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AN ENTROPY PROBLEM OF THE α-CONTINUED FRACTION MAPS

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Abstract

We show that the entropy of the α-continued fraction map w.r.t the absolutely continuous invariant probability measure is strictly less than that of the nearest integer continued fraction map when $0 < \alpha < \frac{3-\sqrt{5}}{2}$. This answers a question by C. Kraaikamp, T. A. Schmidt, and W. Steiner (2012). To prove this result we make use of the notion of the geodesic continued fractions introduced by A. F. Beardon, M. Hockman, and I. Short (2012).

1. Introduction

In this paper, we consider the entropy problem of α-continued fraction maps which is a 1-parameter family of continued fraction maps. The main point of this paper is to apply the idea of the geodesic continued fractions, defined by A. F. Beardon, M. Hockman, and I. Short [3], to the metric theory of the α-continued fractions. With this idea, we give the answer to the question by C. Kraaikamp, T. A. Schmidt, and W. Steiner [12] concerning the maximum value of the entropy of the α-continued fraction maps. We can apply this idea to other 1-parameter families of continued fraction maps.

In 1981, the author [16] introduced the notion of α-continued fraction map for α, $1/2 \le \alpha \le 1$, which is defined as follows:

$$T_\alpha(x) = \begin{cases} \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + 1 - \alpha \right\rfloor & \text{if} \quad x \in [\alpha - 1, \alpha) \setminus \{0\} \\ 0 & \text{if} \quad x = 0. \end{cases}$$

It was shown in [16] that there exists an absolutely continuous ergodic invariant probability measure μ_α for each T_α and the entropy $h(T_\alpha)$ w.r.t. μ_α is given by

$$h(T_\alpha) = \begin{cases} \frac{1}{\log(g+1)} \cdot \frac{\pi^2}{6} & \text{if} \quad \frac{1}{2} \le \alpha \le g \\ \frac{1}{\log(\alpha+1)} \cdot \frac{\pi^2}{6} & \text{if} \quad g < \alpha \le 1 \end{cases} \tag{1}$$

where $g = \frac{\sqrt{5}-1}{2}$. It is easy to see that the definition of T_α can be extended to $0 \le \alpha \le 1/2$ and a number of papers have been working for the behavior of $h(T_\alpha)$ for $0 < \alpha < 1/2$. Here we note that, in the case of $\alpha = 0$, T_α has an absolutely continuous ergodic invariant measure of infinite volume. In 1999, P. Moussa, A. Cassa, and S. Marmi [15] extended the above result to $[\sqrt{2} - 1, 1/2)$, i.e. for $\sqrt{2} - 1 \le \alpha < 1/2$, $h(T_\alpha) = h(T_{1/2})$ also holds. Then, in 2008, L. Luzzi and S. Marmi [14] showed the existence of the absolutely continuous invariant probability measure, which is ergodic, for $0 < \alpha < \sqrt{2} - 1$ and $\lim_{\alpha\to 0} h(T_\alpha) = 0$. They

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also observed by computer simulation that $h(T_\alpha)$ is not monotone on $(0, 1/2)$ as a function of α. After this observation, the author and R. Natsui [18] proved the following : there exist three sequences of sub-intervals $\{I_{1,\ell}\}$, $\{I_{2,\ell}\}$, $\{I_{3,\ell}\}$, $1 \le \ell < \infty$, in $(0, g^2)$ such that for each $\ell \ge 1$ $h(T_\alpha)$ is strictly increasing on $I_{1,\ell}$, constant on $I_{2,\ell}$, and decreasing on $I_{3,\ell}$. After this result, there are several papers concerning the details of the behavior of $h(T_\alpha)$, e.g. [6, 7, 12, 22]. The methods used in these papers were mostly based on the construction of the natural extension of T_α as a planer map and the property which is called the "matching". These ideas were appeared in [23, 16] at first.

Since $h(T_\alpha) = h(T_{1/2})$ for $g^2 \le \alpha \le g$ (see [12]), it is quite natural to ask whether $h(T_\alpha)$, $g^2 \le \alpha \le g$ takes the maximum value (the question in [12, p. 2242]). Because of (1), the question was $h(T_\alpha) < h(T_{1/2})$ holds for $0 < \alpha < g^2$ or not. This is intuitively obvious by the computer simulation in [14]. In this paper, we show that this is certainly true. Our result (Theorem 2 in §3) implies the following:

Main Result (in §3) *The maximum value of the entropy of α-continued fraction maps is* $\frac{1}{\log(g+1)}\frac{\pi^2}{6}$ *and* $h(T_\alpha) = \frac{1}{\log(g+1)}\frac{\pi^2}{6}$ *holds if and only if* $g^2 \le \alpha \le g$.

To prove this result, we do not use neither the natural extension nor the matching property, but use the idea of the geodesic continued fractions introduced in [3]. In this point, the fact "$h(T_\alpha) = \frac{1}{\log(g+1)}\frac{\pi^2}{6}$ holds for $g^2 \le \alpha \le g$" can be proved simpler than that of [12].

To define the geodesic continued fractions, we start with the definition of the Farey graph on $\mathbb{Q} \cup \{\infty\}$. Suppose two rational numbers $r_j = \frac{s_j}{t_j}$, $j = 1, 2$, $s_j, t_j \in \mathbb{Z}$, $(s_j, t_j) = 1$, satisfy the condition $s_1 t_2 - t_1 s_2 = \pm 1$. Then we call r_1 and r_2 are adjacent and consider that there is an edge which connects r_1 and r_2. In this way we have a graph on the set of rational numbers with $\infty = 1/0$. We call this graph the Farey graph. We see that this is a connected graph defined on $\mathbb{Q} \cup \{\infty\}$. For any pair of rational numbers r_1 and r_2 the minimum path which connects them is said to be a geodesic path. For $A \in \mathrm{GL}(2, \mathbb{Z})$, if rational numbers r_1 and r_2 are adjacent then $A(r_1)$ and $A(r_2)$ are also adjacent. Here $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ acts on $\mathbb{Q} \cup \{\infty\}$ as a linear fractional transformation : $x \mapsto \frac{ax+b}{cx+d}$. Thus the image of the Farey graph by any $A \in \mathrm{GL}(2, \mathbb{Z})$ is also the Farey graph. If a rational number r has a continued fraction expansion

$$r = a_0 + \frac{\varepsilon_1 |}{| a_1} + \frac{\varepsilon_2 |}{| a_2} + \frac{\varepsilon_3 |}{| a_3} + \cdots + \frac{\varepsilon_k |}{| a_k}, \quad a_i \in \mathbb{N},\ \varepsilon_i = \pm 1 \text{ for } 1 \le i \le k,\ a_0 \in \mathbb{Z}$$

and there is no continued fraction expansion of r of this form such that its expansion length is less than k, we see that the length of the geodesic path from ∞ to r is equal to $k + 1$. We explain this fact in the next section more precisely. In general, there are a number of geodesic paths from ∞ to a fixed rational number r, see [3]. We combine the notion of the geodesic path and α-continued fraction expansion of a generic point (irrational number) x to prove our main result, which is given in §3. It should be noted that in our method we use the existence of the Legndre constant of the nearest integer continued fractions in addition to the condition for continued fraction maps being geodesic type. Finally, in §4, we give some examples of a 1-parameter family of continued fraction maps for which the same method works.

2. Some definitions and notations

We start with some basic definitions. We fix $\alpha \in (0,1)$. We define

$$\varepsilon_{\alpha,n}(x) = \text{sgn}\,(T_\alpha^{n-1}(x))$$

and

$$a_{\alpha,n}(x) = \begin{cases} \left\lfloor \frac{1}{T_\alpha^{n-1}(x)} + 1 - \alpha \right\rfloor & \text{if} \quad T_\alpha^{n-1}(x) \neq 0 \\ 0 & \text{if} \quad T_\alpha^{n-1}(x) = 0 \end{cases}$$

for $n \geq 1$. Then we have

$$x = \frac{\varepsilon_{\alpha,1}(x)|}{|a_{\alpha,1}(x)} + \frac{\varepsilon_{\alpha,2}(x)|}{|a_{\alpha,2}(x)} + \cdots + \frac{\varepsilon_{\alpha,n}(x)|}{|a_{\alpha,n}(x)} + \cdots$$

for $x \in [\alpha-1,\alpha) \setminus \{0\}$ and the right side terminates at some positive integer n if and only if x is a rational number. We call this expansion the α-continued fraction expansion of x. We put

$$\begin{pmatrix} p_{\alpha,n-1}(x) & p_{\alpha,n}(x) \\ q_{\alpha,n-1}(x) & q_{\alpha,n}(x) \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_{\alpha,1}(x) \\ 1 & a_{\alpha,1}(x) \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_{\alpha,2}(x) \\ 1 & a_{\alpha,2}(x) \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon_{\alpha,n}(x) \\ 1 & a_{\alpha,n}(x) \end{pmatrix} \tag{2}$$

for $n \geq 1$ and $a_{\alpha,n}(x) \neq 0$ and

$$\begin{pmatrix} p_{\alpha,-1}(x) & p_{\alpha,0}(x) \\ q_{\alpha,-1}(x) & q_{\alpha,0}(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From (2), we have

$$\frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)} = \frac{\varepsilon_{\alpha,1}(x)|}{|a_{\alpha,1}(x)} + \frac{\varepsilon_{\alpha,2}(x)|}{|a_{\alpha,2}(x)} + \cdots + \frac{\varepsilon_{\alpha,n}(x)|}{|a_{\alpha,n}(x)}, \quad n \geq 1$$

and call $\frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)}$ the n-th convergent (of the α-continued fraction expansion) of x. Also from (2), we see $|p_{\alpha,n-1}(x)q_{\alpha,n}(x) - q_{\alpha,n-1}(x)p_{\alpha,n}(x)| = 1$, which means $\frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)}$ and $\frac{p_{\alpha,n-1}(x)}{q_{\alpha,n-1}(x)}$ are adjacent. Thus the α-continued fraction expansion of x gives a path from ∞ to $\frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)}$ on the Farey graph for every $n \geq 1$:

$$\infty \to 0 \to \frac{p_{\alpha,1}(x)}{q_{\alpha,1}(x)} \to \frac{p_{\alpha,2}(x)}{q_{\alpha,2}(x)} \to \cdots \to \frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)}.$$

In general we consider $\begin{pmatrix} p_{\alpha,n-1} & p_{\alpha,n} \\ q_{\alpha,n-1} & q_{\alpha,n} \end{pmatrix}$ by (2) when $\varepsilon_{\alpha,1}, \ldots, \varepsilon_{\alpha,n}$ and $a_{\alpha,1}, \ldots, a_{\alpha,n}$ are given without x. For given sequences of ± 1 and positive integers, $\hat{\varepsilon}_{\alpha,1}, \ldots, \hat{\varepsilon}_{\alpha,n}$ and $b_{\alpha,1}$, $b_{\alpha,2}, \ldots, b_{\alpha,n}$, respectively, we denote by $\left\langle \begin{matrix} \hat{\varepsilon}_{\alpha,1} \\ b_{\alpha,1} \end{matrix}, \begin{matrix} \hat{\varepsilon}_{\alpha,2} \\ b_{\alpha,2} \end{matrix}, \ldots, \begin{matrix} \hat{\varepsilon}_{\alpha,n} \\ b_{\alpha,n} \end{matrix} \right\rangle$ the associated cylinder set, i.e.

$$\left\langle \begin{matrix} \hat{\varepsilon}_{\alpha,1} \\ b_{\alpha,1} \end{matrix}, \begin{matrix} \hat{\varepsilon}_{\alpha,2} \\ b_{\alpha,2} \end{matrix}, \ldots, \begin{matrix} \hat{\varepsilon}_{\alpha,n} \\ b_{\alpha,n} \end{matrix} \right\rangle$$
$$= \{x \in [\alpha-1,\,\alpha) : \varepsilon_{\alpha,1}(x) = \hat{\varepsilon}_{\alpha,1}, \ldots, \varepsilon_{\alpha,n}(x) = \hat{\varepsilon}_{\alpha,n}, a_{\alpha,1}(x) = b_{\alpha,1}, \ldots, a_{\alpha,n}(x) = b_{\alpha,n}\}.$$

A sequence

$$\begin{pmatrix} \hat{\varepsilon}_{\alpha,1} \\ b_{\alpha,1} \end{pmatrix}, \begin{pmatrix} \hat{\varepsilon}_{\alpha,2} \\ b_{\alpha,2} \end{pmatrix}, \ldots, \begin{pmatrix} \hat{\varepsilon}_{\alpha,n} \\ b_{\alpha,n} \end{pmatrix}$$

is said to be admissible if the associated cylinder set has an inner point, here we note that any cylinder set is an interval (or the empty set). The transpose of (2) shows that

$$\frac{q_{\alpha,n-1}(x)}{q_{\alpha,n}(x)} = \frac{1}{|a_{\alpha,n}(x)} + \frac{\varepsilon_{\alpha,n}(x)|}{|a_{\alpha,n-1}(x)} + \cdots + \frac{\varepsilon_{\alpha,2}(x)|}{|a_{\alpha,1}(x)}.$$

For a given rational number r with $r = \frac{\varepsilon_{\alpha,1}|}{|a_{\alpha,1}} + \frac{\varepsilon_{\alpha,2}|}{|a_{\alpha,2}} + \cdots + \frac{\varepsilon_{\alpha,n}|}{|a_{\alpha,n}}$,
$\begin{pmatrix} \varepsilon_{\alpha,1} \\ a_{\alpha,1} \end{pmatrix}, \begin{pmatrix} \varepsilon_{\alpha,2} \\ a_{\alpha,2} \end{pmatrix}, \ldots, \begin{pmatrix} \varepsilon_{\alpha,n} \\ a_{\alpha,n} \end{pmatrix}$ is said to be geodesic if the path

$$\infty \to 0 \to \frac{p_{\alpha,1}}{q_{\alpha,1}} \to \frac{p_{\alpha,2}}{q_{\alpha,2}} \to \frac{p_{\alpha,3}}{q_{\alpha,3}} \to \cdots \to \frac{p_{\alpha,n}}{q_{\alpha,n}} = r$$

is a geodesic path from ∞ to r.

DEFINITION 1. The α-continued fraction expansion of $x \in [\alpha - 1, \alpha) \setminus \{0\}$ is said to be geodesic if for any $n \geq 1$,

$$\begin{pmatrix} \varepsilon_{\alpha,1}(x) \\ a_{\alpha,1}(x) \end{pmatrix}, \begin{pmatrix} \varepsilon_{\alpha,2}(x) \\ a_{\alpha,2}(x) \end{pmatrix}, \ldots, \begin{pmatrix} \varepsilon_{\alpha,n}(x) \\ a_{\alpha,n}(x) \end{pmatrix}$$

is geodesic. Moreover, T_α is said to be geodesic type if any α-continued fraction expansion of $x \in [\alpha - 1, \alpha) \setminus \{0\}$ is geodesic.

For any $\varepsilon_{\alpha,k} = \pm 1$ and positive integers $a_{\alpha,k}$, $1 \leq k \leq n$, we see

$$\begin{aligned} &\frac{\varepsilon_{\alpha,1}(x)|}{|a_{\alpha,1}(x)} + \frac{\varepsilon_{\alpha,2}(x)|}{|a_{\alpha,2}(x)} + \cdots + \frac{\varepsilon_{\alpha,n}(x)|}{|a_{\alpha,n}(x)} \\ =&\frac{1}{|\varepsilon_{\alpha,1}(x)a_{\alpha,1}(x)} + \frac{1}{|\varepsilon_{\alpha,1}(x)\varepsilon_{\alpha,2}(x)a_{\alpha,2}(x)} + \cdots + \frac{1}{|\Pi_{k=1}^{n}\varepsilon_{\alpha,k}(x) \cdot a_{\alpha,n}(x)}. \end{aligned}$$

Thus the α-continued fraction expansion can be rewritten to the ICF expansion in [3]. In this way, we have a simple version of [3, Theorem 1.3].

Lemma 1. *A continued fraction*

$$\frac{\varepsilon_{\alpha,1}|}{|a_{\alpha,1}} + \frac{\varepsilon_{\alpha,2}|}{|a_{\alpha,2}} + \cdots + \frac{\varepsilon_{\alpha,n}|}{|a_{\alpha,n}} \qquad (\varepsilon_{\alpha,k} = \pm 1,\ a_{\alpha,k} > 0,\ 1 \leq k \leq n)$$

is geodesic if and only if the following two conditions hold :

1. *For any* $1 \leq k \leq n$, $a_{\alpha,k} \neq 1$.
2. *For any* $1 \leq k < \ell \leq n$,

$$\begin{pmatrix} \varepsilon_{\alpha,k} \\ a_{\alpha,k} \end{pmatrix}, \begin{pmatrix} \varepsilon_{\alpha,k+1} \\ a_{\alpha,k+1} \end{pmatrix}, \ldots, \begin{pmatrix} \varepsilon_{\alpha,\ell} \\ a_{\alpha,\ell} \end{pmatrix}$$

is not equal to

$$\left(\frac{-1}{2}, \underbrace{\frac{-1}{3}, \ldots, \frac{-1}{3}}_{\ell-1}, \frac{-1}{2}\right), \tag{3}$$

$\left(\text{if } \ell = k+1, \text{ then this means} \left(\frac{-1}{2}, \frac{-1}{2}\right)\right)$.

If $a_{\alpha,n} = 1$ appears, then it is easy to see that T_α can not be geodesic type since

$$\frac{1|}{|a} + \frac{1|}{|1} + \frac{1|}{|b} = \frac{1\ \ |}{|a+1} + \frac{-1\ \ |}{|b+1}$$

holds, (see [11, 16] for the related discussions). We never have $(\varepsilon_{\alpha,k}(x), a_{\alpha,k}(x)) = (-1, 1)$ for $x \in [\alpha-1, \alpha)$ and $k \geq 1$ $(0 < \alpha < 1)$. If (3) appears for some $\ell \geq 1$, then we can also shorten it, (see Lemma 3 in §3).

The following theorem is a direct consequence of Lemma 1.

Theorem 1. *The α-continued fraction map T_α is geodesic type if and only if $g^2 \leq \alpha \leq g$ with $g = \frac{\sqrt{5}-1}{2}$.*

Proof. It is easy to see that $a_{\alpha,n}(x) = 1$ is only possible for $\alpha > g$, otherwise $(0 < \alpha \leq g)$ $a_{\alpha,n}(x) \geq 2$ for any $x \in [\alpha-1, \alpha) \setminus \{0\}$ and $n \geq 1$. This implies that T_α is not geodesic type if $\alpha > g$. It is also easy to see that for $\alpha = g^2$

$$\alpha - 1 = \frac{-1|}{|\ 2} + \frac{-1|}{|\ 3} + \frac{-1|}{|\ 3} + \cdots + \frac{-1|}{|\ 3} + \cdots .$$

This shows that for $\alpha \geq g^2$, the sequence of the form

$$\left(\frac{-1}{2}, \frac{-1}{3}, \frac{-1}{3}, \ldots, \frac{-1}{3}, \frac{-1}{2}\right)$$

never appear in any α-continued fractions. This means that T_α is geodesic type for $g^2 \leq \alpha \leq g$. On the other hand, if $0 < \alpha < g^2$, then there exists $\ell \geq 0$ such that

$$\alpha - 1 = \frac{-1|}{|\ 2} + \frac{-1|}{|\ 3} + \frac{-1|}{|\ 3} + \cdots + \frac{-1|}{|\ 3} + \cdots + \frac{-1|}{|\ 2} + \frac{\varepsilon_{\alpha,\ell+2}(\alpha-1)|}{|a_{\alpha,\ell+2}(\alpha-1)} + \frac{\varepsilon_{\alpha,\ell+3}(\alpha-1)|}{|a_{\alpha,\ell+3}(\alpha-1)} + \cdots .$$

Hence we see from Lemma 1 that T_α is not geodesic type. This completes the proof of this theorem. □

3. Main Result

First we show the following basic lemma, which is proved in [16] for $1/2 \leq \alpha \leq 1$ and can be easily extended to $0 < \alpha \leq 1$.

Lemma 2. *For a.e. $x \in [\alpha-1, \alpha)$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log q_{\alpha,n}(x) = -\int_{[\alpha-1,\alpha)} \log |x| d\mu_\alpha .$$

Proof. Since the proof is exactly the same as in [16, Proposition 2] in the case of $1/2 \le \alpha \le 1$, we only give a sketch of the proof for $0 < \alpha < 1/2$. It is easy to see that

$$\frac{1}{2q_{\alpha,n+1}^2(x)} < \left| x - \frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)} \right| < \frac{1}{(1-\alpha)q_{\alpha,n}^2(x)} \tag{4}$$

for any $x \in [\alpha-1, \alpha) \setminus \{0\}$ and $n \ge 1$. For each α, $0 < \alpha < 1/2$, there exists a positive integer m such that if

$$\Bigg(\underbrace{\begin{matrix} -1 \\ 2 \end{matrix}, \begin{matrix} -1 \\ 2 \end{matrix}, \dots, \begin{matrix} -1 \\ 2 \end{matrix}}_{\ell} \Bigg)$$

is admissible, then $\ell \le m$. This shows that there exist $D > 1$ and $C_1 > 0$ such that

$$q_{\alpha,n}(x) > C_1 \cdot D^n \text{ and } |p_{\alpha,n}| > C_1 \cdot D^n$$

for any $n \ge 1$ and $x \in [\alpha - 1, \alpha)$. Thus, from (4), we can find a constant $C_2 > 0$ such that

$$\left| \log|x| - \log\left| \frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)} \right| \right| \le C_2 D^{-2n}$$

for $n \ge 1$. Thus we have

$$\left| \log|T_\alpha^{\ell-1}(x)| - \log\left| \frac{\varepsilon_{\alpha,\ell}(x)|}{|a_{\alpha,\ell}(x)} + \frac{\varepsilon_{\alpha,\ell+1}(x)|}{|a_{\alpha,\ell+1}(x)} + \cdots + \frac{\varepsilon_{\alpha,n}(x)|}{|a_{\alpha,n}(x)} \right| \right| \le C_2 \cdot D^{-2(n+1-\ell)}.$$

for $1 \le \ell \le n$. From (2), we see

$$\frac{1}{q_{\alpha,n}(x)} = \Pi_{k=1}^n \varepsilon_{\alpha,k}(x) \Pi_{\ell=1}^n \left(\frac{\varepsilon_{\alpha,\ell}(x)|}{|a_{\alpha,\ell}(x)} + \frac{\varepsilon_{\alpha,\ell+1}(x)|}{|a_{\alpha,\ell+1}(x)} + \cdots + \frac{\varepsilon_{\alpha,n}(x)|}{|a_{\alpha,n}(x)} \right).$$

Hence we see

$$-\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \log\left|T_\alpha^{\ell-1}(x)\right| = \lim_{n\to\infty} \frac{1}{n} \log q_{\alpha,n}(x)$$

when one of the limits exists. By the ergodic theorem, the limit of the left side exists for a.e. $x \in [\alpha - 1, \alpha)$. Thus we get the assertion of the theorem. □

REMARK. Here we recall that $h(T_\alpha) = -2 \int_{[\alpha-1,\alpha)} \log|x| d\mu_\alpha$, see [14] for example. Thus we have

$$h(T_\alpha) = 2 \lim_{n\to\infty} \log q_{\alpha,n}(x) \qquad a.e. \tag{5}$$

Now we use the following, which concerns the second condition of Lemma 1.

Lemma 3. *We have*

$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}^k \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}^k \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ k \ge 0,$$

which is equivalent to

$$\frac{-1|}{|\,2} + \underbrace{\frac{-1|}{|\,3} + \cdots + \frac{-1|}{|\,3}}_{k} + \frac{-1|}{|\,2} = -1 + \frac{1|}{|3} + \underbrace{\frac{-1|}{|\,3} + \cdots + \frac{-1|}{|\,3}}_{k}.$$

Proof. Proof.It is easy to see that

$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

holds. Then we get the assertion by induction. □

This lemma shows that when

$$\left(\begin{matrix} -1 \\ 2 \end{matrix}, \begin{matrix} -1 \\ 3 \end{matrix}, \ldots, \begin{matrix} -1 \\ 3 \end{matrix}, \begin{matrix} -1 \\ 2 \end{matrix} \right)$$

appears, we can shorten the length of the continued fraction expansion one size. Then we can apply the ergodic theorem to get the following:

Theorem 2. 1. *For $g^2 \le \alpha \le 1/2$, we have $h(T_\alpha) = h(T_{1/2})$.*
2. *For any $0 < \alpha < g^2$, we have $h(T_\alpha) < h(T_{1/2}) = \frac{1}{\log(g+1)} \frac{\pi^2}{6}$.*

Proof. In this proof, we compare the increasing rate of $q_{\alpha,n}(x)$ and that of $q_{1/2,n}(x)$. Because of (5), $h(T_\alpha) < h(T_{1/2})$ or $h(T_\alpha) = h(T_{1/2})$ is equivalent to $\lim_{n\to\infty} \frac{1}{n} \log q_{\alpha,n}(x) < \lim_{n\to\infty} \frac{1}{n} \log q_{1/2,n}(x)$ or $\lim_{n\to\infty} \frac{1}{n} \log q_{\alpha,n}(x) = \lim_{n\to\infty} \frac{1}{n} \log q_{1/2,n}(x)$ for a "typical" irrational number x, respectively. First we show the second assertion of the theorem. Then we see that the first assertion follows by the similar way. Since $\alpha < g^2$, there exists a nonnegative integer k such that

$$\left(\begin{matrix} -1 \\ 2 \end{matrix}, \underbrace{\begin{matrix} -1 \\ 3 \end{matrix}, \ldots, \begin{matrix} -1 \\ 3 \end{matrix}}_{k}, \begin{matrix} -1 \\ 2 \end{matrix} \right)$$

is an admissible sequence. This follows from the expansions of $-g^2$ and $g^2 - 1$:

$$-g^2 = \frac{-1|}{|\,3} + \frac{-1|}{|\,3} + \cdots + \frac{-1|}{|\,3} + \cdots, \qquad g^2 - 1 = \frac{-1|}{|\,2} + \frac{-1|}{|\,3} + \frac{-1|}{|\,3} + \cdots + \frac{-1|}{|\,3} + \cdots.$$

Then, for $\alpha < g^2$, we can find k since T_α is order preserving and expanding on $[\alpha - 1, \frac{-1}{2+\alpha})$ and on $[\frac{-1}{2+\alpha}, -g^2]$. Suppose that

$$\begin{aligned} &\left(\begin{matrix} \varepsilon_{\alpha,n+1}(x) \\ a_{\alpha,n+1}(x) \end{matrix}, \begin{matrix} \varepsilon_{\alpha,n+2}(x) \\ a_{\alpha,n+2}(x) \end{matrix}, \ldots, \begin{matrix} \varepsilon_{\alpha,n+k+1}(x) \\ a_{\alpha,n+k+1}(x) \end{matrix}, \begin{matrix} \varepsilon_{\alpha,n+k+2}(x) \\ a_{\alpha,n+k+2}(x) \end{matrix} \right) \\ &= \left(\begin{matrix} -1 \\ 2 \end{matrix}, \underbrace{\begin{matrix} -1 \\ 3 \end{matrix}, \ldots, \begin{matrix} -1 \\ 3 \end{matrix}}_{k}, \begin{matrix} -1 \\ 2 \end{matrix} \right) \end{aligned}$$

for $x \in [\alpha - 1, \alpha)$. Then

$$T_\alpha^n(x) = \frac{-1|}{|\,2} + \underbrace{\frac{-1|}{|\,3} + \cdots + \frac{-1|}{|\,3}}_{k} + \frac{-1|}{|\,2} + \frac{\varepsilon_{\alpha,n+k+3}(x)|}{|\,a_{\alpha,n+k+3}(x)} + \cdots$$

and the path

$$\frac{p_{\alpha,n-1}(x)}{q_{\alpha,n-1}(x)} \to \frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)} \to \frac{p_{\alpha,n+1}(x)}{q_{\alpha,n+1}(x)} \to \cdots \to \frac{p_{\alpha,n+k+2}(x)}{q_{\alpha,n+k+2}(x)}$$

is mapped to

$$\infty \to 0 \to \frac{-1}{2} \to \frac{-3}{5} \to \cdots \to \frac{p}{q} \tag{6}$$

with

$$\frac{p}{q} = \frac{-1|}{|\,2} + \frac{-1|}{|\,3} + \cdots + \frac{-1|}{|\,3} + \frac{-1|}{|\,2}$$

by the linear fractional transformation associated with A :

$$A = \begin{pmatrix} 0 & \varepsilon_{\alpha,1}(x) \\ 1 & a_{\alpha,1}(x) \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon_{\alpha,n}(x) \\ 1 & a_{\alpha,n}(x) \end{pmatrix}. \tag{7}$$

The path (6) is of length $k+3$. From Lemma 3, there is a path from ∞ to $\frac{p}{q}$ of length $k+2$:

$$\infty \to -1 \to \frac{-2}{3} \to \frac{-5}{8} \to \cdots \to \frac{p}{q}.$$

We map this path by the linear fractional transformation defined by (7) and get a path from $\frac{p_{\alpha,n-1}(x)}{q_{\alpha,n-1}(x)}$ to $\frac{p_{\alpha,n+k+2}(x)}{q_{\alpha,n+k+2}(x)}$ of length $k+2$, which reduces by one length. Now we consider a "typical" x as follows. We choose $x \in [-1/2, \alpha) = [\alpha - 1, \alpha) \cap [-1/2, 1/2)$ so that

1.
$$\lim_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \le n \le N \ : \ T_\alpha^n(x) \in \left\langle \frac{-1}{2}, \underbrace{\frac{-1}{3}, \ldots, \frac{-1}{3}}_{k}, \frac{-1}{2} \right\rangle \right\}$$
$$= \mu_\alpha \left(\left\langle \frac{-1}{2}, \underbrace{\frac{-1}{3}, \ldots, \frac{-1}{3}}_{k}, \frac{-1}{2} \right\rangle \right),$$

2.
$$\lim_{N\to\infty} \frac{1}{N} \log q_{\alpha,N}(x) = \frac{h(T_\alpha)}{2},$$

3.
$$\lim_{N\to\infty} \frac{1}{N} \log q_{1/2,N}(x) = \frac{h(T_{1/2})}{2},$$

4. There are infinitely many $n \ge 1$ such that $a_{\alpha,n} > 10$.

Choosing x with the above four requirements is possible since (T_α, μ_α) and $(T_{1/2}, \mu_{1/2})$ are ergodic. Here we recall the fact that the right side of the condition 1 is positive, e.g. see [14]. For the validation of the requirements 1 and 4, we use the individual ergodic theorem and for 2 and 3, we use the Shannon-McMillan-Breiman-Chung theorem of entropy. [1] We put

$$\eta = \mu_\alpha\left(\left\langle \frac{-1}{2}, \underbrace{\frac{-1}{3}, \ldots, \frac{-1}{3}}_{k}, \frac{-1}{2} \right\rangle\right).$$

From the choice of x, there exists a subsequence of natural numbers $(N_k;\ k \geq 1)$ such that $a_{\alpha,N_k+1}(x) > 10$. For those N_k

$$\left| x - \frac{p_{\alpha,N_k}(x)}{q_{\alpha,N_k}(x)} \right| \leq \frac{1}{8q^2_{\alpha,N_k}(x)}.$$

Now we use the existence of the Legendre constant of the nearest integer continued fractions ([11]):

$$\text{if} \quad \left| x - \frac{p}{q} \right| < \frac{3-\sqrt{5}}{2}\frac{1}{q^2} \quad \text{then} \quad \frac{p}{q} = \frac{p_{1/2,n}}{q_{1/2,n}} \quad \text{for some} \quad n \geq 0$$

and see that

$$\frac{p_{\alpha,N_k}(x)}{q_{\alpha,N_k}(x)} = \frac{p_{1/2,M_k}(x)}{q_{1/2,M_k}(x)} \tag{8}$$

for some $M_k \geq 1$. Since $T_{1/2}$ is geodesic type, $M_k + 1$ is the length of a geodesic path from ∞ to $\frac{p_{\alpha,N_k}(x)}{q_{\alpha,N_k}(x)}$. Then the discussion in the above shows that for any ε, $\eta > \varepsilon > 0$, there exists k_0 such that

$$M_k \leq N_k\,(1 - (\eta - \varepsilon)) \tag{9}$$

holds for any $k \geq k_0$. Now

$$\frac{h(T_\alpha)}{2} = \lim_{k\to\infty} \frac{1}{N_k} \log q_{\alpha,N_k}(x) = \lim_{k\to\infty} \frac{M_k}{N_k} \cdot \frac{1}{M_k} \log q_{1/2,M_k}(x) = \lim_{k\to\infty} \frac{M_k}{N_k} \cdot \frac{h(T_{1/2})}{2}.$$

From (9), we see $\lim_{k\to\infty} \frac{M_k}{N_k} < 1$. Consequently we have $h(T_\alpha) < h(T_{1/2})$.

If $g^2 \leq \alpha \leq 1/2$, then (8) implies $N_k = M_k$, which shows the first assertion of the theorem. In this case, we only need the requirements 2, 3, and 4. □

4. Some other examples

Here are some other examples of 1-parameter family of continued fraction maps.

1. In 1981, S. Tanaka and S. Ito introduced another type of α-continued fraction map S_α for $1/2 \leq \alpha \leq 1$ ([23]) :

[1] We note that Shannon and McMillan discussed the convergence in probability, then L. Breiman [2] proved a.e. convergence in the case of finitely many states, and K. L. Chung [8] extended it to the infinitely many states concerning this theorem.

$$S_\alpha(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor & \text{if} \quad x \in [\alpha - 1, \alpha) \setminus \{0\} \\ 0 & \text{if} \quad x = 0. \end{cases}$$

The definition of S_α can be extended to $0 \le \alpha \le 1$. However S_α and $S_{1-\alpha}$ can be identified by $x \mapsto 1 - x$ and thus it is enough to consider the case $1/2 \le \alpha \le 1$. S. Tanaka and S. Ito constructed the natural extension map for each S_α, $1/2 \le \alpha \le g$ and showed that the entropy value w.r.t. the absolutely continuous invariant measure is constant on $[1/2, g]$. It is easy to see that 1 appears in the partial coefficients of this type of α-continued fractions if and only if $\alpha > g$. Thus by [3, Theorem 3.1], we have the following

Theorem 3. *Tanaka-Ito's α-continued fraction map S_α is geodesic type if and only if $g^2 \le \alpha \le g$.*

To prove this theorem, we use the fact that the digit 1 appears if and only if $\alpha > g$. More about the detail of the behavior of this class of continued fraction maps, we refer to C. Carminati, N. D. S. Langeveld, and W. Steiner [5] and H. Nakada and W. Steiner [19].

The same holds for the map V_α, $0 < \alpha < 1$:

$$V_\alpha(x) = \left\lfloor \frac{1}{x} + \alpha \right\rfloor - \frac{1}{x}$$

where $0 \ne x \in [\alpha - 1, \alpha)$ and $V_\alpha(0) = 0$. $g^2 \le \alpha \le g$. We see that V_α and $V_{1-\alpha}$ can be identified. In this case, $\pm 2, \mp 3, \pm 3, \mp 3, \ldots, \pm 3, \mp 2$ appears if and only if $\alpha > g$ if $1/2 \le \alpha \le 1$. Then, again, by [3, Theorem 3.1], we have the following

Theorem 4. *The continued fraction map V_α is geodesic type if and only if $g^2 \le \alpha \le g$.*

The same discussion is also possible for Katok-Ugarcovici's (a, b) continued fractions (see [10]).

2. In F. P. Boca and C. Merriman [1], they introduced a 1-parameter family of continued fractions with odd partial coefficients, with the parameter $\frac{\sqrt{5}-1}{2} \le \alpha \le \frac{\sqrt{5}+1}{2}$. In this case, the coefficients ± 1 appear for any parameter value α. The corresponding graph is not the Farey graph but its subgraph and each map associated to $\frac{\sqrt{5}-1}{2} \le \alpha \le \frac{\sqrt{5}+1}{2}$ is geodesic on this subgraph. We can extend the notion of geodesic to this subgraph. As [1] says, one can extend this type of continued fraction maps to the parameter value below $\frac{\sqrt{5}-1}{2}$ to 0. Then it turns out that for $\alpha < \frac{3-\sqrt{5}}{2}$, the associated map it is not geodesic type anymore since $\frac{1|}{|k} + \frac{-1|}{|1} + \frac{-1|}{|3} + \frac{-1|}{|1} = \frac{1|}{|k-2}$ or
$$\frac{1|}{|k} + \frac{-1|}{|1} + \underbrace{\frac{-1|}{|3} + \frac{-1|}{|3} + \cdots \frac{-1|}{|3}}_{p} + \cdots + \frac{-1|}{|1} = \frac{1|}{|k-2} + \frac{1|}{|3} + \underbrace{\frac{-1|}{|3} + \cdots + \frac{-1|}{|3}}_{p-2}$$
for some $k \ge 2$ can appear.
3. For the set of cusps of Hecke group of index k(≥ 3), we can define a graph similar

to the Farey graph, see [21]. Then we can get the geodesic and entropy results for α-Rosen continued fraction maps introduced by K. Dajani, C. Kraaikamp, and W. Steiner [9] and C. Kraaikamp, T. A. Schmidt, and I. Smeets [13]. However to discuss the metric theory of α-Rosen continued fraction maps, we have to show the existence of invariant measures with their ergodicity etc for all possible values of α. We will discuss the detail on another occasion.

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