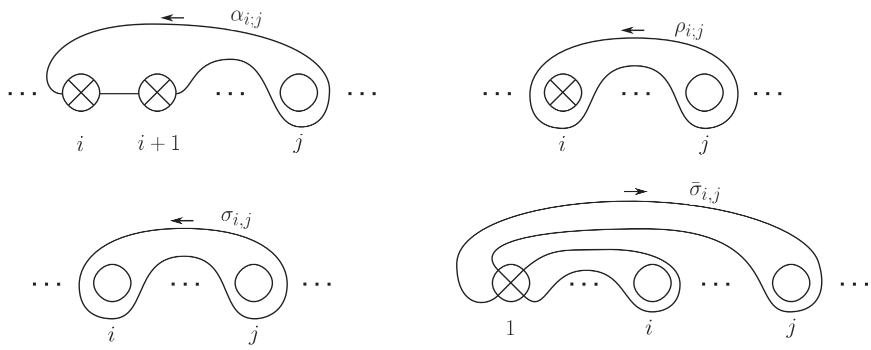


Fig. 5. The simple closed curves $\alpha_{i;j}$, $\rho_{i;j}$, $\sigma_{i,j}$ and $\bar{\sigma}_{i,j}$ on $N_{g,n}$.

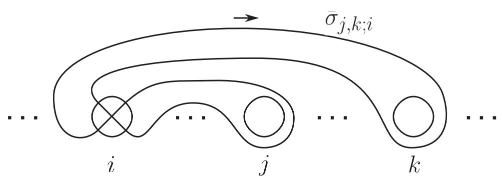


Fig. 6. The simple closed curve $\bar{\sigma}_{j,k;i}$ on $N_{g,n}$.

Epstein [3] show that $\mathcal{M}(N_{1,1})$ is trivial. For a group G and $g_1, g_2 \in G$, we define $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. Stukow gave the following finite presentation for $\mathcal{M}(N_{g,1})$ when $g = 2$ in [18], and when $g \geq 3$ in [20] by rewriting the finite presentation in [17].

Theorem 3.1 ([3], [18], [20]). *$\mathcal{M}(N_{1,1})$ is the trivial group. $\mathcal{M}(N_{2,1})$ has the presentation*

$$\mathcal{M}(N_{2,1}) \quad = \quad \langle a_1, y \mid y a_1 y^{-1} = a_1^{-1} \rangle.$$

If $g \geq 3$, then $\mathcal{M}(N_{g,1})$ admits a presentation with generators $a_1, \ldots, a_{g-1}, y$, and b for $g \geq 4$. The defining relations are

(A1) $[a_i, a_j] = 1$ *for $g \geq 4$, $|i - j| > 1$,*
(A2) $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ *for $i = 1, \ldots, g-2$,*
(A3) $[a_i, b] = 1$ *for $g \geq 4$, $i \neq 4$,*
(A4) $a_4 b a_4 = b a_4 b$ *for $g \geq 5$,*
(A5) $(a_2 a_3 a_4 b)^{10} = (a_1 a_2 a_3 a_4 b)^6$ *for $g \geq 5$,*
(A6) $(a_2 a_3 a_4 a_5 a_6 b)^{12} = (a_1 a_2 a_3 a_4 a_5 a_6 b)^9$ *for $g \geq 7$,*
(A9a) $[b_2, b] = 1$ *for $g = 6$,*
(A9b) $[a_{g-5}, b_{\frac{g-2}{2}}] = 1$ *for $g \geq 8$ even,*
where $b_0 = a_1$, $b_1 = b$ and
$b_{i+1} = (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3} b_i)^5 (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3})^{-6}$
for $1 \leq i \leq \frac{g-4}{2}$,
(B1) $y(a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1}) = (a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1}) y$ *for $g \geq 4$,*
(B2) $y(a_2 a_1 y^{-1} a_2^{-1} y a_1 a_2) y = a_1 (a_2 a_1 y^{-1} a_2^{-1} y a_1 a_2) a_1$,
(B3) $[a_i, y] = 1$ *for $g \geq 4$, $i = 3, \ldots, g-1$,*
(B4) $a_2 (y a_2 y^{-1}) = (y a_2 y^{-1}) a_2$,
(B5) $y a_1 = a_1^{-1} y$,

(B6) $byby^{-1} = \{a_1a_2a_3(y^{-1}a_2y)a_3^{-1}a_2^{-1}a_1^{-1}\}\{a_2^{-1}a_3^{-1}(ya_2y^{-1})a_3a_2\}$ *for* $g \geq 4$,
(B7) $[(a_4a_5a_3a_4a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1}), b] = 1$ *for* $g \geq 6$,
(B8) $\{(ya_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})b(a_4a_3a_2a_1y^{-1})\}\{(a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})b^{-1}(a_4a_3a_2a_1)\}$
$= \{(a_4^{-1}a_3^{-1}a_2^{-1})y(a_2a_3a_4)\}\{a_3^{-1}a_2^{-1}y^{-1}a_2a_3\}\{a_2^{-1}ya_2\}y^{-1}$ *for* $g \geq 5$.

For $n \geq 2$, we have the following finite presentation for $\mathcal{M}(N_{g,n})$ and give a proof in Section 5.4.

Proposition 3.2. *For* $g \geq 1$ *and* $n \geq 2$, $\mathcal{M}(N_{g,n})$ *has the presentation which is obtained from the finite presentation for* $\mathcal{M}(N_{g,1})$ *in* Theorem 3.1 *by adding generators* d_i $(i = 1, \dots, n-1)$, $a_{i;j}$ $(1 \leq i \leq g-1, 1 \leq j \leq n-1)$, $r_{i,j}$ $(1 \leq i \leq g, 1 \leq j \leq n-1)$, $s_{i,j}$ $(1 \leq i < j \leq n-1)$, *and* $\bar{s}_{i,j}$ $(1 \leq i < j \leq n-1)$, *the relations*

(D0) $[d_j, a_i] = [d_j, y] = [d_j, b] = [d_j, d_l] = [d_j, a_{i;k}] = [d_j, r_{i';k}] = [d_j, s_{l,t}] = [d_j, \bar{s}_{l,t}] = 1$

for $1 \leq i \leq g-1$, $1 \leq i' \leq g-1$, $1 \leq j, k \leq n-1$, *and* $1 \leq l < t \leq n-1$, *and the following relations for* $1 \leq j, k \leq n-1$, $1 \leq l, t < k$, *and any possible* $1 \leq i, m \leq g$:

(D1a) $a_m(a_{i;k}a_i^{-1})a_m^{-1} = \begin{cases} (a_{i;k}a_i^{-1})(a_{i-1;k}a_{i-1}^{-1}) & \text{for } m = i-1, \\ (a_{i+1;k}a_{i+1}^{-1})^{-1}(a_{i;k}a_i^{-1}) & \text{for } m = i+1, \\ a_{i;k}a_i^{-1} & \text{for } m \neq i-1, i+1, \end{cases}$

(D1b) $y(a_{i;k}a_i^{-1})y^{-1} = \begin{cases} (a_{1;k}a_1^{-1})^{-1}r_{2;k}r_{1;k}d_{n-1}^{-2} & \text{for } i = 1, \\ (a_{2;k}a_2^{-1})r_{1;k}d_k^{-1} & \text{for } i = 2, \\ a_{i;k}a_i^{-1} & \text{for } i \geq 3, \end{cases}$

(D1c) $b(a_{i;k}a_i^{-1})b^{-1} =$
$\begin{cases} \{(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1})\}^{-1}(a_{1;k}a_1^{-1})\{(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1})\} & \text{for } i = 1, \\ \{(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1})\}^{-1}(a_{2;k}a_2^{-1})\{(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1})\} & \text{for } i = 2, \\ (a_{1;k}a_1^{-1})^{-1}(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1}) & \text{for } i = 3, \\ (a_{4;k}a_4^{-1})(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1})d_k^{-1} & \text{for } i = 4, \\ a_{i;k}a_i^{-1} & \text{for } i \geq 5, \end{cases}$

(D1d) $a_{m;l}(a_{i;k}a_i^{-1})a_{m;l}^{-1} =$
$\begin{cases} [(s_{l,k}d_l^{-1})^{-1}, (a_{m;k}a_m^{-1})^{-1}]^{-1}(a_{i;k}a_i^{-1})[(s_{l,k}d_l^{-1})^{-1}, (a_{m;k}a_m^{-1})^{-1}] \\ \text{for } m \leq i-2, \\ [(a_{i-1;k}a_{i-1}^{-1})^{-1}, (s_{l,k}d_l^{-1})^{-1}](a_{i;k}a_i^{-1})(s_{l,k}d_l^{-1})(a_{i-1;k}a_{i-1}^{-1})d_k^{-1} \\ \text{for } m = i-1, \\ \{(s_{l,k}d_l^{-1})(a_{i;k}a_i^{-1})\}^{-1}(a_{i;k}a_i^{-1})\{(s_{l,k}d_l^{-1})(a_{i;k}a_i^{-1})\} & \text{for } m = i, \\ (a_{i+1;k}a_{i+1}^{-1})^{-1}(s_{l,k}d_l^{-1})^{-1}(a_{i;k}a_i^{-1})d_k & \text{for } m = i+1, \\ a_{i;k}a_i^{-1} & \text{for } m \geq i+2, \end{cases}$

(D1e) $r_{m;l}(a_{i;k}a_i^{-1})r_{m;l}^{-1} =$
$\begin{cases} [(s_{l,k}d_l^{-1})^{-1}, r_{m;k}^{-1}]^{-1}(a_{i;k}a_i^{-1})[(s_{l,k}d_l^{-1})^{-1}, r_{m;k}^{-1}] & \text{for } m \leq i-1, \\ \{r_{i;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{i;k}\}(s_{l,k}d_l^{-1})(a_{i;k}a_i^{-1}) \\ (\bar{s}_{l,k;i}d_l^{-1})^{-1}\{r_{i;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{i;k}\}^{-1}d_k^{-2} & \text{for } m = i, \\ r_{i+1;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{i+1;k}(\bar{s}_{l,k;i+1}d_l^{-1})(a_{i;k}a_i^{-1})d_k^2 & \text{for } m = i+1, \\ a_{i;k}a_i^{-1} & \text{for } m \geq i+2, \end{cases}$

(D1f) $s_{l,t}(a_{i;k}a_i^{-1})s_{l,t}^{-1} = a_{i;k}a_i^{-1}$,

(D1g) $\bar{s}_{l,t}(a_{i;k}a_i^{-1})\bar{s}_{l,t}^{-1} =$

$$\begin{cases} [(\bar{s}_{l,k}d_l^{-1})^{-1}, s_{t,k}d_t^{-1}]^{-1}(s_{l,k}d_l^{-1})(a_{1;k}a_1^{-1})r_{1;k}(\bar{s}_{t,k}d_t^{-1})r_{1;k}^{-1} \\ (s_{l,k}d_l^{-1})^{-1}r_{1;k}(\bar{s}_{t,k}d_t^{-1})^{-1}r_{1;k}^{-1}[(\bar{s}_{l,k}d_l^{-1})^{-1}, (s_{t,k}d_t^{-1})] \quad \textit{for } i = 1, \\ \{[\bar{s}_{l,k}d_l^{-1}, (s_{t,k}d_t^{-1})^{-1}][s_{l,k}d_l^{-1}, r_{1;k}(\bar{s}_{t,k}d_t^{-1})^{-1}r_{1;k}^{-1}]\}(a_{i;k}a_i^{-1}) \\ \{[\bar{s}_{l,k}d_l^{-1}, (s_{t,k}d_t^{-1})^{-1}][s_{l,k}d_l^{-1}, r_{1;k}(\bar{s}_{t,k}d_t^{-1})^{-1}r_{1;k}^{-1}]\}^{-1} \quad \textit{for } i \geq 2, \end{cases}$$

(D2a) $$a_m r_{i;k} a_m^{-1} = \begin{cases} r_{i;k}r_{i-1;k}(a_{i-1;k}a_{i-1}^{-1})^{-1}r_{i;k}(a_{i-1;k}a_{i-1}^{-1})d_k \\ \textit{for } m = i-1, \\ (a_{i;k}a_i^{-1})^{-1}r_{i+1;k}^{-1}(a_{i;k}a_i^{-1}) \quad \textit{for } m = i, \\ r_{i;k} \quad \textit{for } m \neq i-1, i, \end{cases}$$

(D2b) $$y r_{i;k} y^{-1} = \begin{cases} \{(a_{1;k}a_1^{-1})^{-1}r_{2;k}r_{1;k}\}^{-1}r_{1;k}^{-1}\{(a_{1;k}a_1^{-1})^{-1}r_{2;k}r_{1;k}\} \\ \textit{for } i = 1, \\ (a_{1;k}a_1^{-1})r_{1;k}(a_{1;k}a_1^{-1})^{-1}r_{2;k}r_{1;k}d_{n-1} \quad \textit{for } i = 2, \\ r_{i;k} \quad \textit{for } i \geq 3, \end{cases}$$

(D2c) $$b r_{i;k} b^{-1} = \begin{cases} (a_{1;k}a_1^{-1})^{-1}(a_{3;k}a_3^{-1})^{-1}(a_{2;k}a_2^{-1})^{-1} \\ r_{4;k}^{-1}(a_{3;k}a_3^{-1})r_{3;k}^{-1}(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1}) \quad \textit{for } i = 1, \\ \{(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1})\}^{-1}(a_{1;k}a_1^{-1})(a_{3;k}a_3^{-1})r_{2;k} \\ r_{1;k}(a_{1;k}a_1^{-1})^{-1}r_{2;k}(a_{2;k}a_2^{-1})^{-1}r_{3;k}(a_{3;k}a_3^{-1})^{-1} \\ r_{4;k}(a_{2;k}a_2^{-1})\{(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1})\} \quad \textit{for } i = 2, \\ \{(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1})\}^{-1}r_{4;k}^{-1} \\ (a_{3;k}a_3^{-1})r_{3;k}^{-1}(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})r_{1;k}^{-1} \\ (a_{2;k}a_2^{-1})^{-1}r_{3;k} \\ (a_{3;k}a_3^{-1})^{-1}(a_{1;k}a_1^{-1})^{-1}\{(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1})\} \quad \textit{for } i = 3, \\ r_{4;k}(a_{2;k}a_2^{-1}) \\ r_{1;k}(a_{1;k}a_1^{-1})^{-1}r_{2;k}(a_{2;k}a_2^{-1})^{-1}r_{3;k}(a_{3;k}a_3^{-1})^{-1} \\ r_{4;k}(a_{3;k}a_3^{-1})(a_{1;k}a_1^{-1}) \quad \textit{for } i = 4, \\ r_{i;k} \quad \textit{for } i \geq 5, \end{cases}$$

(D2d) $$a_{m;l} r_{i;k} a_{m;l}^{-1} = \begin{cases} [(s_{l,k}d_l^{-1})^{-1}, (a_{m;k}a_m^{-1})^{-1}]^{-1}r_{i;k}[(s_{l,k}d_l^{-1})^{-1}, (a_{m;k}a_m^{-1})^{-1}] \\ \textit{for } m \leq i-2, \\ \{(s_{l,k}d_l^{-1})(a_{i-1;k}a_{i-1}^{-1})\}^{-1}(a_{i-1;k}a_{i-1}^{-1})(s_{l,k}d_l^{-1})r_{i;k} \\ r_{i-1;k}(a_{i-1;k}a_{i-1}^{-1})^{-1}r_{i;k}(\bar{s}_{l,k;i}d_l^{-1})\{(s_{l,k}d_l^{-1})(a_{i-1;k}a_{i-1}^{-1})\} \\ \textit{for } m = i-1, \\ (a_{i;k}a_i^{-1})^{-1}(s_{l,k}d_l^{-1})^{-1}r_{i+1;k}^{-1}(a_{i;k}a_i^{-1})(\bar{s}_{l,k;i}d_l^{-1})^{-1} \quad \textit{for } m = i, \\ r_{i;k} \quad \textit{for } m \geq i+1, \end{cases}$$

(D2e) $$r_{m;l} r_{i;k} r_{m;l}^{-1} = \begin{cases} [(s_{l,k}d_l^{-1})^{-1}, r_{m;k}^{-1}]^{-1}r_{i;k}[(s_{l,k}d_l^{-1})^{-1}, r_{m;k}^{-1}] \\ \textit{for } m \leq i-1, \\ \{(s_{l,k}d_l^{-1})r_{i;k}\}^{-1}r_{i;k}\{(s_{l,k}d_l^{-1})r_{i;k}\} \quad \textit{for } m = i, \\ r_{i;k} \quad \textit{for } m \geq i+1, \end{cases}$$

(D2f) $s_{l,t} r_{i;k} s_{l,t}^{-1} = r_{i;k}$,

(D2g) $$\bar{s}_{l,t} r_{i;k} \bar{s}_{l,t}^{-1} = \begin{cases} [s_{t,k}d_t^{-1}, (\bar{s}_{l,k}d_l^{-1})^{-1}][r_{1;k}(\bar{s}_{t,k}d_t^{-1})r_{1;k}^{-1}, (s_{l,k}d_l^{-1})^{-1}]r_{1;k} \quad \textit{for } i = 1, \\ \{[\bar{s}_{l,k}d_l^{-1}, (s_{t,k}d_t^{-1})^{-1}][s_{l,k}d_l^{-1}, r_{1;k}(\bar{s}_{t,k}d_t^{-1})^{-1}r_{1;k}^{-1}]\}^{-1}r_{1;k} \\ \{[\bar{s}_{l,k}d_l^{-1}, (s_{t,k}d_t^{-1})^{-1}][s_{l,k}d_l^{-1}, r_{1;k}(\bar{s}_{t,k}d_t^{-1})^{-1}r_{1;k}^{-1}]\} \quad \textit{for } i \geq 2, \end{cases}$$

(D3a) $a_m(s_{j,k}d_j^{-1})a_m^{-1} = s_{j,k}d_j^{-1}$, (D3b) $y(s_{j,k}d_j^{-1})y^{-1} = s_{j,k}d_j^{-1}$,

(D3c) $b(s_{j,k}d_j^{-1})b^{-1} = s_{j,k}d_j^{-1}$,

(D3d) $a_{m;l}(s_{j,k}d_j^{-1})a_{m;l}^{-1} =$

$$\begin{cases} [(s_{l,k}d_l^{-1})^{-1}, (a_{m;k}a_m^{-1})^{-1}]^{-1}(s_{j,k}d_j^{-1})[(s_{l,k}d_l^{-1})^{-1}, (a_{m;k}a_m^{-1})^{-1}] \\ \textit{for } l > j, \\ (a_{m;k}a_m^{-1})^{-1}(s_{j,k}d_j^{-1})(a_{m;k}a_m^{-1}) \quad \textit{for } l = j, \\ s_{j,k}d_j^{-1} \quad \textit{for } l = j, \end{cases}$$

(D3e) $r_{m;l}(s_{j,k}d_j^{-1})r_{m;l}^{-1} =$

$$\begin{cases} [(s_{l,k}d_l^{-1})^{-1}, r_{m;k}^{-1}]^{-1}(s_{j,k}d_j^{-1})[(s_{l,k}d_l^{-1})^{-1}, r_{m;k}^{-1}] \quad \textit{for } l > j, \\ r_{m;k}^{-1}(s_{j,k}d_j^{-1})r_{m;k} \quad \textit{for } l = j, \\ s_{j,k}d_j^{-1} \quad \textit{for } l < j, \end{cases}$$

(D3f) $s_{l,t}(s_{j,k}d_j^{-1})s_{l,t}^{-1} =$

$$\begin{cases} \{(s_{t,k}d_t^{-1})(s_{j,k}d_j^{-1})\}^{-1}(s_{j,k}d_j^{-1})\{(s_{t,k}d_t^{-1})(s_{j,k}d_j^{-1})\} \quad \textit{for } l = j, \\ [(s_{t,k}d_t^{-1})^{-1}, (s_{l,k}d_l^{-1})^{-1}]^{-1}(s_{j,k}d_j^{-1})[(s_{t,k}d_t^{-1})^{-1}, (s_{l,k}d_l^{-1})^{-1}] \\ \textit{for } l < j < t, \\ (s_{l,k}d_l^{-1})^{-1}(s_{j,k}d_j^{-1})(s_{l,k}d_l^{-1}) \quad \textit{for } t = j, \\ s_{j,k}d_j^{-1} \quad \textit{for the other cases}, \end{cases}$$

(D3g) $\bar{s}_{l,t}(s_{j,k}d_j^{-1})\bar{s}_{l,t}^{-1} =$

$$\begin{cases} \{[\bar{s}_{l,k}d_l^{-1}, (s_{t,k}d_t^{-1})^{-1}][s_{l,k}d_l^{-1}, r_{1;k}(\bar{s}_{t,k}d_t^{-1})^{-1}r_{1;k}^{-1}]\}^{-1}(s_{j,k}d_j^{-1}) \\ \{[\bar{s}_{l,k}d_l^{-1}, (s_{t,k}d_t^{-1})^{-1}][s_{l,k}d_l^{-1}, r_{1;k}(\bar{s}_{t,k}d_t^{-1})^{-1}r_{1;k}^{-1}]\} \quad \textit{for } l > j, \\ \{[\bar{s}_{j,k}d_j^{-1}, (s_{t,k}d_t^{-1})^{-1}](s_{j,k}d_j^{-1})r_{1;k}(\bar{s}_{t,k}d_t^{-1})^{-1}r_{1;k}^{-1}\}^{-1}(s_{j,k}d_j^{-1}) \\ \{[\bar{s}_{j,k}d_j^{-1}, (s_{t,k}d_t^{-1})^{-1}](s_{j,k}d_j^{-1})r_{1;k}(\bar{s}_{t,k}d_t^{-1})^{-1}r_{1;k}^{-1}\} \quad \textit{for } l = j, \\ [(\bar{s}_{l,k}d_l^{-1})^{-1}, s_{t,k}d_t^{-1}]^{-1}(s_{j,k}d_j^{-1})[(\bar{s}_{l,k}d_l^{-1})^{-1}, s_{t,k}d_t^{-1}] \quad \textit{for } l < j < t, \\ \{(\bar{s}_{l,k}d_l^{-1})(s_{j,k}d_j^{-1})^{-1}\}^{-1}(s_{j,k}d_j^{-1})\{(\bar{s}_{l,k}d_l^{-1})(s_{j,k}d_j^{-1})^{-1}\} \quad \textit{for } t = j, \\ s_{j,k}d_j^{-1} \quad \textit{for } t < j, \end{cases}$$

(D4a) $a_m(\bar{s}_{j,k}d_j^{-1})a_m^{-1} =$

$$\begin{cases} \{r_{1;k}^{-1}r_{2;k}^{-1}(a_{1;k}a_1^{-1})\}^{-1}(\bar{s}_{j,k}d_j^{-1})\{r_{1;k}^{-1}r_{2;k}^{-1}(a_{1;k}a_1^{-1})\} \quad \textit{for } m = 1, \\ \bar{s}_{j,k}d_j^{-1} \quad \textit{for } m \geq 2, \end{cases}$$

(D4b) $y(\bar{s}_{j,k}d_j^{-1})y^{-1} = \{r_{1;k}^{-1}(a_{1;k}a_1^{-1})^{-2}r_{2;k}r_{1;k}\}^{-1}(\bar{s}_{j,k}d_j^{-1})\{r_{1;k}^{-1}(a_{1;k}a_1^{-1})^{-2}r_{2;k}r_{1;k}\}$,

(D4c) $b(\bar{s}_{j,k}d_j^{-1})b^{-1} =$

$$\{r_{1;k}^{-1}(a_{2;k}a_2^{-1})^{-1}r_{4;k}^{-1}(a_{3;k}a_3^{-1})r_{3;k}^{-1}(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})\}^{-1}(\bar{s}_{j,k}d_j^{-1})$$
$$\{r_{1;k}^{-1}(a_{2;k}a_2^{-1})^{-1}r_{4;k}^{-1}(a_{3;k}a_3^{-1})r_{3;k}^{-1}(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})\},$$

(D4d) $a_{m;l}(\bar{s}_{j,k}d_j^{-1})a_{m;l}^{-1} =$

$$\begin{cases}
\{r_{1;k}^{-1}r_{2;k}^{-1}(a_{1;k}a_1^{-1})(\bar{s}_{l,k}d_l^{-1})^{-1}\}^{-1}(\bar{s}_{j,k}d_j^{-1})\{r_{1;k}^{-1}r_{2;k}^{-1}(a_{1;k}a_1^{-1})(\bar{s}_{l,k}d_l^{-1})^{-1}\} \\
\textit{for } m=1, l<j, \\
\{(\bar{s}_{l,k}d_l^{-1})^{-1}r_{1;k}^{-1}r_{2;k}^{-1}(a_{1;k}a_1^{-1})\}^{-1}(\bar{s}_{j,k}d_j^{-1})\{(\bar{s}_{l,k}d_l^{-1})^{-1}r_{1;k}^{-1}r_{2;k}^{-1}(a_{1;k}a_1^{-1})\} \\
\textit{for } m=1, l>j, \\
\{(a_{m-1;k}a_{m-1}^{-1})r_{m-1;k}\cdots(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})\}^{-1}(\bar{s}_{j,k;m+1}d_j^{-1}) \\
\{(a_{m-1;k}a_{m-1}^{-1})r_{m-1;k}\cdots(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})\} \quad \textit{for } m\geq 2, l=j, \\
\{(a_{m-1;k}a_{m-1}^{-1})r_{m-1;k}\cdots(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})\}^{-1} \\
\{(\bar{s}_{l,k;m}d_l^{-1})^{-1}r_{m;k}^{-1}(\bar{s}_{l,k;m+1}d_l^{-1})\}^{-1}(\bar{s}_{j,k;m}d_j^{-1}) \\
\{(\bar{s}_{l,k;m}d_l^{-1})^{-1}r_{m;k}^{-1}(\bar{s}_{l,k;m+1}d_l^{-1})\} \\
\{(a_{m-1;k}a_{m-1}^{-1})r_{m-1;k}\cdots(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})\} \quad \textit{for } m\geq 2, l>j, \\
\bar{s}_{j,k}d_j^{-1} \quad \textit{for the other cases},
\end{cases}$$

(D4e) $r_{m;l}(\bar{s}_{j,k}d_j^{-1})r_{m;l}^{-1} =$

$$\begin{cases}
\{r_{1;k}^{-1}(\bar{s}_{l,k}d_l^{-1})^{-1}(s_{l,k}d_l^{-1})r_{1;k}\}^{-1}(\bar{s}_{j,k}d_j^{-1}) \\
\{r_{1;k}^{-1}(\bar{s}_{l,k}d_l^{-1})^{-1}(s_{l,k}d_l^{-1})r_{1;k}\} \quad \textit{for } m=1, l<j, \\
\{(s_{j,k}d_j^{-1})r_{1;k}\}^{-1}(\bar{s}_{j,k}d_j^{-1})\{(s_{j,k}d_j^{-1})r_{1;k}\} \quad \textit{for } m=1, l=j, \\
\{(\bar{s}_{l,k}d_l^{-1})^{-1}r_{1;k}^{-1}(s_{l,k}d_l^{-1})r_{1;k}\}^{-1}(\bar{s}_{j,k}d_j)^{-1} \\
\{(\bar{s}_{l,k}d_l^{-1})^{-1}r_{1;k}^{-1}(s_{l,k}d_l^{-1})r_{1;k}\} \quad \textit{for } m=1, l>j, \\
\{(a_{m-1;k}a_{m-1}^{-1})r_{m-1;k}\cdots(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})\}^{-1} \\
(\bar{s}_{j,k;m}d_j^{-1}) \\
\{(a_{m-1;k}a_{m-1}^{-1})r_{m-1;k}\cdots(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})\} \quad \textit{for } m\geq 2, l=j, \\
\{(a_{m-1;k}a_{m-1}^{-1})r_{m-1;k}\cdots(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})\}^{-1} \\
\{(\bar{s}_{l,k;m}d_l^{-1})^{-1}r_{m;k}^{-1}(\bar{s}_{l,k;m}d_l^{-1})\}^{-1} \\
(\bar{s}_{j,k;m}d_j^{-1}) \\
\{(\bar{s}_{l,k;m}d_l^{-1})^{-1}r_{m;k}^{-1}(\bar{s}_{l,k;m}d_l^{-1})\} \\
\{(a_{m-1;k}a_{m-1}^{-1})r_{m-1;k}\cdots(a_{2;k}a_2^{-1})r_{2;k}^{-1}(a_{1;k}a_1^{-1})\} \quad \textit{for } m\geq 2, l>j, \\
\bar{s}_{j,k}d_j^{-1} \quad \textit{for } m\geq 2, l<j,
\end{cases}$$

(D4f) $s_{l,t}(\bar{s}_{j,k}d_j^{-1})s_{l,t}^{-1} =$

$$\begin{cases}
(\bar{s}_{l,k}d_l^{-1})^{-1}(\bar{s}_{j,k}d_j^{-1})(\bar{s}_{l,k}d_l^{-1}) \quad \textit{for } t=j, \\
[(\bar{s}_{t,k}d_t^{-1})^{-1},(\bar{s}_{l,k}d_l^{-1})^{-1}]^{-1}(\bar{s}_{j,k}d_j^{-1}) \\
[(\bar{s}_{t,k}d_t^{-1})^{-1},(\bar{s}_{l,k}d_l^{-1})^{-1}] \quad \textit{for } l<j<t, \\
\{(\bar{s}_{t,k}d_t^{-1})(\bar{s}_{j,k}d_j^{-1})\}^{-1}(\bar{s}_{j,k}d_j^{-1})\{(\bar{s}_{t,k}d_t^{-1})(\bar{s}_{j,k}d_j^{-1})\} \quad \textit{for } l=j, \\
\bar{s}_{j,k}d_j^{-1} \quad \textit{for the other cases},
\end{cases}$$

(D4g) $\bar{s}_{l,t}(\bar{s}_{j,k}d_j^{-1})\bar{s}_{l,t}^{-1} =$

$$\begin{cases}
[(\bar{s}_{t,k}d_t^{-1}),r_{1;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{1;k}]^{-1}(\bar{s}_{j,k}d_j^{-1}) \\
[(\bar{s}_{t,k}d_t^{-1}),r_{1;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{1;k}] \quad \textit{for } t<j, \\
\{r_{1;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{1;k}\}(\bar{s}_{j,k}d_j^{-1})\{r_{1;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{1;k}\}^{-1} \quad \textit{for } t=j, \\
(s_{t,k}d_t^{-1})(\bar{s}_{j,k}d_j^{-1})(s_{t,k}d_t^{-1})^{-1} \quad \textit{for } l=j, \\
[(\bar{s}_{l,k}d_l^{-1})^{-1},(s_{t,k}d_t^{-1})]^{-1}(\bar{s}_{j,k}d_j^{-1})[(\bar{s}_{l,k}d_l^{-1})^{-1},(s_{t,k}d_t^{-1})] \quad \textit{for } l>j, \\
\bar{s}_{j,k}d_j^{-1} \quad \textit{for } l<j<t.
\end{cases}$$

4. Infinite presentation for $\mathcal{M}(N_{g,n})$

The main theorem in this paper is as follows:

Theorem 4.1. *For $g \geq 1$ and $n \geq 0$, $\mathcal{M}(N_{g,n})$ has the following presentation:*
generators: $\{t_{c;+_c}, t_{c;-_c} \mid c :$ *two-sided s.c.c. on* $N_{g,n}\}$
$\cup\{Y_{\mu,\alpha} \mid \mu :$ *one-sided s.c.c. on* $N_{g,n}$, $\alpha :$ *s.c.c. on* $N_{g,n}$, $|\mu \cap \alpha| = 1\}$.
Denote the generating set by X.
relations:

(0) (i) $t_{c;\theta_c} = 1$ *when* $\theta_c \in \{+_c, -_c\}$ *and c bounds a disk or a Möbius band in* $N_{g,n}$,
(ii) $t_{c;+_c} = t_{c^{-1};+_c} = t_{c;-_c}^{-1}$,
(iii) $Y_{\mu,\alpha} = Y_{\mu,\alpha^{-1}}^{-1} = Y_{\mu^{-1},\alpha}$,

(I) *All the braid relations*

$$\begin{cases} \text{(i)} & f t_{c;\theta} f^{-1} = t_{f(c);f_*(\theta)} \quad \text{for } f \in X, \\ \text{(ii)} & f Y_{\mu,\alpha} f^{-1} = Y_{f(\mu),f(\alpha)} \quad \text{for } f \in X, \end{cases}$$

(II) *All the 2-chain relations,*
(III) *All the lantern relations,*
(IV) *All the relations* in Lemma 2.9, *i.e.* $Y_{\mu,\alpha\beta} = Y_{\mu,\alpha} Y_{\mu,\beta}$,
(V) *All the relations in* Lemma 2.8, *i.e.* $Y_{\mu,\alpha} = t_{\delta_1;\theta_1} t_{\delta_2;\theta_2}^{-1}$ *for one-sided* α.

The second author [16] proved Theorem 4.1 when $g \geq 1$ and $n \in \{0, 1\}$. The presentation in Theorem 3.1 of [16] is different from the presentation in Theorem 4.1 since we do not distinguish $t_{c;+_c}$, $t_{c^{-1};+_c}$ and $t_{c;-_c}^{-1}$, and also do not distinguish $Y_{\mu,\alpha}$, $Y_{\mu,\alpha^{-1}}^{-1}$ and $Y_{\mu^{-1},\alpha}$ in [16]. However, these presentation are equivalent by Relation (0)(ii) and (0)(iii). In fact, we can apply the proof of Theorem 3.1 in [16] to the presentation in Theorem 4.1. In (I) and (IV) one can substitute the right hand side of (V) for each generator $Y_{\mu,\alpha}$ with one-sided α. Then one can remove the generators $Y_{\mu,\alpha}$ with one-sided α and relations (V) from the presentation.

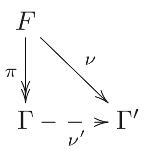
We denote by G the group which has the presentation in Theorem 4.1 throughout this section. Set $X^{\pm} := X \cup \{x^{-1} \mid x \in X\}$, where X is the generating set in Theorem 4.1. By Relation (I) in Theorem 4.1, we have the following lemma.

Lemma 4.2. *For $f \in G$, suppose that $f = f_1 f_2 \dots f_k$, where $f_1, f_2, \dots, f_k \in X^{\pm}$. Then we have*

$$\begin{cases} \text{(i)} & f t_{c;\theta} f^{-1} = t_{f(c);f_*(\theta)}, \\ \text{(ii)} & f Y_{\mu,\alpha} f^{-1} = Y_{f(\mu),f(\alpha)}. \end{cases}$$

The next lemma follows from an argument of the combinatorial group theory (for instance, see [10, Lemma 4.2.1, p42]).

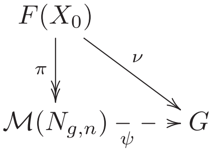
Lemma 4.3. *For groups Γ, Γ' and F, a surjective homomorphism $\pi : F \to \Gamma$ and a homomorphism $\nu : F \to \Gamma'$, we define a map $\nu' : \Gamma \to \Gamma'$ by $\nu'(x) := \nu(\widetilde{x})$ for $x \in \Gamma$, where $\widetilde{x} \in F$ is a lift of x with respect to π (see the diagram below).*
Then if $\ker \pi \subset \ker \nu$, *$\nu'$ is well-defined and a homomorphism.*



We start the proof of Theorem 4.1. When $n \in \{0, 1\}$, we proved Theorem 4.1 in [16].

Assume $g \geq 1$ and $n \geq 2$. Then we obtain Theorem 4.1 if $\mathcal{M}(N_{g,n})$ is isomorphic to G. Let $\varphi : G \to \mathcal{M}(N_{g,n})$ be the surjective homomorphism defined by $\varphi(t_{c;+_c}) := t_{c;+_c}$, $\varphi(t_{c;-_c}) := t_{c;-_c}$ and $\varphi(Y_{\mu,\alpha}) := Y_{\mu,\alpha}$.

Denote by $X_0 \subset \mathcal{M}(N_{g,n})$ the generating set of the finite presentation for $\mathcal{M}(N_{g,n})$ in Proposition 3.2. Let $F(X_0)$ be the free group which is freely generated by X_0 and let $\pi : F(X_0) \to \mathcal{M}(N_{g,n})$ be the natural projection. We define the homomorphism $\nu : F(X_0) \to G$ by $\nu(a_i) := a_i$, $\nu(b) := b$, $\nu(y) := y$, $\nu(a_{i;j}) := a_{i;j}$, $\nu(r_{i;j}) := r_{i;j}$, $\nu(s_{i,j}) := s_{i,j}$ and $\nu(\bar{s}_{i,j}) := \bar{s}_{i,j}$, and a map $\psi = \nu' : \mathcal{M}(N_{g,n}) \to G$ by $\psi(a_i^{\pm 1}) := a_i^{\pm 1}$, $\psi(b^{\pm 1}) := b^{\pm 1}$, $\psi(y^{\pm 1}) := y^{\pm 1}$, $\psi(a_{i;j}^{\pm 1}) := a_{i;j}^{\pm 1}$, $\psi(r_{i;j}^{\pm 1}) := r_{i;j}^{\pm 1}$, $\psi(s_{i,j}^{\pm 1}) := s_{i,j}^{\pm 1}$, $\psi(\bar{s}_{i,j}^{\pm 1}) := \bar{s}_{i,j}^{\pm 1}$ and $\psi(f) := \nu(\widetilde{f})$ for the other $f \in \mathcal{M}(N_{g,n})$, where $\widetilde{f} \in F(X_0)$ is a lift of f with respect to π (see the diagram below).



If ψ is a homomorphism, $\varphi \circ \psi = \mathrm{id}_{\mathcal{M}(N_{g,n})}$ by the definition of φ and ψ. Thus it is sufficient to show that ψ is a homomorphism and surjective for proving that ψ is isomorphism.

4.1. Proof that ψ is a homomorphism. By Lemma 4.3, if the relations of the presentation in Proposition 3.2 are obtained from Relations (0), (I), (II), (III), (IV) and (V) in Theorem 4.1, then ψ is well-defined and a homomorphism.

Let N be the subsurface of $N_{g,n}$ as in Figure 7. N is diffeomorphic to $N_{g,1}$ and includes simple closed curves $\alpha_1, \dots, \alpha_{g-1}$, μ_1 and β. We regard $\mathcal{M}(N)$ as a subgroup of $\mathcal{M}(N_{g,n})$. Relations (A1), ..., (A9b) and (B1), ..., (B8) of the presentation for $\mathcal{M}(N_{g,n})$ in Proposition 3.2 are relations of $\mathcal{M}(N) \cong \mathcal{M}(N_{g,1})$. By Theorem 3.1 in [16], Relations (A1), ..., (A9b) and (B1), ..., (B8) are obtained from Relations (0), (I), (II), (III), (IV) and (V).

By Proposition 5.13 in Section 5.5, we show that Relations (D0), (D1a)-(D4g) in Proposition 3.2 are obtained from Relations (I) and (III) in Theorem 4.1. We have proved that ψ is a homomorphism.

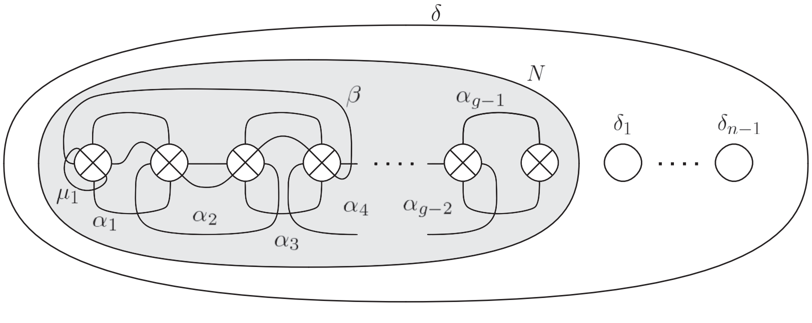


Fig. 7. The subsurface N of $N_{g,n}$ which is diffeomorphic to $N_{g,1}$.

4.2. Surjectivity of ψ. For some convenience, we write $t_{c;+_c} = t_c$ in this subsection. We show that there exist the inverse images of t_c's and $Y_{\mu,\alpha}$'s with respect to ψ for cases below, to prove the surjectivity of ψ.

(1) t_c; c is non-separating and $N_{g,n} - c$ is non-orientable,

(2) t_c; c is non-separating and $N_{g,n} - c$ is orientable,
(3) t_c; c is separating,
(4) $Y_{\mu,\alpha}$; α is two-sided and $N_{g,n} - \alpha$ is non-orientable,
(5) $Y_{\mu,\alpha}$; α is two-sided and $N_{g,n} - \alpha$ is orientable,
(6) $Y_{\mu,\alpha}$; α is one-sided.

Set $X_0^{\pm} := X_0 \cup \{x^{-1} \mid x \in X_0\}$, where X_0 is the generating set in Proposition 3.2. For a simple closed curve c on $N_{g,n}$, we denote by $(N_{g,n})_c$ the surface obtained from $N_{g,n}$ by cutting $N_{g,n}$ along c and denote by Σ the component of $(N_{g,n})_c$ which does not include δ.

Simple closed curves c, μ, and α for generators of type (1), (2), (4), (5), and (6) are mapped in N by a product of elements in $X_0^{\pm}$ using Relations (I) in Theorem 4.1 since $X_0^{\pm}$ is a generating set of $\mathcal{M}(N_{g,n})$. Hence, by similar arguments in Section 3.2 of [16] for the case of $N_{g,1}$, there exist their inverse images with respect to ψ. We note that we use the existence for the inverse images of generators of type (3) for the proof of the existence for the inverse images of generators of type (6).

Case (3) where Σ is diffeomorphic to $\Sigma_{0,m+1}$ for $m \geq 0$. We proceed by induction on $m \geq 0$. When $m = 0$, t_c is trivial by Relation (0) in Theorem 4.1. When $m = 1$, $c = \delta_i^{\varepsilon'}$ for some $1 \leq i \leq n-1$ and $\varepsilon' \in \{-1, 1\}$. Hence d_i is the inverse image of t_c.

When $m = 2$, there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \cdots, f_k \in X_0^{\pm}$ which satisfies either $c = f(\sigma_{i,j}^{\varepsilon'})$ or $c = f(\bar{\sigma}_{i,j}^{\varepsilon'})$ for some $1 \leq i < j \leq n-1$ and $\varepsilon' \in \{-1, 1\}$, where $\sigma_{i,j}$ and $\bar{\sigma}_{i,j}$ are simple closed curves on $N_{g,n}$ as in Figure 5. Thus, if $c = f(\sigma_{i,j}^{\varepsilon'})$, we have

$$\psi(f s_{i,j} f^{-1}) = f_1 f_2 \cdots f_k s_{i,j} f_k^{-1} \cdots f_2^{-1} f_1^{-1} \overset{\text{Lem. 4.2}}{=} t_{f(\sigma_{i,j})}^{\varepsilon} = t_c^{\varepsilon},$$

where ε is 1 or -1. Thus $f s_{i,j}^{\varepsilon} f^{-1} \in \mathcal{M}(N_{g,n})$ is the inverse image of $t_c \in G$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$. By a similar argument, when $c = f(\bar{\sigma}_{i,j}^{\varepsilon'})$, $f \bar{s}_{i,j}^{\varepsilon} f^{-1} \in \mathcal{M}(N_{g,n})$ is also the inverse image of $t_c \in G$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$.

For $m \geq 3$, there exists a simple closed curve c' on Σ such that c' separates Σ into Σ' and Σ'' which are diffeomorphic to $\Sigma_{0,4}$ and $\Sigma_{0,m-1}$, respectively, and $c \subset \Sigma'$. By using a lantern relation on Σ', there exist simple closed curves $c_1 = c', c_2, \ldots, c_6$ on Σ' such that $t_c = t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_6}^{\varepsilon_6} \in G$ for some $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6 \in \{-1, 1\}$. Since each c_i $(i = 1, 2, \ldots, 6)$ bounds a subsurface of $N_{g,n}$ which does not include c and is diffeomorphic to Σ_{0,m_i+1} for some $m_i < m$, by the inductive assumption, there exist the inverse images $h_1, \ldots, h_6 \in \mathcal{M}(N_{g,n})$ of $t_{c_1}, \ldots, t_{c_6} \in G$ with respect to ψ, respectively. Thus $h_1^{\varepsilon_1} h_2^{\varepsilon_2} \cdots h_6^{\varepsilon_6} \in \mathcal{M}(N_{g,n})$ is the inverse image of t_c with respect to ψ.

Case (3) where Σ is diffeomorphic to $\Sigma_{h,m+1}$ for $h \geq 1$ $m \geq 0$. In this case, there exists a simple closed curve c' on Σ such that c' separates Σ into Σ' and Σ'' which are diffeomorphic to $\Sigma_{h,2}$ and $\Sigma_{0,m+1}$, respectively. Then there exists a $2h+1$-chain $c_1, c_2, \ldots, c_{2h+1}$ on Σ' such that $\mathcal{N}_{N_{g,n}}(c_1 \cup c_2 \cup \cdots \cup c_{2h+1}) = \Sigma'$. By the chain relation, we have $(t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_{2h+1}}^{\varepsilon_{2h+1}})^{2h+2} = t_c t_{c'}^{\varepsilon'}$ for some $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2h+1}, \varepsilon' \in \{-1, 1\}$. Then we show that the relation $(t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_{2h+1}}^{\varepsilon_{2h+1}})^{2h+2} = t_c t_{c'}^{\varepsilon'}$ holds in G as follows: let $\iota : \Sigma \hookrightarrow N_{g,n}$ be the inclusion and let G' be the group whose presentation has all Dehn twists along simple closed curves on Σ as generators and Relations (0′), (I′), (II), and (III) in Theorem 2.5. By Theorem 2.5, $\mathcal{M}(\Sigma)$ is isomorphic to G', and we have the homomorphism $G' \to G$ defined by the correspondence of $t_{c;+_c}$ to $t_{\iota(c);\iota_*(+_c)}$. Since the Dehn twists appeared in the $2h+1$-chain relation $(t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_{2h+1}}^{\varepsilon_{2h+1}})^{2h+2} =$

$t_c t_{c'}^{\varepsilon'}$ are supported on the oriented subsurface Σ of $N_{g,n}$, we regard the $2h+1$-chain relation as a relation of $\mathcal{M}(\Sigma)$. Thus, by the composition $\iota_* : \mathcal{M}(\Sigma_{h,m}) \to G$ of the isomorphism $\mathcal{M}(\Sigma_{h,m}) \to G'$ and the homomorphism $G' \to G$, the relation $(t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_{2h+1}}^{\varepsilon_{2h+1}})^{2h+2} = t_c t_{c'}^{\varepsilon'}$ holds in G.

Since $t_{c_1}, t_{c_2}, \ldots, t_{c_{2h+1}}$ are Dehn twists of type (1) and c' bounds Σ'', the elements $t_{c_1}, t_{c_2}, \ldots, t_{c_{2h+1}}, t_{c'} \in G$ have the inverse images $h_1, h_2, \ldots, h_{2h+1}, h' \in \mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Then we have

$$\psi((h_1^{\varepsilon_1} h_2^{\varepsilon_2} \ldots h_{2h+1}^{\varepsilon_{2h+1}})^{2h+2}(h')^{-\varepsilon'}) = (t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_{2h+1}}^{\varepsilon_{2h+1}})^{2h+2} t_{c'}^{-\varepsilon'} = t_c.$$

Thus $((h_1^{\varepsilon_1} h_2^{\varepsilon_2} \ldots h_{2h+1}^{\varepsilon_{2h+1}})^{2h+2}(h')^{-\varepsilon'} \in \mathcal{M}(N_{g,n})$ is the inverse image of $t_c \in G$ with respect to ψ.

Case (3) where Σ is diffeomorphic to $N_{h,m+1}$ for $h \geq 1$ $m \geq 0$. We proceed by induction on $m \geq 0$. When $m = 0$, by similar arguments in Section 3.2 in [16], there exists an inverse image of $t_c \in G$ with respect to ψ.

When $m = 1$, we proceed by induction on $h \geq 1$. When $h = 1$, there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \cdots, f_k \in X_0^{\pm}$ such that $c = f(\rho_{1;j}^{\varepsilon'})$ for some $1 \leq j \leq n-1$ and $\varepsilon' \in \{-1, 1\}$. By a similar argument in the case where Σ is diffeomorphic to $\Sigma_{0,m+1}$, we can obtain the inverse image of t_c with respect to ψ. Suppose $h \geq 2$. Then there exist simple closed curves c_1 and c_2 on Σ such that $c_1 \sqcup c_2$ separates Σ into Σ', Σ'' and Σ''' which are diffeomorphic to $\Sigma_{0,4}$, $N_{1,1}$ and $N_{h-1,1}$, respectively, and $c \subset \Sigma'$. By using a lantern relation on Σ', there exist simple closed curves $c_3, \ldots, c_6$ on Σ' such that $t_c = t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} t_{c_3}^{\varepsilon_3} \cdots t_{c_6}^{\varepsilon_6} \in G$ for some $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_6 \in \{-1, 1\}$. Since each c_i $(i = 1, \ldots, 6)$ is a boundary component of a subsurface of Σ which is diffeomorphic to an orientable surface, $N_{h_i,1}$ for some $h_i \leq h$ or $N_{h_i,2}$ for some $h_i < h$, by the inductive assumption, there exis inverse images $h_1, \ldots, h_6 \in \mathcal{M}(N_{g,n})$ of $t_{c_1}, \ldots, t_{c_6} \in G$ with respect to ψ, respectively. Thus $h_1^{\varepsilon_1} h_2^{\varepsilon_2} \cdots h_6^{\varepsilon_6} \in \mathcal{M}(N_{g,n})$ is the inverse image of t_c with respect to ψ.

Suppose $m \geq 2$. Then there exist simple closed curves c_1 and c_2 on Σ such that $c_1 \sqcup c_2$ separates Σ into Σ', Σ'' and Σ''' which are diffeomorphic to $\Sigma_{0,4}$, $\Sigma_{0,m}$ and $N_{h,1}$, respectively, and $c \subset \Sigma'$. By using a lantern relation on Σ', there exist simple closed curves $c_3, \ldots, c_6$ on Σ' such that $t_c = t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} t_{c_3}^{\varepsilon_3} \cdots t_{c_6}^{\varepsilon_6} \in G$ for some $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_6 \in \{-1, 1\}$. Since each c_i $(i = 1, \ldots, 6)$ is a boundary component of a subsurface of Σ which is diffeomorphic to an orientable surface or N_{h,m_i+1} for some $m_i < m$, by the inductive assumption, there exist inverse images $h_1, \ldots, h_6 \in \mathcal{M}(N_{g,n})$ of $t_{c_1}, \ldots, t_{c_6} \in G$ with respect to ψ, respectively. Thus $h_1^{\varepsilon_1} h_2^{\varepsilon_2} \cdots h_6^{\varepsilon_6} \in \mathcal{M}(N_{g,n})$ is the inverse image of t_c with respect to ψ.

We have completed the proof of Theorem 4.1.

5. Proof of Proposition 3.2 and preliminaries for the proof

In this section, we give a proof of Proposition 3.2 which is used in the proof of Theorem 4.1. The proof is given in Section 5.4 and 5.5. For giving the proof, we prepare Section 5.1, 5.2, and 5.3.

5.1. Extended lantern relations. Let S be a connected compact surface and let D be a disk on int$S$ with the center point x_0. Then we have the point pushing map (defined in Section 2.2) $j_{x_0} : \pi_1(S, x_0) \to \mathcal{M}(S, x_0)$. We take an orientation $\theta_{\partial D} \in \{+_{\partial D}, -_{\partial D}\}$ of $\mathcal{N}_S(\partial D)$. For a two-sided simple loop γ on S based at x_0, we take the orientation $\theta_\gamma \in \{+_\gamma, -_\gamma\}$ of

$\mathcal{N}_S(\gamma)$ which is induced by $\theta_{\partial D}$. Denote by c_1 (resp. c_2) the boundary component of $\mathcal{N}_S(\gamma)$ on the right (resp. left) side of γ with respect to θ_γ, and by $\theta_i \in \{+_{c_i}, -_{c_i}\}$ ($i = 1, 2$) the orientation of $\mathcal{N}_S(c_i)$ which is induced by θ_γ. We regard γ as an element of $\pi_1(S, x_0)$. Then we have a well-known relation

$$j_{x_0}(\gamma) = t_{c_1;\theta_1} t_{c_2;\theta_2}^{-1}.$$

Let $\mathcal{L}^+ = \mathcal{L}^+(S, x_0)$ be the subset of $\pi_1(S, x_0)$ which consists of elements represented by two-sided simple loops. Then we define a map

$$\Delta = \Delta_{x_0} : \mathcal{L}^+ \to \mathcal{M}(S - \mathrm{int}D)$$

as follows. For any two-sided simple loop γ on S based at x_0, we take $\mathcal{N}_S(\gamma)$ whose interior contains D. Then we take c_1, c_2, θ_1 and θ_2 as above. Define the inclusion $\iota : S - \mathrm{int}D \to S$ and $\tilde{c}_i := \iota^{-1}(c_i)$ for $i = 1, 2$. Then we define

$$\Delta(\gamma) := t_{\tilde{c}_1;\theta_{\tilde{c}_1}} t_{\tilde{c}_2;\theta_{\tilde{c}_2}}^{-1} \in \mathcal{M}(S - \mathrm{int}D),$$

where $\theta_{\tilde{c}_i}$ is the orientation of $\mathcal{N}_{S-\mathrm{int}D}(\tilde{c}_i)$ ($i = 1, 2$) which is induced by θ_i.

Lemma 5.1 and 5.3 below are obtained from an argument in Section 3 of [9].

Lemma 5.1. *Let $\Delta = \Delta_{x_0} : \mathcal{L}^+ \to \mathcal{M}(S - \mathrm{int}D)$ be the map defined as above. Suppose that $\alpha, \beta \in \mathcal{L}^+$ are represented by two-sided simple loops such that they tangentially intersect only at x_0, and the product $\alpha\beta$ also lies in $\mathcal{L}^+$. Then we have*

$$\Delta(\alpha)\Delta(\beta) = \Delta(\alpha\beta) t_{\partial D;\theta_{\partial D}}^{\varepsilon},$$

where $\varepsilon = 1$ if α and β are counterclockwise around x_0 as on the left-hand side of Figure 8 *and $\varepsilon = -1$ if α and β are clockwise around x_0 as on the right-hand side of* Figure 8.

We call the relations in Lemma 5.1 *Relations (L+)* when $\varepsilon = 1$ and *Relations (L-)* when $\varepsilon = -1$ (see Figure 8). By Lemma 2.8, Relations (L+) and (L-) are original lantern relations (see for instance Section 5.1.1 in [4]). In other words, we have the following lemma.

Lemma 5.2. *Relations (L+) and (L-) coincide with Relations* (III) *in* Theorem 4.1.

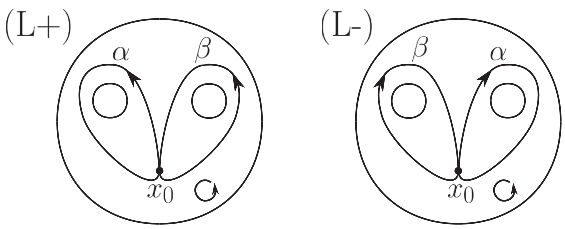


Fig. 8. Oriented subsurfaces $\mathcal{N}_S(\alpha \cup \beta)$.

Lemma 5.3. *Let $\Delta = \Delta_{x_0} : \mathcal{L}^+ \to \mathcal{M}(S - \mathrm{int}D)$ be the map defined as above. Suppose that $\alpha, \beta \in \mathcal{L}^+$ are represented by two-sided simple loops such that they transversely intersect only at x_0. Then we have*

$$\Delta(\alpha)\Delta(\beta) = \Delta(\alpha\beta).$$

We call the relations in Lemma 5.3 *Relations (L0)*. We have the following lemma.

Lemma 5.4. *Relations (L0) are obtained from the braid relations* (i).

Proof. Suppose that $\alpha, \beta \in \mathcal{L}^+$ are represented by two-sided simple loops such that they transversely intersect only at x_0. We take a representative of $\alpha\beta \in \pi_1(S, x_0)$ by a simple loop γ and also take the orientations of $\mathcal{N}_S(\alpha \cup \beta) \subset S$ and $\mathcal{N}_S(\alpha \cup \beta) - \mathrm{int}D \subset S - \mathrm{int}D$ which is induced by the orientation of $\mathcal{N}_{S-\mathrm{int}D}(\partial D)$. Define boundary components $a_1 \sqcup a_2 = \partial\mathcal{N}_S(\alpha)$, $b_1 \sqcup b_2 = \partial\mathcal{N}_S(\beta)$ and $c_1 \sqcup c_2 = \partial\mathcal{N}_S(\gamma)$ such that a_1, b_1 and c_1 are on the right-hand side of α, β and γ, respectively. We consider the case where the algebraic intersection number, with respect to the orientation of $\mathcal{N}_S(\alpha \cup \beta)$, of α and β is 1 and orientations of $\mathcal{N}_{S-\mathrm{int}D}(\tilde{a}_i)$, $\mathcal{N}_{S-\mathrm{int}D}(\tilde{b}_i)$ and $\mathcal{N}_{S-\mathrm{int}D}(\tilde{c}_i)$ are compatible with the orientation of $\mathcal{N}_S(\alpha \cup \beta)$. Figure 9 expresses this situation. Then we have $\Delta(\alpha) = t_{\tilde{a}_1}t_{\tilde{a}_2}^{-1}$, $\Delta(\beta) = t_{\tilde{b}_1}t_{\tilde{b}_2}^{-1}$ and $\Delta(\gamma) = t_{\tilde{c}_1}t_{\tilde{c}_2}^{-1}$. For the other cases, we can prove this lemma by an argument similar to the following argument.

Since $t_{\tilde{a}_2}^{-1}(\tilde{b}_i) = \tilde{c}_i$ for $i = 1, 2$, we have

$$\begin{aligned}
\Delta(\alpha\beta) &= t_{\tilde{c}_1}t_{\tilde{c}_2}^{-1} \\
&\overset{(\mathrm{I})}{=} t_{\tilde{a}_2}^{-1}(t_{\tilde{b}_1}t_{\tilde{b}_2}^{-1})t_{\tilde{a}_2} \\
&= t_{\tilde{a}_2}^{-1}t_{\tilde{b}_1}t_{\tilde{b}_2}^{-1}t_{\tilde{a}_2} \cdot (t_{\tilde{b}_1}t_{\tilde{b}_2}^{-1})^{-1}t_{\tilde{b}_1}t_{\tilde{b}_2}^{-1} \\
&= \underline{t_{\tilde{a}_2}^{-1}(t_{\tilde{b}_1}t_{\tilde{b}_2}^{-1})t_{\tilde{a}_2}(t_{\tilde{b}_1}t_{\tilde{b}_2}^{-1})^{-1}} \cdot t_{\tilde{b}_1}t_{\tilde{b}_2}^{-1} \\
&\overset{(\mathrm{I})}{=} t_{\tilde{a}_2}^{-1}t_{\tilde{a}_1} \cdot t_{\tilde{b}_1}t_{\tilde{b}_2}^{-1} \\
&= \Delta(\alpha)\Delta(\beta).
\end{aligned}$$

We have the lemma. □

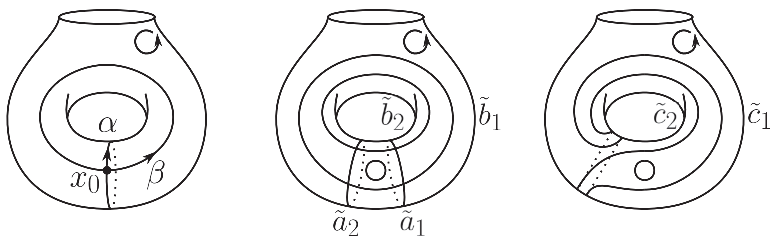


Fig. 9. Two-sided simple loops α and β on $\mathcal{N}_S(\alpha \cup \beta)$ such that their algebraic intersection number is 1 (on the left side), and simple closed curves $\tilde{a}_i$, $\tilde{b}_i$ and $\tilde{c}_i$ on $\mathcal{N}_S(\alpha \cup \beta) - \mathrm{int}D$ for $i = 1, 2$ (on the center and the right side).

5.2. Generators for the subgroup of the fundamental group generated by two-sided loops. Recall that we take a model of $N_{g,n}$ as in Figure 4 for $n \geq 1$. Assume $n \geq 2$. We regard $N_{g,n-1}$ as the surface obtained by regluing D_{g+n-1} and $N_{g,n}$. Put the center point x_0 of D_{g+n-1}. Let $\pi_1(N_{g,n-1})^+$ be the subgroup of $\pi_1(N_{g,n-1}, x_0)$ which consists of elements that is represented by loops such that the pushing maps along their loops preserve a local orientation of x_0. Let $x_1, \ldots, x_g, y_1, \ldots, y_{n-2}$ be loops on $N_{g,n-1}$ based at x_0 as in Figure 10 and we regard $N_{g,n-1}$ as the surface in Figure 10 for some conveniences. Note that x_i^2 for

$1 \leq i \leq g$, $x_{i+1}x_i$ for $1 \leq i \leq g-1$ and $x_1^{-1}y_ix_1$ for $1 \leq i \leq n-2$ are elements of $\pi_1(N_{g,n-1})^+$. x_i^2, $x_{i+1}x_i$ and $x_1^{-1}y_ix_1$ are represented by loops as in Figure 11. Since $\pi_1(N_{g,n-1}, x_0)$ is the free group which is freely generated by $x_1, \ldots, x_g, y_1, \ldots, y_{n-2}$, $\pi_1(N_{g,n-1})^+$ is also isomorphic to a free group. We have the following lemma.

Lemma 5.5. *For $g \geq 1$ and $n \geq 2$, $\pi_1(N_{g,n-1})^+$ is the free group which is freely generated by $x_1^2, \ldots, x_g^2, x_2x_1, \ldots, x_gx_{g-1}, y_1, \ldots, y_{n-2}, x_1^{-1}y_1x_1, \ldots, x_1^{-1}y_{n-2}x_1$.*

Proof. We use the Reidemeister-Schreier method (for instance see [10]) for $\pi_1(N_{g,n-1})^+ \subset \pi_1(N_{g,n-1}, x_0)$ to obtain the generators for $\pi_1(N_{g,n-1})^+$. Since $\pi_1(N_{g,n-1})^+$ is an index 2 subgroup of $\pi_1(N_{g,n-1}, x_0)$ and the non-trivial element of the quotient group $\pi_1(N_{g,n-1}, x_0)/\pi_1(N_{g,n-1})^+$ is represented by x_1, the set $U := \{1, x_1\} \subset \pi_1(N_{g,n-1})^+$ is a Schreier transversal for $\pi_1(N_{g,n-1})^+$ in $\pi_1(N_{g,n-1}, x_0)$. Set $X := \{x_1, \ldots, x_g, y_1, \ldots, y_{n-2}\}$. For any word w in X, denote by $\overline{w}$ the element of U whose equivalence class in $\pi_1(N_{g,n-1}, x_0)/\pi_1(N_{g,n-1})^+$ is the same as that of w. Then $\pi_1(N_{g,n-1})^+$ is the free group which is freely generated by

$$\begin{aligned} B &= \{\overline{xu}^{-1}xu \mid x \in X, u \in U, xu \notin U\} \\ &= \{x_ix_1,\ x_1^{-1}x_j,\ y_k,\ x_1^{-1}y_kx_1 \mid i = 1, \ldots, g,\ j = 2, \ldots, g,\ k = 1, \ldots, n-2\}. \end{aligned}$$

Put $z_1 := x_1^2$, $z_i := (x_ix_1)(x_1^{-1}x_i)$ for $i = 2, \ldots, g$, $w_1 := x_2x_1$ and $w_i := (x_{i+1}x_1)(x_1^{-1}x_i)$ for $i = 2, \ldots, g-1$ as words in B. By using the Tietze transformations (for instance see [10, Proposition 4.4.5, p46]) and relations $(x_1^{-1}x_i) = (x_ix_1)^{-1}z_i$ and $(x_{i+1}x_1) = w_i(x_1^{-1}x_i)^{-1}$ for $i \geq 2$, we have isomorphisms

$$\begin{aligned} & \langle B \mid \rangle \\ \cong\ & \langle B \cup \{z_i, w_j \mid i = 1, \ldots g,\ j = 1, \ldots, g-1\} \mid \\ & z_1 = x_1^2,\ z_i = (x_ix_1)(x_1^{-1}x_i),\ w_1 = x_2x_1,\ w_j = (x_{j+1}x_1)(x_1^{-1}x_j)\rangle \\ \cong\ & \langle B \cup \{z_i, w_j \mid i = 1, \ldots g,\ j = 1, \ldots, g-1\} \mid \\ & z_1 = x_1^2,\ w_1 = x_2x_1,\ x_ix_1 = w_{i-1}z_{i-1}^{-1} \cdots w_2z_2^{-1}w_1, \\ & x_1^{-1}x_{j+1} = w_1^{-1}z_2w_2^{-1} \cdots z_jw_j^{-1}z_{j+1}\rangle \\ \cong\ & \langle \{z_i,\ w_j,\ y_k,\ x_1^{-1}y_kx_1 \mid i = 1, \ldots g,\ j = 1, \ldots, g-1,\ k = 1, \ldots, n-2\} \mid \rangle. \end{aligned}$$

Note that $z_i = x_i^2$ and $w_i = x_{i+1}x_i$ as elements of $\pi_1(N_{g,n-1})^+$. We get this lemma. □

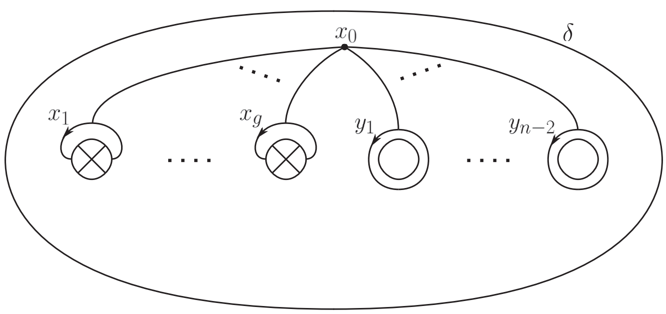


Fig. 10. Loops $x_1, \ldots, x_g, y_1, \ldots, y_{n-2}$ on $N_{g,n-1}$ based at x_0.

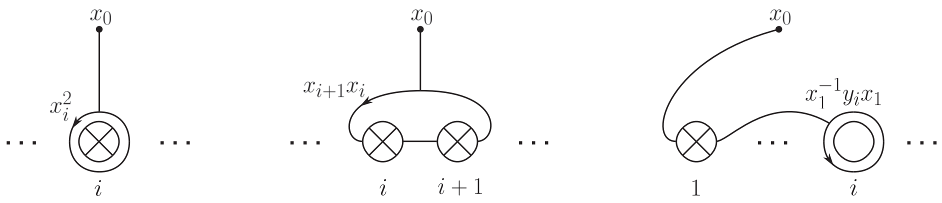


Fig. 11. Loops x_i^2, $x_{i+1}x_i$, $x_1^{-1}y_ix_1$ on $N_{g,n-1}$ based at x_0.

5.3. Group presentations and short exact sequence. Let G be a group and let $H = \langle X \mid R\rangle$, $Q = \langle Y \mid S\rangle$ be presented groups which have the exact sequence

$$1 \longrightarrow H \stackrel{\iota}{\longrightarrow} G \stackrel{\nu}{\longrightarrow} Q \longrightarrow 1.$$

We take a lift $\tilde{y} \in G$ of $y \in Q$ with respect to ν for each $y \in Q$. Then we put $\widetilde{X} := \{\iota(x) \mid x \in X\} \subset G$ and $\widetilde{Y} := \{\tilde{y} \mid y \in Y\} \subset G$. Denote by $\tilde{r}$ the word in $\widetilde{X}$ which is obtained from $r \in R$ by replacing each $x \in X$ by $\iota(x)$ and denote by $\tilde{s}$ the word in $\widetilde{Y}$ which is obtained from $s \in S$ by replacing each $y \in Y$ by $\tilde{y}$. We note that $\tilde{r} = 1$ in G. For each $s \in S$, since $\tilde{s} \in G$ is an element of $\ker\nu$, there exists a word v_s in $\widetilde{X}$ such that $\tilde{s} = v_s$ in G. Since $\iota(H)$ is a normal subgroup of G, for each $x \in X$ and $y \in Y$, $\tilde{y}\iota(x)\tilde{y}^{-1}$ is an element of $\iota(H)$. Hence there exists a word $w_{x,y}$ in $\widetilde{X}$ such that $\tilde{y}\iota(x)\tilde{y}^{-1} = w_{x,y}$ in G. The next lemma follows from an argument of the combinatorial group theory (for instance, see [10, Proposition 10.2.1, p139]).

Lemma 5.6. *In this situation above, the group G has the following presentation:*
generators: $\{\iota(x), \tilde{y} \mid x \in X, y \in Y\}$.
relations:

(A) $\tilde{r} = 1$ *for* $r \in R$,
(B) $\tilde{s} = v_s$ *for* $s \in S$,
(C) $\tilde{y}\iota(x)\tilde{y}^{-1} = w_{x,y}$ *for* $x \in X$, $y \in Y$.

5.4. Proof of Proposition 3.2. Assume $g \geq 1$ and $n \geq 2$. Let $\iota : N_{g,n} \hookrightarrow N_{g,n-1}$ be the natural inclusion obtained by regluing $N_{g,n}$ and the 2-disk D_{g+n-1} with the base point x_0, and let $\mathcal{M}^+(N_{g,n-1}, x_0)$ be the subgroup of $\mathcal{M}(N_{g,n-1}, x_0)$ whose elements preserve a local orientation of x_0. For $n \geq 2$, the forgetful homomorphism $\mathcal{F}\colon \mathcal{M}(N_{g,n-1}, x_0) \to \mathcal{M}(N_{g,n-1})$ induces the following exact sequence

$$1 \longrightarrow \pi_1(N_{g,n-1})^+ \stackrel{j_{x_0}}{\longrightarrow} \mathcal{M}^+(N_{g,n-1}, x_0) \stackrel{\mathcal{F}}{\longrightarrow} \mathcal{M}(N_{g,n-1}) \longrightarrow 1. \tag{5.1}$$

For the case $(g, n) = (1, 2)$, the group $\pi_1(N_{1,1})^+$ is generated by x_1^2 by Lemma 5.5. Since the image $j_{x_0}(x_1^2) \in \mathcal{M}^+(N_{1,1}, x_0)$ coincides with the Dehn twist t_δ along $\delta = \partial N_{1,1}$ and t_δ is an infinite order element of $\mathcal{M}^+(N_{1,1}, x_0)$, the homomorphism j_{x_0} is injective. For the other cases, since $\pi_1(N_{g,n-1}, x_0)$ is isomorphic to a free group, the center of $\pi_1(N_{g,n-1}, x_0)$ is trivial. Thus the homomorphism j_{x_0} is injective by [1, Corollary 1.2].

The inclusion $\iota : N_{g,n} \hookrightarrow N_{g,n-1}$ induces the surjective homomorphism $\iota_* : \mathcal{M}(N_{g,n}) \to \mathcal{M}^+(N_{g,n-1}, x_0)$. By Theorem 3.6 in [19], we have the exact sequence

$$1 \longrightarrow \mathbb{Z}[d_{n-1}] \longrightarrow \mathcal{M}(N_{g,n}) \stackrel{\iota_*}{\longrightarrow} \mathcal{M}^+(N_{g,n-1}, x_0) \longrightarrow 1. \tag{5.2}$$

The proof of Proposition 3.2 is proceeded by the induction for $n \geq 1$ and applying the inductive steps to the exact sequences 5.1 and 5.2. For some conveniences, we denote simply $\iota_*(a_i) = a_i$, $\iota_*(y) = y$, $\iota_*(b) = b$, $\iota_*(d_i) = d_i$, $\iota_*(a_{i,j}) = a_{i,j}$, $\iota_*(r_{i,j}) = r_{i,j}$, $\iota_*(s_{i,j}) = s_{i,j}$, $\iota_*(\bar{s}_{i,j}) = \bar{s}_{i,j}$, $\iota_*(\bar{s}_{j,k;i}) = \bar{s}_{j,k;i} \in \mathcal{M}^+(N_{g,n-1}, x_0)$, and we can check the following:

$$
\begin{aligned}
\mathcal{F}(a_i) &= a_i && \text{for } 1 \le i \le g-1,\\
\mathcal{F}(b) &= b, &&\\
\mathcal{F}(y) &= y, &&\\
\mathcal{F}(d_i) &= d_i && \text{for } 1 \le i \le n-2,\\
\mathcal{F}(a_{i;j}) &= a_{i;j} && \text{for } 1 \le i \le g-1 \text{ and } 1 \le j \le n-2,\\
\mathcal{F}(r_{i;j}) &= r_{i;j} && \text{for } 1 \le i \le g \text{ and } 1 \le j \le n-2,\\
\mathcal{F}(s_{i,j}) &= s_{i,j} && \text{for } 1 \le i < j \le n-2,\\
\mathcal{F}(\bar{s}_{i,j}) &= \bar{s}_{i,j} && \text{for } 1 \le i < j \le n-2,\\
\mathcal{F}(\bar{s}_{j,k;i}) &= \bar{s}_{j,k;i} && \text{for } 2 \le i \le g \text{ and } 1 \le j < k \le n-2.
\end{aligned}
$$

Thus, throughout this section, we take the lifts of a_i, y, b, d_i, $a_{i,j}$, $r_{i,j}$, $s_{i,j}$, $\bar{s}_{i,j}$, $\bar{s}_{j,k;i} \in \mathcal{M}^+(N_{g,n-1}, x_0)$ (resp. $\in \mathcal{M}(N_{g,n-1})$) with respect to $\iota_* : \mathcal{M}(N_{g,n}) \to \mathcal{M}^+(N_{g,n-1}, x_0)$ (resp. $\mathcal{F} : \mathcal{M}(N_{g,n-1}, x_0) \to \mathcal{M}(N_{g,n-1})$) by a_i, y, b, d_i, $a_{i,j}$, $r_{i,j}$, $s_{i,j}$, $\bar{s}_{i,j}$, $\bar{s}_{j,k;i} \in \mathcal{M}(N_{g,n})$ (resp. $\in \mathcal{M}^+(N_{g,n-1}, x_0)$), respectively.

Recall that $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. First, we compute the conjugacy action of the lifts of the generators for $\mathcal{M}(N_{g,n-1})$ on $\pi_1(N_{g,n-1})^+$. Let $x_{i+1}x_i$ $(1 \le i \le g-1)$, x_i^2 $(1 \le i \le g)$, y_j and $x_1^{-1} y_j x_1$ $(1 \le j \le n-2)$ be generators for $\pi_1(N_{g,n-1})^+$ in Lemma 5.5 (see Figure 10 and 11). Remark that

$$x_i^{-1} y_j x_i = x_i^{-2}(x_i x_{i-1}) \cdots x_2^{-2}(x_2 x_1)(x_1^{-1} y_j x_1)(x_2 x_1)^{-1} x_2^2 \cdots (x_i x_{i-1})^{-1} x_i^2$$

for $2 \le i \le g$. Then we have the following lemma.

Lemma 5.7. *For the elements a_m $(1 \le m \le g-1)$, y, b, $a_{m;l}$ $(1 \le m \le g-1, 1 \le l \le n-2)$, $r_{m,l}$ $(1 \le m \le g, 1 \le l \le n-2)$, $s_{l,t}$ $(1 \le l < t \le n-2)$, $\bar{s}_{l,t}$ $(1 \le l < t \le n-2)$, and d_l $(1 \le l \le n-2)$ in $\mathcal{M}^+(N_{g,n-1}, x_0)$ and the generators $x_{i+1}x_i$ $(1 \le i \le g-1)$, x_i^2 $(1 \le i \le g)$, y_j and $x_1^{-1} y_j x_1$ $(1 \le j \le n-2)$ for $\pi_1(N_{g,n-1})^+$ in* Lemma 5.5, *we have the following formulas:*

(D1a)″ $$a_m(x_{i+1}x_i) = \begin{cases} (x_{i+1}x_i)(x_i x_{i-1}) & \textit{for } m = i-1,\\ (x_{i+2}x_{i+1})^{-1}(x_{i+1}x_i) & \textit{for } m = i+1,\\ x_{i+1}x_i & \textit{for } m \ne i-1, i+1, \end{cases}$$

(D1b)″ $$y(x_{i+1}x_i) = \begin{cases} (x_2 x_1)^{-1} x_2^2 x_1^2 & \textit{for } i = 1,\\ (x_3 x_2) x_1^2 & \textit{for } i = 2,\\ x_{i+1}x_i & \textit{for } i \ge 3, \end{cases}$$

(D1c)″ $b(x_{i+1}x_i) =$
$$\begin{cases} \{(x_4x_3)(x_2x_1)\}^{-1}(x_2x_1)\{(x_4x_3)(x_2x_1)\} & \textit{for } i = 1,\\ \{(x_4x_3)(x_2x_1)\}^{-1}(x_3x_2)\{(x_4x_3)(x_2x_1)\} & \textit{for } i = 2,\\ (x_2x_1)^{-1}(x_4x_3)(x_2x_1) & \textit{for } i = 3,\\ (x_5x_4)(x_4x_3)(x_2x_1) & \textit{for } i = 4,\\ x_{i+1}x_i & \textit{for } i \ge 5, \end{cases}$$

(D1d)″ $a_{m;l}(x_{i+1}x_i) =$

$$\begin{cases} [y_l^{-1},(x_{m+1}x_m)^{-1}]^{-1}(x_{i+1}x_i)[y_l^{-1},(x_{m+1}x_m)^{-1}] & \text{for } m \le i-2, \\ [(x_ix_{i-1})^{-1},y_l^{-1}](x_{i+1}x_i)y_l(x_ix_{i-1}) & \text{for } m = i-1, \\ \{y_l(x_{i+1}x_i)\}^{-1}(x_{i+1}x_i)\{y_l(x_{i+1}x_i)\} & \text{for } m = i, \\ (x_{i+2}x_{i+1})^{-1}y_l^{-1}(x_{i+1}x_i) & \text{for } m = i+1, \\ x_{i+1}x_i & \text{for } m \ge i+2, \end{cases}$$

(D1e)″ $r_{m;l}(x_{i+1}x_i) =$
$$\begin{cases} [y_l^{-1},x_m^{-2}]^{-1}(x_{i+1}x_i)[y_l^{-1},x_m^{-2}] & \text{for } m \le i-1, \\ \{x_i^{-2}y_l^{-1}x_i^2\}y_l(x_{i+1}x_i)x_i^{-1}y_l^{-1}x_i\{x_i^{-2}y_l^{-1}x_i^2\}^{-1} & \text{for } m = i, \\ x_{i+1}^{-2}y_l^{-1}x_{i+1}^2 \cdot x_{i+1}^{-1}y_lx_{i+1}(x_{i+1}x_i) & \text{for } m = i+1, \\ x_{i+1}x_i & \text{for } m \ge i+2, \end{cases}$$

(D1f)″ $s_{l,t}(x_{i+1}x_i) = x_{i+1}x_i$,

(D1g)″ $\bar{s}_{l,t}(x_{i+1}x_i) =$
$$\begin{cases} [y_l^{-1},y_t]^{-1}y_l(x_2x_1)x_1^2(x_1^{-1}y_tx_1)x_1^{-2}y_l^{-1}x_1^2(x_1^{-1}y_tx_1)^{-1}x_1^{-2} \\ [(x_1^{-1}y_lx_1)^{-1},y_t] \quad \text{for } i = 1, \\ \{[x_1^{-1}y_lx_1,y_t^{-1}][y_l,x_1^2(x_1^{-1}y_tx_1)^{-1}x_1^{-2}]\}(x_{i+1}x_i) \\ \{[x_1^{-1}y_lx_1,y_t^{-1}][y_l,x_1^2(x_1^{-1}y_tx_1)^{-1}x_1^{-2}]\}^{-1} \quad \text{for } i \ge 2, \end{cases}$$

(D1h)″ $d_l(x_{i+1}x_i) = x_{i+1}x_i$,

(D2a)″ $$a_m(x_i^2) = \begin{cases} x_i^2x_{i-1}^2(x_ix_{i-1})^{-1}x_i^2(x_ix_{i-1}) & \text{for } m = i-1, \\ (x_{i+1}x_i)^{-1}x_{i+1}^{-2}(x_{i+1}x_i) & \text{for } m = i, \\ x_i^2 & \text{for } m \ne i-1, i, \end{cases}$$

(D2b)″ $$y(x_i^2) = \begin{cases} \{(x_2x_1)^{-1}x_2^2x_1^2\}^{-1}x_1^{-2}\{(x_2x_1)^{-1}x_2^2x_1^2\} & \text{for } i = 1, \\ (x_2x_1)x_1^2(x_2x_1)^{-1}x_2^2x_1^2 & \text{for } i = 2, \\ x_i^2 & \text{for } i \ge 3, \end{cases}$$

(D2c)″ $$b(x_i^2) = \begin{cases} (x_2x_1)^{-1}(x_4x_3)^{-1}(x_3x_2)^{-1}x_4^{-2}(x_4x_3)x_3^{-2}(x_3x_2)x_2^{-2}(x_2x_1) \\ \text{for } i = 1, \\ \{(x_4x_3)(x_2x_1)\}^{-1}(x_2x_1)(x_4x_3)x_2^2x_1^2(x_2x_1)^{-1}x_2^2(x_3x_2)^{-1} \\ x_3^2(x_4x_3)^{-1}x_4^2(x_3x_2)\{(x_4x_3)(x_2x_1)\} \quad \text{for } i = 2, \\ \{(x_4x_3)(x_2x_1)\}^{-1}x_4^{-2}(x_4x_3)x_3^{-2}(x_3x_2)x_2^{-2}(x_2x_1)x_1^{-2} \\ (x_3x_2)^{-1}x_3^2(x_4x_3)^{-1}(x_2x_1)^{-1}\{(x_4x_3)(x_2x_1)\} \quad \text{for } i = 3, \\ x_4^2(x_3x_2)x_1^2(x_2x_1)^{-1}x_2^2(x_3x_2)^{-1}x_3^2(x_4x_3)^{-1}x_4^2(x_4x_3)(x_2x_1) \\ \text{for } i = 4, \\ x_i^2 \quad \text{for } i \ge 5, \end{cases}$$

(D2d)″ $a_{m;l}(x_i^2) =$
$$\begin{cases} [y_l^{-1},(x_{m+1}x_m)^{-1}]^{-1}x_i^2[y_l^{-1},(x_{m+1}x_m)^{-1}] & \text{for } m \le i-2, \\ \{y_l(x_ix_{i-1})\}^{-1}(x_ix_{i-1})y_lx_i^2x_{i-1}^2(x_ix_{i-1})^{-1}x_i^2(x_i^{-1}y_lx_i)\{y_l(x_ix_{i-1})\} \\ \text{for } m = i-1, \\ (x_{i+1}x_i)^{-1}y_l^{-1}x_{i+1}^{-2}(x_{i+1}x_i)(x_i^{-1}y_lx_i)^{-1} & \text{for } m = i, \\ x_i^2 & \text{for } m \ge i+1, \end{cases}$$

(D2e)″ $$r_{m;l}(x_i^2) = \begin{cases} [y_l^{-1},x_m^{-2}]^{-1}x_i^2[y_l^{-1},x_m^{-2}] & \text{for } m \le i-1, \\ \{y_lx_i^2\}^{-1}x_i^2\{y_lx_i^2\} & \text{for } m = i, \\ x_i^2 & \text{for } m \ge i+1, \end{cases}$$

(D2f)″ $s_{l,t}(x_i^2) = x_i^2$,

(D2g)″ $\bar{s}_{l,t}(x_i^2) =$

$$\begin{cases} [y_t,(x_1^{-1}y_lx_1)^{-1}][x_1^2(x_1^{-1}y_tx_1)x_1^{-2},y_l^{-1}]x_1^2 & \textit{for } i=1,\\ \{[x_1^{-1}y_lx_1,y_t^{-1}][y_l,x_1^2(x_1^{-1}y_tx_1)^{-1}x_1^{-2}]\}^{-1} & \\ x_1^2\{[x_1^{-1}y_lx_1,y_t^{-1}][y_l,x_1^2(x_1^{-1}y_tx_1)^{-1}x_1^{-2}]\} & \textit{for } i\geq 2, \end{cases}$$

(D2h)″ $d_l(x_i^2)=x_i^2$,

(D3a)″ $a_m(y_j)=y_j$,

(D3b)″ $y(y_j)=y_j$,

(D3c)″ $b(y_j)=y_j$,

(D3d)″ $a_{m;l}(y_j)=$

$$\begin{cases} [y_l^{-1},(x_{m+1}x_m)^{-1}]^{-1}y_j[y_l^{-1},(x_{m+1}x_m)^{-1}] & \textit{for } l>j,\\ (x_{m+1}x_m)^{-1}y_j(x_{m+1}x_m) & \textit{for } l=j,\\ y_j & \textit{for } l<j, \end{cases}$$

(D3e)″ $r_{m;l}(y_j)=$

$$\begin{cases} [y_l,x_m^{-2}]^{-1}y_j[y_l,x_m^{-2}] & \textit{for } l>j,\\ x_m^{-2}y_jx_m^2 & \textit{for } l=j,\\ y_j & \textit{for } l<j, \end{cases}$$

(D3f)″ $s_{l,t}(y_j)=$

$$\begin{cases} \{y_ty_j\}^{-1}y_j\{y_ty_j\} & \\ \textit{for } l=j, & \\ [y_t^{-1},y_l^{-1}]^{-1}y_j[y_t^{-1},y_l^{-1}] & \textit{for } l<j<t,\\ y_l^{-1}y_jy_l & \textit{for } t=j,\\ y_j & \textit{for the other cases}, \end{cases}$$

(D3g)″ $\bar{s}_{l,t}(y_j)=$

$$\begin{cases} \{[x_1^{-1}y_lx_1,y_t^{-1}][y_l,x_1^2(x_1^{-1}y_tx_1)^{-1}x_1^{-2}]\}^{-1}y_j & \\ \{[x_1^{-1}y_lx_1,y_t^{-1}][y_l,x_1^2(x_1^{-1}y_tx_1)^{-1}x_1^{-2}]\} & \textit{for } l>j,\\ \{[x_1^{-1}y_jx_1,y_t^{-1}]y_jx_1^2(x_1^{-1}y_tx_1)^{-1}x_1^{-2}\}^{-1}y_j & \\ \{[x_1^{-1}y_jx_1,y_t^{-1}]y_jx_1^2(x_1^{-1}y_tx_1)^{-1}x_1^{-2}\} & \textit{for } l=j,\\ [(x_1^{-1}y_lx_1)^{-1},y_t]^{-1}y_j[(x_1^{-1}y_lx_1)^{-1},y_t] & \textit{for } l<j<t,\\ \{(x_1^{-1}y_lx_1)y_j^{-1}\}^{-1}y_j\{(x_1^{-1}y_lx_1)y_j^{-1}\} & \textit{for } t=j,\\ y_j & \textit{for } t<j, \end{cases}$$

(D3h)″ $d_l(y_j)=y_j$,

(D4a)″ $a_m(x_1^{-1}y_jx_1)=$

$$\begin{cases} \{x_1^{-2}x_2^{-2}(x_2x_1)\}^{-1}(x_1^{-1}y_jx_1)\{x_1^{-2}x_2^{-2}(x_2x_1)\} & \textit{for } m=1,\\ x_1^{-1}y_jx_1 & \textit{for } m\geq 2, \end{cases}$$

(D4b)″ $y(x_1^{-1}y_jx_1)=\{x_1^{-2}(x_2x_1)^{-2}x_2^2x_1^2\}^{-1}(x_1^{-1}y_jx_1)\{x_1^{-2}(x_2x_1)^{-2}x_2^2x_1^2\}$,

(D4c)″ $b(x_1^{-1}y_jx_1)=\{x_1^{-2}(x_3x_2)^{-1}x_4^{-2}(x_4x_3)x_3^{-2}(x_3x_2)x_2^{-2}(x_2x_1)\}^{-1}(x_1^{-1}y_jx_1)$
$\{x_1^{-2}(x_3x_2)^{-1}x_4^{-2}(x_4x_3)x_3^{-2}(x_3x_2)x_2^{-2}(x_2x_1)\}$,

(D4d)″ $a_{m;l}(x_1^{-1}y_jx_1)=$

$$\begin{cases} \{x_1^{-2}x_2^{-2}(x_2x_1)(x_1^{-1}y_lx_1)^{-1}\}^{-1}(x_1^{-1}y_jx_1) \\ \{x_1^{-2}x_2^{-2}(x_2x_1)(x_1^{-1}y_lx_1)^{-1}\} \quad \textit{for } m=1, l<j, \\ \{(x_1^{-1}y_lx_1)^{-1}x_1^{-2}x_2^{-2}(x_2x_1)\}^{-1}(x_1^{-1}y_jx_1) \\ \{(x_1^{-1}y_lx_1)^{-1}x_1^{-2}x_2^{-2}(x_2x_1)\} \quad \textit{for } m=1, l>j, \\ \{(x_mx_{m-1})x_{m-1}^2\cdots(x_3x_2)x_2^{-2}(x_2x_1)\}^{-1}(x_{m+1}^{-1}y_jx_{m+1}) \\ \{(x_mx_{m-1})x_{m-1}^2\cdots(x_3x_2)x_2^{-2}(x_2x_1)\} \quad \textit{for } m\geq 2, l=j, \\ \{(x_mx_{m-1})x_{m-1}^2\cdots(x_3x_2)x_2^{-2}(x_2x_1)\}^{-1}\{(x_m^{-1}y_lx_m)^{-1}x_m^{-2}(x_{m+1}^{-1}y_lx_{m+1})\}^{-1} \\ (x_m^{-1}y_jx_m)\{(x_m^{-1}y_lx_m)^{-1}x_m^{-2}(x_{m+1}^{-1}y_lx_{m+1})\} \\ \{(x_mx_{m-1})x_{m-1}^2\cdots(x_3x_2)x_2^{-2}(x_2x_1)\} \quad \textit{for } m\geq 2, l>j, \\ x_1^{-1}y_jx_1 \quad \textit{for the other cases,} \end{cases}$$

(D4e)″ $r_{m;l}(x_1^{-1}y_jx_1) =$

$$\begin{cases} \{x_1^{-2}(x_1^{-1}y_lx_1)^{-1}(y_l)x_1^2\}^{-1}(x_1^{-1}y_jx_1)\{x_1^{-2}(x_1^{-1}y_lx_1)^{-1}y_lx_1^2\} \\ \textit{for } m=1, l<j, \\ \{y_jx_1^2\}^{-1}(x_1^{-1}y_jx_1)\{y_jx_1^2\} \quad \textit{for } m=1, l=j, \\ \{(x_1^{-1}y_lx_1)^{-1}x_1^{-2}y_lx_1^2\}^{-1}(x_1^{-1}y_jx_1)^{-1}\{(x_1^{-1}y_lx_1)^{-1}x_1^{-2}y_lx_1^2\} \\ \textit{for } m=1, l>j, \\ \{(x_mx_{m-1})x_{m-1}^2\cdots(x_3x_2)x_2^{-2}(x_2x_1)\}^{-1}(x_m^{-1}y_jx_m) \\ \{(x_mx_{m-1})x_{m-1}^2\cdots(x_3x_2)x_2^{-2}(x_2x_1)\} \quad \textit{for } m\geq 2, l=j, \\ \{(x_mx_{m-1})x_{m-1}^2\cdots(x_3x_2)x_2^{-2}(x_2x_1)\}^{-1}\{(x_m^{-1}y_lx_m)^{-1}x_m^{-2}(x_m^{-1}y_lx_m)\}^{-1} \\ (x_m^{-1}y_jx_m)\{(x_m^{-1}y_lx_m)^{-1}x_m^{-2}(x_m^{-1}y_lx_m)\} \\ \{(x_mx_{m-1})x_{m-1}^2\cdots(x_3x_2)x_2^{-2}(x_2x_1)\} \quad \textit{for } m\geq 2, l>j, \\ x_1^{-1}y_jx_1 \quad \textit{for } m\geq 2, l<j, \end{cases}$$

(D4f)″ $s_{l,t}(x_1^{-1}y_jx_1) =$

$$\begin{cases} (x_1^{-1}y_lx_1)^{-1}(x_1^{-1}y_jx_1)(x_1^{-1}y_lx_1) \quad \textit{for } t=j, \\ [(x_1^{-1}y_tx_1)^{-1},(x_1^{-1}y_lx_1)^{-1}]^{-1}(x_1^{-1}y_jx_1) \\ [(x_1^{-1}y_tx_1)^{-1},(x_1^{-1}y_lx_1)^{-1}] \quad \textit{for } l<j<t, \\ \{(x_1^{-1}y_tx_1)(x_1^{-1}y_jx_1)\}^{-1}(x_1^{-1}y_jx_1)\{(x_1^{-1}y_tx_1)(x_1^{-1}y_jx_1)\} \quad \textit{for } l=j, \\ x_1^{-1}y_jx_1 \quad \textit{for the other cases,} \end{cases}$$

(D4g)″ $\bar{s}_{l,t}(x_1^{-1}y_jx_1) =$

$$\begin{cases} [(x_1^{-1}y_tx_1),x_1^{-2}y_l^{-1}x_1^2]^{-1}(x_1^{-1}y_jx_1) \\ [(x_1^{-1}y_tx_1),x_1^{-2}y_l^{-1}x_1^2] \quad \textit{for } t<j, \\ \{x_1^{-2}y_l^{-1}x_1^2\}(x_1^{-1}y_jx_1)\{x_1^{-2}y_l^{-1}x_1^2\}^{-1} \quad \textit{for } t=j, \\ y_t(x_1^{-1}y_jx_1)y_t^{-1} \quad \textit{for } l=j, \\ [(x_1^{-1}y_lx_1)^{-1},y_t]^{-1}(x_1^{-1}y_jx_1)[(x_1^{-1}y_lx_1)^{-1},y_t] \quad \textit{for } l>j, \\ x_1^{-1}y_jx_1 \quad \textit{for } l<j<t, \end{cases}$$

(D4h)″ $d_l(x_1^{-1}y_jx_1) = x_1^{-1}y_jx_1$.

Sketch of the proof. Let X_1 be the subset of $\mathcal{M}^+(N_{g,n-1}, x_0)$ consists of the elements a_m ($1 \leq m \leq g-1$), y, b, $a_{m;l}$ ($1 \leq m \leq g-1, 1 \leq l \leq n-1$), $r_{m,l}$ ($1 \leq m \leq g, 1 \leq l \leq n-1$), $s_{l,t}$ ($1 \leq l < t \leq n-1$), $\bar{s}_{l,t}$ ($1 \leq l < t \leq n-1$), and d_l ($1 \leq l \leq n-2$) in $\mathcal{M}^+(N_{g,n-1}, x_0)$, and X_2 the generating set for $\pi_1(N_{g,n-1})^+$ in Lemma 5.5. For each elements $x \in X_2$ and $f \in X_1$, we construct a product $w'_{x,f}$ of elements in X_2 such that $f(x) = w'_{x,f}$. For example, for $x_{i+1}x_i \in X_2$ and $r_{i;l} \in X_1$, $r_{i;l}(x_{i+1}x_i)$ is represented by the loop as on the right-hand side of Figure 12. Thus we have

$$r_{i;l}(x_{i+1}x_i) \quad = \quad \{x_i^{-2}y_l^{-1}x_i^2\}y_l(x_{i+1}x_i)x_i^{-1}y_l^{-1}x_i\{x_i^{-2}y_l^{-1}x_i^2\}^{-1} \in \pi_1(N_{g,n-1})^+,$$

and show the formula (D1e)″ for $m = i$. Table 1 indicates the corresponding codes of formulas $f(x) = w'_{x,f}$. □

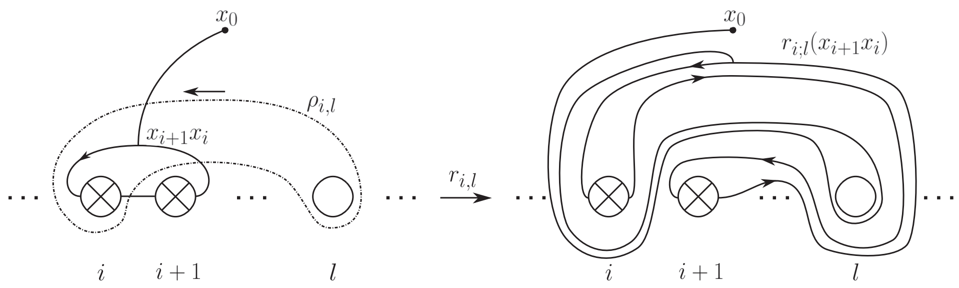


Fig. 12. Loop $r_{i;l}(x_{i+1}x_i)$ on $N_{g,n-1}$ based at x_0 for $1 \leq i \leq n-2$.

Table 1. Codes of formulas in Lemma 5.7.

$f \in X_1$ \ $x \in X_2$	a_m	y	b	$a_{m;l}$	$r_{m;l}$	$s_{l,t}$	$\bar{s}_{l,t}$	d_l
$x_{i+1}x_i$	(D1a)″	(D1b)″	(D1c)″	(D1d)″	(D1e)″	(D1f)″	(D1g)″	(D1h)″
x_i^2	(D2a)″	(D2b)″	(D2c)″	(D2d)″	(D2e)″	(D2f)″	(D2g)″	(D2h)″
y_j	(D3a)″	(D3b)″	(D3c)″	(D3d)″	(D3e)″	(D3f)″	(D3g)″	(D3h)″
$x_1^{-1}y_jx_1$	(D4a)″	(D4b)″	(D4c)″	(D4d)″	(D4e)″	(D4f)″	(D4g)″	(D4h)″

Applying Lemma 5.6 to the exact sequence 5.1, we have the following lemma.

Lemma 5.8. *Assume that $g \geq 1$ and $n \geq 2$. If $\mathcal{M}(N_{g,n-1})$ has the finite presentation in* Proposition 3.2*, then $\mathcal{M}^+(N_{g,n-1}, x_0)$ admits the presentation which is obtained from the finite presentation for $\mathcal{M}(N_{g,n-1})$ in* Proposition 3.2 *by adding generators $a_{i;n-1}$ $(1 \leq i \leq g-1)$, $r_{i,n-1}$ $(1 \leq i \leq g)$, $s_{i,n-1}$ $(1 \leq i \leq n-2)$, and $\bar{s}_{i,n-1}$ $(1 \leq i \leq n-2)$, and the following relations for $1 \leq j \leq n-2$, $1 \leq l, t \leq n-2$, and any possible $1 \leq i, m \leq g$:*

$$\text{(D1a)}' \quad a_m(a_{i;n-1}a_i^{-1})a_m^{-1} = \begin{cases} (a_{i;n-1}a_i^{-1})(a_{i-1;n-1}a_{i-1}^{-1}) & \text{for } m = i-1, \\ (a_{i+1;n-1}a_{i+1}^{-1})^{-1}(a_{i;n-1}a_i^{-1}) & \text{for } m = i+1, \\ a_{i;n-1}a_i^{-1} & \text{for } m \neq i-1, i+1, \end{cases}$$

$$\text{(D1b)}' \quad y(a_{i;n-1}a_i^{-1})y^{-1} = \begin{cases} (a_{1;n-1}a_1^{-1})^{-1}r_{2;n-1}r_{1;n-1} & \text{for } i = 1, \\ (a_{2;n-1}a_2^{-1})r_{1;n-1} & \text{for } i = 2, \\ a_{i;n-1}a_i^{-1} & \text{for } i \geq 3, \end{cases}$$

$$\text{(D1c)}' \quad b(a_{i;n-1}a_i^{-1})b^{-1} =$$

$$\begin{cases} \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}(a_{1;n-1}a_1^{-1})\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \\ \text{for } i = 1, \\ \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}(a_{2;n-1}a_2^{-1})\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \\ \text{for } i = 2, \\ (a_{1;n-1}a_1^{-1})^{-1}(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1}) \quad \text{for } i = 3, \\ (a_{4;n-1}a_4^{-1})(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1}) \quad \text{for } i = 4, \\ a_{i;n-1}a_i^{-1} \quad \text{for } i \geq 5, \end{cases}$$

(D1d)′ $a_{m;l}(a_{i;n-1}a_i^{-1})a_{m;l}^{-1} =$

$$\begin{cases} [(s_{l,n-1}d_l^{-1})^{-1}, (a_{m;n-1}a_m^{-1})^{-1}]^{-1}(a_{i;n-1}a_i^{-1}) \\ [(s_{l,n-1}d_l^{-1})^{-1}, (a_{m;n-1}a_m^{-1})^{-1}] \quad \textit{for } m \leq i-2, \\ [(a_{i-1;n-1}a_{i-1}^{-1})^{-1}, (s_{l,n-1}d_l^{-1})^{-1}](a_{i;n-1}a_i^{-1})(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1}) \\ \textit{for } m = i-1, \\ \{(s_{l,n-1}d_l^{-1})(a_{i;n-1}a_i^{-1})\}^{-1}(a_{i;n-1}a_i^{-1})\{(s_{l,n-1}d_l^{-1})(a_{i;n-1}a_i^{-1})\} \\ \textit{for } m = i, \\ (a_{i+1;n-1}a_{i+1}^{-1})^{-1}(s_{l,n-1}d_l^{-1})^{-1}(a_{i;n-1}a_i^{-1}) \quad \textit{for } m = i+1, \\ a_{i;n-1}a_i^{-1} \quad \textit{for } m \geq i+2, \end{cases}$$

(D1e)′ $r_{m;l}(a_{i;n-1}a_i^{-1})r_{m;l}^{-1} =$

$$\begin{cases} [(s_{l,n-1}d_l^{-1})^{-1}, r_{m;n-1}^{-1}]^{-1}(a_{i;n-1}a_i^{-1})[(s_{l,n-1}d_l^{-1})^{-1}, r_{m;n-1}^{-1}] \\ \textit{for } m \leq i-1, \\ \{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}(s_{l,n-1}d_l^{-1})(a_{i;n-1}a_i^{-1}) \\ (\bar{s}_{l,n-1;i}d_l^{-1})^{-1}\{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}^{-1} \quad \textit{for } m = i, \\ r_{i+1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i+1;n-1}(\bar{s}_{l,n-1;i+1}d_l^{-1})(a_{i;n-1}a_i^{-1}) \\ \textit{for } m = i+1, \\ a_{i;n-1}a_i^{-1} \quad \textit{for } m \geq i+2, \end{cases}$$

(D1f)′ $s_{l,t}(a_{i;n-1}a_i^{-1})s_{l,t}^{-1} = a_{i;n-1}a_i^{-1},$

(D1g)′ $\bar{s}_{l,t}(a_{i;n-1}a_i^{-1})\bar{s}_{l,t}^{-1} =$

$$\begin{cases} [(\bar{s}_{l,n-1}d_l^{-1})^{-1}, s_{t,n-1}d_t^{-1}]^{-1}(s_{l,n-1}d_l^{-1})(a_{1;n-1}a_1^{-1})r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})r_{1;n-1}^{-1} \\ (s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})^{-1}r_{1;n-1}^{-1}[(\bar{s}_{l,n-1}d_l^{-1})^{-1}, (s_{t,n-1}d_t^{-1})] \quad \textit{for } i = 1, \\ \{[\bar{s}_{l,n-1}d_l^{-1}, (s_{t,n-1}d_t^{-1})^{-1}][s_{l,n-1}d_l^{-1}, r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})^{-1}r_{1;n-1}^{-1}]\} \\ (a_{i;n-1}a_i^{-1}) \\ \{[\bar{s}_{l,n-1}d_l^{-1}, (s_{t,n-1}d_t^{-1})^{-1}][s_{l,n-1}d_l^{-1}, r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})^{-1}r_{1;n-1}^{-1}]\}^{-1} \\ \textit{for } i \geq 2, \end{cases}$$

(D1h)′ $d_l(a_{i;n-1}a_i^{-1})d_l^{-1} = a_{i;n-1}a_i^{-1},$

(D2a)′ $$a_m r_{i;n-1}a_m^{-1} = \begin{cases} r_{i;n-1}r_{i-1;n-1}(a_{i-1;n-1}a_{i-1}^{-1})^{-1}r_{i;n-1}(a_{i-1;n-1}a_{i-1}^{-1}) \\ \textit{for } m = i-1, \\ (a_{i;n-1}a_i^{-1})^{-1}r_{i+1;n-1}^{-1}(a_{i;n-1}a_i^{-1}) \quad \textit{for } m = i, \\ r_{i;n-1} \quad \textit{for } m \neq i-1, i, \end{cases}$$

(D2b)′ $$y r_{i;n-1}y^{-1} = \begin{cases} \{(a_{1;n-1}a_1^{-1})^{-1}r_{2;n-1}r_{1;n-1}\}^{-1}r_{1;n-1}^{-1} \\ \{(a_{1;n-1}a_1^{-1})^{-1}r_{2;n-1}r_{1;n-1}\} \quad \textit{for } i = 1, \\ (a_{1;n-1}a_1^{-1})r_{1;n-1}(a_{1;n-1}a_1^{-1})^{-1}r_{2;n-1}r_{1;n-1} \quad \textit{for } i = 2, \\ r_{i;n-1} \quad \textit{for } i \geq 3, \end{cases}$$

$$(\mathrm{D2c})' \quad br_{i;n-1}b^{-1} = \begin{cases} (a_{1;n-1}a_1^{-1})^{-1}(a_{3;n-1}a_3^{-1})^{-1}(a_{2;n-1}a_2^{-1})^{-1} \\ r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1}) \\ \text{for } i=1, \\ \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}(a_{1;n-1}a_1^{-1})(a_{3;n-1}a_3^{-1})r_{2;n-1} \\ r_{1;n-1}(a_{1;n-1}a_1^{-1})^{-1}r_{2;n-1}(a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1} \\ (a_{3;n-1}a_3^{-1})^{-1}r_{4;n-1}(a_{2;n-1}a_2^{-1})\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \\ \text{for } i=2, \\ \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}r_{4;n-1}^{-1} \\ (a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})r_{1;n-1}^{-1} \\ (a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1}(a_{3;n-1}a_3^{-1})^{-1}(a_{1;n-1}a_1^{-1})^{-1} \\ \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \quad \text{for } i=3, \\ r_{4;n-1}(a_{2;n-1}a_2^{-1})r_{1;n-1}(a_{1;n-1}a_1^{-1})^{-1}r_{2;n-1}(a_{2;n-1}a_2^{-1})^{-1} \\ r_{3;n-1}(a_{3;n-1}a_3^{-1})^{-1}r_{4;n-1}(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1}) \\ \text{for } i=4, \\ r_{i;n-1} \quad \text{for } i\geq 5, \end{cases}$$

$$(\mathrm{D2d})' \quad a_{m;l}r_{i;n-1}a_{m;l}^{-1} =$$
$$\begin{cases} [(s_{l,n-1}d_l^{-1})^{-1},(a_{m;n-1}a_m^{-1})^{-1}]^{-1}r_{i;n-1}[(s_{l,n-1}d_l^{-1})^{-1},(a_{m;n-1}a_m^{-1})^{-1}] \\ \text{for } m\leq i-2, \\ \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}(a_{i-1;n-1}a_{i-1}^{-1})(s_{l,n-1}d_l^{-1})r_{i;n-1} \\ r_{i-1;n-1}(a_{i-1;n-1}a_{i-1}^{-1})^{-1}r_{i;n-1}(\bar{s}_{l,n-1;i}d_l^{-1})\{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\} \\ \text{for } m=i-1, \\ (a_{i;n-1}a_i^{-1})^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i+1;n-1}^{-1}(a_{i;n-1}a_i^{-1})(\bar{s}_{l,n-1;i}d_l^{-1})^{-1} \\ \text{for } m=i, \\ r_{i;n-1} \quad \text{for } m\geq i+1, \end{cases}$$

$$(\mathrm{D2e})' \quad r_{m;l}r_{i;n-1}r_{m;l}^{-1} = \begin{cases} [(s_{l,n-1}d_l^{-1})^{-1},r_{m;n-1}^{-1}]^{-1}r_{i;n-1}[(s_{l,n-1}d_l^{-1})^{-1},r_{m;n-1}^{-1}] \\ \text{for } m\leq i-1, \\ \{(s_{l,n-1}d_l^{-1})r_{i;n-1}\}^{-1}r_{i;n-1}\{(s_{l,n-1}d_l^{-1})r_{i;n-1}\} \\ \text{for } m=i, \\ r_{i;n-1} \quad \text{for } m\geq i+1, \end{cases}$$

$$(\mathrm{D2f})' \quad s_{l,t}r_{i;n-1}s_{l,t}^{-1} = r_{i;n-1},$$

$$(\mathrm{D2g})' \quad \bar{s}_{l,t}r_{i;n-1}\bar{s}_{l,t}^{-1} =$$
$$\begin{cases} [s_{t,n-1}d_t^{-1},(\bar{s}_{l,n-1}d_l^{-1})^{-1}][r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})r_{1;n-1}^{-1},(s_{l,n-1}d_l^{-1})^{-1}]r_{1;n-1} \\ \text{for } i=1, \\ \{[\bar{s}_{l,n-1}d_l^{-1},(s_{t,n-1}d_t^{-1})^{-1}][s_{l,n-1}d_l^{-1},r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})^{-1}r_{1;n-1}^{-1}]\}^{-1} \\ r_{1;n-1}\{[\bar{s}_{l,n-1}d_l^{-1},(s_{t,n-1}d_t^{-1})^{-1}][s_{l,n-1}d_l^{-1},r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})^{-1}r_{1;n-1}^{-1}]\} \\ \text{for } i\geq 2, \end{cases}$$

$$(\mathrm{D2h})' \quad d_lr_{i;n-1}d_l^{-1} = r_{i;n-1},$$

$$(\mathrm{D3a})' \quad a_m(s_{j,n-1}d_j^{-1})a_m^{-1} = s_{j,n-1}d_j^{-1},$$

$$(\mathrm{D3b})' \quad y(s_{j,n-1}d_j^{-1})y^{-1} = s_{j,n-1}d_j^{-1},$$

$$(\mathrm{D3c})' \quad b(s_{j,n-1}d_j^{-1})b^{-1} = s_{j,n-1}d_j^{-1},$$

$$(\mathrm{D3d})' \quad a_{m;l}(s_{j,n-1}d_j^{-1})a_{m;l}^{-1} =$$

$$\begin{cases} [(s_{l,n-1}d_l^{-1})^{-1},(a_{m;n-1}a_m^{-1})^{-1}]^{-1}(s_{j,n-1}d_j^{-1}) \\ [(s_{l,n-1}d_l^{-1})^{-1},(a_{m;n-1}a_m^{-1})^{-1}] \quad \textit{for } l>j, \\ (a_{m;n-1}a_m^{-1})^{-1}(s_{j,n-1}d_j^{-1})(a_{m;n-1}a_m^{-1}) \quad \textit{for } l=j, \\ s_{j,n-1}d_j^{-1} \quad \textit{for } l=j, \end{cases}$$

(D3e)′ $r_{m;l}(s_{j,n-1}d_j^{-1})r_{m;l}^{-1}=$

$$\begin{cases} [(s_{l,n-1}d_l^{-1})^{-1},r_{m;n-1}^{-1}]^{-1}(s_{j,n-1}d_j^{-1})[(s_{l,n-1}d_l^{-1})^{-1},r_{m;n-1}^{-1}] \\ \textit{for } l>j, \\ r_{m;n-1}^{-1}(s_{j,n-1}d_j^{-1})r_{m;n-1} \quad \textit{for } l=j, \\ s_{j,n-1}d_j^{-1} \quad \textit{for } l<j, \end{cases}$$

(D3f)′ $s_{l,t}(s_{j,n-1}d_j^{-1})s_{l,t}^{-1}=$

$$\begin{cases} \{(s_{t,n-1}d_t^{-1})(s_{j,n-1}d_j^{-1})\}^{-1}(s_{j,n-1}d_j^{-1})\{(s_{t,n-1}d_t^{-1})(s_{j,n-1}d_j^{-1})\} \\ \textit{for } l=j, \\ [(s_{t,n-1}d_t^{-1})^{-1},(s_{l,n-1}d_l^{-1})^{-1}]^{-1}(s_{j,n-1}d_j^{-1}) \\ [(s_{t,n-1}d_t^{-1})^{-1},(s_{l,n-1}d_l^{-1})^{-1}] \quad \textit{for } l<j<t, \\ (s_{l,n-1}d_l^{-1})^{-1}(s_{j,n-1}d_j^{-1})(s_{l,n-1}d_l^{-1}) \quad \textit{for } t=j, \\ s_{j,n-1}d_j^{-1} \quad \textit{for the other cases}, \end{cases}$$

(D3g)′ $\bar{s}_{l,t}(s_{j,n-1}d_j^{-1})\bar{s}_{l,t}^{-1}=$

$$\begin{cases} \{[\bar{s}_{l,n-1}d_l^{-1},(s_{t,n-1}d_t^{-1})^{-1}][s_{l,n-1}d_l^{-1},r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})^{-1}r_{1;n-1}^{-1}]\}^{-1} \\ (s_{j,n-1}d_j^{-1}) \\ \{[\bar{s}_{l,n-1}d_l^{-1},(s_{t,n-1}d_t^{-1})^{-1}][s_{l,n-1}d_l^{-1},r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})^{-1}r_{1;n-1}^{-1}]\} \\ \textit{for } l>j, \\ \{[\bar{s}_{j,n-1}d_j^{-1},(s_{t,n-1}d_t^{-1})^{-1}](s_{j,n-1}d_j^{-1})r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})^{-1}r_{1;n-1}^{-1}\}^{-1} \\ (s_{j,n-1}d_j^{-1}) \\ \{[\bar{s}_{j,n-1}d_j^{-1},(s_{t,n-1}d_t^{-1})^{-1}](s_{j,n-1}d_j^{-1})r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})^{-1}r_{1;n-1}^{-1}\} \\ \textit{for } l=j, \\ [(\bar{s}_{l,n-1}d_l^{-1})^{-1},s_{t,n-1}d_t^{-1}]^{-1}(s_{j,n-1}d_j^{-1})[(\bar{s}_{l,n-1}d_l^{-1})^{-1},s_{t,n-1}d_t^{-1}] \\ \textit{for } l<j<t, \\ \{(\bar{s}_{l,n-1}d_l^{-1})(s_{j,n-1}d_j^{-1})^{-1}\}^{-1}(s_{j,n-1}d_j^{-1})\{(\bar{s}_{l,n-1}d_l^{-1})(s_{j,n-1}d_j^{-1})^{-1}\} \\ \textit{for } t=j, \\ s_{j,n-1}d_j^{-1} \quad \textit{for } t<j, \end{cases}$$

(D3h)′ $d_l(s_{j,n-1}d_j^{-1})d_l^{-1}=s_{j,n-1}d_j^{-1}$,

(D4a)′ $a_m(\bar{s}_{j,n-1}d_j^{-1})a_m^{-1}=$

$$\begin{cases} \{r_{1;n-1}^{-1}r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\}^{-1}(\bar{s}_{j,n-1}d_j^{-1})\{r_{1;n-1}^{-1}r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\} \\ \textit{for } m=1, \\ \bar{s}_{j,n-1}d_j^{-1} \quad \textit{for } m\geq 2, \end{cases}$$

(D4b)′ $y(\bar{s}_{j,n-1}d_j^{-1})y^{-1}=\{r_{1;n-1}^{-1}(a_{1;n-1}a_1^{-1})^{-2}r_{2;n-1}r_{1;n-1}\}^{-1}(\bar{s}_{j,n-1}d_j^{-1})$
$\{r_{1;n-1}^{-1}(a_{1;n-1}a_1^{-1})^{-2}r_{2;n-1}r_{1;n-1}\}$,

(D4c)′ $b(\bar{s}_{j,n-1}d_j^{-1})b^{-1}=\{r_{1;n-1}^{-1}(a_{2;n-1}a_2^{-1})^{-1}r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}$
$(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\}^{-1}(\bar{s}_{j,n-1}d_j^{-1})\{r_{1;n-1}^{-1}(a_{2;n-1}a_2^{-1})^{-1}$
$r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\}$,

(D4d)′ $a_{m;l}(\bar{s}_{j,n-1}d_j^{-1})a_{m;l}^{-1}=$

$$\begin{cases} \{r_{1;n-1}^{-1}r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})(\bar{s}_{l,n-1}d_l^{-1})^{-1}\}^{-1}(\bar{s}_{j,n-1}d_j^{-1}) \\ \{r_{1;n-1}^{-1}r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})(\bar{s}_{l,n-1}d_l^{-1})^{-1}\} \quad \textit{for } m=1, l<j, \\ \{(\bar{s}_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}^{-1}r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\}^{-1}(\bar{s}_{j,n-1}d_j^{-1}) \\ \{(\bar{s}_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}^{-1}r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\} \quad \textit{for } m=1, l>j, \\ \{(a_{m-1;n-1}a_{m-1}^{-1})r_{m-1;n-1}\cdots(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\}^{-1} \\ (\bar{s}_{j,n-1;m+1}d_j^{-1}) \\ \{(a_{m-1;n-1}a_{m-1}^{-1})r_{m-1;n-1}\cdots(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\} \\ \textit{for } m\geq 2, l=j, \\ \{(a_{m-1;n-1}a_{m-1}^{-1})r_{m-1;n-1}\cdots(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\}^{-1} \\ \{(\bar{s}_{l,n-1;m}d_l^{-1})^{-1}r_{m;n-1}^{-1}(\bar{s}_{l,n-1;m+1}d_l^{-1})\}^{-1}(\bar{s}_{j,n-1;m}d_j^{-1}) \\ \{(\bar{s}_{l,n-1;m}d_l^{-1})^{-1}r_{m;n-1}^{-1}(\bar{s}_{l,n-1;m+1}d_l^{-1})\} \\ \{(a_{m-1;n-1}a_{m-1}^{-1})r_{m-1;n-1}\cdots(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\} \\ \textit{for } m\geq 2, l>j, \\ \bar{s}_{j,n-1}d_j^{-1} \quad \textit{for the other cases}, \end{cases}$$

(D4e)′ $r_{m;l}(\bar{s}_{j,n-1}d_j^{-1})r_{m;l}^{-1} =$

$$\begin{cases} \{r_{1;n-1}^{-1}(\bar{s}_{l,n-1}d_l^{-1})^{-1}(s_{l,n-1}d_l^{-1})r_{1;n-1}\}^{-1}(\bar{s}_{j,n-1}d_j^{-1}) \\ \{r_{1;n-1}^{-1}(\bar{s}_{l,n-1}d_l^{-1})^{-1}(s_{l,n-1}d_l^{-1})r_{1;n-1}\} \quad \textit{for } m=1, l<j, \\ \{(s_{j,n-1}d_j^{-1})r_{1;n-1}\}^{-1}(\bar{s}_{j,n-1}d_j^{-1})\{(s_{j,n-1}d_j^{-1})r_{1;n-1}\} \\ \textit{for } m=1, l=j, \\ \{(\bar{s}_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})r_{1;n-1}\}^{-1}(\bar{s}_{j,n-1}d_j)^{-1} \\ \{(\bar{s}_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})r_{1;n-1}\} \quad \textit{for } m=1, l>j, \\ \{(a_{m-1;n-1}a_{m-1}^{-1})r_{m-1;n-1}\cdots(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\}^{-1} \\ (\bar{s}_{j,n-1;m}d_j^{-1}) \\ \{(a_{m-1;n-1}a_{m-1}^{-1})r_{m-1;n-1}\cdots(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\} \\ \textit{for } m\geq 2, l=j, \\ \{(a_{m-1;n-1}a_{m-1}^{-1})r_{m-1;n-1}\cdots(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\}^{-1} \\ \{(\bar{s}_{l,n-1;m}d_l^{-1})^{-1}r_{m;n-1}^{-1}(\bar{s}_{l,n-1;m}d_l^{-1})\}^{-1} \\ (\bar{s}_{j,n-1;m}d_j^{-1}) \\ \{(\bar{s}_{l,n-1;m}d_l^{-1})^{-1}r_{m;n-1}^{-1}(\bar{s}_{l,n-1;m}d_l^{-1})\} \\ \{(a_{m-1;n-1}a_{m-1}^{-1})r_{m-1;n-1}\cdots(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})\} \\ \textit{for } m\geq 2, l>j, \\ \bar{s}_{j,n-1}d_j^{-1} \quad \textit{for } m\geq 2, l<j, \end{cases}$$

(D4f)′ $s_{l,t}(\bar{s}_{j,n-1}d_j^{-1})s_{l,t}^{-1} =$

$$\begin{cases} (\bar{s}_{l,n-1}d_l^{-1})^{-1}(\bar{s}_{j,n-1}d_j^{-1})(\bar{s}_{l,n-1}d_l^{-1}) \quad \textit{for } t=j, \\ [(\bar{s}_{t,n-1}d_t^{-1})^{-1},(\bar{s}_{l,n-1}d_l^{-1})^{-1}]^{-1}(\bar{s}_{j,n-1}d_j^{-1}) \\ [(\bar{s}_{t,n-1}d_t^{-1})^{-1},(\bar{s}_{l,n-1}d_l^{-1})^{-1}] \quad \textit{for } l<j<t, \\ \{(\bar{s}_{t,n-1}d_t^{-1})(\bar{s}_{j,n-1}d_j^{-1})\}^{-1}(\bar{s}_{j,n-1}d_j^{-1})\{(\bar{s}_{t,n-1}d_t^{-1})(\bar{s}_{j,n-1}d_j^{-1})\} \\ \textit{for } l=j, \\ \bar{s}_{j,n-1}d_j^{-1} \quad \textit{for the other cases}, \end{cases}$$

(D4g)′ $\bar{s}_{l,t}(\bar{s}_{j,n-1}d_j^{-1})\bar{s}_{l,t}^{-1} =$

$$\begin{cases} [(\bar{s}_{t,n-1}d_t^{-1}), r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}]^{-1}(\bar{s}_{j,n-1}d_j^{-1}) \\ [(\bar{s}_{t,n-1}d_t^{-1}), r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}] \quad \textit{for } t < j, \\ \{r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}\}(\bar{s}_{j,n-1}d_j^{-1})\{r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}\}^{-1} \\ \textit{for } t = j, \\ (s_{t,n-1}d_t^{-1})(\bar{s}_{j,n-1}d_j^{-1})(s_{t,n-1}d_t^{-1})^{-1} \quad \textit{for } l = j, \\ [(\bar{s}_{l,n-1}d_l^{-1})^{-1}, (s_{t,n-1}d_t^{-1})]^{-1}(\bar{s}_{j,n-1}d_j^{-1})[(\bar{s}_{l,n-1}d_l^{-1})^{-1}, (s_{t,n-1}d_t^{-1})] \\ \textit{for } l > j, \\ \bar{s}_{j,n-1}d_j^{-1} \quad \textit{for } l < j < t, \end{cases}$$

(D4h)′ $d_l(\bar{s}_{j,n-1}d_j^{-1})d_l^{-1} = \bar{s}_{j,n-1}d_j^{-1}$.

Proof. Assume that $g \geq 1$, $n \geq 2$, and $\mathcal{M}(N_{g,n-1})$ has the finite presentation in Proposition 3.2. The presentation for $\mathcal{M}(N_{g,n-1})$ has generators a_i, d_j, $a_{i;j}$, $r_{i',j}$, $s_{k,l}$, $\bar{s}_{k,l}$ $(1 \leq i \leq g-1, 1 \leq i' \leq g, 1 \leq j \leq n-2, 1 \leq k < l \leq n-2)$, y for $g \geq 2$, and b for $g \geq 4$. By Lemma 5.5, the group $\pi_1(N_{g,n-1})^+$ is the free group which is freely generated by $x_{i+1}x_i$ $(1 \leq i \leq g-1)$, x_i^2 $(1 \leq i \leq g)$, y_j $(1 \leq j \leq n-2)$, and $x_1^{-1}y_jx_1$ $(1 \leq j \leq n-2)$. Remark that

$$(5.3) \qquad \begin{cases} j_{x_0}(x_{i+1}x_i) = a_{i;n-1}a_i^{-1}, \\ j_{x_0}(x_i^2) = r_{i;n-1}, \\ j_{x_0}(y_j) = s_{j,n-1}d_j^{-1}, \\ j_{x_0}(x_1^{-1}y_jx_1) = \bar{s}_{j,n-1}d_j^{-1} \end{cases}$$

by Lemma 2.8. Since we take lifts of a_i, d_j, $a_{i;j}$, $r_{i',j}$, $s_{k,l}$, $\bar{s}_{k,l}$, y, $b \in \mathcal{M}(N_{g,n-1})$ with respect to $\mathcal{F}|_{\mathcal{M}^+(N_{g,n-1},x_0)} : \mathcal{M}^+(N_{g,n-1}, x_0) \to \mathcal{M}(N_{g,n-1})$ by a_i, d_j, $a_{i;j}$, $r_{i',j}$, $s_{k,l}$, $\bar{s}_{k,l}$, y, $b \in \mathcal{M}^+(N_{g,n-1}, x_0)$, respectively, applying Lemma 5.6 to the exact sequence 5.1, we obtain the presentation for $\mathcal{M}^+(N_{g,n-1}, x_0)$ whose generators are

(1) a_i, d_j, $a_{i;j}$, $r_{i',j}$, $s_{k,l}$, $\bar{s}_{k,l}$ $(1 \leq i \leq g-1, 1 \leq i' \leq g, 1 \leq j \leq n-2, 1 \leq k < l \leq n-2)$, y for $g \geq 2$, and b for $g \geq 4$,

(2) $r_{i;n-1}$ $(1 \leq i \leq g)$, $a_{i;n-1}a_i^{-1}$ $(1 \leq i \leq g-1)$, $s_{i,n-1}d_i^{-1}$ $(1 \leq i \leq n-2)$, and $\bar{s}_{i,n-1}d_i^{-1}$ $(1 \leq i \leq n-2)$.

Denote by X_1 the set of generators in (1) and by X_2 the set of generators in (2). Since $\pi_1(N_{g,n-1})^+$ is the free group, the defining relations are obtained as follows:

(1) for any relation $v_1^{\varepsilon_1} \cdots v_k^{\varepsilon_k} = w_1^{\delta_1} \cdots w_l^{\delta_l}$ of the presentation for $\mathcal{M}(N_{g,n-1})$ in Proposition 3.2 and the lifts $\tilde{v}_i, \tilde{w}_j \in \mathcal{M}^+(N_{g,n-1}, x_0)$ of $v_i, w_j \in \mathcal{M}(N_{g,n-1})$ with respect to $\mathcal{F}|_{\mathcal{M}^+(N_{g,n-1},x_0)}$, respectively, there exists a product $v_{(w_1^{\delta_1}\cdots w_l^{\delta_l})^{-1}v_1^{\varepsilon_1}\cdots v_k^{\varepsilon_k}}$ of elements in X_2 such that

$$\tilde{v}_1^{\varepsilon_1} \cdots \tilde{v}_k^{\varepsilon_k} = \tilde{w}_1^{\delta_1} \cdots \tilde{w}_l^{\delta_l} v_{(w_1^{\delta_1}\cdots w_l^{\delta_l})^{-1}v_1^{\varepsilon_1}\cdots v_k^{\varepsilon_k}},$$

(2) for $x \in X_2$ and $f \in X_1$, there exists a product $w_{x,f}$ of elements in X_2 such that

$$fxf^{-1} = w_{x,f}.$$

Note that the generators of this presentation consist of a_i, d_j, $a_{i;j'}$, $r_{i',j'}$, $s_{k,l}$, $\bar{s}_{k,l}$ $(1 \leq i \leq g-1, 1 \leq i' \leq g, 1 \leq j \leq n-2, 1 \leq j' \leq n-1, 1 \leq k < l \leq n-1)$, y for $g \geq 2$, and b for $g \geq 4$ essentially, the generators in (1) are lifts of the generators of the presentation for $\mathcal{M}(N_{g,n-1})$

in Proposition 3.2, and the generators in (2) are the images of the generators for $\pi_1(N_{g,n-1})^+$ in Lemma 5.5.

We calculate each $v_{(w_1^{\delta_1}\cdots w_l^{\delta_l})^{-1}v_1^{\varepsilon_1}\cdots v_k^{\varepsilon_k}}$ and $w_{x,f}$ in the relations (1) and (2) above. We take the subsurface N' of $N_{g,n}$ which is diffeomorphic to $N_{g,n-1}$ as in Figure 13. By the definition of the elements in $X_1 \subset \mathcal{M}^+(N_{g,n-1}, x_0)$, every simple closed curve which appears in X_1 is isotopic to a simple closed curve on intN' relative to x_0. We regard generators of the presentation for $\mathcal{M}(N_{g,n-1})$ as elements of $\mathcal{M}(N')$. In particular, the inclusion $\iota' : N' \hookrightarrow N_{g,n}$ induces the injective homomorphism $\iota'_* : \mathcal{M}(N') \to \mathcal{M}(N_{g,n})$. By using the composition $\iota_* \circ \iota'_* : \mathcal{M}(N') \to \mathcal{M}^+(N_{g,n-1}, x_0)$, we can show that $v_{(w_1^{\delta_1}\cdots w_l^{\delta_l})^{-1}v_1^{\varepsilon_1}\cdots v_k^{\varepsilon_k}} = 1$ for each relation $v_1^{\varepsilon_1}\cdots v_k^{\varepsilon_k} = w_1^{\delta_1}\cdots w_l^{\delta_l}$ of the presentation for $\mathcal{M}(N_{g,n-1})$.

For the relation (2) above, we construct $w_{x,f}$ as follows. We take $x \in X_2$ and $f \in X_1$. By the formulas 5.3, there exists an element $w \in \pi_1(N_{g,n-1})^+$ which lies in the generating set in Lemma 5.5 such that $x = j_{x_0}(w)$. Since $fj_{x_0}(w)f^{-1} = j_{x_0}(f(w))$ for any $w \in \pi_1(N_{g,n-1}, x_0)$ and $f \in \mathcal{M}^+(N_{g,n-1}, x_0)$, we have

$$fxf^{-1} = fj_{x_0}(w)f^{-1} = j_{x_0}(f(w)).$$

By Lemma 5.7, we have a product $w_1^{\delta_1}\cdots w_l^{\delta_l}$ of generators for $\pi_1(N_{g,n-1})^+$ in Lemma 5.5 such that $f(w) = w_1^{\delta_1}\cdots w_l^{\delta_l}$. Therefore we have the relation

$$fxf^{-1} = j_{x_0}(w_1)^{\delta_1}\cdots j_{x_0}(w_l)^{\delta_l}.$$

Applying the formulas 5.3 to the equation above, $fj_{x_0}(w)f^{-1}$ is equal to a product of elements in X_2 and we obtain $w_{x,f}$. We can check that if the formula $f(w) = w_1^{\delta_1}\cdots w_l^{\delta_l}$ is one of the cords (D1a)″-(D4h)″ in Lemma 5.7, then the obtained relation $fj_{x_0}(x)f^{-1} = w_{x,f}$ coincides with one of Relations (D1a)′-(D4h)′ in Lemma 5.8. For example, for $a_{i;n-1}a_i^{-1} \in X_2$ and $r_{i;l} \in X_1$, we have $a_{i;n-1}a_i^{-1} = j_{x_0}(x_{i+1}x_i)$. Recall that $r_{i;l}(x_{i+1}x_i)$ is represented by the loop as on the right-hand side of Figure 12. By the formula (D1e)″ for $m = i$ in Lemma 5.7, we have

$$r_{i;l}(x_{i+1}x_i) = \{x_i^{-2}y_l^{-1}x_i^2\}y_l(x_{i+1}x_i)x_i^{-1}y_l^{-1}x_i\{x_i^{-2}y_l^{-1}x_i^2\}^{-1} \in \pi_1(N_{g,n-1})^+.$$

Recall that $\bar{s}_{j,k;i}$ is the Dehn twist along the simple closed curve $\bar{\sigma}_{j,k;i} = \iota(\bar{\sigma}_{j,k;i})$ and $\bar{\sigma}_{j,k;i}$ is defined in Figure 6. Thus we show that

$$\begin{aligned} r_{i;l}(a_{i;n-1}a_i^{-1})r_{i;l}^{-1} &= \{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}(s_{l,n-1}d_l^{-1})(a_{i;n-1}a_i^{-1}) \\ &\quad (\bar{s}_{l,n-1;i}d_l^{-1})^{-1}\{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}^{-1}. \end{aligned}$$

This relation is Relation (D1e)′ for $m = i$. Therefore $\mathcal{M}^+(N_{g,n-1}, x_0)$ has the presentation which is obtained from the finite presentation for $\mathcal{M}(N_{g,n-1})$ by adding generators $a_{i;n-1}$ for $1 \leq i \leq g-1$, $r_{i;n-1}$ for $1 \leq i \leq g$, $s_{i,n-1}$ and $\bar{s}_{i,n-1}$ for $1 \leq i \leq n-2$, and Relations (D1a)′-(D4h)′. □

Let $v_1^{\varepsilon_1}\cdots v_k^{\varepsilon_k} = w_1^{\delta_1}\cdots w_l^{\delta_l}$ be a relation in $\mathcal{M}^+(N_{g,n-1}, x_0)$, and $\tilde{v}_i \in \mathcal{M}(N_{g,n})$ (resp. $\tilde{w}_j \in \mathcal{M}(N_{g,n})$) a lift of v_i (resp. w_j) with respect to $\iota_* : \mathcal{M}(N_{g,n}) \to \mathcal{M}^+(N_{g,n-1}, x_0)$. By the exact sequence 5.2, there exists an integer $\varepsilon \in \mathbb{Z}$ such that we have the following relation in $\mathcal{M}(N_{g,n})$:

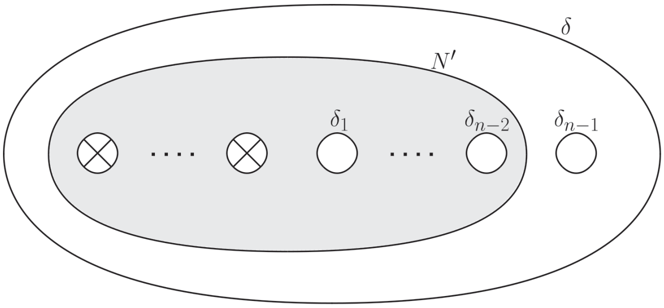


Fig. 13. The subsurface N' of $N_{g,n}$ which is diffeomorphic to $N_{g,n-1}$.

$$\tilde{v}_1^{\varepsilon_1}\cdots\tilde{v}_k^{\varepsilon_k}=\tilde{w}_1^{\delta_1}\cdots\tilde{w}_l^{\delta_l}d_{n-1}^{\varepsilon}.$$

We call the integer $\varepsilon\in\mathbb{Z}$ above the *index* of the relation $v_1^{\varepsilon_1}\cdots v_k^{\varepsilon_k}=w_1^{\delta_1}\cdots w_l^{\delta_l}$ in $\mathcal{M}^+(N_{g,n-1},x_0)$, and the relation $\tilde{v}_1^{\varepsilon_1}\cdots\tilde{v}_k^{\varepsilon_k}=\tilde{w}_1^{\delta_1}\cdots\tilde{w}_l^{\delta_l}d_{n-1}^{\varepsilon}$ in $\mathcal{M}(N_{g,n})$ the *relation which is obtained from the relation* $v_1^{\varepsilon_1}\cdots v_k^{\varepsilon_k}=w_1^{\delta_1}\cdots w_l^{\delta_l}$ *with the index* ε. Note that the index depends on the choice of lifts $\tilde{v}_i$ and $\tilde{w}_i$. Recall that we take lifts of a_i, y, b, d_i, $a_{i,j}$, $r_{i,j}$, $s_{i,j}$, $\bar{s}_{i,j}$, $\bar{s}_{j,k;i}\in\mathcal{M}^+(N_{g,n-1},x_0)$ with respect to $\iota_*:\mathcal{M}(N_{g,n})\to\mathcal{M}^+(N_{g,n-1},x_0)$ by a_i, y, b, d_i, $a_{i,j}$, $r_{i,j}$, $s_{i,j}$, $\bar{s}_{i,j}$, $\bar{s}_{j,k;i}\in\mathcal{M}(N_{g,n})$, respectively. We remark that the defining relations of the presentation for $\mathcal{M}^+(N_{g,n-1},x_0)$ in Lemma 5.8 are Relations (A1)-(B8), (D0) for $1\leq j,k,t\leq n-2$, (D1a)-(D4g) for $1\leq k\leq n-2$ in Proposition 3.2, and (D1a)′-(D4g)′ in Lemma 5.8. We prepare the following lemma.

Lemma 5.9. *The indices for the relations of the presentation for* $\mathcal{M}^+(N_{g,n-1},x_0)$ *in* Lemma 5.8 *are as follows:*

$$\varepsilon=\begin{cases}2 & \text{for Relation (D1e)}' \text{ for } m=i+1,\\ 1 & \text{for Relations (D1d)}' \text{ for } m=i+1,\ \text{(D2a)}' \text{ for } m=i-1,\\ & \text{and (D2b)}' \text{ for } i=2,\\ -1 & \text{for Relations (D1b)}' \text{ for } i=2,\ \text{(D1c)}' \text{ for } i=4,\\ & \text{and (D1d)}' \text{ for } m=i-1\\ -2 & \text{for Relations (D1b)}' \text{ for } i=1,\ \text{and (D1e)}' \text{ for } m=i,\\ 0 & \text{for the other cases.}\end{cases}$$

We prove Lemma 5.9 in Section 5.5. Applying Lemma 5.6 to the exact sequence 5.2, we have the following lemma.

Lemma 5.10. *Assume that* $g\geq 1$ *and* $n\geq 2$. *If* $\mathcal{M}^+(N_{g,n-1},x_0)$ *has the finite presentation in* Lemma 5.8, *then* $\mathcal{M}(N_{g,n})$ *admits the presentation which is obtained from the finite presentation for* $\mathcal{M}^+(N_{g,n-1},x_0)$ *in* Lemma 5.8 *by adding the generator* d_{n-1} *and the relations*

(D0)′ $[d_{n-1},a_i]=[d_{n-1},y]=[d_{n-1},b]=[d_{n-1},d_l]=[d_{n-1},a_{i;k}]=[d_{n-1},r_{j;k}]=[d_{n-1},s_{l,t}]$
$=[d_{n-1},\bar{s}_{l,t}]=1$

for $1\leq i\leq g-1$, $1\leq j\leq g$, $1\leq k\leq n-1$, *and* $1\leq l<t\leq n-1$, *and replacing Relations* (D1a)′-(D4h)′ *by the relations which are obtained from Relations* (D1a)′-(D4h)′ *with the following indices:*

$$\varepsilon = \begin{cases} 2 & \text{for Relation (D1e)}' \text{ for } m = i+1, \\ 1 & \text{for Relations (D1d)}' \text{ for } m = i+1, \text{ (D2a)}' \text{ for } m = i-1, \\ & \text{and (D2b)}' \text{ for } i = 2, \\ -1 & \text{for Relations (D1b)}' \text{ for } i = 2, \text{ (D1c)}' \text{ for } i = 4, \\ & \text{and (D1d)}' \text{ for } m = i-1 \\ -2 & \text{for Relations (D1b)}' \text{ for } i = 1, \text{ and (D1e)}' \text{ for } m = i, \\ 0 & \text{for the other cases.} \end{cases}$$

Proof. Assume that $g \geq 1$, $n \geq 2$ and $\mathcal{M}^+(N_{g,n-1}, x_0)$ has the finite presentation in Lemma 5.8. Let X be the subset of $\mathcal{M}(N_{g,n-1})$ which consists of a_i, d_j, $a_{i;j'}$, $r_{i',j'}$, $s_{k,l}$, $\bar{s}_{k,l}$ $(1 \leq i \leq g-1, 1 \leq i' \leq g, 1 \leq j \leq n-2, 1 \leq j' \leq n-1, 1 \leq k < l \leq n-1)$, y for $g \geq 2$, and b for $g \geq 4$. We remark that the presentation for $\mathcal{M}^+(N_{g,n-1}, x_0)$ in Lemma 5.8 has the generating set $\iota_*(X) \subset \mathcal{M}^+(N_{g,n-1}, x_0)$ and Relations (A1)-(B8), (D0) for $1 \leq j, k, t \leq n-2$, (D1a)-(D4g) for $1 \leq k \leq n-2$ in Proposition 3.2, and (D1a)′-(D4g)′ in Lemma 5.8. Every element in $\mathcal{M}(N_{g,n})$ commutes with d_{n-1}. By applying Lemma 5.6 to the exact sequence 5.2, we obtain the presentation for $\mathcal{M}(N_{g,n-1})$ whose generating set is $X \cup \{d_{n-1}\}$ and the defining relations as follows:

(1) For each relation $v_1^{\varepsilon_1} \cdots v_k^{\varepsilon_k} = w_1^{\delta_1} \cdots w_l^{\delta_l}$ of the finite presentation for $\mathcal{M}^+(N_{g,n-1}, x_0)$ in Lemma 5.8 and the lift $\tilde{v}_i \in \mathcal{M}(N_{g,n})$ (resp. $\tilde{w}_j \in \mathcal{M}(N_{g,n})$) of v_i (resp. w_j) with respect to $\iota_* : \mathcal{M}(N_{g,n}) \to \mathcal{M}^+(N_{g,n-1}, x_0)$, there exists $\varepsilon \in \mathbb{Z}$ such that

$$\tilde{v}_1^{\varepsilon_1} \cdots \tilde{v}_k^{\varepsilon_k} = \tilde{w}_1^{\delta_1} \cdots \tilde{w}_l^{\delta_l} d_{n-1}^{\varepsilon}.$$

(2) For each $x \in X$,

$$[d_{n-1}, x] = 1.$$

The relations (2) above correspond to the added relations (D0)′. To determine the relation (1) above, we need to compute the index $\varepsilon \in \mathbb{Z}$ for each relation of the finite presentation for $\mathcal{M}^+(N_{g,n-1}, x_0)$ in Lemma 5.8. By Lemma 5.9, their indices are determined and coincide with ones in Lemma 5.10. Therefore we have proved Lemma 5.10. □

Proof of Proposition 3.2. We proceed by induction on $n \geq 1$. The base case is of $n = 1$. The finite presentation for $\mathcal{M}(N_{g,1})$ is given by Theorem 3.1.

Assume $n \geq 2$.By the inductive hypothesis, $\mathcal{M}(N_{g,n-1})$ admits the presentation in Proposition 3.2. By Lemma 5.8, the group $\mathcal{M}^+(N_{g,n-1}, x_0)$ admits the presentation whose generators are a_i, d_j, $a_{i;j'}$, $r_{i',j'}$, $s_{k,l}$, $\bar{s}_{k,l}$ $(1 \leq i \leq g-1, 1 \leq i' \leq g, 1 \leq j \leq n-2, 1 \leq j' \leq n-1, 1 \leq k < l \leq n-1)$, y for $g \geq 2$, and b for $g \geq 4$, and the defining relations are Relations (A1)-(B8), (D0) for $1 \leq j, k, t \leq n-2$, (D1a)-(D4g) for $1 \leq k \leq n-2$ in Proposition 3.2, and (D1a)′-(D4g)′ in Lemma 5.8. By Lemma 5.10, $\mathcal{M}(N_{g,n})$ admits the presentation whose generators are a_i, d_j, $a_{i;j'}$, $r_{i',j'}$, $s_{k,l}$, $\bar{s}_{k,l}$ $(1 \leq i \leq g-1, 1 \leq i' \leq g, 1 \leq j \leq n-1, 1 \leq j' \leq n-1, 1 \leq k < l \leq n-1)$, y for $g \geq 2$, and b for $g \geq 4$, and the defining relations are Relations (A1)-(B8), (D0), (D1a)-(D4g) above, (D0)′, and the relations which are obtained from Relations (D1a)′-(D4h)′ with the indices as in Lemma 5.10. The generating set of this presentation for $\mathcal{M}(N_{g,n})$ coincides with the generating set of the presentation for $\mathcal{M}(N_{g,n})$ in Proposition 3.2. We can check that Relation (D0)′ coincides with Relation (D0) for either $j = n-1$, $k = n-1$, or $t = n-1$, the relations which are obtained from Relations (D1a)′-(D4h)′ with the indices above coincide with Relations (D1a)-(D4h) for $k = n - 1$, respectively. Therefore the presentation for $\mathcal{M}(N_{g,n})$ above is equal to the presentation for $\mathcal{M}(N_{g,n})$ in Proposition 3.2 and we have completed the proof of Proposition 3.2. □

5.5. Computing indices. In this subsection, we prove Lemma 5.9. We compute the index $\varepsilon \in \mathbb{Z}$ for each relation of the finite presentation for $\mathcal{M}^+(N_{g,n-1}, x_0)$ in Lemma 5.8. Recall that the defining relations of the presentation for $\mathcal{M}^+(N_{g,n-1}, x_0)$ in Lemma 5.8 are Relations (A1)-(B8), (D0) for $1 \leq j, k, t \leq n-2$, (D1a)-(D4g) for $1 \leq k \leq n-2$ in Proposition 3.2, and (D1a)′-(D4g)′ in Lemma 5.8. First we compute the indices for Relations (A1)-(B8), (D0), and (D1a)-(D4g) above. These relations come from the defining relations of the presentation for $\mathcal{M}(N_{g,n-1})$ in Proposition 3.2. We have the following lemma.

Lemma 5.11. *The indices for Relations* (A1)-(B8)*,* (D0)*, and* (D1a)-(D4g) *in the presentation for $\mathcal{M}^+(N_{g,n-1}, x_0)$ in* Lemma 5.8 *are zero.*

Proof. Recall that N' is the subsurface of $N_{g,n}$ as in Figure 13 which is diffeomorphic to $N_{g,n-1}$, and we have the inclusions $N' \overset{\iota'}{\hookrightarrow} N_{g,n} \overset{\iota}{\hookrightarrow} N_{g,n-1}$. The inclusion relations induce the sequence of homomorphisms $\mathcal{M}(N') \overset{\iota'_*}{\hookrightarrow} \mathcal{M}(N_{g,n}) \overset{\iota_*}{\to} \mathcal{M}^+(N_{g,n-1}, x_0)$. By the definition of defining simple closed curves of the generators for $\mathcal{M}^+(N_{g,n-1}, x_0)$ (see the beginning of Section 5.4 and Figure 4, 5, and 6), the mapping classes a_i, y, b, d_k, $a_{i,k}$, $r_{i,k}$, $s_{i,k}$, $\bar{s}_{i,k}$, $\bar{s}_{j,k;i} \in \mathcal{M}^+(N_{g,n-1}, x_0)$ are represented by diffeomorphisms on $N_{g,n-1}$ which are supported on N'. Note that at the beginning of Section 5.4, we took the lifts of a_i, y, b, d_k, $a_{i,k}$, $r_{i,k}$, $s_{i,k}$, $\bar{s}_{i,k}$, $\bar{s}_{j,k;i} \in \mathcal{M}^+(N_{g,n-1}, x_0)$ with respect to $\iota_* : \mathcal{M}(N_{g,n}) \to \mathcal{M}^+(N_{g,n-1}, x_0)$ by a_i, y, b, d_k, $a_{i,k}$, $r_{i,k}$, $s_{i,k}$, $\bar{s}_{i,k}$, $\bar{s}_{j,k;i} \in \mathcal{M}(N_{g,n})$, respectively. We can check that the choice of lifts are natural for the homomorphisms $\mathcal{M}(N') \overset{\iota'_*}{\hookrightarrow} \mathcal{M}(N_{g,n}) \overset{\iota_*}{\to} \mathcal{M}^+(N_{g,n-1}, x_0)$. Thus, for each relation $v_1^{\varepsilon_1} \cdots v_k^{\varepsilon_k} = w_1^{\delta_1} \cdots w_l^{\delta_l}$ of the finite presentation for $\mathcal{M}^+(N_{g,n-1}, x_0)$ in Lemma 5.8 and the lift $\tilde{v}_i \in \mathcal{M}(N_{g,n})$ (resp. $\tilde{w}_j \in \mathcal{M}(N_{g,n})$) of v_i (resp. w_j) with respect to $\iota_* : \mathcal{M}(N_{g,n}) \to \mathcal{M}^+(N_{g,n-1}, x_0)$, we have the relation $\tilde{v}_1^{\varepsilon_1} \cdots \tilde{v}_k^{\varepsilon_k} = \tilde{w}_1^{\delta_1} \cdots \tilde{w}_l^{\delta_l}$ in $\mathcal{M}(N')$. Therefore, the indices for Relations (A1)-(B8), (D0), and (D1a)-(D4g) in the presentation for $\mathcal{M}^+(N_{g,n-1}, x_0)$ in Lemma 5.8 are zero, and we have completed Lemma 5.11. □

For the completion of the proof of Lemma 5.9, calculations of indices for Relation (D1a)′-(D4g)′ remain. By using the braid relations, we have the following lemma.

Lemma 5.12. *The indices for the following relations are zero:*
Relations (D1a)′ *for $m \neq i-1, i+1$,* (D1b)′ *for $i \geq 3$,* (D1c)′ *for $i \neq 4$,* (D1d)′ *for $m \leq i-2$, $m = i$, and $m \geq i+2$,* (D1e)′ *for $m \leq i-1$, and $m \geq i+2$,* (D1f)′*,* (D1g)′ *for $i \geq 2$,* (D1h)′*,* (D2a)′ *for $m \neq i-1$,* (D2b)′ *for $i \neq 2$,* (D2c)′ *for $i \geq 5$,* (D2d)′ *for $m \leq i-2$, and $m \geq i+1$,* (D2e)′*,* (D2f)′*,* (D2g)′ *for $i \geq 2$,* (D2h)′*, and* (D3a)′-(D4h)′.

Proof. Let X_1 and X_2 be the subsets of $\mathcal{M}^+(N_{g,n-1}, x_0)$ as in the proof of Lemma 5.8. Each relation above is to be a following form: there exist elements $f \in X_1$, $x \in X_2$, and $h \in \mathcal{M}^+(N_{g,n-1}, x_0)$ such that

$$fxf^{-1} = hxh^{-1}.$$

Recall that we have the inclusion $N_{g,n} = N_{g,n-1} - \mathrm{int}D_{g+n-1} \overset{\iota}{\hookrightarrow} N_{g,n-1}$ and the point x_0 lies in the interior of the disk D_{g+n-1}. The element x is either a Dehn twist or a product of two Dehn twists along disjoint simple closed curves on the subsurface $N_{g,n} \subset N_{g,n-1}$. Hence we put either $x = t_c$ or $x = t_{c_1}t_{c_2}^{-1}$ for some simple closed curve c or a pair of disjoint simple closed curves c_1 and c_2 on $N_{g,n} \subset N_{g,n-1}$ in Figure 4, 5, and 6. In each case, by the braid relations, the relation above coincides with either the relation $t_{f(c)} = t_{h(c)}$ or $t_{f(c_1)}t_{f(c_2)}^{-1} = t_{h(c_1)}t_{h(c_2)}^{-1}$ in $\mathcal{M}^+(N_{g,n-1}, x_0)$. These relations mean that the simple closed curve $f(c)$ (resp. the pair $(f(c_1), f(c_2))$) is isotopic to the simple closed curve $t_{h(c)}$ (resp. the pair $(h(c_1), h(c_2))$) in $N_{g,n-1} - x_0$. We can take the isotopy as one which fixes the disk D_{g+n-1} pointwise. Hence the relations $t_{f(c)} = t_{h(c)}$ and $t_{f(c_1)}t_{f(c_2)}^{-1} = t_{h(c_1)}t_{h(c_2)}^{-1}$ hold in $\mathcal{M}(N_{g,n})$. Therefore the index of the relation $fxf^{-1} = hxh^{-1}$ is zero and we have completed the proof of Lemma 5.12. □

Proof of Lemma 5.9. By Lemma 5.11 and 5.12, remaining cases are for the following 19 relations: Relators (D1a)′ for $m = i-1, i+1$, (D1b)′ for $i = 1, 2$, (D1c)′ for $i = 4$, (D1d)′ for $m = i-1, i+1$, (D1e)′ for $m = i, i+1$, (D1g)′ for $i = 1$, (D2a)′ for $m = i-1$, (D2b)′ for $i = 2$, (D2c)′ for $i = 1, 2, 3, 4$, (D2d)′ for $m = i-1, i$ and (D2g)′ for $i = 1$. We compute the indices by using Relations (L+), (L-), and (L0) (see Lemma 5.1 and 5.3), and the braid relations. The indices for Relations (D1a)′ for $m = i-1, i+1$ and (D1b)′ for $i = 2$ are computed by a single braid relation and one of Relations (L+), (L-) and (L0). For instance, for Relation (D1a)′ when $m = i-1$, we have

$$\begin{aligned}(a_{i,n-1}a_i^{-1})(a_{i-1;n-1}a_{i-1}^{-1}) &= \Delta(x_{i+1}x_i)\Delta(x_ix_{i-1})\\ &\overset{\text{(L0)}}{=} \Delta((x_{i+1}x_i)(x_ix_{i-1}))\\ &= \Delta(a_{i-1}(x_{i+1}x_i))\\ &\overset{\text{(I)}}{=} a_{i-1}(a_{i;n-1}a_i^{-1})a_{i-1}^{-1}\end{aligned}$$

in $\mathcal{M}(N_{g,n})$. Thus the index for Relations (D1a)′ for $m = i-1$ is $\varepsilon = 0$. By similar arguments, we can show that the indices are $\varepsilon = 0$ for Relations (D1a)′ when $m = i+1$ and $\varepsilon = -1$ Relation (D1b)′ when $i = 2$. By easy calculations, we show that the indices are $\varepsilon = -2$ for Relations (D1b)′ when $i = 1$, $\varepsilon = -1$ for (D1c)′ when $i = 4$, $\varepsilon = 1$ for (D1d)′ when $m = i+1$, $\varepsilon = 1$ for (D2a)′ when $m = i-1$, $\varepsilon = 1$ for (D2b)′ when $i = 2$, $\varepsilon = 0$ for (D2d)′ when $m = i$.

For the other cases, computations of indices require observations by figures. As examples, we compute the indices for Relation (D1e)′ when $m = i$ and Relation (D2c)′ when $i = 2$ by using figures.

For the other cases, we give computations of indices by only deformations of the expressions. We define

$$\begin{aligned}\bar{y}_i &:= x_1^{-1}y_ix_1 \quad \text{for } i = 1, \dots, n-2,\\ \bar{y}_{i;j} &:= \{(x_2x_1)^{-1}x_2^2 \cdots (x_jx_{j-1})^{-1}x_j^2\}^{-1}\bar{y}_i\{(x_2x_1)^{-1}x_2^2 \cdots (x_jx_{j-1})^{-1}x_j^2\}\\ &\quad\ \text{for } 1 \le i \le n-2,\ 2 \le j \le g.\end{aligned}$$

Remark that $\bar{y}_{i;j} = x_j^{-1}y_ix_j$.

For Relation (D1e)′ when $m = i$, we have the following relation in $\mathcal{M}(N_{g,n})$ by Figure 14:

$$\underline{\Delta(y_l)\Delta((x_{i+1}x_i))}\Delta(\bar{y}_{l;i})^{-1} \overset{\text{(L+)}}{=} \underline{\Delta(y_l(x_{i+1}x_i))\Delta(\bar{y}_{l;i}^{-1})}d_{n-1} \overset{\text{(L+)}}{=} \Delta(y_l(x_{i+1}x_i)\bar{y}_{l;i}^{-1})d_{n-1}^2.$$

Note that $\Delta(r_{i;l}(x_{i+1}x_i)) = r_{i;l}(a_{i;n-1}a_i^{-1})r_{i;l}^{-1}$ by the braid relation. Hence we have

$$\begin{aligned}
& \{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}(s_{l,n-1}d_l^{-1})(a_{i;n-1}a_i^{-1})(\bar{s}_{l,n-1;i}d_l^{-1})^{-1} \\
& \{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}^{-1} \\
= \ & \{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}\Delta(y_l)\Delta((x_{i+1}x_i))\Delta(\bar{y}_{l;i})^{-1} \\
& \{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}^{-1} \\
= \ & \{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}\Delta(y_l(x_{i+1}x_i)\bar{y}_{l;i}^{-1})d_{n-1}^2 \\
& \{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}^{-1} \\
\overset{(I)}{=} \ & \Delta(\{r_{i;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i;n-1}\}(y_l(x_{i+1}x_i)\bar{y}_{l;i}^{-1}))d_{n-1}^2 \\
= \ & \Delta(\{x_i^{-2}y_l^{-1}x_i^2\}y_l(x_{i+1}x_i)\bar{y}_{l;i}^{-1}\{x_i^{-2}y_l^{-1}x_i^2\}^{-1})d_{n-1}^2 \\
= \ & \Delta(r_{i;l}(x_{i+1}x_i))d_{n-1}^2 \\
\overset{(I)}{=} \ & r_{i;l}(a_{i;n-1}a_i^{-1})r_{i;l}^{-1}d_{n-1}^2.
\end{aligned}$$

Thus $\varepsilon = -2$ for Relation (D1e)′ when $m = i$.

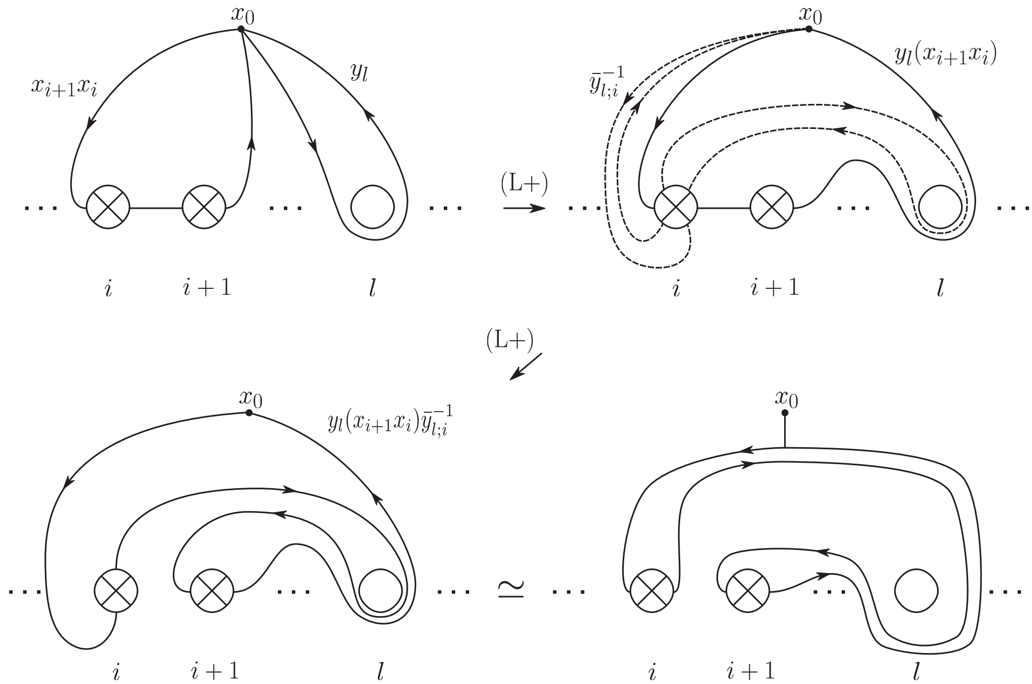


Fig. 14. On the upper side, we explain that the relation $\Delta(y_l)\Delta((x_{i+1}x_i)) = \Delta((y_l(x_{i+1}x_i))d_{n-1}$ is obtained from Relation (L+). Similarly, the arrow from the upper right side to lower left side explain that the relation $\Delta((y_l(x_{i+1}x_i))\Delta(\bar{y}_{l;i}^{-1}) = \Delta(y_l(x_{i+1}x_i)\bar{y}_{l;i}^{-1})d_{n-1}$ is obtained from Relation (L+). "≃" means a deformation of the loop by a homotopy fixing x_0.

For Relation (D2c)′ when $i = 2$, we note that $b(x_2^2)$ is represented by a loop as on the lower right side of Figure 20 and $\Delta(b(x_2^2)) = br_{2;n-1}b^{-1}$ by the braid relation. By Figure 15, we have

$$\begin{aligned}
& (a_{1;n-1}a_1^{-1})(a_{3;n-1}a_3^{-1})r_{2;n-1}r_{1;n-1}(a_{1;n-1}a_1^{-1})^{-1} \\
= \ & (a_{1;n-1}a_1^{-1})\Delta(x_4x_3)\underline{\Delta(x_2^2)\Delta(x_1^2)}(a_{1;n-1}a_1^{-1})^{-1}
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{(L+)}}{=} (a_{1;n-1}a_1^{-1})\underline{\Delta(x_4x_3)\Delta(x_2^2x_1^2)}(a_{1;n-1}a_1^{-1})^{-1}d_{n-1}\\
&\overset{\text{(L+)}}{=} (a_{1;n-1}a_1^{-1})\Delta((x_4x_3)x_2^2x_1^2)(a_{1;n-1}a_1^{-1})^{-1}d_{n-1}^2\\
&\overset{\text{(I)}}{=} \Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2))d_{n-1}^2
\end{aligned}$$

in $\mathcal{M}(N_{g,n})$. By Figure 16 and Figure 17, we have the following relation in $\mathcal{M}(N_{g,n})$.

$$\begin{aligned}
& (a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1}(a_{3;n-1}a_3^{-1})^{-1}r_{4;n-1}(a_{2;n-1}a_2^{-1})\\
= \;& (a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1}r_{4;n-1}r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})^{-1}r_{4;n-1}(a_{2;n-1}a_2^{-1})\\
= \;& (a_{2;n-1}a_2^{-1})^{-1}\underline{\Delta(x_3^2)\Delta(x_4^2)}\underline{r_{4;n-1}^{-1}\Delta(x_4x_3)^{-1}r_{4;n-1}}(a_{2;n-1}a_2^{-1})\\
\overset{\text{(L+),(I)}}{=} \;& (a_{2;n-1}a_2^{-1})^{-1}\underline{\Delta(x_3^2x_4^2)\Delta(r_{4;n-1}^{-1}((x_4x_3)^{-1}))}(a_{2;n-1}a_2^{-1})d_{n-1}\\
\overset{\text{(L-)}}{=} \;& (a_{2;n-1}a_2^{-1})^{-1}\Delta(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1}))(a_{2;n-1}a_2^{-1})\\
\overset{\text{(I)}}{=} \;& \Delta((a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1}))).
\end{aligned}$$

Let ζ_1 and ζ_2 be simple closed curves on $N_{g,n}$ as in Figure 18. Since $\iota_*(t_{\zeta_1}) \in \mathcal{M}^+(N_{g,n-1}, x_0)$ fixes x_2^2 and $\Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)) = t_{\zeta_1}t_{\zeta_2}^{-1}$, we have $\Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2))(x_2^2) = t_{\zeta_2}^{-1}(x_2^2)$. We remark that the loop as on the upper right side of Figure 19 is homotopic to the loop as on the lower right side of Figure 19 by a homotopy fixing x_0 as in Figure 19. By the relations above and Figure 20, we have

$$\begin{aligned}
& \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}\underline{(a_{1;n-1}a_1^{-1})(a_{3;n-1}a_3^{-1})r_{2;n-1}r_{1;n-1}}\\
& \underline{(a_{1;n-1}a_1^{-1})^{-1}}r_{2;n-1}(a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1}(a_{3;n-1}a_3^{-1})^{-1}r_{4;n-1}\\
& (a_{2;n-1}a_2^{-1})\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}\\
\overset{\text{(L+),(I)}}{=} \;& \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}\Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2))r_{2;n-1}\\
& \underline{(a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1}(a_{3;n-1}a_3^{-1})^{-1}r_{4;n-1}(a_{2;n-1}a_2^{-1})}\\
& \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}d_{n-1}^2\\
\overset{\text{(L+),(L-),(I)}}{=} \;& \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}\Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2))r_{2;n-1}\\
& \Delta((a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1})))\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}\\
& d_{n-1}^2\\
= \;& \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}\underline{\Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2))\Delta(x_2^2)}\\
& \underline{\Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2))^{-1}}\Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2))\\
& \Delta((a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1})))\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}\\
& d_{n-1}^2\\
\overset{\text{(I)}}{=} \;& \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}\Delta(t_{\zeta_2}^{-1}(x_2^2))\underline{\Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2))}\\
& \underline{\Delta((a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1})))}\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}\\
& d_{n-1}^2\\
\overset{\text{(L-)}}{=} \;& \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}\underline{\Delta(t_{\zeta_2}^{-1}(x_2^2))}
\end{aligned}$$

$$
\begin{aligned}
& \frac{\Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)\cdot(a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1})))}{\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}d_{n-1}} \\
\overset{\text{(L-)}}{=} \quad & \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}\Delta(t_{\zeta_2}^{-1}(x_2^2)\cdot \\
& (a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)\cdot(a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1}))) \\
& \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \\
\overset{\text{(I)}}{=} \quad & \Delta(\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}(t_{\zeta_2}^{-1}(x_2^2)\cdot \\
& (a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)\cdot(a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1})))) \\
= \quad & \Delta(b(x_2^2)) \\
\overset{\text{(I)}}{=} \quad & br_{2;n-1}b^{-1}.
\end{aligned}
$$

Thus $\varepsilon = 0$ for Relation (D2c)′ when $i = 2$.

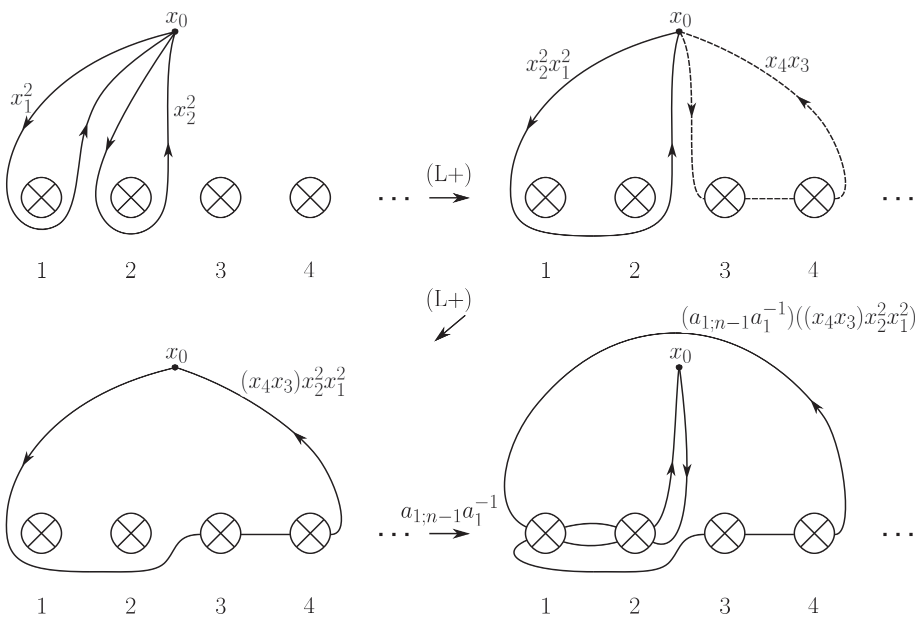


Fig. 15. Relations $\Delta(x_2^2)\Delta(x_1^2) = \Delta(x_2^2x_1^2)d_{n-1}$ and $\Delta(x_4x_3)\Delta(x_1^2x_2^2) = \Delta((x_4x_3)x_2^2x_1^2)d_{n-1}$ and loop $(a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)$.

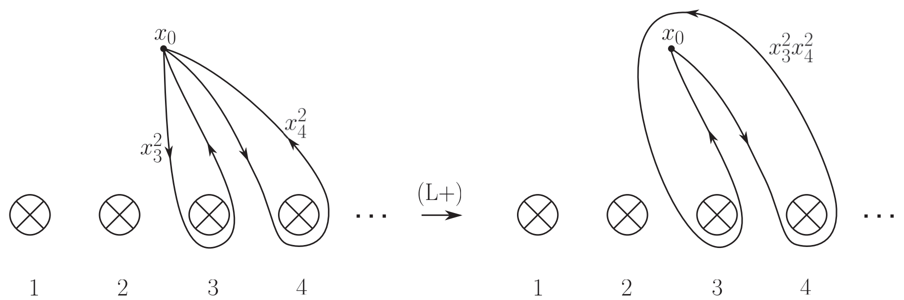


Fig. 16. Relation $\Delta(x_3^2)\Delta(x_4^2) = \Delta(x_3^2x_4^2)d_{n-1}$.

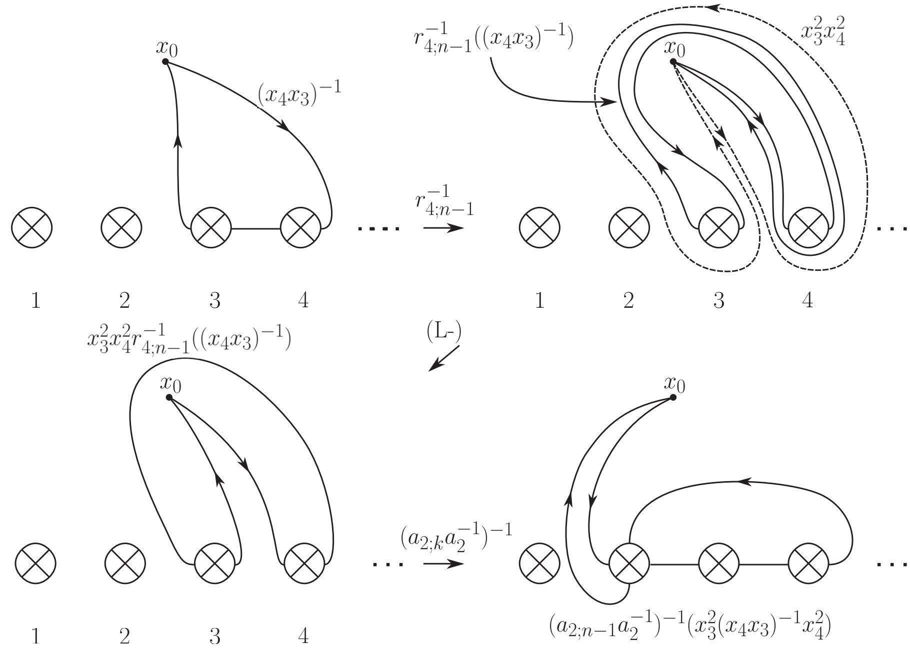


Fig. 17. Relation $\Delta(x_3^2x_4^2)\Delta(r^{-1}_{4;n-1}((x_4x_3)^{-1})) = \Delta(x_3^2x_4^2r^{-1}_{4;n-1}((x_4x_3)^{-1}))d^{-1}_{n-1}$ and loops $r^{-1}_{4;n-1}((x_4x_3)^{-1})$ and $(a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r^{-1}_{4;n-1}((x_4x_3)^{-1}))$.

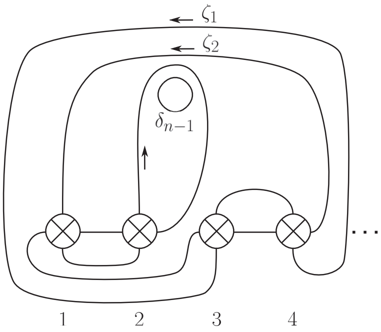


Fig. 18. Simple closed curves ζ_1 and ζ_2 on $N_{g,n}$.

For Relation (D1d)′ when $m = i-1$, we have

$$
\begin{aligned}
& [(a_{i-1;n-1}a_{i-1}^{-1})^{-1}, (s_{l,n-1}d_l^{-1})^{-1}](a_{i;n-1}a_i^{-1})(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1}) \\
= \; & \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}\Delta(x_ix_{i-1})\underline{\Delta(y_l)\Delta(x_{i+1}x_i)}\{(s_{l,n-1}d_l^{-1}) \\
& (a_{i-1;n-1}a_{i-1}^{-1})\} \\
\overset{\text{(L+)}}{=} \; & \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}\underline{\Delta(x_ix_{i-1})\Delta(y_l(x_{i+1}x_i))}\{(s_{l,n-1}d_l^{-1}) \\
& (a_{i-1;n-1}a_{i-1}^{-1})\}d_{n-1} \\
\overset{\text{(L0)}}{=} \; & \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}\Delta((x_ix_{i-1})y_l(x_{i+1}x_i))\{(s_{l,n-1}d_l^{-1}) \\
& (a_{i-1;n-1}a_{i-1}^{-1})\}d_{n-1} \\
\overset{\text{(I)}}{=} \; & \Delta(\{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}((x_ix_{i-1})y_l(x_{i+1}x_i)))d_{n-1}
\end{aligned}
$$

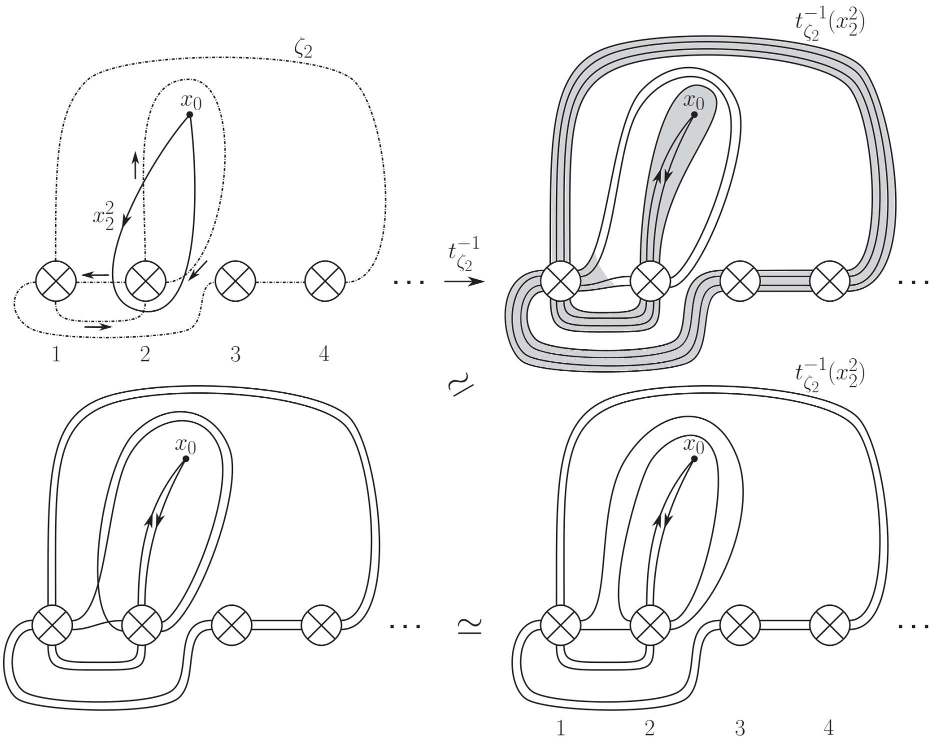


Fig. 19. Loop $t_{\zeta_2}^{-1}(x_2^2)$.

$$
\begin{aligned}
&= \quad \Delta(a_{i-1;l}(x_{i+1}x_i))d_{n-1} \\
&\overset{(I)}{=} \quad a_{i-1;l}(a_{i;n-1}a_i^{-1})a_{i-1;l}^{-1}d_{n-1}.
\end{aligned}
$$

Thus $\varepsilon = -1$ for Relation (D1d)′ when $m = i - 1$.

For Relation (D1e)′ when $m = i + 1$, we have

$$
\begin{aligned}
& r_{i+1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{i+1;n-1}(\bar{s}_{l,n-1;i+1}d_l^{-1})(a_{i;n-1}a_i^{-1}) \\
= \; & \underline{r_{i+1;n-1}^{-1}\Delta(y_l^{-1})r_{i+1;n-1}}\,\underline{\Delta(\bar{y}_{l;i+1})\Delta(x_{i+1}x_i)} \\
\overset{(I),(L-)}{=} \; & \Delta(r_{i+1;n-1}^{-1}(y_l^{-1}))\Delta(\bar{y}_{l;i+1}(x_{i+1}x_i))d_{n-1}^{-1} \\
\overset{(L-)}{=} \; & \Delta(r_{i+1;n-1}^{-1}(y_l^{-1})\bar{y}_{l;i+1}(x_{i+1}x_i))d_{n-1}^{-2} \\
= \; & \Delta(r_{i+1;l}(x_{i+1}x_i))d_{n-1}^{-2} \\
\overset{(I)}{=} \; & r_{i+1;l}(a_{i;n-1}a_i^{-1})r_{i+1;l}^{-1}d_{n-1}^{-2}.
\end{aligned}
$$

Thus $\varepsilon = -2$ for Relation (D1e)′ when $m = i + 1$.

For Relation (D1g)′ when $i = 1$, we have

$$
\begin{aligned}
& [(\bar{s}_{l,n-1}d_l^{-1})^{-1}, s_{t,n-1}d_t^{-1}]^{-1}(s_{l,n-1}d_l^{-1})(a_{1;n-1}a_1^{-1})r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})r_{1;n-1}^{-1} \\
& (s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})^{-1}r_{1;n-1}^{-1}[(\bar{s}_{l,n-1}d_l^{-1})^{-1}, (s_{t,n-1}d_t^{-1})] \\
= \quad & [(\bar{s}_{l,n-1}d_l^{-1})^{-1}, s_{t,n-1}d_t^{-1}]^{-1}\underline{\Delta(y_l)\Delta(x_2x_1)}\,\underline{r_{1;n-1}\Delta(\bar{y}_t)r_{1;n-1}^{-1}}\Delta(y_l)^{-1} \\
& \underline{r_{1;n-1}\Delta(\bar{y}_t)^{-1}r_{1;n-1}^{-1}}[(\bar{s}_{l,n-1}d_l^{-1})^{-1}, (s_{t,n-1}d_t^{-1})]
\end{aligned}
$$

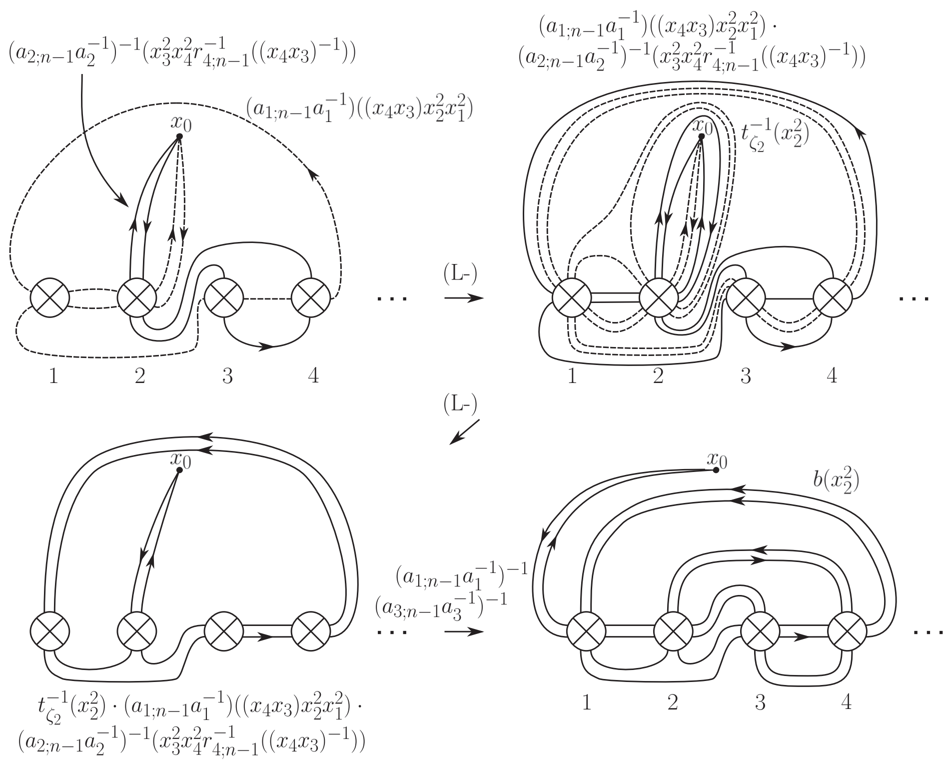


Fig. 20. Relations $\Delta((a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2))\Delta((a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1}))) = \Delta(a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)\cdot(a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1})))d_{n-1}^{-1}$ and $\Delta(t_{\zeta_2}^{-1}(x_2^2))\Delta(a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)\cdot(a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1}))) = \Delta(t_{\zeta_2}^{-1}(x_2^2)\cdot a_{1;n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)\cdot(a_{2;n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4;n-1}^{-1}((x_4x_3)^{-1})))d_{n-1}^{-1}$ and loop $b(x_2^2)$.

$$
\begin{aligned}
&\overset{\text{(L+),(I)}}{=} [(\bar{s}_{l,n-1}d_l^{-1})^{-1}, s_{t,n-1}d_t^{-1}]^{-1}\Delta(y_l(x_2x_1))\\
&\qquad \underline{\Delta(r_{1;n-1}(\bar{y}_t))\Delta(y_l)^{-1}\Delta(r_{1;n-1}(\bar{y}_t))^{-1}}[(\bar{s}_{l,n-1}d_l^{-1})^{-1}, (s_{t,n-1}d_t^{-1})]d_{n-1}\\
&\overset{\text{(I)}}{=} [(\bar{s}_{l,n-1}d_l^{-1})^{-1}, s_{t,n-1}d_t^{-1}]^{-1}\underline{\Delta(y_l(x_2x_1))\Delta(\Delta(r_{1;n-1}(\bar{y}_t))(y_l))^{-1}}\\
&\qquad [(\bar{s}_{l,n-1}d_l^{-1})^{-1}, (s_{t,n-1}d_t^{-1})]d_{n-1}\\
&\overset{\text{(L-)}}{=} [(\bar{s}_{l,n-1}d_l^{-1})^{-1}, s_{t,n-1}d_t^{-1}]^{-1}\Delta(y_l(x_2x_1)\Delta(r_{1;n-1}(\bar{y}_t))(y_l)^{-1})\\
&\qquad [(\bar{s}_{l,n-1}d_l^{-1})^{-1}, (s_{t,n-1}d_t^{-1})]\\
&\overset{\text{(I)}}{=} \Delta([(\bar{s}_{l,n-1}d_l^{-1})^{-1}, s_{t,n-1}d_t^{-1}]^{-1}(y_l(x_2x_1)\Delta(r_{1;n-1}(\bar{y}_t))(y_l)^{-1}))\\
&= \Delta(\bar{s}_{l;t}(x_{i+1}x_i))\\
&\overset{\text{(I)}}{=} \bar{s}_{l;t}(a_{i;n-1}a_i^{-1})\bar{s}_{l;t}^{-1}.
\end{aligned}
$$

Thus $\varepsilon = 0$ for Relation (D1g)′ when $i = 1$.

For Relation (D2c)′ when $i = 1$, we have

$$(a_{1;n-1}a_1^{-1})^{-1}(a_{3;n-1}a_3^{-1})^{-1}(a_{2;n-1}a_2^{-1})^{-1}r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})$$

$$
\begin{aligned}
& r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1})r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1}) \\
= \quad & (a_{1;n-1}a_1^{-1})^{-1}(a_{3;n-1}a_3^{-1})^{-1}\underline{\Delta(x_3x_2)^{-1}\Delta(x_4^2)^{-1}}(a_{3;n-1}a_3^{-1}) \\
& \underline{r_{3;n-1}^{-1}\Delta(x_3x_2)r_{3;n-1}}\underline{\Delta(x_3^2)^{-1}\Delta(x_2^2)^{-1}}(a_{1;n-1}a_1^{-1}) \\
\overset{\text{(L-),(I)}}{=} \quad & (a_{1;n-1}a_1^{-1})^{-1}\underline{(a_{3;n-1}a_3^{-1})^{-1}\Delta((x_3x_2)^{-1}x_4^{-2})(a_{3;n-1}a_3^{-1})} \\
& \underline{\Delta(r_{3;n-1}^{-1}(x_3x_2))\Delta(x_3^{-2}x_2^{-2})}(a_{1;n-1}a_1^{-1})d_{n-1}^{-2} \\
\overset{\text{(I),(L+)}}{=} \quad & (a_{1;n-1}a_1^{-1})^{-1}\underline{\Delta((a_{3;n-1}a_3^{-1})^{-1}((x_3x_2)^{-1}x_4^{-2}))\Delta(r_{3;n-1}^{-1}(x_3x_2)x_3^{-2}x_2^{-2})} \\
& (a_{1;n-1}a_1^{-1})d_{n-1}^{-1} \\
\overset{\text{(L+)}}{=} \quad & (a_{1;n-1}a_1^{-1})^{-1}\Delta((a_{3;n-1}a_3^{-1})^{-1}((x_3x_2)^{-1}x_4^{-2})r_{3;n-1}^{-1}(x_3x_2)x_3^{-2}x_2^{-2}) \\
& (a_{1;n-1}a_1^{-1}) \\
\overset{\text{(I)}}{=} \quad & \Delta((a_{1;n-1}a_1^{-1})^{-1}((a_{3;n-1}a_3^{-1})^{-1}((x_3x_2)^{-1}x_4^{-2})r_{3;n-1}^{-1}(x_3x_2)x_3^{-2}x_2^{-2})) \\
= \quad & \Delta(b(x_1^2)) \\
\overset{\text{(I)}}{=} \quad & b(r_{1;n-1})b^{-1}.
\end{aligned}
$$

Thus $\varepsilon = 0$ for Relation (D2c)′ when $i = 1$.

For Relation (D2c)′ when $i = 3$, we have

$$
\begin{aligned}
& \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1}) \\
& r_{2;n-1}^{-1}(a_{1;n-1}a_1^{-1})r_{1;n-1}^{-1}(a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1}(a_{3;n-1}a_3^{-1})^{-1} \\
& (a_{1;n-1}a_1^{-1})^{-1}\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \\
= \quad & \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1}) \\
& \underline{r_{2;n-1}^{-1}\Delta(x_2x_1)r_{2;n-1}}\underline{\Delta(x_2^2)^{-1}\Delta(x_1^2)^{-1}}(a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1}(a_{3;n-1}a_3^{-1})^{-1} \\
& \Delta(x_2x_1)^{-1}\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \\
\overset{\text{(I),(L-)}}{=} \quad & \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1}) \\
& \underline{\Delta(r_{2;n-1}^{-1}(x_2x_1))\Delta(x_2^{-2}x_1^{-2})}(a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1}(a_{3;n-1}a_3^{-1})^{-1} \\
& \Delta(x_2x_1)^{-1}\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}d_{n-1}^{-1} \\
\overset{\text{(L+)}}{=} \quad & \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}r_{4;n-1}^{-1}\underline{(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1})} \\
& \underline{\Delta(r_{2;n-1}^{-1}(x_2x_1)x_2^{-2}x_1^{-2})(a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1}(a_{3;n-1}a_3^{-1})^{-1}} \\
& \Delta(x_2x_1)^{-1}\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \\
\overset{\text{(I)}}{=} \quad & \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1} \\
& \underline{r_{4;n-1}^{-1}\Delta(\{(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1})\}(r_{2;n-1}^{-1}(x_2x_1)x_2^{-2}x_1^{-2}))r_{4;n-1}} \\
& \underline{\Delta(x_4^2)^{-1}\Delta(x_2x_1)^{-1}}\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \\
\overset{\text{(I),(L-)}}{=} \quad & \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1} \\
& \underline{\Delta(\{r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1})\}(r_{2;n-1}^{-1}(x_2x_1)x_2^{-2}x_1^{-2}))} \\
& \underline{\Delta(x_4^{-2}(x_2x_1)^{-1})}\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{(L+)}}{=} && \{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}\\
&&& \Delta(\{r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1})\}(r_{2;n-1}^{-1}(x_2x_1)x_2^{-2}x_1^{-2})x_4^{-2}\\
&&& (x_2x_1)^{-1})\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}\\
&\overset{\text{(I)}}{=} && \Delta(\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1}(\{r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1})\}\\
&&& (r_{2;n-1}^{-1}(x_2x_1)x_2^{-2}x_1^{-2})x_4^{-2}(x_2x_1)^{-1}))\\
&= && \Delta(b(x_3^2))\\
&\overset{\text{(I)}}{=} && b(r_{3;n-1})b^{-1}.
\end{aligned}
$$

Thus $\varepsilon = 0$ for Relation (D2c)′ when $i = 3$.

For Relation (D2c)′ when $i = 4$, we have

$$
\begin{aligned}
&&& r_{4;n-1}(a_{2;n-1}a_2^{-1})r_{1;n-1}(a_{1;n-1}a_1^{-1})^{-1}r_{2;n-1}\\
&&& (a_{2;n-1}a_2^{-1})^{-1}r_{3;n-1}(a_{3;n-1}a_3^{-1})^{-1}r_{4;n-1}(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\\
&= && \Delta(x_4^2)\Delta(x_3x_2)\Delta(x_1^2)(a_{1;n-1}a_1^{-1})^{-1}\Delta(x_2^2)(a_{1;n-1}a_1^{-1})\Delta(x_2x_1)^{-1}\\
&&& (a_{2;n-1}a_2^{-1})^{-1}\Delta(x_3^2)(a_{2;n-1}a_2^{-1})\ \Delta(x_3x_2)^{-1}\Delta(x_4x_3)^{-1}\\
&&& r_{4;n-1}\Delta(x_4x_3)\Delta(x_2x_1)\\
&\overset{\text{(I),(L0)}}{=} && \Delta(x_4^2)\Delta(x_3x_2)\Delta(x_1^2)\Delta((a_{1;n-1}a_1^{-1})^{-1}(x_2^2))\Delta(x_2x_1)^{-1}\\
&&& \Delta((a_{2;n-1}a_2^{-1})^{-1}(x_3^2))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}\Delta(x_4x_3)\Delta(x_2x_1)\\
&\overset{\text{(L+)}}{=} && \Delta(x_4^2(x_3x_2)x_1^2)\Delta((a_{1;n-1}a_1^{-1})^{-1}(x_2^2))\Delta(x_2x_1)^{-1}\\
&&& \Delta((a_{2;n-1}a_2^{-1})^{-1}(x_3^2))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}\Delta(x_4x_3x_2x_1)d_{n-1}^3\\
&\overset{\text{(L-)}}{=} && \Delta(x_4^2(x_3x_2)x_1^2)\Delta((a_{1;n-1}a_1^{-1})^{-1}(x_2^2))\Delta(x_2x_1)^{-1}\\
&&& \Delta((a_{2;n-1}a_2^{-1})^{-1}(x_3^2)(x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}\Delta(x_4x_3x_2x_1)d_{n-1}^2\\
&\overset{\text{(L0)}}{=} && \Delta(x_4^2(x_3x_2)x_1^2)\Delta((a_{1;n-1}a_1^{-1})^{-1}(x_2^2))\Delta((x_2x_1)^{-1}\\
&&& (a_{2;n-1}a_2^{-1})^{-1}(x_3^2)(x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}\Delta(x_4x_3x_2x_1)d_{n-1}^2\\
&\overset{\text{(L-)}}{=} && \Delta(x_4^2(x_3x_2)x_1^2)\Delta((a_{1;n-1}a_1^{-1})^{-1}(x_2^2)(x_2x_1)^{-1}\\
&&& (a_{2;n-1}a_2^{-1})^{-1}(x_3^2)(x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}\Delta(x_4x_3x_2x_1)d_{n-1}\\
&\overset{\text{(L-)}}{=} && \Delta(x_4^2(x_3x_2)x_1^2(a_{1;n-1}a_1^{-1})^{-1}(x_2^2)(x_2x_1)^{-1}(a_{2;n-1}a_2^{-1})^{-1}(x_3^2)(x_3x_2)^{-1}\\
&&& (x_4x_3)^{-1})\Delta(x_4x_3x_2x_1)\ \Delta(x_4x_3x_2x_1)^{-1}\Delta(x_4^2)\Delta(x_4x_3x_2x_1)\\
&\overset{\text{(L+),(I)}}{=} && \Delta(x_4^2(x_3x_2)x_1^2(a_{1;n-1}a_1^{-1})^{-1}(x_2^2)(x_2x_1)^{-1}(a_{2;n-1}a_2^{-1})^{-1}(x_3^2)(x_3x_2)^{-1}\\
&&& (x_4x_3)^{-1}x_4x_3x_2x_1)\Delta(\Delta(x_4x_3x_2x_1)^{-1}(x_4^2))d_{n-1}\\
&\overset{\text{(L-)}}{=} && \Delta(x_4^2(x_3x_2)x_1^2(a_{1;n-1}a_1^{-1})^{-1}(x_2^2)(x_2x_1)^{-1}(a_{2;n-1}a_2^{-1})^{-1}(x_3^2)(x_3x_2)^{-1}\\
&&& (x_4x_3)^{-1}x_4x_3x_2x_1\Delta(x_4x_3x_2x_1)^{-1}(x_4^2))\\
&= && \Delta(b(x_4^2))\\
&\overset{\text{(I)}}{=} && b(r_{4;n-1})b^{-1}.
\end{aligned}
$$

Thus $\varepsilon = 0$ for Relation (D2c)′ when $i = 4$.

For Relation (D2d)′ when $m = i-1$, we have

$$
\begin{aligned}
& \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}(a_{i-1;n-1}a_{i-1}^{-1})(s_{l,n-1}d_l^{-1})r_{i;n-1}r_{i-1;n-1} \\
& (a_{i-1;n-1}a_{i-1}^{-1})^{-1}r_{i;n-1}(\bar{s}_{l,n-1;i}d_l^{-1})\{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\} \\
= \; & \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}(a_{i-1;n-1}a_{i-1}^{-1})\underline{\Delta(y_l)\Delta(x_i^2)\Delta(x_{i-1}^2)} \\
& (a_{i-1;n-1}a_{i-1}^{-1})^{-1}\Delta(x_i^2)(\bar{s}_{l,n-1;i}d_l^{-1})\{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\} \\
\overset{\text{(L+)}}{=} \; & \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}\underline{(a_{i-1;n-1}a_{i-1}^{-1})\Delta(y_l x_i^2 x_{i-1}^2)(a_{i-1;n-1}a_{i-1}^{-1})^{-1}} \\
& \Delta(x_i^2)(\bar{s}_{l,n-1;i}d_l^{-1})\{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}d_{n-1}^2 \\
\overset{\text{(I)}}{=} \; & \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}\Delta((a_{i-1;n-1}a_{i-1}^{-1})(y_l x_i^2 x_{i-1}^2))\Delta(\bar{y}_{l;i}) \\
& \underline{(\bar{s}_{l,n-1;i}d_l^{-1})^{-1}\Delta(x_i^2)(\bar{s}_{l,n-1;i}d_l^{-1})}\{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}d_{n-1}^2 \\
\overset{\text{(I)}}{=} \; & \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}\underline{\Delta((a_{i-1;n-1}a_{i-1}^{-1})(y_l x_i^2 x_{i-1}^2))\Delta(\bar{y}_{l;i})} \\
& \Delta((\bar{s}_{l,n-1;i}d_l^{-1})^{-1}(x_i^2))\{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}d_{n-1}^2 \\
\overset{\text{(L-)}}{=} \; & \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}\underline{\Delta((a_{i-1;n-1}a_{i-1}^{-1})(y_l x_i^2 x_{i-1}^2)\bar{y}_{l;i})} \\
& \underline{\Delta((\bar{s}_{l,n-1;i}d_l^{-1})^{-1}(x_i^2))}\{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}d_{n-1} \\
\overset{\text{(L-)}}{=} \; & \{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}\Delta((a_{i-1;n-1}a_{i-1}^{-1})(y_l x_i^2 x_{i-1}^2)\bar{y}_{l;i} \\
& (\bar{s}_{l,n-1;i}d_l^{-1})^{-1}(x_i^2))\{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\} \\
\overset{\text{(I)}}{=} \; & \Delta(\{(s_{l,n-1}d_l^{-1})(a_{i-1;n-1}a_{i-1}^{-1})\}^{-1}((a_{i-1;n-1}a_{i-1}^{-1})(y_l x_i^2 x_{i-1}^2)\bar{y}_{l;i} \\
& (\bar{s}_{l,n-1;i}d_l^{-1})^{-1}(x_i^2))) \\
= \; & \Delta(a_{i-1;n-1}(x_i^2)) \\
\overset{\text{(I)}}{=} \; & a_{i-1;n-1}(r_{i;n-1})a_{i-1;n-1}^{-1}.
\end{aligned}
$$

Thus $\varepsilon = 0$ for Relation (D2d)′ when $m = i-1$.

For Relation (D2g)′ when $i = 1$, we have

$$
\begin{aligned}
& [s_{t,n-1}d_t^{-1},(\bar{s}_{l,n-1}d_l^{-1})^{-1}][r_{1;n-1}(\bar{s}_{t,n-1}d_t^{-1})r_{1;n-1}^{-1},(s_{l,n-1}d_l^{-1})^{-1}]r_{1;n-1} \\
= \; & \underline{(s_{t,n-1}d_t^{-1})\Delta(\bar{y}_l)^{-1}(s_{t,n-1}d_t^{-1})^{-1}}\Delta(\bar{y}_l)\Delta(x_1^2)(\bar{s}_{t,n-1}d_t^{-1}) \\
& \underline{r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}\Delta(\bar{y}_t)^{-1}r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})r_{1;n-1}} \\
\overset{\text{(I)}}{=} \; & \Delta((s_{t,n-1}d_t^{-1})(\bar{y}_l))^{-1}\Delta(\bar{y}_l)\Delta(\bar{y}_t)\underline{(\bar{s}_{t,n-1}d_t^{-1})^{-1}\Delta(x_1^2)(\bar{s}_{t,n-1}d_t^{-1})} \\
& \Delta(r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}(\bar{y}_t))^{-1} \\
\overset{\text{(I)}}{=} \; & \Delta((s_{t,n-1}d_t^{-1})(\bar{y}_l))^{-1}\underline{\Delta(\bar{y}_l)\Delta(\bar{y}_t)}\Delta((\bar{s}_{t,n-1}d_t^{-1})^{-1}(x_1^2)) \\
& \Delta(r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}(\bar{y}_t))^{-1} \\
\overset{\text{(L-)}}{=} \; & \Delta((s_{t,n-1}d_t^{-1})(\bar{y}_l))^{-1}\underline{\Delta(\bar{y}_l\bar{y}_t)\Delta((\bar{s}_{t,n-1}d_t^{-1})^{-1}(x_1^2))} \\
& \Delta(r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}(\bar{y}_t))^{-1}d_{n-1}^{-1} \\
\overset{\text{(L-)}}{=} \; & \underline{\Delta((s_{t,n-1}d_t^{-1})(\bar{y}_l))^{-1}\Delta(\bar{y}_l\bar{y}_t(\bar{s}_{t,n-1}d_t^{-1})^{-1}(x_1^2))}
\end{aligned}
$$

$$
\begin{aligned}
&\Delta(r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}(\bar{y}_t))^{-1}d_{n-1}^{-2}\\
\overset{\text{(L+)}}{=}\ & \frac{\Delta((s_{t,n-1}d_t^{-1})(\bar{y}_l)^{-1}\bar{y}_l\bar{y}_t(\bar{s}_{t,n-1}d_t^{-1})^{-1}(x_1^2))}{\Delta(r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}(\bar{y}_t))^{-1}}d_{n-1}^{-1}\\
\overset{\text{(L+)}}{=}\ & \Delta((s_{t,n-1}d_t^{-1})(\bar{y}_l)^{-1}\bar{y}_l\bar{y}_t(\bar{s}_{t,n-1}d_t^{-1})^{-1}(x_1^2)r_{1;n-1}^{-1}(s_{l,n-1}d_l^{-1})^{-1}r_{1;n-1}(\bar{y}_t)^{-1})\\
=\ & \Delta(\bar{s}_{l,t}(x_1^2))\\
\overset{\text{(I)}}{=}\ & \bar{s}_{l,t}(r_{1;n-1})\bar{s}_{l,t}^{-1}.
\end{aligned}
$$

Thus $\varepsilon = 0$ for Relation (D2g)′ when $i = 1$.Therefore we have completed the proof of Lemma 5.9. □

As a corollary of the proof of Lemma 5.9 and 5.12, we have the following proposition.

Proposition 5.13. *Relations* (D0), (D1a)-(D4g) *of the finite presentation for* $\mathcal{M}(N_{g,n})$ *in* Proposition 3.2 *are obtained from Relations* (I) *and* (III) *in* Theorem 4.1.

Proof. Relations (D0) in Proposition 3.2 are clearly obtained from Relations (I) in Theorem 4.1. We first consider Relations (D1a)-(D4g) for $k = n - 1$. We remark that Relations (D1a)-(D4g) for $k = n - 1$ coincide with the relations which are obtained from Relations (D1a)′-(D4h)′ in Lemma 5.8 with the indices as in Lemma 5.9. For Relations (D1a)-(D4g) which are obtained from some of Relations (D1a)′-(D4h)′ listed in Lemma 5.12, by the argument in the proof of Lemma 5.12, we show that these relations are obtained from Relations (I). For the other relations, that are Relations (D1a)-(D4g) which are obtained from Relations (D1a)′-(D4h)′ discussed in the proof of Lemma 5.9, by the argument and deformations of the expressions in the proof of Lemma 5.9, we show that these relations are obtained from Relations (I), (L+), (L-), and (L0). By Lemma 5.2 and 5.4, we show that Relations (L0) are obtained from Relations (I) in Theorem 4.1 and Relations (L+) and (L-) coincide with Relations (III) in Theorem 4.1.

We take any $1 \leq k' \leq n - 2$. Let $N(k')$ be the subsurface of $N_{g,n}$ as in Figure 21, and $N'(k')$ the surface which is obtained by regluing $N(k')$ and the 2-disk $D_{g+k'}$ with the base point x_0'. Since $N(k')$ and $N'(k')$ are diffeomorphic to $N_{g,k'+1}$ and $N_{g,k'}$, respectively, we regard the inclusion relation $N(k') \subset N'(k')$ as $N_{g,k'+1} \subset N_{g,k'}$. The mapping classes a_i, y, b, d_l, $a_{i,l}$, $r_{i,l}$, $s_{i,l}$, $\bar{s}_{i,l}$, $\bar{s}_{j,l;i} \in \mathcal{M}(N_{g,n})$ for $1 \leq l \leq k'$ are represented by diffeomorphisms on $N_{g,n}$ which are supported on $N(k') = N_{g,k'+1}$. Thus Relations (D1a)-(D4g) for $k = k'$ of the presentation for $\mathcal{M}(N_{g,n})$ in Proposition 3.2 are regarded as relations in $\mathcal{M}(N_{g,k'+1}) \subset \mathcal{M}(N_{g,n})$. These relations clearly coincide with Relations (D1a)-(D4g) for $k = k'$ of the presentation for $\mathcal{M}(N_{g,k'+1})$ in Proposition 3.2, and also the relations which are obtained from Relations (D1a)′-(D4g)′ of the presentation for $\mathcal{M}^+(N_{g,k'}, x_0')$ in Lemma 5.8 with the indices as in Lemma 5.9. By the argument in the proof of Lemma 5.9 and Lemma 5.12 for the case $n-1 = k'$, we show that Relations (D1a)-(D4g) for $k = k'$ of the presentation for $\mathcal{M}(N_{g,k'+1})$ in Proposition 3.2 are obtained from Relations (I) and (III) of the presentation for $\mathcal{M}(N_{g,k'+1})$ in Theorem 4.1. By the natural inclusion $\mathcal{M}(N_{g,k'+1}) \subset \mathcal{M}(N_{g,n})$, Relations (I) and (III) in $\mathcal{M}(N_{g,k'+1})$ are to be Relations (I) and (III) in $\mathcal{M}(N_{g,n})$. Therefore we have completed the proof of Proposition 5.13. □

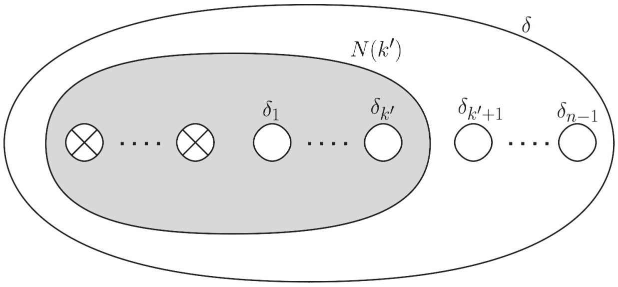


Fig. 21. The subsurface $N(k')$ of $N_{g,n}$ which is diffeomorphic to $N_{g,k'+1}$.

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References

[1] J.S. Birman: *Mapping class groups and their relationship to braid groups*, Comm. Pure Appl. Math. **22** (1969), 213–238.
[2] J.S. Birman and D.R.J. Chillingworth: *On the homeotopy group of a non-orientable surface*, Proc. Camb. Philos. Soc. **71** (1972), 437–448.
[3] D.B.A. Epstein: *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966), 83-107.
[4] B. Farb and D. Margalit: A Primer on Mapping Class Groups, Princeton University Press, Princeton, NJ, 2012.
[5] S. Gervais: *Presentation and central extensions of mapping class groups*, Trans. Amer. Math. Soc. **348** (1996), 3097–3132.
[6] S. Gervais: *A finite presentation of the mapping class group of a punctured surface*, Topology **40** (2001), 703–725.
[7] L. Harer: *The second homology group of the mapping class groups of orientable surfaces*, Invent. Math. **72** (1983), 221–239.
[8] A. Hatcher and W. Thurston: *A presentation for the mapping class group of a closed orientable surface*, Topology **19** (1980), 221–237.
[9] D.L. Johnson: *The structure of the Torelli group. I. A finite set of generators for $\mathcal{I}$*, Ann. of Math. (2) **118** (1983), 423–442.
[10] D.L. Johnson: Presentations of Groups, London Math. Soc. Stud. Texts **15**, Cambridge University Press, Cambridge, 1990.
[11] M. Korkmaz: *Mapping class groups of nonorientable surfaces*, Geom. Dedicata. **89** (2002), 109–133.
[12] C. Labruère and L. Paris: *Presentations for the punctured mapping class groups in terms of Artin groups*, Algebr. Geom. Topol. **1** (2001), 73–114.
[13] W.B.R. Lickorish: *Homeomorphisms of non-orientable two-manifolds*, Proc. Camb. Philos. Soc. **59** (1963), 307–317.
[14] W.B.R. Lickorish: *On the homeomorphisms of a non-orientable surface*, Proc. Camb. Philos. Soc. **61** (1965), 61–64.
[15] F. Luo: *A presentation of the mapping class groups*, Math. Res. Lett. **4** (1997), 735–739.
[16] G. Omori: *An infinite presentation for the mapping class group of a non-orientable surface*, Algebr. Geom. Topol. **17** (2017), 419–437.
[17] L. Paris and B. Szepietowski: *A presentation for the mapping class group of a nonorientable surface*, Bull. Soc. Math. France **143** (2015), 503–566.

[18] M. Stukow: *Dehn twists on nonorientable surfaces*, Fund. Math. **189** (2006), 117–147.
[19] M. Stukow: *Commensurability of geometric subgroups of mapping class groups*, Geom. Dedicata **143** (2009), 117–142.
[20] M. Stukow: *A finite presentation for the mapping class group of a nonorientable surface with Dehn twists and one crosscap slide as generators*, J. Pure Appl. Algebra **218** (2014), 2226–2239.
[21] B. Szepietowski: *Crosscap slides and the level 2 mapping class group of a nonorientable surface*, Geom. Dedicata **160** (2012), 169–183.
[22] B. Wajnryb: *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. **45** (1989), 157–174.

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