Inaba, K.
Osaka J. Math.
59 (2022), 403–416

JOIN THEOREM FOR REAL ANALYTIC SINGULARITIES

Kazumasa INABA

(Received November 6, 2020, revised February 9, 2021)

Abstract

Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^p, \mathbf{0}_p)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^p, \mathbf{0}_p)$ be analytic germs of independent variables, where $n, m \geq p \geq 2$. In this paper, we assume that f_1, f_2 and $f = f_1 + f_2$ satisfy a_f-condition. Then we show that the tubular Milnor fiber of f is homotopy equivalent to the join of tubular Milnor fibers of f_1 and f_2. If $p = 2$, the monodromy of the tubular Milnor fibration of f is equal to the join of the monodromies of the tubular Milnor fibrations of f_1 and f_2 up to homotopy.

1. Introduction

Let $g : (\mathbb{R}^N, \mathbf{0}_N) \to (\mathbb{R}^p, \mathbf{0}_p)$ be an analytic germ, where $N \geq p \geq 2, \mathbf{0}_N$ and $\mathbf{0}_p$ are the origins of $\mathbb{R}^N$ and $\mathbb{R}^p$ respectively. Take a positive real number ε_0 sufficiently small if necessary. Assume that for any $0 < \varepsilon \leq \varepsilon_0$, there exists a positive real number δ such that $\delta \ll \varepsilon$ and

$$g : B^N_\varepsilon \cap g^{-1}(D^p_\delta \setminus \{\mathbf{0}_p\}) \to D^p_\delta \setminus \{\mathbf{0}_p\}$$

is a locally trivial fibration, where $B^N_\varepsilon = \{\mathbf{x} \in \mathbb{R}^N \mid \|\mathbf{x}\| \leq \varepsilon\}$ and $D^p_\delta = \{\mathbf{w} \in \mathbb{R}^p \mid \|\mathbf{w}\| \leq \delta\}$. In this paper, B^N_ε is used for the disk in the defining euclidean space. The isomorphism class of the above fibration does not depend on the choice of ε and δ. This map is called the *tubular Milnor fibration of* g. If f_1 and f_2 are holomorphic functions of independent variables, the following theorem is known.

Theorem 1 (Join theorem)**.** *Let $f_1 : (\mathbb{C}^n, \mathbf{0}_{2n}) \to (\mathbb{C}, \mathbf{0}_2)$ and $f_2 : (\mathbb{C}^m, \mathbf{0}_{2m}) \to (\mathbb{C}, \mathbf{0}_2)$ be holomorphic functions of independent variables $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{w} = (w_1, \dots, w_m)$. Set $f(\mathbf{z}, \mathbf{w}) = f_1(\mathbf{z}) + f_2(\mathbf{w})$. Then the Milnor fiber of f is homotopy equivalent to the join of the Milnor fibers of f_1 and f_2 and the monodromy of f is equal to the join of the monodromies of f_1 and f_2 up to homotopy.*

Join theorem is algebraically proved by M. Sebastiani and R. Thom for isolated singularities [31]. M. Oka showed this for weighted homogeneous singularities [23]. For general complex singularities, this is proved by K. Sakamoto [30]. In [14], L. H. Kauffman and W. D. Neumann studied fiber structures and Seifert forms of links defined by tame isolated singularities of real analytic germs of independent variables. In this paper, we study Join theorem for more general real analytic singularities.

To show the existence of Milnor fibrations for real analytic singularities, we consider stratifications of analytic sets. Let ε be a small positive real number. Let $g : (\mathbb{R}^N, \mathbf{0}_N) \to$

2020 Mathematics Subject Classification. Primary 32S55; Secondary 32S40, 57K45.

$(\mathbb{R}^p, \mathbf{0}_p)$ be a smooth map and $\mathcal{S}$ be a stratification of $B^N_\varepsilon \cap g^{-1}(\mathbf{0}_p)$. The map g satisfies *a_f-condition* if $B^N_\varepsilon \setminus g^{-1}(\mathbf{0}_p)$ has no critical point and satisfies the following condition: For any sequence $p_\nu \in B^N_\varepsilon \setminus g^{-1}(\mathbf{0}_p)$ such that

$$T_{p_\nu} g^{-1}(g(p_\nu)) \to \tau, \quad p_\nu \to p_\infty \in M,$$

where $M \in \mathcal{S}$, we have $T_{p_\infty} M \subset \tau$. A stratification $\mathcal{S}$ is called *Whitney (a)-regular* if for any pair of strata (S_1, S_2) of $\mathcal{S}$ and any point $p \in S_1 \cap \overline{S_2}$, (S_1, S_2) satisfies the following condition: For any sequence $q_\nu \in S_2$ satisfying

$$q_\nu \to p, \quad T_{q_\nu} S_2 \to T,$$

we have $T_p S_1 \subset T$. We say ε is *an a_f-stable radius for g with respect to $\mathcal{S}$* if it satisfies the following: Each sphere $S^{N-1}_{\varepsilon'}, 0 < \varepsilon' \leq \varepsilon$ intersects transversely with any stratum of $\mathcal{S}$ and $\mathbf{0}_p$ is the only critical value of $g : B^N_\varepsilon \to \mathbb{R}^p$.

Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^p, \mathbf{0}_p)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^p, \mathbf{0}_p)$ be analytic germs, where $n, m \geq p \geq 2$. Set $V(f_1) = f_1^{-1}(\mathbf{0}_p) \cap B^n_\varepsilon$ and $V(f_2) = f_2^{-1}(\mathbf{0}_p) \cap B^m_\varepsilon$ for $0 < \varepsilon \ll 1$. We denote a stratification of $V(f_1)$ (resp. $V(f_2)$) by $\mathcal{S}_1$ (resp. $\mathcal{S}_2$). Assume that f_1 and f_2 satisfy the following conditions:

(a-i) $V(f_j)$ has codimension p at the origin, f_j has an isolated value at the origin and f_j is locally surjective on $V(f_j)$ near the origin for $j = 1, 2$,

(a-ii) f_j satisfies a_f-condition with respect to $\mathcal{S}_j$ for $j = 1, 2$.

Here $g : (\mathbb{R}^N, \mathbf{0}_N) \to (\mathbb{R}^p, \mathbf{0}_p)$ is *locally surjective near the origin* if there exists a positive real number ε so that for any $\mathbf{x} \in V(g) \cap B^N_\varepsilon$, there exists an open neighborhood W of $\mathbf{x}$ so that $\mathbf{0}_p$ is an interior point of the image $g(W)$. Since $V(f_1)$ and $V(f_2)$ are real analytic sets, we may assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are Whitney stratifications. See [10] for further information. We take ε sufficiently small if necessary. Then the sphere ∂B^n_ε (resp. ∂B^m_ε) intersects M_1 (resp. M_2) transversely for any $M_1 \in \mathcal{S}_1$ and $M_2 \in \mathcal{S}_2$. See [19, Corollary 2.9] and the proof of [3, Lemma 3.2].

Take a common a_f-stable radius ε for f_1 and f_2 and take a sufficiently small δ, $0 < \delta \ll \varepsilon$ so that $f_j^{-1}(\eta)$ intersects transversely with the sphere of radius ε for $j = 1, 2$. See Lemma 1 below for the existence of such a δ. Hereafter we use $0 < \delta \ll \varepsilon$ in this sense.

We assume that ε is a common a_f-stable radius for f_j with respect to $\mathcal{S}_j$ for $j = 1, 2$ and take $U_1 = U_1(\varepsilon, \delta)$ and $U_2 = U_2(\varepsilon, \delta)$ for $0 < \delta \ll \varepsilon$. Here $U_j(\varepsilon, \delta) = \{\mathbf{x} \in B^{n_j}_\varepsilon \mid \|f_j(\mathbf{x})\| \leq \delta\}$ with $n_1 = n$ and $n_2 = m$. By the above conditions and the Ehresmann fibration theorem [33], we may assume that

$$f_j : U_j \setminus V(f_j) \to D^p_\delta \setminus \{\mathbf{0}_p\}$$

is a locally trivial fibration for $j = 1, 2$. We call these fibrations *stable tubular Milnor fibrations of f_j* for $j = 1, 2$.

Let $f : (\mathbb{R}^n \times \mathbb{R}^m, \mathbf{0}_{n+m}) \to (\mathbb{R}^p, \mathbf{0}_p)$ be the analytic germ defined by $f = f_1 + f_2$. Put $V(f) = f^{-1}(0) \cap (U_1 \times U_2)$. By [1, Proposition 5.2], f also satisfies the conditions (a-i) and (a-ii) with respect the stratification $\mathcal{S}$ for f which will be defined in Section 2. See Section 2.1. The main theorem of this paper is the following.

Theorem 2. *Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^p, \mathbf{0}_p)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^p, \mathbf{0}_p)$ be analytic germs of independent variables, where $n, m \geq p \geq 2$. Assume that f_1 and f_2 satisfy the conditions* (a-i) *and* (a-ii). *Set* $f = f_1 + f_2$. *Then the fiber of the tubular Milnor fibration of* f *is homotopy equivalent to the join of the fibers of the tubular Milnor fibrations of* f_1 *and* f_2.

Moreover, if $p = 2$, *the monodromy of the tubular Milnor fibration of* f *is equal to the join of the monodromies of* f_1 *and* f_2 *up to homotopy.*

Moreover, we assume that f_1, f_2 and f satisfy the following condition:

(a-iii) there exists a positive real number r' such that

$$P/|P| : \partial B_r^N \setminus K_P \to S^{p-1}$$

is a locally trivial fibration and this fibration is isomorphic to the tubular Milnor fibration of P, where $K_P = \partial B_r^N \cap P^{-1}(0)$ and $0 < r \leq r'$ for $(P, N) = (f_1, n), (f_2, m), (f, n+m)$.

The fibration in (a-iii) is called *the spherical Milnor fibration of* P. By using Theorem 2 and the condition (a-iii), we have

Corollary 1. *Let* $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^p, \mathbf{0}_p)$ *and* $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^p, \mathbf{0}_p)$ *be analytic germs in* Theorem 2. *Assume that* f_1, f_2 *and* $f = f_1 + f_2$ *satisfy the condition (a-iii). Then the fiber of the spherical Milnor fibration of* f *is homotopy equivalent to the join of the fibers of the spherical Milnor fibrations of* f_1 *and* f_2.

If p is equal to 2, analytic germs which satisfy the above conditions were studied by Oka [25, 26]. Let $(\rho_1, \rho_2) : (\mathbb{R}^{2n}, \mathbf{0}_{2n}) \to (\mathbb{R}^2, \mathbf{0}_2)$ be an analytic map germ with real $2n$-variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. Then (ρ_1, ρ_2) is represented by a complex-valued function of variables $\mathbf{z} = (z_1, \ldots, z_n)$ and $\bar{\mathbf{z}} = (\bar{z}_1, \ldots, \bar{z}_n)$ as

$$P(\mathbf{z}, \bar{\mathbf{z}}) := \rho_1\left(\frac{\mathbf{z} + \bar{\mathbf{z}}}{2}, \frac{\mathbf{z} - \bar{\mathbf{z}}}{2\sqrt{-1}}\right) + \sqrt{-1}\rho_2\left(\frac{\mathbf{z} + \bar{\mathbf{z}}}{2}, \frac{\mathbf{z} - \bar{\mathbf{z}}}{2\sqrt{-1}}\right).$$

Here any complex variable z_j of $\mathbb{C}^n$ is represented by $x_j + \sqrt{-1}y_j$ and $\bar{z}_j$ is the complex conjugate of z_j for $j = 1, \ldots, n$. Then a map $P : (\mathbb{C}^n, \mathbf{0}_{2n}) \to (\mathbb{C}, \mathbf{0}_2)$ is called a *mixed function map*. For mixed weighted homogeneous singularities, Join theorem is proved by J. L. Cisneros-Molina [4]. Oka introduced the notion of Newton boundaries of mixed functions and the concept of strong non-degeneracy. If P is a convenient strongly non-degenerate mixed function or a strongly non-degenerate mixed function which is locally tame along vanishing coordinate subspaces, then P satisfies the conditions (a-i), (a-ii) and (a-iii). See [25, 26, 8].

We study the topology of Milnor fibrations of join type. If a mixed function P satisfies the condition (a-iii) and the origin is an isolated singularity of P, the Seifert form is determined by the spherical Milnor fibration of P. Note that the Seifert form is a topological invariant of fibrations. Then we calculate Seifert forms defined by joins of Milnor fibrations of 1-variable mixed functions in Corollary 5. This is a generalization of [29, Corollary 3].

We also study homotopy types of fibered links defined by isolated singularities of join type. In [20, 21, 22], W. Neumann and L. Rudolph defined the enhanced Milnor number and the enhancement to the Milnor number of a fibered link. These are invariants of homotopy types of fibered links in S^{2k+1}. If $k = 1$, it is shown that for any $d \in \mathbb{Z}$, there exists a mixed polynomial P such that the enhancement to the Milnor number of K_P is equal to d [11]. If k is greater than 1, the enhanced Milnor number is represented by $((-1)^{k+1}\ell, r)$, where $\ell \in \mathbb{N}$

and $r \in \{0, 1\}$. Note that there exists a complex polynomial Q such that the enhanced Milnor number determined by the Milnor fibration of Q is equal to $((-1)^{k+1}\ell, 0)$ for $\ell \in \mathbb{N}$ and $k \geq 2$. We show that there exists a mixed polynomial of join type such that the enhanced Milnor number of a link defined by a mixed polynomial is equal to $((-1)^{k+1}\ell, 1)$ for $\ell \in \mathbb{N}$ and $k \geq 2$.

This paper is organized as follows. In Section 2 we give some Join type statements, the definition of zeta functions of monodromies and strongly non-degenerate mixed functions. In Section 3 we prove Theorem 2 and Corollary 1. In Section 4 we consider Join theorem of Seifert forms of links defined by 1-variable mixed polynomials. In Section 5 we study homotopy types of Milnor fibrations defined by mixed polynomial of Join type.

2. Preliminaries

2.1. Join type statements. Let $g : (\mathbb{R}^N, \mathbf{0}_N) \to (\mathbb{R}^p, \mathbf{0}_p)$ be an analytic germ which satisfies the conditions (a-i) and (a-ii) with respect to a Whitney regular stratification $\mathcal{S}$. By using the same argument in [26, Proposition 11], we can show the following lemma.

Lemma 1. *Assume that an analytic germ $g : (\mathbb{R}^N, \mathbf{0}_N) \to (\mathbb{R}^p, \mathbf{0}_p)$ satisfies the conditions* (a-i) *and* (a-ii)*. Take an a_f-stable radius r_0 for g. For any positive real number r_1 which satisfies $r_1 \leq r_0$, there exists a positive real number $\tilde{\delta}$ such that $g^{-1}(\eta)$ intersects transversely with the sphere S_r^{N-1} for $r_1 \leq r \leq r_0$ and $0 < \|\eta\| \leq \tilde{\delta}$.*

Corollary 2. *Take an a_f-stable radius r_0 for g and take any $\tilde{\delta}_1 \leq \tilde{\delta}$. Then for any $r_1 \leq r \leq r_0$, the isomorphism class of the tubular Milnor fibration $g : U(r, \tilde{\delta}_1) \setminus g^{-1}(\mathbf{0}_p) \to D^p_{\tilde{\delta}_1} \setminus \{\mathbf{0}_p\}$ is independent of the choice of r and $\tilde{\delta}_1$.*

Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^p, \mathbf{0}_p)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^p, \mathbf{0}_p)$ be analytic germs which satisfy the conditions (i) and (ii), where $n, m \geq p \geq 2$. Let $f : (\mathbb{R}^n \times \mathbb{R}^m, \mathbf{0}_{n+m}) \to (\mathbb{R}^p, \mathbf{0}_p)$ be the analytic germ defined by $f = f_1 + f_2$. Put $V(f) = f^{-1}(0) \cap (U_1(\varepsilon_0, \delta) \times U_2(\varepsilon_0, \delta))$. We take the stratification $\mathcal{S}$ of $V(f)$ as follows:

$$\mathcal{S} : (\mathcal{S}_1 \times \mathcal{S}_2) \sqcup (V(f) \setminus (V(f_1) \times V(f_2))),$$

where $\mathcal{S}_j$ is a stratification of $V(f_j)$ in Section 1 for $j = 1, 2$. By [9, p.12], we may assume that $\mathcal{S}_1 \times \mathcal{S}_2$ is Whitney (a)-regular. By using the stratification $\mathcal{S}$ of $V(f)$, R. N. Araújo dos Santos, Y. Chen and M. Tibăr showed the following lemma.

Lemma 2 ([1, Proposition 5.2])**.** *Let $f_1 : (\mathbb{R}^n, \mathbf{0}_n) \to (\mathbb{R}^p, \mathbf{0}_p)$ and $f_2 : (\mathbb{R}^m, \mathbf{0}_m) \to (\mathbb{R}^p, \mathbf{0}_p)$ be analytic germs which satisfy the conditions* (a-i) *and* (a-ii)*, where $n, m \geq p \geq 2$. Then the analytic germ $f = f_1 + f_2 : (\mathbb{R}^n \times \mathbb{R}^m, \mathbf{0}_{n+m}) \to (\mathbb{R}^p, \mathbf{0}_p)$ also satisfies the conditions* (a-i) *and* (a-ii).

By Lemma 1 and Lemma 2, we have

Corollary 3. *Take ε_0 sufficiently small so that we assume that ε_0 is also an a_f-stable radius for f with respect to the above $\mathcal{S}$. Then f satisfies the conditions* (a-i) *and* (a-ii) *on B_ε^{n+m} for $0 < \varepsilon \leq \varepsilon_0$. Moreover, there exists a positive real number δ such that*

$$f : B_\varepsilon^{n+m} \cap f^{-1}(D^p_\delta \setminus \{\mathbf{0}_p\}) \to D^p_\delta \setminus \{\mathbf{0}_p\}$$

is a locally trivial fibration.

2.2. Divisors and Zeta functions of monodromies. Take 1-variable polynomials $q_1(t)$ and $q_2(t)$ with $q_1(0) = q_2(0) = 0$. Set $q_1(t) = a\prod_{j=1}^{k}(t-\alpha_i)$ and $q_2(t) = b\prod_{j=1}^{\ell}(t-\beta_j)$, where $a, b, \alpha_i, \beta_j \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, \ell$. Then we define the *divisor of* $q_1(t)/q_2(t)$ by

$$\left(\frac{q_1(t)}{q_2(t)}\right) = \sum_{i=1}^{k}\langle\alpha_i\rangle - \sum_{j=1}^{\ell}\langle\beta_j\rangle \in \mathbb{Z}(\mathbb{C}^*),$$

where $\mathbb{Z}(\mathbb{C}^*)$ is the group ring of $\mathbb{C}^*$.

Let F be the fiber of the spherical Milnor fibration of $g : (\mathbb{R}^{2n}, \mathbf{0}_{2n}) \to (\mathbb{R}^2, \mathbf{0}_2)$ and $h : F \to F$ be the monodromy of this fibration. Set $P_j(t) = \det(\mathrm{Id} - th_{*,j})$, where $h_{*,j} : H_j(F, \mathbb{Q}) \to H_j(F, \mathbb{Q})$ is an isomorphism induced by h. Then the *zeta function* $\zeta(t)$ *of the monodromy* is defined by

$$\zeta(t) = \prod_{j=0}^{2n-2} P_j(t)^{(-1)^{j+1}}.$$

See [19, Section 9] and [24, Chapter I]. Assume that g satisfies the following properties:

(a) $\mathbf{0}_{2n}$ is an isolated singularity of g,

(b) F has a homotopy type of a finite CW-complex of dimension $\leq n-1$,

(c) F is $(n-2)$-connected.

Then the zeta function $\zeta(t)$ is equal to $P_{n-1}(t)^{(-1)^n}/(t-1)$ and the *reduced zeta function* is defined by $\tilde{\zeta}(t) = (t-1)\zeta(t)$.

2.3. Strongly non-degenerate mixed functions. In this subsection, we introduce a class of mixed functions which admit tubular Milnor fibrations and spherical Milnor fibrations given by Oka in [25]. Let $P(\mathbf{z}, \bar{\mathbf{z}})$ be a mixed function, i.e., $P(\mathbf{z}, \bar{\mathbf{z}})$ is a function expanded in a convergent power series of variables $\mathbf{z} = (z_1, \ldots, z_n)$ and $\bar{\mathbf{z}} = (\bar{z}_1, \ldots, \bar{z}_n)$

$$P(\mathbf{z}, \bar{\mathbf{z}}) := \sum_{\nu,\mu} c_{\nu,\mu}\mathbf{z}^{\nu}\bar{\mathbf{z}}^{\mu},$$

where $\mathbf{z}^{\nu} = z_1^{\nu_1}\cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \ldots, \nu_n)$ (respectively $\bar{\mathbf{z}}^{\mu} = \bar{z}_1^{\mu_1}\cdots\bar{z}_n^{\mu_n}$ for $\mu = (\mu_1, \ldots, \mu_n)$). The *Newton polygon* $\Gamma_+(P; \mathbf{z}.\bar{\mathbf{z}})$ is defined by the convex hull of

$$\bigcup_{(\nu,\mu)}\{(\nu+\mu) + \mathbb{R}_+^n \mid c_{\nu,\mu} \neq 0\},$$

where $\nu + \mu$ is the sum of the multi-indices of $\mathbf{z}^{\nu}\bar{\mathbf{z}}^{\mu}$, i.e., $\nu + \mu = (\nu_1 + \mu_1, \ldots, \nu_n + \mu_n)$. *The Newton boundary* $\Gamma(P; \mathbf{z}, \bar{\mathbf{z}})$ is the union of compact faces of $\Gamma_+(P; \mathbf{z}, \bar{\mathbf{z}})$. The *strongly non-degeneracy* is defined from the Newton boundary as follows: let $\Delta_1, \ldots, \Delta_m$ be the faces of $\Gamma(P; \mathbf{z}, \bar{\mathbf{z}})$. For each face Δ_k, the *face function* $P_{\Delta_k}(\mathbf{z}, \bar{\mathbf{z}})$ is defined by $P_{\Delta_k}(\mathbf{z}, \bar{\mathbf{z}}) := \sum_{(\nu+\mu)\in\Delta_k} c_{\nu,\mu}\mathbf{z}^{\nu}\bar{\mathbf{z}}^{\mu}$. If $P_{\Delta_k}(\mathbf{z}, \bar{\mathbf{z}}) : \mathbb{C}^{*n} \to \mathbb{C}$ has no critical point, and P_{Δ_k} is surjective if $\dim\Delta_k \geq 1$, we say that $P(\mathbf{z}, \bar{\mathbf{z}})$ is *strongly non-degenerate* for Δ_k, where $\mathbb{C}^{*n} = \{\mathbf{z} = (z_1, \ldots, z_n) \mid z_j \neq 0, j = 1, \ldots, n\}$. If $P(\mathbf{z}, \bar{\mathbf{z}})$ is strongly non-degenerate for any Δ_k for $k = 1, \ldots, m$, we say that $P(\mathbf{z}, \bar{\mathbf{z}})$ is *strongly non-degenerate*. If $P((0, \ldots, 0, z_j, 0, \ldots, 0), (0, \ldots, 0, \bar{z}_j, 0, \ldots, 0)) \not\equiv 0$ for each $j = 1, \ldots, n$, then we say that $P(\mathbf{z}, \bar{\mathbf{z}})$ is *convenient*. Oka showed that a convenient strongly non-degenerate mixed function $P(\mathbf{z}, \bar{\mathbf{z}})$ has both tubular and spherical Milnor fibrations and also two fibrations are isomorphic [25].

Theorem 3 ([25, 26, 8])**.** *Let $P(\mathbf{z},\bar{\mathbf{z}}) : (\mathbb{C}^n, \mathbf{0}_{2n}) \to (\mathbb{C}, \mathbf{0}_2)$ be a convenient strongly non-degenerate mixed function. Then $\mathbf{0}_{2n}$ is an isolated singularity of P and P satisfies the conditions* (a-i)*,* (a-ii) *and* (a-iii).

Let f_t be an analytic family of convenient strongly non-degenerate mixed polynomials such that the Newton boundary of f_t is constant for $0 \le t \le 1$. C. Eyral and M. Oka showed that the topological type of $(V(f_t), \mathbf{0}_{2n})$ is constant for any t and their tubular Milnor fibrations are equivalent [8].

3. Proof of Theorem 2

Assume that f_1 and f_2 satisfy the conditions (a-i) and (a-ii) in Section 1. Then the proof of Theorem 2 is analogous to the holomorphic case [30]. We assume that ε_0 is a common a_f-stable radius of f_1, f_2 and $f_1 + f_2$. Taking $\delta_0 \ll \varepsilon_0$ sufficiently small and put

$$U_1 := U_1(\varepsilon_0, \delta_0), \quad U_2 := U_2(\varepsilon_0, \delta_0).$$

Set $X_{\mathbf{t}} = f_1^{-1}(\mathbf{t}) \cap U_1, Y_{\mathbf{t}} = f_2^{-1}(\mathbf{t}) \cap U_2$ and $Z_{\mathbf{t}} = f^{-1}(\mathbf{t}) \cap (U_1 \times U_2)$. We fix a point $\mathbf{t} \in \mathbb{R}^p$ with $0 < \|\mathbf{t}\| \ll \delta_0$ and define the map

$$F_1 : Z_{\mathbf{t}} \to A_{\mathbf{t}} \text{ as } (\mathbf{x}, \mathbf{y}) \mapsto f_1(\mathbf{x}),$$

where $A_{\mathbf{t}} = \{\mathbf{w} \in \mathbb{R}^p \mid \|\mathbf{w}\| \le \delta_0, \|\mathbf{t} - \mathbf{w}\| \le \delta_0\}$.

Lemma 3. *The restriction map $F_1 : Z_{\mathbf{t}} \setminus F_1^{-1}(\{\mathbf{0}_p, \mathbf{t}\}) \to A_{\mathbf{t}} \setminus \{\mathbf{0}_p, \mathbf{t}\}$ is a locally trivial fibration.*

Proof. From the tubular Milnor fibrations of f_1 and f_2, for each $\mathbf{w} \in A_{\mathbf{t}} \setminus \{\mathbf{0}_p, \mathbf{t}\}$, we may find a neighborhood $V_{\mathbf{w}} \subset A_{\mathbf{t}} \setminus \{\mathbf{0}_p, \mathbf{t}\}$ of $\mathbf{w}$ such that there exist local trivializations

$$\phi_1 : V_{\mathbf{w}} \times X_{\mathbf{w}} \xrightarrow{\cong} f_1^{-1}(V_{\mathbf{w}}) \cap U_1, \quad \phi_2 : V_{\mathbf{t}-\mathbf{w}} \times Y_{\mathbf{t}-\mathbf{w}} \xrightarrow{\cong} f_2^{-1}(V_{\mathbf{t}-\mathbf{w}}) \cap U_2,$$

where $V_{\mathbf{t}-\mathbf{w}} = \{\mathbf{t} - \mathbf{w} \mid \mathbf{w} \in V_{\mathbf{w}}\} \subset A_{\mathbf{t}} \setminus \{\mathbf{0}_p, \mathbf{t}\}$. We define the map on $V_{\mathbf{w}} \times F_1^{-1}(\mathbf{w}) = V_{\mathbf{w}} \times (X_{\mathbf{w}} \times Y_{\mathbf{t}-\mathbf{w}})$ as follows:

$$\psi : V_{\mathbf{w}} \times (X_{\mathbf{w}} \times Y_{\mathbf{t}-\mathbf{w}}) \to F_1^{-1}(V_{\mathbf{w}}), \quad (\mathbf{w}', \mathbf{x}, \mathbf{y}) \mapsto (\phi_1(\mathbf{w}', \mathbf{x}), \phi_2(\mathbf{w}', \mathbf{y})).$$

Since ϕ_1 and ϕ_2 are local trivializations, ψ is a continuous map. For any $(\mathbf{x}', \mathbf{y}') \in F_1^{-1}(V_{\mathbf{w}})$, we put $(\mathbf{w}', \mathbf{x}) = \phi_1^{-1}(\mathbf{x}')$ and $(\mathbf{t} - \mathbf{w}', \mathbf{y}) = \phi_2^{-1}(\mathbf{y}')$. Then $\psi^{-1}(\mathbf{x}', \mathbf{y}') = (\mathbf{w}', \mathbf{x}, \mathbf{y})$ and thus we see that ψ^{-1} is a continuous map. Thus ψ is a homeomorphism. This shows the local triviality of F_1. □

Lemma 4. *Let J be the line segment with endpoints $\mathbf{0}_p$ and $\mathbf{t}$. The inclusion $F_1^{-1}(J) \hookrightarrow Z_{\mathbf{t}}$ is a homotopy equivalence.*

Proof. Since $Z_{\mathbf{t}}$ is semi-analytic, there is a triangulation of $Z_{\mathbf{t}}$ such that $F_1^{-1}(J)$ is a subcomplex [17]. Since $Z_{\mathbf{t}}$ is compact, by using the local triviality of F_1 and the partition of unity, $Z_{\mathbf{t}}$ is deformed into a regular neighborhood of $F_1^{-1}(J)$. Thus $F_1^{-1}(J)$ and $Z_{\mathbf{t}}$ are homotopy equivalent. See [28, Chapter 3]. □

Let $\pi : U_1 \times U_2 \to (U_1/V(f_1)) \times (U_2/V(f_2))$ be the identification map where we use the quotient topology for $U_1/V(f_1)$ and $U_2/V(f_2)$.

Lemma 5. *The identification map $\pi : F_1^{-1}(J) \to \pi(F_1^{-1}(J))$ is a homotopy equivalence.*

Proof. The semi-analytic set $V(f_j)$ has a conic structure for $j = 1, 2$ by [19, Theorem 2.10] and [3], i.e.,

$$V(f_1) \cong \mathrm{Cone}(V(f_1) \cap S_{\varepsilon_0}^{n-1}) = ([0,1] \times (V(f_1) \cap S_{\varepsilon_0}^{n-1}))/(\{0\} \times (V(f_1) \cap S_{\varepsilon_0}^{n-1})),$$
$$V(f_2) \cong \mathrm{Cone}(V(f_2) \cap S_{\varepsilon_0}^{m-1}) = ([0,1] \times (V(f_2) \cap S_{\varepsilon_0}^{m-1}))/(\{0\} \times (V(f_2) \cap S_{\varepsilon_0}^{m-1})).$$

So $V(f_1)$ and $V(f_2)$ retract to the origins of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively by strong deformation retracts. We can construct deformation retractions from $F_1^{-1}(\mathbf{0}_p) = V(f_1) \times Y_{\mathbf{t}}$ to $\{\mathbf{0}_n\} \times Y_{\mathbf{t}}$ and from $F_1^{-1}(\mathbf{t}) = X_{\mathbf{t}} \times V(f_2)$ to $X_{\mathbf{t}} \times \{\mathbf{0}_m\}$. By applying a triangulation of $F_1^{-1}(J)$ and using the homotopy extension property of a polyhedral pair [32, p.118], the above homotopies can extend to a homotopy $H_s : F_1^{-1}(J) \to F_1^{-1}(J)$, where $0 \le s \le 1$ so that

$$H_0 = \mathrm{id}_{F_1^{-1}(J)}, \quad H_1(F_1^{-1}(\mathbf{0}_p) \cup F_1^{-1}(\mathbf{t})) = \{\mathbf{0}_n\} \times Y_{\mathbf{t}} \cup X_{\mathbf{t}} \times \{\mathbf{0}_m\}.$$

Let $\tilde{H}_s : \pi(F_1^{-1}(J)) \to \pi(F_1^{-1}(J))$ be the homotopy which satisfies $\pi(H_s(\mathbf{x}, \mathbf{y})) = \tilde{H}_s(\pi(\mathbf{x}, \mathbf{y}))$, where $(\mathbf{x}, \mathbf{y}) \in F_1^{-1}(J)$ and $0 \le s \le 1$. Note that $\pi(F_1^{-1}(J)) \setminus (\{\mathbf{0}_n\} \times Y_{\mathbf{t}} \cup X_{\mathbf{t}} \times \{\mathbf{0}_m\}) = F_1^{-1}(J) \setminus (F_1^{-1}(\mathbf{0}_p) \cup F_1^{-1}(\mathbf{t}))$. The map $\varphi : \pi(F_1^{-1}(J)) \to F_1^{-1}(J)$ is defined by

$$\varphi \mid_{\pi(F_1^{-1}(J)) \setminus (\{\mathbf{0}_n\} \times Y_{\mathbf{t}} \cup X_{\mathbf{t}} \times \{\mathbf{0}_m\})} = H_1 \mid_{F_1^{-1}(J) \setminus (F_1^{-1}(\mathbf{0}_p) \cup F_1^{-1}(\mathbf{t}))},$$
$$\varphi(\{\mathbf{0}_n\} \times Y_{\mathbf{t}}) = \{\mathbf{0}_n\} \times Y_{\mathbf{t}}, \quad \varphi(X_{\mathbf{t}} \times \{\mathbf{0}_m\}) = X_{\mathbf{t}} \times \{\mathbf{0}_m\}.$$

Then φ is continuous and $H_1 = \varphi \circ \pi$. By the definition of $\tilde{H}_s$, $\pi \circ \varphi = \tilde{H}_1$. Thus the identification map π is a homotopy equivalence. □

Lemma 6. *Let $X_{\mathbf{t}} * Y_{\mathbf{t}}$ be the join of $X_{\mathbf{t}}$ and $Y_{\mathbf{t}}$. Then $X_{\mathbf{t}} * Y_{\mathbf{t}}$ is homeomorphic to $\pi(F_1^{-1}(J))$.*

Proof. Put $I = [0, 1]$. By the local trivialities of the tubular Milnor fibrations of f_1 and f_2, there exist homeomorphisms

$$\tilde{\phi}_1 : (I \setminus \{0\}) \times X_{\mathbf{t}} \to f_1^{-1}(J \setminus \{\mathbf{0}_p\}) \cap U_1, \quad \tilde{\phi}_2 : (I \setminus \{0\}) \times Y_{\mathbf{t}} \to f_2^{-1}(J \setminus \{\mathbf{0}_p\}) \cap U_2$$

such that $f_1(\tilde{\phi}_1(s, \mathbf{x})) = f_2(\tilde{\phi}_2(s, \mathbf{y})) = s\mathbf{t}$ for $0 < s \le 1$. We define the map

$$\Phi : X_{\mathbf{t}} \times I \times Y_{\mathbf{t}} \to \pi(F_1^{-1}(J)) \text{ as } (\mathbf{x}, s, \mathbf{y}) \mapsto \pi(\tilde{\phi}_1(s, \mathbf{x}), \tilde{\phi}_2(1 - s, \mathbf{y})),$$

where $\tilde{\phi}_1(0, \mathbf{x}) = \mathbf{0}_n$ and $\tilde{\phi}_2(0, \mathbf{y}) = \mathbf{0}_m$. Since $V(f_1)$ and $V(f_2)$ have conic structures, Φ is a continuous map. Let $\Psi : X_{\mathbf{t}} * Y_{\mathbf{t}} \to \pi(F_1^{-1}(J))$ be the map defined by $\Psi([\mathbf{x}, s, \mathbf{y}]) = \Phi(\mathbf{x}, s, \mathbf{y})$, where $[\mathbf{x}, s, \mathbf{y}]$ is the equivalence class of $(\mathbf{x}, s, \mathbf{y})$. By the definition of Φ and conic structures of $V(f_1)$ and $V(f_2)$, Ψ is a continuous and bijective map. Thus Ψ is a homeomorphism. □

Lemma 7. *The fiber $Z_{\mathbf{t}}$ is homotopy equivalent to $f^{-1}(\mathbf{t}) \cap B_{\varepsilon'}^{n+m}$, where $0 < \varepsilon' \ll 1$.*

Proof. Let ε_0 be a common a_f-stable radius of f_1, f_2 and $f = f_1 + f_2$. Put

$$U(\varepsilon_0, \delta_0) := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+m} \mid \|(\mathbf{x}, \mathbf{y})\| \le \varepsilon_0, \|f_1(\mathbf{x}) + f_2(\mathbf{y})\| \le \delta_0\},$$

where $0 < \delta_0 \ll \varepsilon_0$. As the families $\{U_1(\varepsilon_0, \delta_0) \times U_2(\varepsilon_0, \delta_0)\}_{0 < \delta_0 \ll \varepsilon_0}$ and $\{U(\varepsilon_0, \delta_0)\}_{0 < \delta_0 \ll \varepsilon_0}$ are cofinal systems of neighborhoods of the origin $\mathbf{0}_{n+m}$ respectively for ε_0 which is an a_f-stable radius of f_1, f_2, f and $0 < \delta_0 \ll \varepsilon_0$. Thus we can choose positive real numbers $\varepsilon_3 < \varepsilon_2 < \varepsilon_1 < \varepsilon_0$ and sufficiently small $\delta_3 < \delta_2 < \delta_1 < \delta_0$ with $\delta_j \ll \varepsilon_j$ for $j = 0, \ldots, 3$ so that

$$\begin{aligned}U_1(\varepsilon_1,\delta_1)\times U_2(\varepsilon_1,\delta_1)\supset U(\varepsilon_1,\delta_1)\supset U_1(\varepsilon_2,\delta_2)\times U_2(\varepsilon_2,\delta_2)\\ \supset U(\varepsilon_2,\delta_2)\supset U_1(\varepsilon_3,\delta_3)\times U_2(\varepsilon_3,\delta_3)\supset U(\varepsilon_3,\delta_3).\end{aligned}$$

By Corollary 2, the inclusions

$$\begin{aligned}&\iota_j : U_1(\varepsilon_j,\delta_j)\times U_2(\varepsilon_j,\delta_j)\to U_1(\varepsilon_{j-1},\delta_{j-1})\times U_2(\varepsilon_{j-1},\delta_{j-1}),\\ &\iota'_j : U(\varepsilon_j,\delta_j)\to U(\varepsilon_{j-1},\delta_{j-1})\end{aligned}$$

are isomorphisms for $j = 1,\dots,3$. The homotopies of the above sequence can be defined by ι_j and ι'_j for $j = 1,\dots,3$. Thus by a standard homotopy argument, we can see that the inclusion maps in the above sequence are homotopy equivalences. Take $\mathbf{t}\in\mathbb{R}^p$ which satisfies $\|\mathbf{t}\|\le\delta_3$. Then we see also the restriction of the homotopies of the above sequence to $f^{-1}(\mathbf{t})$ is also homotopy equivalences. □

Proof of Theorem 2. By using Lemma 4, Lemma 5 and Lemma 6, we can show that $X_{\mathbf{t}} * Y_{\mathbf{t}}$ is homotopy equivalent to $Z_{\mathbf{t}}$. By Lemma 7, the fiber of the tubular Milnor fibration of f is homotopy equivalent to $X_{\mathbf{t}} * Y_{\mathbf{t}}$.

If $p = 2$, set

$$E = \{(\mathbf{x},\mathbf{y})\in U_1\times U_2 \mid 0 < \|f(\mathbf{x},\mathbf{y})\|\le\rho\},$$

where $0<\rho\ll\varepsilon$. Then the map $\tilde{f} : \pi(E)\to D^2_\rho\setminus\{\mathbf{0}_2\}$ is defined by $\tilde{f}(\pi(\mathbf{x},\mathbf{y})) = f(\mathbf{x},\mathbf{y})$. By the local trivialities of f_1 and f_2, there are continuous one-parameter families of homeomorphisms

$$\alpha_\theta : U_1\setminus V(f_1)\to U_1\setminus V(f_1),\ \ \beta_\theta : U_2\setminus V(f_2)\to U_2\setminus V(f_2)$$

such that $f_1(\alpha_\theta(\mathbf{x})) = e^{i\theta}f_1(\mathbf{x})$ and $f_2(\beta_\theta(\mathbf{y})) = e^{i\theta}f_2(\mathbf{y})$, where $\theta\in[0,2\pi]$. Then we define the map $\gamma_\theta : \pi(E)\to\pi(E)$ as follows:

$$\gamma_\theta(\pi(\mathbf{x},\mathbf{y})) = \begin{cases}\pi(\alpha_\theta(\mathbf{x}),\beta_\theta(\mathbf{y})) & \mathbf{x}\in U_1\setminus V(f_1), \mathbf{y}\in U_2\setminus V(f_2)\\ \pi(\mathbf{0}_n,\beta_\theta(\mathbf{y})) & \mathbf{x}\in V(f_1), \mathbf{y}\in U_2\setminus V(f_2)\\ \pi(\alpha_\theta(\mathbf{x}),\mathbf{0}_m) & \mathbf{x}\in U_1\setminus V(f_1), \mathbf{y}\in V(f_2).\end{cases}$$

Note that $\{\gamma_\theta\}$ is well-defined and a continuous one-parameter family of homeomorphisms such that $\tilde{f}(\gamma_\theta(\mathbf{z})) = e^{i\theta}\tilde{f}(\mathbf{z})$, where $\mathbf{z}\in\pi(E)$ and $\theta\in[0,2\pi]$. Hence $\{\gamma_\theta\}$ gives the local triviality of $\tilde{f}$. Then the monodromy of $\tilde{f}$ can be identified with $\alpha_{2\pi}*\beta_{2\pi}$ up to homotopy. Here the map $\alpha_{2\pi}*\beta_{2\pi}$ is defined by

$$\alpha_{2\pi}*\beta_{2\pi}([\mathbf{x},s,\mathbf{y}]) = [\alpha_{2\pi}(\mathbf{x}),s,\beta_{2\pi}(\mathbf{y})],$$

where $[\mathbf{x},s,\mathbf{y}]\in X_{\mathbf{t}} * Y_{\mathbf{t}}$.

By Lemma 7, the fiber of $\tilde{f}$ is homotopy equivalent to the fiber of f. Since $D^2_\rho\setminus\{\mathbf{0}_2\}$ is a CW-complex and $\tilde{f}^{-1}(\mathbf{t})$ is homotopy equivalent to $f^{-1}(\mathbf{t})$ for any $\mathbf{t}\in D^2_\rho\setminus\{\mathbf{0}_2\}$, $\tilde{f}$ is fiber homotopy equivalent to f by [5]. Then the monodromy of the tubular Milnor fibration of f is equal to $\alpha_{2\pi}*\beta_{2\pi}$. □

Proof of Corollary 1. By Theorem 2 and the condition (a-iii), the fiber of the spherical Milnor fibration of f is homotopy equivalent to $X_{\mathbf{t}} * Y_{\mathbf{t}}$, where $0<\|\mathbf{t}\|\ll 1$. By the condition (a-iii), $X_{\mathbf{t}}$ and $Y_{\mathbf{t}}$ are diffeomorphic to the fibers of the spherical Milnor fibrations of f_1 and f_2 respectively. This completes the proof. □

Let F_j be the fiber of the spherical Milnor fibration of f_j which satisfies the assumptions in Section 2.2 for $j = 1, 2$. By [18], the reduced homology $\tilde{H}_{n+m-1}(F_1 * F_2)$ satisfies

$$\tilde{H}_{n+m-1}(F_1 * F_2) = \sum_{i+j=n+m-2} \tilde{H}_i(F_1, \mathbb{Z}) \otimes \tilde{H}_j(F_2, \mathbb{Z}) + \sum_{i'+j'=n+m-3} \mathrm{Tor}(\tilde{H}_{i'}(F_1, \mathbb{Z}), \tilde{H}_{j'}(F_2, \mathbb{Z})).$$

Let F be the fiber of the spherical Milnor fibration of $f = f_1 + f_2$ and $\tau : F \to F_1 * F_2$ be the homotopy equivalence in Theorem 2. Then f also satisfies the assumptions in Section 2.2 and we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{H}_{n+m-1}(F, \mathbb{Z}) & \xrightarrow{\gamma_*} & \tilde{H}_{n+m-1}(F, \mathbb{Z}) \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ \tilde{H}_{n-1}(F_1, \mathbb{Z}) \otimes \tilde{H}_{m-1}(F_2, \mathbb{Z}) & \xrightarrow{\alpha_* \otimes \beta_*} & \tilde{H}_{n-1}(F_1, \mathbb{Z}) \otimes \tilde{H}_{m-1}(F_2, \mathbb{Z}) \end{array},$$

where α_*, β_* and γ_* are the linear transformations induced by the monodromy of the spherical Milnor fibrations of f_1, f_2 and f respectively. Since the eigenvalues of the linear transformation $\alpha_* \otimes \beta_* : \tilde{H}_{n-1}(F_1, \mathbb{Z}) \otimes \tilde{H}_{m-1}(F_2, \mathbb{Z}) \to \tilde{H}_{n-1}(F_1, \mathbb{Z}) \otimes \tilde{H}_{m-1}(F_2, \mathbb{Z})$ are given by the product of the eigenvalues of α_* and β_*, we obtain the following corollary.

Corollary 4. *Assume that f_1 and f_2 satisfy the assumptions in* Section 2.2. *Let $\tilde{\zeta}_1(t), \tilde{\zeta}_2(t)$ and $\tilde{\zeta}(t)$ of the reduced zeta functions defined by α_*, β_* and γ_* respectively. Then the divisors of the reduced zeta functions are related by*

$$\left(\tilde{\zeta}(t)\right) = \left(\tilde{\zeta}_1(t)\right) \cdot \left(\tilde{\zeta}_2(t)\right).$$

4. Seifert forms of simple links defined by mixed functions

Let K be a *link* in the $(2k+1)$-sphere S^{2k+1}, i.e., K is an oriented codimension-two closed smooth submanifold in S^{2k+1}. A link K is said to be *fibered* if there exists a trivialization $K \times D^2 \to N(K)$ of a tubular neighborhood $N(K)$ of K in S^{2k+1} and a fibration of the link exterior $E(K) = S^{2k+1} \setminus \mathrm{Int}(N(K))$, $\xi_1 : E(K) \to S^1$ such that $\xi_0 | \partial N(K) = \xi_1 | \partial N(K)$, where $\xi_0 : N(K) \to D^2$ is a trivialization $K \times D^2 \to N(K)$ composed with the second factor. This fibration is also called an *open book decomposition* of S^{2k+1}. A fiber of ξ_1 is called a *fiber surface of the fibration of K*. If $f(\mathbf{z}, \bar{\mathbf{z}})$ is convenient strongly non-degenerate, K_f is a fibered link by [25].

We assume that a fibered link K in S^{2k+1} is $(k-2)$-connected and its fiber surface F is $(k-1)$-connected. Then K is called a *simple fibered link*. Let $\alpha, \beta \in \tilde{H}_k(F; \mathbb{Z})$ and a and b be cycles on F representing α and β respectively. Set

$$L_K(\alpha, \beta) := \mathrm{link}(a^+, b),$$

where a^+ is a pushed off of a to the positive side of F by a transverse vector field and $\mathrm{link}(a^+, b)$ is the linking number of a^+ and b. The *Seifert form L_K* of K is the non-singular bilinear form

$$L_K : \tilde{H}_k(F; \mathbb{Z}) \times \tilde{H}_k(F; \mathbb{Z}) \to \mathbb{Z}$$

on the k-th homology group $\tilde{H}_k(F; \mathbb{Z})$ with respect to a choice of basis of $\tilde{H}_k(F; \mathbb{Z})$. By [14], we can show the following proposition.

Proposition 1 ([29, 14])**.** *Let $f_1 : (\mathbb{C}^n, \mathbf{0}_{2n}) \to (\mathbb{C}, \mathbf{0}_2)$ and $f_2 : (\mathbb{C}^m, \mathbf{0}_{2m}) \to (\mathbb{C}, \mathbf{0}_2)$ be mixed function germs of independent variables which satisfy the conditions* (a-i)*,* (a-ii) *and* (a-iii)*. Suppose that the origin is an isolated singularity of f_j and K_{f_j} is a simple fibered link for $j = 1, 2$. Then L_{K_f} is congruent to $(-1)^{nm} L_{K_{f_1}} \otimes L_{K_{f_2}}$.*

Kauffman and Neumann studied Seifert forms of non-simple fibered links. See [14].

Let $A = (a_{i,j})$ and A' be integral unimodular matrices. We say that A' is an *extension of A* if A' is congruent to

$$\left(\begin{array}{ccc|c} a_{1,1} & \dots & a_{1,n} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} & 0 \\ \hline b_1 & \dots & b_n & \varepsilon \end{array}\right),$$

where n is the rank of A, $b_i \in \mathbb{Z}, i = 1, \dots, n$ and $\varepsilon = \pm 1$. Let K and K' be simple fibered links in S^{2k+1}. Set F and F' to be the fiber surfaces of K and K' respectively. If a fiber surface F is obtained from F' by a plumbing of a Hopf band, the Seifert form of F is an extension of the Seifert form of F' (cf. [16]). If a fiber surface is obtained from a disk by successive plumbings of Hopf bands then its Seifert form becomes a unimodular lower triangular matrix for a suitable choice of the basis. D. Lines studied high dimensional fibered knots by using plumbings [15, 16].

Proposition 2 ([15])**.** *Let F be the fiber surface of a simple fibered link K in S^{2k+1}, where $k \geq 3$. Then F is obtained from a disk by successive plumbings of Hopf bands if and only if it admits a unimodular lower triangular Seifert form. For $k = 1$, the above condition is necessary.*

Example 1. Let f_1 be a 2-variable complex polynomial which has an isolated singularity at the origin and $f_2(\mathbf{w}) = \sum_{j=1}^{m} w_j^2$, where $m \geq 2$. In [13, Corollary 1.2], the Milnor fiber of f_1 is obtained from a disk by successive plumbings of Hopf bands. By Proposition 1 and Proposition 2, the Milnor fiber of $f_1 + f_2$ is also obtained from a disk by successive plumbings of Hopf bands.

For mixed singularities, we can find a 2-variable mixed polynomial f_1' such that the Milnor fiber of $f_1' + f_2$ cannot be obtained from a disk by successive plumbings of Hopf bands. We will give an explicit example below. Suppose that $\alpha_j \neq \alpha_{j'}$ $(j \neq j')$. Then we define a mixed polynomial as follows:

$$f_1'(\mathbf{z}) := (z_1 + \alpha_1 z_2)(z_1 + \alpha_2 z_2)\overline{(z_1 + \alpha_3 z_2)}.$$

Note that f_1' is a convenient strongly non-degenerate mixed polynomial. Thus f_1' satisfies the conditions (a-i), (a-ii) and (a-iii). See also [27]. By [12, Lemma 1], the Seifert form of $K_{f_1'}$ is equal to

$$A_1 := \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}.$$

Since the Seifert form of K_{f_2} is equal to $(-1)^{\frac{m(m-1)}{2}}$ [6, Proposition 2.2], by Proposition 1, the Seifert form of $K_{f'}$ is equal to

$$\Gamma_1 := \epsilon A_1 = (-1)^{\frac{m(m-1)}{2}} \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix},$$

where $f' = f_1' + f_2$ and $\epsilon = (-1)^{\frac{m(m-1)}{2}}$. We show the following assertion.

ASSERTION 1. The Milnor fibers of f_1' and f' for $m \geq 2$ cannot be obtained by series of plumbings from a disk.

Proof. By the assertion of Proposition 2.4 of [15], it is enough to show that the corresponding Seifert form of $K_{f'}$ cannot be a lower triangle unimodular matrix. Let e_1, e_2 be the basis of the homology group of the Milnor fiber of f' which gives Γ_1 and take a unimodular matrix $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $a_1a_4 - a_2a_3 = \pm 1$.

We show that the assertion by contradiction. Consider the base change $e_1' = a_1e_1 + a_2e_2, e_2' = a_3e_1 + a_4e_2$ and consider the corresponding Seifert form Γ_1' which should be a lower triangle unimodular matrix. Let $\tilde{e}_j$ and $\tilde{e}_j'$ be cycles of the Milnor fiber representing e_j and e_j' for $j = 1, 2$. By the definition of e_1, e_2, e_1' and e_2', we have

$$\begin{aligned} \mathrm{link}(\tilde{e}_1'^{+}, \tilde{e}_2') &= \epsilon\{a_1a_3\mathrm{link}(\tilde{e}_1^{+}, \tilde{e}_1) + a_1a_4\mathrm{link}(\tilde{e}_1^{+}, \tilde{e}_2) + a_2a_3\mathrm{link}(\tilde{e}_2^{+}, \tilde{e}_1) + a_2a_4\mathrm{link}(\tilde{e}_2^{+}, \tilde{e}_2)\} \\ &= \epsilon(2a_2a_4 - (a_1a_4 + a_2a_3)) \\ &= \epsilon(2a_2a_4 - (2a_2a_3 \pm 1)). \end{aligned}$$

If Γ_1' is a lower triangle unimodular matrix, we have the equality $2(a_2a_4 - a_2a_3) \mp 1 = 0$ which has no integer solution. Thus the Seifert form of $K_{f'}$ cannot be a lower triangle unimodular matrix. □

By Proposition 1, Proposition 2 and Assertion 1, the Milnor fiber of f' cannot be obtained from a disk by successive plumbings of Hopf bands.

By using the notion of strongly non-degenerate mixed functions and Proposition 1, we show a generalization of [29, Corollary 3].

Corollary 5. *Let $f_j(z_j)$ be a strongly non-degenerate mixed polynomial of 1-variable z_j for $j = 1, \dots, n$. Set m_j to be the mapping degree of $f_j/|f_j| : S^1_{\varepsilon_j} = \{z_j \in \mathbb{C} \mid |z_j| = \varepsilon_j\} \to S^1$ and*

$$g_j(z_j) = \begin{cases} z_j^{m_j+\ell_j}\bar{z}_j^{\ell_j} & m_j > 0 \\ z_j^{\ell_j}\bar{z}_j^{-m_j+\ell_j} & m_j < 0 \end{cases},$$

where $0 < \varepsilon_j \ll 1$ for $j = 1, \dots, n$. Suppose that the Newton boundary of f_j is equal to that of g_j for $j = 1, \dots, n$. For any $j \in \{1, \dots, n\}$, assume that there exists an analytic family $f_{j,t}$ of strongly non-degenerate mixed polynomials such that $f_{j,0} = f_j, f_{j,1} = g_j$ and the Newton boundaries of $f_{j,t}$ is constant for $0 \leq t \leq 1$. Then the Milnor fibration of $f(\mathbf{z}) = f_1(z_1) + \cdots + f_n(z_n)$ is equivalent to that of $g(\mathbf{z}) = g_1(z_1) + \cdots + g_n(z_n)$. Set the $(|m| - 1) \times (|m| - 1)$ matrix Λ_m' as follows:

$$\Lambda_m' = \begin{cases} \Lambda_m & m > 0 \\ {}^t\Lambda_m & m < 0 \end{cases},$$

where Λ_m is the $(m - 1) \times (m - 1)$ matrix given by

$$\Lambda_m = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}.$$

Then we have

$$\Gamma_{K_f} \cong (-1)^{\frac{n(n+1)}{2}} \Lambda'_{m_1} \otimes \cdots \otimes \Lambda'_{m_n}.$$

Proof. Since $f_{j,0}$ is strongly non-degenerate, the tubular Milnor fibration of f_t is equivalent to that of f_0 for $0 \leq t \leq 1$ [8, Theorem 3.14]. Thus the tubular Milnor fibrations of f_j is equivalent to that of g_j for $j = 1, \dots, n$. By [25] and Corollary 1, the spherical Milnor fibration of f are equivalent to that of g. By Proposition 1 and [29, Corollary 3], the Seifert form Γ_{K_g} is congruent to

$$(-1)^{\frac{n(n+1)}{2}} \Lambda'_{m_1} \otimes \cdots \otimes \Lambda'_{m_n}.$$

This completes the proof. □

5. Enhanced Milnor numbers of simple links defined by mixed functions

Let K be a fibered link in S^{2k+1}. By gluing ξ_0 and ξ_1, we give a piecewise smooth map $\xi : S^{2k+1} \to D^2$. By [14], ξ can be extended to a continuous map $\Xi : B^{2k+2} \to D^2$ which is a smooth submersion except at $\mathbf{0}_2$ and a corner along $\partial N(K)$. Then we consider the following map:

$$B^{2k+2} \setminus \{\mathbf{0}_{2k+2}\} \to G(2k, 2k+2), \quad \mathbf{x} \mapsto \ker D(\Xi(\mathbf{x})),$$

where $D(\Xi(\mathbf{x}))$ is the differential of Ξ at $\mathbf{x}$ and $G(2k, 2k+2)$ is the Grassman manifold of oriented $2k$-planes in $\mathbb{R}^{2k+2}$. This map defines an element of $\pi_{2k+1}(G(2k, 2k+2))$. Note that $\pi_{2k+1}(G(2k, 2k+2))$ is isomorphic to

$$\begin{cases} \mathbb{Z} \oplus \mathbb{Z} & k = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & k > 1 \end{cases}.$$

The homotopy class of Ξ has the form $((-1)^{k+1}\mu(K), \lambda(K))$. This pair $((-1)^{k+1}\mu(K), \lambda(K))$ is called the *enhanced Milnor number of K* and $\lambda(K)$ is called the *enhancement to the Milnor number*. See [20, 21, 22]. Note that if K is a fibered link coming from an isolated singularity of a complex hypersurface, $\lambda(K)$ always vanishes. By [21], we have

Theorem 4 ([21])**.** *Let* $f_1 : (\mathbb{C}^n, \mathbf{0}_{2n}) \to (\mathbb{C}, \mathbf{0}_2)$ *and* $f_2 : (\mathbb{C}^m, \mathbf{0}_{2m}) \to (\mathbb{C}, \mathbf{0}_2)$ *be mixed function germs of independent variables. Assume that* f_1, f_2 *and* f *satisfy the conditions* (a-i), (a-ii) *and* (a-iii). *Suppose that* $\mathbf{0}_{2n}$ *and* $\mathbf{0}_{2m}$ *are isolated singularities of* f_1 *and* f_2. *Then* $\mu(K_f) = \mu(K_{f_1})\mu(K_{f_2})$ *and* $\lambda(K_f) \equiv \lambda(K_{f_1})\mu(K_{f_2}) + \mu(K_{f_1})\lambda(K_{f_2}) \bmod 2$.

For any $\ell \in \mathbb{N}$ and $k \geq 2$, there exists a $(k+1)$-variables Brieskorn polynomial P such that $((-1)^{k+1}\mu(K_P), \lambda(K_P)) = ((-1)^{k+1}\ell, 0)$. See [2]. By Theorem 4 and [11], we calculate the enhanced Milnor numbers of simple fibered links defined by mixed polynomials of join type as follows.

Theorem 5. *For any $\ell \in \mathbb{N}$, there exists a $(k+1)$-variables mixed polynomial $P = P_1 + P_2$ of join type such that P_1, P_2 and P satisfies the conditions* (a-i), (a-ii) *and* (a-iii) *and K_P is a simple fibered link which satisfies $((-1)^{k+1}\mu(K_P), \lambda(K_P)) = ((-1)^{k+1}\ell, 1)$, where $k \geq 2$.*

Proof. We define a mixed polynomial and a complex polynomial as follows:

$$f_1(\mathbf{z}) = (z_1^p + \alpha_1 z_2)(z_1^p + \alpha_2 z_2)\overline{(z_1^p + \alpha_3 z_2)}, \quad f_2(\mathbf{z}) = z_1^2 + \bar{z}_2^2, \quad f_3(\mathbf{w}) = \textstyle\sum_{i=1}^m w_i^{a_i},$$

where $\alpha_j \neq \alpha_{j'}$ $(j \neq j')$, $a_j \geq 2$ and $m \geq 1$. Then f_j is a convenient strongly non-degenerate mixed polynomial and K_{f_j} is a simple fibered link for $j = 1, 2, 3$. By [7, 11, 19], we have

$$\begin{aligned}
&(\mu(K_{f_1}), \lambda(K_{f_1})) = (2p, 1), \quad (\mu(K_{f_2}), \lambda(K_{f_2})) = (1, 1),\\
&(\mu(K_{f_3}), \lambda(K_{f_3})) = ((a_1 - 1)\cdots(a_m - 1), 0).
\end{aligned}$$

If ℓ is a positive even integer, we set $p = \frac{\ell}{2}, P_1 = f_1$ and $P_2 = f_3$. By Corollary 1, K_P is also a simple fibered link. By Theorem 4, we have

$$(\mu(K_P), \lambda(K_P)) = (\ell(a_1 - 1)\cdots(a_m - 1), (a_1 - 1)\cdots(a_m - 1) \bmod 2).$$

We set $a_i = 2$ for $i = 1, \ldots, m$. Then $((-1)^{k+1}\mu(K_P), \lambda(K_P))$ is equal to $((-1)^{k+1}\ell, 1)$.

If ℓ is a positive odd integer, put $P_1 = f_2$ and $P_2 = f_3$. Then we have

$$(\mu(K_P), \lambda(K_P)) = ((a_1 - 1)\cdots(a_m - 1), (a_1 - 1)\cdots(a_m - 1) \bmod 2).$$

We set $a_1 = \ell + 1$ and $a_i = 2$ for $i = 2, \ldots, m$. Then $((-1)^{k+1}\mu(K_P), \lambda(K_P))$ is equal to $((-1)^{k+1}\ell, 1)$. □

ACKNOWLEDGEMENTS. The author would like to thank Masaharu Ishikawa, Mutsuo Oka and Mihai Tibăr for precious comments and fruitful suggestions. He also thanks to the referee for careful reading of the manuscript and several accurate comments.

References

[1] R.N. Araújo dos Santos, Y. Chen and M. Tibăr, *Singular open book structures from real mappings*, Cent. Eur. J. Math. **11** (2013), 817–828.
[2] E. Brieskorn: *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. **2** (1966), 1–14.
[3] D. Burghelea and A. Verona: *Local homological properties of analytic sets*, Manuscripta Math. **7** (1972), 55–66.
[4] J.L. Cisneros-Molina: *Join theorem for polar weighted homogeneous singularities*; in Singularities II, Contemp. Math. **475**, Amer. Math. Soc., Providence, RI, 2008, 43–59.
[5] A. Dold: *Partitions of unity in the theory of fibrations*, Ann. of Math. **78** (1963), 223–255.
[6] A. Durfee: *Fibered knots and algebraic singularities*, Topology **13** (1974), 47–59.
[7] D. Eisenbud and W. Neumann: Three-Dimensional Link Theory and Invariants of Plane Curve Singularities, Annals of Mathematics Studies **110**, Princeton University Press, Princeton, N.J., 1985.
[8] C. Eyral and M. Oka: *Whitney regularity and Thom condition for families of non-isolated mixed singularities*, J. Math. Soc. Japan. **70** (2018), 1305–1336.

[9] C. Gibson, K. Wirthmüller, A. du Plessis and E. Looijenga: Topological Stability of Smooth Mappings, Lecture Notes in Mathematics **552**, Springer-Verlag, Berlin-New York, 1976.

[10] H. Hironaka: *Subanalytic sets*; in Number theory, Algebraic Geometry and Commutative Algebra, in Honor of Yasuo Akizuki, Kinokuniya, Tokyo, 1973, 453–493.

[11] K. Inaba: *On the enhancement to the Milnor number of a class of mixed polynomials*, J. Math. Soc. Japan, **66** (2014), 25–36.

[12] K. Inaba: *On fibered links of singularities of polar weighted homogeneous mixed polynomials*; in Singularities in Geometry and Topology 2011, Advanced Studies in Pure Mathematics **66** (2015), 81–92.

[13] M. Ishikawa: *Plumbing constructions of connected divides and the Milnor fibers of plane curve singularities*, Indag. Mathem., N.S., **13** (2002), 499–514.

[14] L.H. Kauffman and W.D. Neumann: *Products of knots, branched fibrations and sums of singularities*, Topology **16** (1977), 369–393.

[15] D. Lines: *On odd-dimensional fibred knots obtained by plumbing and twisting*, J. London. Math. Soc. **32** (1985), 557–571.

[16] D. Lines: *Stable plumbing for high odd-dimensional fibred knots*, Canad. Math. Bull. **30** (1987), 429–435.

[17] S. Lojasiewicz: *Triangulation of semi-analytic set*, Ann. Scuola Norm. Sup. Pisa Cl, Sci. **18** (1964), 449–474.

[18] J. Milnor: *Construction of universal bundles, II*, Ann. of Math. **63** (1956), 430–436.

[19] J. Milnor: Singular points of complex hypersurfaces, Annals of Mathematics Studies **61**, Princeton University Press, Princeton, N.J., 1968.

[20] W. Neumann and L. Rudolph: *Unfoldings in knot theory*, Math. Ann. **278** (1987), 409–439.

[21] W. Neumann and L. Rudolph: *The enhanced Milnor number in high dimensions*; in Differential Topology Proceedings, Siegen 1987, Lecture Notes in Math. **1350**, Springer-Verlag, 1988, 109–121.

[22] W. Neumann and L. Rudolph: *Difference index of vectorfields and the enhanced Milnor number*, Topology, **29** (1990), 83–100.

[23] M. Oka: *On the homotopy types of hypersurfaces defined by weighted homogeneous polynomials*, Topology **12** (1973), 19–32.

[24] M. Oka: Non-degenerate complete intersection singularity, Hermann, Paris, 1997.

[25] M. Oka: *Non-degenerate mixed functions*, Kodai Math. J. **33** (2010), 1–62.

[26] M. Oka: *On Milnor fibrations, a_f-condition and boundary stability*, Kodai Math. J. **38** (2015), 581–603.

[27] M. Oka: *On the Milnor fibration for $f(\mathbf{z})\bar{g}(\mathbf{z})$*, Eur. J. Math. **6** (2020), 998–1019.

[28] C.P. Rourke and B.J. Sanderson: Introduction to piecewise-linear topology, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **69**, Springer-Verlag, New York-Heidelberg, 1972.

[29] K. Sakamoto: *The Seifert matrices of Milnor fiberings defined by holomorphic functions*, J. Math. Soc. Japan. **26** (1974), 714–721.

[30] K. Sakamoto: *Milnor fiberings and their characteristic maps*; in Manifolds-Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), Univ. Tokyo Press, 1975, 145–150.

[31] M. Sebastiani and R. Thom: *Un résultat sur la monodromie*, Invent. Math. **13** (1971), 90–96.

[32] E.H. Spanier: Algebraic Topology, McGraw-Hill Book Company, New York, 1966.

[33] J. A. Wolf: *Differentiable fibre spaces and mappings compatible with Riemannian metrics*, Michigan Math. J. **11** (1964), 65–70.

Faculty of Education, Iwate University
18–33 Ueda 3–chome Morioka
Iwate 020–8550
Japan
email: inaba@iwate-u.ac.jp

[*Current Address*:]
Faculty of Science, Okayama University of Science
1–1 Ridai-cho, Kita-ku, Okayama
Okayama 700–0005
Japan
e-mail: inaba@das.ous.ac.jp