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GENUS ZERO PALF STRUCTURES ON THE AKBULUT-YASUI PLUGS

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Abstract

We construct a genus zero PALF structure on each of plugs introduced by Akbulut and Yasui and describe the monodromy as a positive factorization in the mapping class group of a fiber.

1. Introduction

The problem of classifying all differential structures defined on a given 4-manifold is an important problem in understanding the overall picture of a 4-manifold.

Akbulut and Yasui [4] introduced corks and plugs. Corks and plugs are compact Stein surfaces. Matveyev, Curtis-Freedman-Hsiang-Stong, and Akbulut-Matveyev's theorem show that the study of exotic manifold pairs constructed using cork is important for the classification problem of the differential structure of 4-manifolds.

Theorem 1.1 (Matveyev [12], Curtis-Freedman-Hsiang-Stong [7], Akbulut-Matveyev [2])**.** *For every homeomorphic but non-diffeomorphic pair of simply connected closed 4-manifolds, one is obtained from the other by removing a contractible 4-manifold and gluing it via an involution on the boundary. Such a contractible 4-manifold has since been called a Cork. Furthermore, corks and their complements can always be made compact Stein 4-manifolds.*

It is shown by Akbulut and Yasui [6] using cork that an infinite number of exotic Stein surface pairs embedded in X exist for any four-dimensional two-handle body X with $b_2(X) \geq 1$.

The plug generalizes the Gluck twist. The plug is also used to make exotic manifolds, as well as cork.

On the other hand, Loi and Piergallini [11] proved that every compact Stein surface admits a positive allowable Lefschetz fibration over D^2 (a PALF for short). Therefore we can investigate compact Stein surfaces in terms of positive factorizations in mapping class groups (see also Akbulut and Ozbagci [3], Akbulut and Arikan [1]).

Since corks and plugs are Stein surfaces, the study of the relationship between Stein surfaces and mapping class groups using PALFs plays an important role in classifying differential structures on 4-manifolds.

If a PALF is created from a Stein surface by the existing method ([11], [3], [1]), its genus will be large, and it will be complicated and difficult to handle as a mapping class

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group element.

Gompf [8] indicates that the Stein surface is compatible with Kirby calculus. In this paper, we use Kirby calculus to construct PALFs on Akbulut-Yasui plugs realizing the smallest possible fiber genera.

One planar (i.e. genus zero) PALF on the Akbulut cork was made in the previous paper of the author [14], but in this paper, we made an infinite number of planar PALFs on the Akbulut-Yasui plugs. Being planar is also playing an important role in [10].

In this paper, we construct a genus zero PALF structure on each of plugs introduced by Akbulut and Yasui [4] and describe the monodromy as a positive factorization in the mapping class group of a fiber.

Theorem 1.2. *For any $m \geq 1, n \geq 2$, the Akbulut-Yasui plug $(W_{m,n}, f_{m,n})$ admits a genus zero PALF structure. The monodromy of the PALF is described by the factorization $t_{\alpha_{2n+m}} \cdots t_{\alpha_1}$, where t_α is a right-handed Dehn twist along a simple closed curve α on a fiber and $\alpha_{2n+m}, \ldots, \alpha_1$ are simple closed curves shown in* Figure 2.

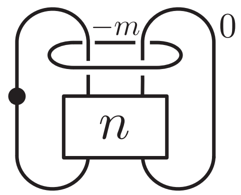


Fig. 1. Kirby diagram for $W_{m,n}$

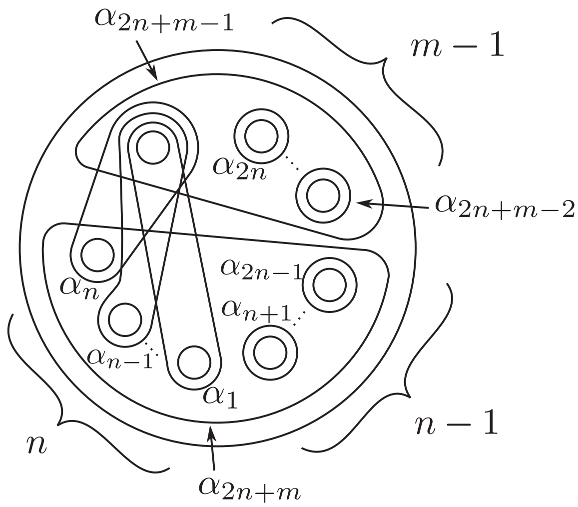


Fig. 2. Vanishing cycles of a genus zero PALF on $W_{m,n}$.

Note that the genus of a PALF on the manifold $W_{m,n}$ in a known way (cf. [3] and [1]) is much more than zero. We obtained similar results for the Akbulut cork W_1 [14]. In the present paper, we construct a genus zero PALF on an infinite number of the Akbulut-Yasui plugs.

2. Preliminaries

2.1. Mapping class groups. In this subsection, we review a precise definition of the mapping class groups of surfaces with boundary and that of Dehn twists along simple closed curves on surfaces.

Definition 2.1. Let F be a compact oriented connected surface with boundary. Let $\mathrm{Diff}^+(F, \partial F)$ be the group of all orientation-preserving self-diffeomorphisms of F fixing the boundary ∂F point-wise. Let $\mathrm{Diff}_0^+(F, \partial F)$ be the subgroup of $\mathrm{Diff}^+(F, \partial F)$ consisting of self-diffeomorphisms isotopic to the identity. The quotient group $\mathrm{Diff}^+(F, \partial F)/\ \mathrm{Diff}_0^+(F, \partial F)$ is called the mapping class group of F and it is denoted by $\mathrm{Map}(F, \partial F)$.

Definition 2.2. A *positive* (*or right-handed*) *Dehn twist* along a simple closed curve α, $t_\alpha : F \to F$ is a diffeomorphism obtained by cutting F along α, twisting $360°$ to the right and regluing.

2.2. PALF.

Definition 2.3. Let M^4 and B^2 be compact oriented smooth manifolds of dimensions 4 and 2. Let $f : M \to B$ be a smooth map. f is called a *positive Lefschetz fibration* over B if it satisfies the following conditions (1) and (2):

(1) There are finitely many critical values $b_1, \ldots, b_m$ of f in the interior of B and there is a unique critical point p_i on each fiber $f^{-1}(b_i)$, and

(2) The map f is locally written as $f(z_1, z_2) = z_1^2 + z_2^2$ with respect to some local complex coordinates around p_i and b_i compatible with the orientations of M and B.

Definition 2.4. A positive Lefschetz fibration is called *allowable* if its all vanishing cycles are homologically non-trivial on the fiber. A positive allowable Lefschetz fibration over D^2 with bounded fibers is called a *PALF* for short.

The following Lemma is useful to prove Theorem 1.2.

Lemma 2.5 (cf. Akbulut-Ozbagci [3, Remark 1])**.** *Suppose that a* 4*-manifold X admits a PALF. If a* 4*-manifold Y is obtained from X by attaching a Lefschetz* 2*-handle, then Y also admits a PALF.*

The Lefschetz 2-handle is defined as follows.

Definition 2.6. Suppose that X admits a PALF. A *Lefschetz* 2*-handle* is a 2-handle attached along a homologically non-trivial simple closed curve in the boundary of X with framing -1 relative to the product framing induced by the fiber structure.

2.3. Stein surfaces. In this subsection, we recall a definition of the Stein surfaces. The question of which smooth 4-manifolds admit Stein structures can be completely reduced to a problem in handlebody theory.

Definition 2.7. A complex manifold is called a *Stein manifold* if it admits a proper biholomorphic embedding to $\mathbb{C}^n$.

DEFINITION 2.8. Let W be a compact manifold with boundary. The manifold W is called a *Stein domain* if it satisfies following condition: There is a Stein manifold X and a plurisubharmonic function $\varphi : X \to [0, \infty)$ such that $W = \varphi^{-1}([0, a])$ for a regular value a of φ.

DEFINITION 2.9. A Stein manifold or a Stein domain is called a *Stein surface* if its complex dimension is 2.

2.4. Plugs. In this subsection, we give the definition of the plug.

DEFINITION 2.10 (AKBULUT-YASUI [4, Definition 2.2.]). Let P be a compact Stein 4-manifold with boundary and $\tau : \partial P \to \partial P$ an involution on the boundary, which cannot extend to any self-homeomorphism of P. We call (P, τ) a *Plug* of X, if $P \subset X$ and X keeps its homeomorphism type and changes its diffeomorphism type when removing P and gluing it via τ. We call (P, τ) a *Plug* if there exists a smooth 4-manifold X such that (P, τ) is a plug of X.

DEFINITION 2.11 (AKBULUT-YASUI [4, Definition 2.3.]). Let $W_{m,n}$ be a smooth 4-manifold given by Figure 1. Let $f_{m,n} : \partial W_{m,n} \to \partial W_{m,n}$ be the obvious involution obtained from first surgering $S^1 \times D^3$ to $D^2 \times S^2$ in the interiors of $W_{m,n}$, then surgering the other imbedded $D^2 \times S^2$ back to $S^1 \times D^2$ (i.e. replacing the dot in Figure 1).

Theorem 2.12 (Akbulut-Yasui [4, Theorem 2.5(2)]). *For $m \geq 1$ and $n \geq 2$, the pair $(W_{m,n}, f_{m,n})$ is a plug.*

3. Proof of Theorem 1.2.

In this section, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $F_{m,n}$ be the compact oriented surface of genus zero with $2n + m$ boundary components and $\alpha_1, \ldots, \alpha_{2n+m}$ the curves on $F_{m,n}$ shown in Figure 4 (a). Note that Figure 2 and Figure 4 (a) show the same PALF. We denote the right-handed Dehn twists along $\alpha_1, \ldots, \alpha_{2n+m}$ by $t_{\alpha_1}, \ldots, t_{\alpha_{2n+m}}$, respectively. Let $f : X_{m,n} \to D^2$ be a Lefschetz fibration over D^2 with monodromy representation $(t_{\alpha_{2n+m}}, \ldots, t_{\alpha_1})$. Since each curve α_i is homologically non-trivial on $F_{m,n}$, we see that f is a PALF with fiber $F_{m,n}$.

We now show that $X_{m,n}$ is diffeomorphic to $W_{m,n}$.

The Kirby diagram for $X_{m,n}$ corresponding to the monodromy representation $(t_{\alpha_{2n+m}}, \ldots,$

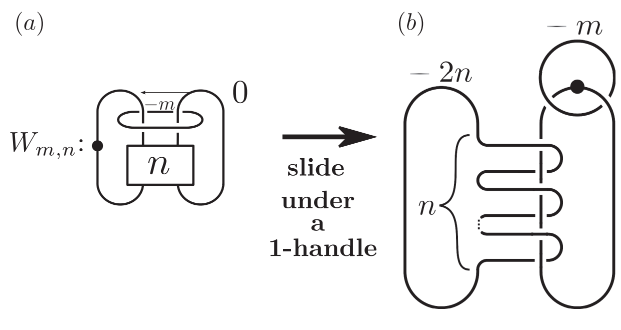
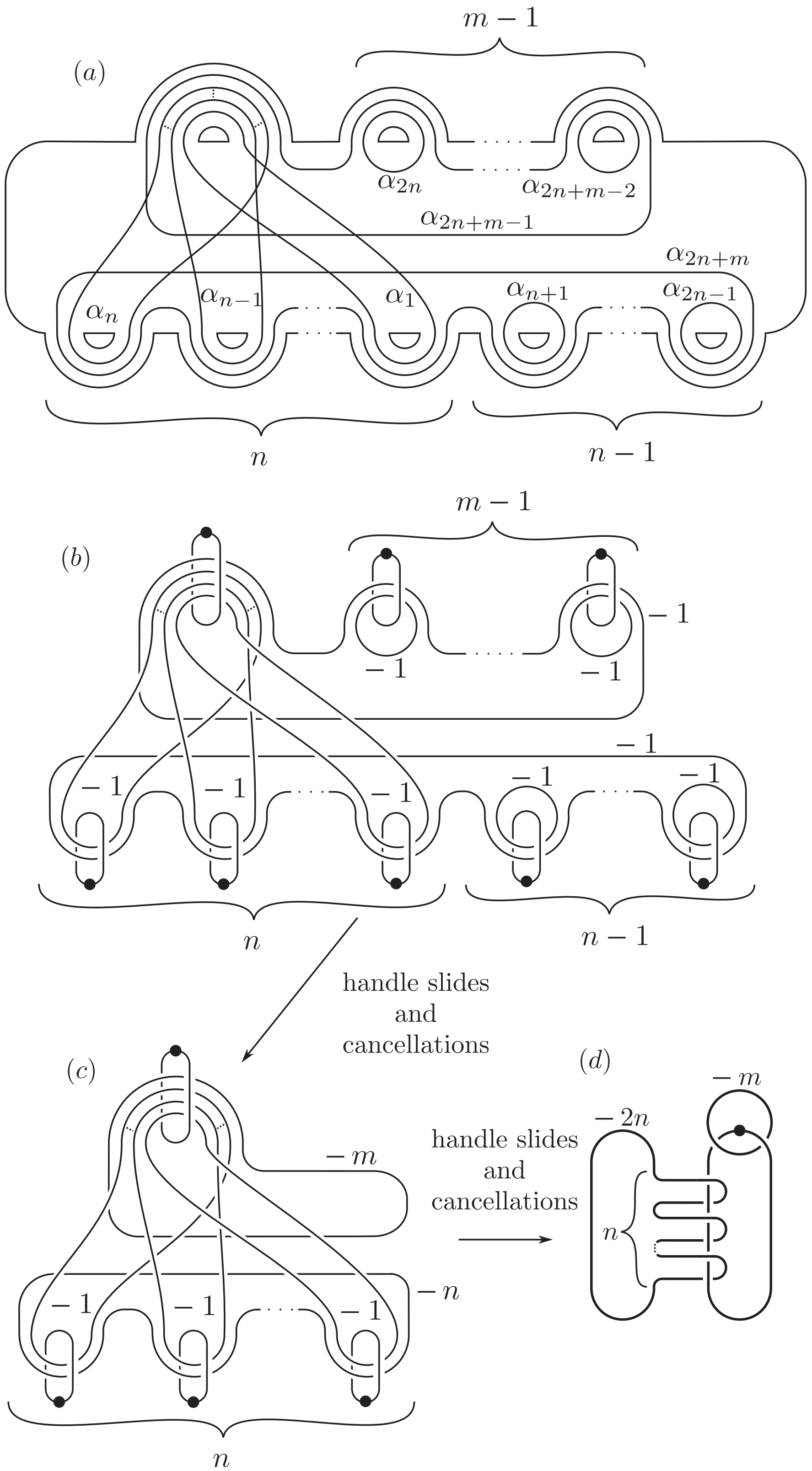


Fig. 3



$m-1$
(a)
α_{2n}
α_{2n+m-2}
α_{2n+m-1}
α_{2n+m}
α_n
α_{n-1}
α_1
α_{n+1}
α_{2n-1}
n
$n-1$
$m-1$
(b)
-1
n
$n-1$
handle slides and cancellations
(c)
$-m$
$-n$
-1
n
(d)
$-m$
$-2n$
handle slides and cancellations
n

Fig. 4

t_{α_1}) is given by Figure 4 (b). We slide the -1-framed 2-handles over -1-framed 2-handles and erase canceling 1-handle/2-handle pairs to get Figure 4 (c). We get Figure 4 (d) by sliding the $-m$-framed 2-handle over -1-framed 2-handles and sliding the $-n$-framed 2-handle over -1-framed 2-handles and erasing canceling 1-handle/2-handle pairs.

The Kirby diagram for $W_{m,n}$ is given by Figure 3 (a). We slide the 0-framed 2-handle under the 1-handle to get Figure 3 (b).

Since Figure 3 (b) and Figure 4 (d) are the same, we conclude that $X_{m,n}$ is diffeomorphic to $W_{m,n}$, which implies the theorem. $\square$

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