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ERGODIC THEOREMS AND EXPONENTIAL DECAY OF SAMPLE PATHS FOR CERTAIN INTERACTING DIFFUSION SYSTEMS

Dedicated to Professor T. Watanabe on occasion of his 60th birthday

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1. Introduction and main results

Let S be a countable space. In the present paper we treat a class of diffusion processes taking values in a suitable subspace of R^S, which are governed by the following stochastic differential equation (SDE):

$$dx_i(t) = \sum_{j\in S} A_{ij}x_j(t)dt + a(x_i(t))dB_i(t), \quad i\in S, \tag{1.1}$$

where $\{B_i(t)\}_{i\in S}$ is an independent system of one-dimensional standard $\{\mathcal{F}_t\}$-Brownian motions defined on a complete probability space with filtration $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

We here assume

(1.2) $A=(A_{ij})$ is an $S\times S$ real matrix satisfying that $A_{ij}\geqq 0$ for $i\neq j$, $-A_{ii}=\sum_{j\neq i} A_{ij}<\infty$, and $\sup_{i\in S} |A_{ii}|<\infty$,

(1.3) $a(u)$: $R\to R$ is a locally 1/2-Hölder continuous function satisfying a linear growth condition: for some $C>0$,

$$|a(u)|\leq C(1+|u|) \quad \text{for } u\in R.$$

The diffusion models described by the SDE (1.1) arise in various fields such as mathematical biology and statistical physics. We here list several examples.

EXAMPLE 1. (Stepping stone model with random drift [10])

$$a(u) = \begin{cases} \sqrt{u(1-u)} & \text{for } 0\leq u\leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 2. (Stepping stone model with radom selection [8])

$$a(u) = \begin{cases} u(1-u) & \text{for } 0\leq u\leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

In these two examples $x=(x_i(t))$ in the SDE (1.1) describes a time evolution of gene frequencies of a specified genotype at each colony, A_{ij} means migration rate from the j-th colony to the i-th one, and $a(u)$ comes from the effect of random sampling drift in the example 1 and random selection in the example 2.

EXAMPLE 3. (Branching diffusion model)

$$a(u)=\begin{cases}\sqrt{u} & \text{for } u\geq 0\\ 0 & \text{otherwise.}\end{cases}$$

EXAMPLE 4. (Scalar field in non-stationary random potential [12]) Let $S=Z^d$,

$$a(u)=\begin{cases}u & \text{for } u\geq 0\\ 0 & \text{otherwise.}\end{cases}$$

$$A_{ij}=\begin{cases}\kappa>0 & \text{if } |i-j|=1,\\ -2d\kappa & \text{if } i=j\\ 0 & \text{otherwise.}\end{cases}$$

EXAMPLE 5. (Ornstein-Uhlenbeck type process)

$$c_1\leq a(u)\leq c_2 \quad \text{for } u\in R \text{ with constants } 0<c_1<c_2<\infty.$$

For the example 1 the ergodic behaviors were extensively studied in [10], [11], whose phenomena are very similar to those of the voter model. For the example 4 with a small $\kappa>0$ it was shown in [12] exponential decay of the sample paths, from which it follows that the extinction occurs in any dimension in the sense of Liggett's book [6], Chap. IX. For a class of diffusion models including the examples 1 and 2 some ergodic behaviors were investigated in [8], and furthermore Cox and Greven [1] recently obtained a complete description of Z^d-translation invariant stationary distributions for the same model in the case $S=Z^d$. For the example 3 we refer [2] which treats a corresponding continuum space model. The example 5 is a generalization of Ornstein-Uhlenbeck process where $a(u)$ is constant.

In this paper we restrict our consideration to the case $S=Z^d$ (d-dimensional cubic lattice space) and $A=(A_{ij})$ is Z^d-translation invariant, namely

$$A_{ij}=A_{0,j-i}\equiv A_{j-i} \quad \text{for } i,j\in Z^d. \tag{1.4}$$

To formulate a diffusion process associated with the SDE (1.1) we first specify the state space as follows.

Let $\gamma=(\gamma_i)$, $i\in Z^d$ be a positive summable sequence over Z^d such that for some $C>0$,

$$\sum_{i\in Z^d}\gamma_i|A_{ij}|\leq C\gamma_j \quad \text{for } j\in Z^d. \tag{1.5}$$

We note that for a given $A=(A_{ij})$ with (1.4) one can easily construct a positive summable sequence $\gamma=(\gamma_i)$ satisfying (1.5).

Let $L^2(\gamma)$ be the Hilbert space of all square γ-summable sequences over Z^d with the Hilbertian norm $|\cdot|_\gamma$ i.e.

$$L^2(\gamma)=\{x=(x_i)_{i\in Z^d}\mid \sum_{i\in Z^d}\gamma_i|x_i|^2=|x|_\gamma^2<\infty\}.$$

Under the assumptions (1.2)-(1.5) it is known that for each $x(0)\in L^2(\gamma)$, there exists a unique strong solution $(x(t)=(x_i(t)), (B_i(t)))$ of the SDE (1.1) such that

$$P(x(t) \text{ is } L^2\text{-}(\gamma)\text{-valued strongly continuous in } t\geq 0)=1. \quad (\text{ct. [8]})$$

The solution defines a diffusion process $(\Omega, \mathcal{F}, \mathcal{F}_t, P^x, x(t))$ taking values in $L^2(\gamma)$, and its transition probability defines a Feller Markov semi-group T_t acting on $C_b(L^2(\gamma))$, (the totality of bounded continous functions defined on $L^2(\gamma)$) such that

$$T_tf-f=\int_0^t T_sLfds \quad \text{for } f\in C_0^2(L^2(\gamma)), \tag{1.6}$$

where $C_0^2(L^2(\gamma))$ stands for the totality of such C^2-functions f defined on $L^2(\gamma)$ with bounded derivatives and Lf being bounded, which depend on finitely many coordinates, and

$$Lf(x)=\frac{1}{2}\sum_{i\in Z^d}a(x_i)^2\frac{\partial^2 f}{\partial x_i^2}+\sum_{i\in Z^d}(\sum_{j\in Z^d}A_{ij}x_j)\frac{\partial f}{\partial x_i}. \tag{1.7}$$

Let $\mathcal{P}=\mathcal{P}(L^2(\gamma))$ be the totality of probability measures on $L^2(\gamma)$ which is equipped with the topology of weak convergence. T_t induces the dual semi-group T_t^* acting on $\mathcal{P}$ by

$$\langle T_t^*\mu, f\rangle=\langle\mu, T_tf\rangle \quad \text{for } f\in C_b(L^2(\gamma)) \tag{1.8}$$

where $\langle\mu, f\rangle=\int_{L^2(\gamma)}f(x)\mu(dx)$.

Let $\mathcal{S}$ be the totality of stationary distributions for the deffusion process $(\Omega, \mathcal{F}, \mathcal{F}_t, P^x, x(t))$; i.e. $\mathcal{S}=\{\mu\in\mathcal{P}\mid T_t^*\mu=\mu \text{ for } t>0\}$.

Note that by (1.5) with (1.4) the Z^d-translation group acts on $L^2(\gamma)$, and let $\mathcal{I}$ be the totality of Z^d-translation invariant probability measures on $L^2(\gamma)$. For $\alpha>0$ we also denote by $\mathcal{I}_\alpha$ the totality of elements of $\mathcal{I}$ which have finite α-order absolute moments, i.e.

$$\mathcal{I}_\alpha=\{\mu\in\mathcal{I}\mid\langle\mu, |x_i|^\alpha\rangle<\infty(i\in Z^d)\}.$$

These sets are closed and convex, so we use the notation $\mathcal{C}_{\text{ext}}$ for a convex set $\mathcal{C}$, which denotes the totality of extremal elements of $\mathcal{C}$. If $\mathcal{C}$ is a compact and covex set, the convex closure of $\mathcal{C}_{ext}$ coincides with $\mathcal{C}$ by Krein-Milman's theorem (cf. [5]). However notice that $\mathcal{S}$, $\mathcal{I}$ and $\mathcal{I}_{\alpha}$ are not compact in general.

In this paper we will first obtain a complete description of Z^d-translation invariant stationary distributions under the following assumption: Let $(\Omega, \mathcal{B}, \boldsymbol{P}_i, \xi_t)$ be the continuous time Markov process taking values in Z^d generated by the infinitesimal matrix A, and let $P_t=(P_t(i,j))$ be its transition probability.

ASSUMPTION [A]

(1.9) $A=(A_{ij}=A_{j-i})$ is irreducible, and the symmetrized Markov process of $(\Omega, \mathcal{B}, \boldsymbol{P}_i, \xi_t)$, which is a Markov process taking values in Z^d generated by $A^S=A+A^*=(A^S_{ij}=A_{i-j}+A_{j-i})$, is transient, and

$$\limsup_{|u|\to\infty} \frac{|a(u)|}{|u|} < G^S(0)^{-1/2} \tag{1.10}$$

where G^S is the potential matrix of the symmetrised Markov process, i.e.

$$G^S(0)=\int_0^\infty P^S_t(0)dt \quad \text{with} \quad P^S_t(i)=P^S_t(j,j+i)=\sum_{k\in Z^d} P_t(i,k)P_t(i+j,k) \quad (i\in Z^d). \tag{1.11}$$

Then we obtain the following result.

Theorem 1.1. *Assume the assumption* [A]. *Then*

(i) *For each* $\theta\in R$, $\lim_{t\to\infty} T^*_t\delta_0=\nu_\theta$ *exists and* $\langle \nu_\theta, x_i\rangle=\theta$ *for* $i\in Z^d$, *where* $\boldsymbol{\theta}=(x_i\equiv\theta)\in L^2(\gamma)$ *and* δ_θ *stands for the Dirac measure at* $\boldsymbol{\theta}$.

(i) $(\mathcal{S}\cap\mathcal{I}_1)_{ext}=\{\nu_\theta | \theta\in R\}$. *For every* $\nu\in\mathcal{S}\cap\mathcal{I}_1$, *there exists an* $m\in\mathcal{P}(R)$ (*the totality of probability measures on* R) *such that*

$$\nu=\int_R \nu_\theta m(d\theta)\,.$$

(iii) *If* $\mu\in\mathcal{I}_1$ *is ergodic with respect to the* Z^d-*translation group, then*

$$\lim_{t\to\infty} T^*_t\mu=\nu_\theta \quad with \quad \theta=\langle\mu, x_i\rangle\,.$$

Moreover for every $\mu\in\mathcal{I}_1$, $\lim_{t\to\infty} T^*_t\mu$ *exists in* $\mathcal{S}\cap\mathcal{I}_1$.

REMARK 1. In addition to the assumption [A], suppose that $a(\theta_0)=0$ for some $\theta_0\in R$. The state space $L^2(\gamma)$ of the diffusion process $(\Omega, \mathcal{F}, \mathcal{F}_t, P^x, x(t))$ contains two invariant subspaces

$$\mathcal{X}_+(\theta_0)=L^2(\gamma)\cap[\theta_0, \infty)^{Z^d} \quad \text{and} \quad \mathcal{X}_-(\theta_0)=L^2(\gamma)\cap(-\infty, \theta_0]^{Z^d}.$$

So, if one takes $\mathcal{X}_+(\theta_0)$ (or $\mathcal{X}_-(\theta_0)$) as the state space of the diffusion process $(\Omega, \mathcal{F}, \mathcal{F}_1, P^x, x(t))$, Theorem 1.1 (ii) can be refined as follows.

(ii)′ $(\mathcal{S}\cap\mathcal{I})_{\text{ext}}=\{\nu_\theta\,|\,\theta\geq\theta_0\}$ (or $\nu_\theta\,|\,\theta\leq\theta_0\}$).

REMARK 2. It is obvious that if $a(\theta)=0$ for $\theta\in R$, then $\nu_\theta=\delta_0$. On the other hand if $a(\theta_1)=a(\theta_2)=0$ for some $\theta_1>\theta_2$, the diffusion process can be restricted to a narrow state space $[\theta_1, \theta_2]^{Z^d}$ rather than $L^2(\gamma)$, then it holds $(\mathcal{S}\cap\mathcal{I})_{\text{ext}}=\{\nu_\theta\,|\,\theta_1\leq\theta\leq\theta_2\}$. Thus Theorem 1 is a generalization of the result by Cox and Greven [1].

REMARK 3. Obviously Theorem 1.1 is applicable to the above examples 1-5 except the example 4, since these fulfill the assumption [A]. But for the example 4 the assumption [A] is fulfilled only for a sufficiently large $\kappa>0$. In fact if $\kappa>0$ is sufficiently small, then a different phenomenon occurs as shown in [12], (also see Theorem 1.2 below).

REMARK 4. In the case that the symmetrized Markov process of $(\Omega, \mathcal{B}, \boldsymbol{P}_i, \xi_1)$ generated by $A^S=A+A^*=(A^S_{ij}=A_{i-j}+A_{j-i})$ is recurrent, assuming the same asumption on A as in Theorem 4.5 of [6], we can prove the extinction for finite mass system under a regularity condition on $a(u)$ together with $a(0)=0$, by modifying the proof of [6]. Furthermore we can show the extincion occurs even for infinite mass system using a duality between finite mass sysetm and infinite mass system when $a(u)=cu$ with a constant c. Although it is plausible that the extinction occurs for infinite system in general, we have no proof for it due to lack of the duality.

We emphasize that the condition (1.10) of the assumption $[A]$ is crucial for for the phenomena of Theorem 1.1. In fact, as shown in [12], for the example 4 with a small $\kappa>0$ the sample path $x_i(t)$ decays exponentially fast as $t\to\infty$ almost surely for each $i\in Z^d$. Accordingly we would like to extend this exponential decay result to more general case with some non-linear coefficient $a(u)$.

ASSUMPTION [B]

Let $\alpha(u): R\to R$ be a locally 1/2 Hölder continuous function satisfying that for some $0<c<C<\infty$

$$c^{1/2}|u|\leq|\alpha(u)|\leq C^{1/2}|u| \quad \text{for } u\in R, \tag{1.12}$$

and set

$$a(u)=\kappa^{-1/2}\alpha(u) \quad \text{with } \kappa>0. \tag{1.13}$$

According to the condition (1.12) the diffusion process associated with the SDE (1.1) is defined in the state space $\mathcal{X}_+(0)=\{x=(x_i)\in L^2(\gamma)\,|\,x_i\geq 0$ for all $i\in Z^d\}$. Then we obtain the following result.

Theorem 1.2. *Suppose that $a(u)$ satisfies the asumption $[B]$. Assume further that $A=(A_{ij}=A_{j-i})$ is Z^d-translation invariant and of finite range, namely,* *there is an* $R_0>0$ *such that*

$$A_i = 0 \quad \text{for } |i| > R_0. \tag{1.14}$$

Then there exists a $c(\kappa)$: $(0, \infty)\to(0, \infty)$ *satisfying* $\lim_{\kappa\to 0} c(\kappa)=0$ *(more precisely,* $c(\kappa)$ *is estimated as* $c(\kappa)=O((\log 1/\kappa)^{-1+\varepsilon})$ *as* $\kappa\to 0$ *for every* $\varepsilon>0$*) such that for every* $\theta>0$*, the sample path* $x(t)=(x_i(t))$ *with* $x(0)=\boldsymbol{\theta}$ *satisfies*

$$P^{\theta}\Big(-\frac{\kappa C}{2}\leq \liminf_{t\to\infty}\frac{1}{t}\log x_i(t)\leq\limsup_{t\to\infty}\frac{1}{t}\log x_i(t)\leq-\kappa\Big(\frac{c}{2}-c(\kappa)\Big) \quad \text{for } i\in Z^d\Big)=1.$$

We remark that it is an easy task to extend Theorem 1.2 to the case with spatially inhomogeneous $(a_i(u))$ and $A=(A_{ij})$ under some uniformity condition. Furthermore as a corollary of Theorem 1.2 we obtain

Corollary 1.3. *Under the situation of Theorem* 1.2, *if* $\kappa>0$ *sufficiently small, then* $\mathcal{S}\cap\mathcal{I}_1=\{\delta_0\}$, *and* $\lim_{t\to\infty} T_t^*\mu=\delta_0$ *for every* $\mu\in\mathcal{I}_1$.

Although our diffusion processes associated with (1.1) are non-linear models, the proof of Theorem 1.1 is very similar to that for linear systems treated in Chapter IX of Liggett's book [6], which relies mainly upon second moment calcuations and a coupling technique. (See also [1], [7]). For the proof of Theorem 1.2 we essentially follow the idea of [12] pp, 61–62 to use Feynman-Kac's formula. However the arguments of [12] are so crude, and furthermore an approximation procedure by discrete time process should be improved even in the linear case. Therefore it would be worthy to present the details in our setting refining the proofs of [12].

2. Proof of Theorem 1.1

Recalling the assumption (1.10) we have positive constans C and D such that

$$|a(u)|^2\leq C+D|u|^2 \quad \text{for } u\in R, \tag{2.1}$$

$$D<G^S(0)^{-1}, \tag{2.2}$$

from which we obtain the following second moment estimates.

Lemma 2.1. *Let* $\mu\in\mathcal{I}_2$. *Then* $T_t^*\mu\in\mathcal{I}_2$ *for* $t>0$, *and* $\langle T_t^*\mu, |x_i|^2\rangle$ *is bounded in* $t\geq 0$ *for* $i\in Z^d$.

Proof. It is easy to see that $T_t^*\mu\in\mathcal{I}_2$ for $t>0$, and $f_{ij}(t)=E^{\mu}(x_i(t)x_j(t))=\langle T_t^*\mu, x_ix_j\rangle$ satisfies

$$(2.3)\qquad \frac{d}{dt}f_{ij}(t)=(A^1+A^2)f(t)_{ij}+\delta_{ij}E^{\mu}(a(x_i(t))^2)$$
$$=(A^1+A^2)f(t)_{ij}+D\delta_{ij}f_{ij}(t)+\delta_{ij}h(t)\,,$$

where $(A^1+A^2)f_{ij}=\sum_{k\in Z^d}A_{ik}f_{kj}+\sum_{k\in Z^d}A_{jk}f_{ik}$,

$h(t)=E^{\mu}(a(x_0(t))^2-Dx_0(t)^2)$, and δ_{ij} stands for the Kronecker symbol.

Let $(\zeta_t, \boldsymbol{P}_{(i,j)})$ be the direct product process taking values in $Z^d\times Z^d$ of two copies of the Markov processes $(\xi_t, \boldsymbol{P}_i)$, which has the generator A^1+A^2 and the tranition probability

$$P_t^{(2)}((i,j),(k,l))=P_t(i,k)P_t(j,l)\quad \text{for } (i,j),(k,l)\in Z^d.$$

Noting by (2.1) and (2.2) that $h(t)$ is locally bounded in $t\geq 0$ and $h(t)\leq C$ for every $t\geq 0$, one can apply Feynman-Kac's formula to get

$$f_{ij}(t)=\boldsymbol{E}_{(i,j)}(f_{\zeta_t}(0)\exp(\int_0^t DI_\Delta(\zeta_s)ds))+\int_0^t \boldsymbol{E}_{(i,j)}(\exp(\int_0^t DI_\Delta(\zeta_s)ds);$$
$$\zeta_r\in\Delta)\,h(t-r)dr\,,$$

hence

$$(2.4)\quad f_{ii}(t)\leq f_{ii}(0)\,\boldsymbol{E}_{(i,j)}(\exp(\int_0^t DI_\Delta(\zeta_s)ds))+C\int_0^t\boldsymbol{E}_{(i,j)}(\exp(\int_0^t DI_\Delta(\zeta_s)ds);\zeta_r\in\Delta)dr,$$

where $\Delta=\{(i,j)\in Z^d\times Z^d\,|\,i=j\}$.
Using Taylor's expansion for the exponential function it is easy to estimate as follows:

$$\int_0^t \boldsymbol{E}_{(i,j)}(\exp(\int_0^t DI_\Delta(\zeta_s)ds);\zeta_r\in\Delta)$$
$$=\sum_{n=0}^{\infty}D^n\int_0^t dr\int\cdots\int_{0<t_1<\cdots<t_n<r} P^s_{t_1}(0)P^s_{t_2-t_1}(0)\cdots P^s_{t_n-t_{n-1}}(0)\,P^s_{r-t_n}(0)dt_1dt_2\cdots dt_n$$
$$\leq\sum_{n=0}^{\infty}D^nG^s(0)^{n+1}<\infty,\quad \text{(by (2.2))}$$

and

$$\boldsymbol{E}_{(i,j)}(\exp(\int_0^t DI_\Delta(\zeta_s)ds))$$
$$\leq\sum_{n=0}^{\infty}D^nG^s(0)^n<\infty.$$

Hence by (2.4), $\langle T_t^*\mu, |x_i|^2\rangle=f_{ii}(t)$ is bounded in $t\geq 0$ for $i\in Z^d$.

For $\theta\in R$, set

$$\mathcal{I}_2(\theta)=\{\mu\in\mathcal{I}_2 \lim_{t\to\infty}\langle\mu, |\sum_{j\in Z^d}P_t(i,j)x_j-\theta|^2\rangle=0 \quad \text{for } i\in Z^d\}.$$

Lemma 2.2. *Let $\mu\in\mathcal{I}_2(\theta)$ for $\theta\in R$. Then*
(i) $T_t^*\mu\in\mathcal{I}_2(\theta)$ *for* $t>0$,
(ii) *If* $\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}T_s^*\mu\, ds=\nu$ *for some* $\{t_n\}$ *with* $t_n\uparrow\infty$, *then* $\nu\in\mathcal{I}_2(\theta)$.

Proof. (i) is easy. For (ii) use (2.3) and the Z^d-translation invariance of μ to get

$$f_{ij}(t)=\sum_k\sum_l P_t(i,k)P_t(j,l)f_{kl}(0)+\int_0^t P_{t-s}^s(j-i)E^{\mu}(a(x_0(s))^2)\, ds. \tag{2.5}$$

Using the assumption of (ii) and (2.5) to together with $\mu\in\mathcal{I}_2(\theta)$ we see

$$\begin{aligned}
&\langle\nu, |\sum_j P_t(i,j)x_j-\theta|^2\rangle\\
&\le \liminf_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\langle T_s^*\mu, |\sum_j P_t(i,j)x_j-\theta|^2\rangle\, ds\\
&=\liminf_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\{\sum_k\sum_l P_t(i,k)P_t(i,l)f_{kl}(s)-2\theta\sum_k P_t(i,k)\langle T_s^*\mu, x_k\rangle+\theta^2\}\, ds\\
&=\liminf_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\{\sum_k\sum_l P_{t+s}(i,k)P_{t+s}(i,l)f_{kl}(0)+\\
&\qquad\qquad\int_0^s P_{t+s-r}^s(0)E^{\mu}(a(x_0(r))^2)dr-\theta^2\}\, ds\\
&\le \sup_{s>0}E^{\mu}(a(x_0(s))^2)\int_t^{\infty}P_r^s(0)dr\\
&\to 0 \text{ as } t\to\infty,
\end{aligned}$$

which concludes $\nu\in\mathcal{I}_2(\theta)$.

Following [1] let us introduce a diffusion process taking values in $L^2(\gamma)\times L^2(\gamma)$ which is governed by the following SDE:

$$\begin{cases} dx_i^1(t)=\sum_{j\in Z^d}A_{ij}x_j^1(t)\, dt+a(x_i^1(t))dB_i(t)\\ dx_i^2(t)=\sum_{j\in Z^d}A_{ij}x_j^2(t)\, dt+a(x_i^2(t))dB_i(t)\\ \bar{x}(0)=(x^1, x^2)\in L^2(\gamma)\times L^2(\gamma), \quad (i\in Z^d). \end{cases} \tag{2.6}$$

By using a method of one-diemnsional SDE, one can show if $x^1\ge x^2$, then $x^1(t)\ge x^2(t)$ holds for every $t>0$ with probability one. This means that the diffusion porcess $(x(t), P_x)$ associated with the SDE (1.1) has monotonicity property.
We denote by $(\bar{x}(t)=(x_i^1(t), x_i^2(t)), \bar{P}^{\bar{x}})$ the diffusion process associated with (2.6).

Clearly each component process of $(\bar{x}(t), \bar{P}^{\bar{x}})$ is equivalent to the original diffusion process $(x(t), P^x)$. Let denote by $\bar{T}_t$, $\bar{T}_t^*$, $\bar{\mathcal{S}}$, $\bar{\mathcal{I}}$, $\bar{\mathcal{I}}_\alpha$ the corresponding semi-group, the dual smei-group, the totality of stationary distributions, the totality of Z^d-translation invariant probability measures on $L^2(\gamma)\times L^2(\gamma)$ and so on.

Lemma 2.3. *Let* $\lambda\in\bar{\mathcal{S}}\subset\bar{\mathcal{I}}_1$. *Then*

$$\lambda(\textit{either } x_i^1\geq x_i^2 \textit{ for all } i\in Z^d \textit{ or } x_i^1\leq x_i^2 \textit{ for all } i\in Z^d) = 1\,.$$

Proof. Let $(\bar{x}(t), \bar{P}^\lambda)$ be a stationary Markov process with λ as its marginal distribution associated with (2.6), namely, $\bar{x}(t)$ is a soluiton of (2.6) with λ as the law of $\bar{x}(0)$. Setting $\Delta_i(t)=x_i^1(t)-x_i^2(t)$, and applying Ito's formula to $f(u)=|u|$, we have

$$\bar{E}^\lambda(|\Delta_i(t)|)-\bar{E}^\lambda(|\Delta_i(0)|)=\int_0^t\sum_j A_{ij}\bar{E}^\lambda(\operatorname{sgn}\Delta_i(s))\Delta_j(s))ds \tag{2.7}$$

where $\operatorname{sgn} u=\begin{cases}1 \text{ for } u>0,\\ 0 \text{ for } u=0,\\ -1 \text{ for } u<0.\end{cases}$ (cf. [9], pp. 404.)

By the stationarity of λ,

$$\sum_{j\neq i}A_{ij}\langle\lambda, (\operatorname{sgn}\Delta_i)\Delta_j-|\Delta_i|\rangle=0\,,$$

Since $\langle\lambda, |\Delta_i|\rangle$ is constant in i by the translation invariance of λ, we see

$$|\Delta_j|=(\operatorname{sgn}\Delta_i)\Delta_j \quad \lambda\text{-a.e. if } A_{ij}>0\,, \tag{2.8}$$

which implies that when $A_{ij}>0$, if $\Delta_i\geq 0$, then $\Delta_j\geq 0$, and if $\Delta_i\leq 0$, then $\Delta_j\leq 0$ λ-a.s. Therefore combining this with the irreducibility of A we obtain the desired property of λ.

Lemma 2.4. *For* $\mu\in\mathcal{I}_1$ *and* $\nu\in\mathcal{I}_1$,

$$\bar{E}^{\mu\times\nu}(|x_i^1(t)-x_i^2(t)|) \textit{ is non-increasing in } t>0.$$

Proof. In the same way to get (2.7), for $s<t$,

$$\begin{aligned}&\bar{E}^{\mu\times\nu}(|\Delta_i(t)|)-\bar{E}^{\mu\times\nu}(|\Delta_i(s)|)\\ &=\int_s^t\sum_j A_{ij}\bar{E}^{\mu\times\nu}((\operatorname{sgn}\Delta_i(s))\Delta_j(s)-|\Delta_i(s)|)ds\leq 0\,,\end{aligned}$$

since $\bar{E}^{\mu\times\nu}(|\Delta_i(s)|)$ is independent of i.

Lemma 2.5. *Let* $\theta\in R$. *For* $\mu\in\mathcal{I}_2(\theta)$ *and* $\nu\in\mathcal{I}_2(\theta)$, $\lim_{t\to\infty}\bar{E}^{\mu\times\nu}(|x_i^1(t)-x_i^2(t)|)=0$ *for* $i\in Z^d$.

Proof. Using Lemma 2.1 one can esaily see that $\{\frac{1}{t}\int_0^t \bar{T}_s^*\mu\times\nu\, ds\}$ is a tight family as $t\to\infty$. Let λ be a limit point, i.e. $\lambda=\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\bar{T}_s^*\mu\times\nu\, ds$ for some $\{t_n\}$ with $t_n\uparrow\infty$. Then $\lambda\in\bar{\mathcal{S}}\cap\bar{\mathcal{I}}_2$, so by Lemma 2.3,

(2.9) $\lambda(\text{either } x_i^1\geq x_i^2 \text{ for all } i\in Z^d \text{ or } x_i^1\leq x_i^2 \text{ for all } i\in Z^d)=1.$

Hence, by (2.4)

$$\begin{aligned}&\lim_{t\to\infty}\bar{E}^{\mu\times\nu}(|x_i^1(t)-x_i^2(t)|)\\ &=\lim_{n\to\infty}\langle\frac{1}{t_n}\int_z^{t_n}\bar{T}_s^*\mu\times\nu\, ds,\ |x_i^1-x_i^2|\rangle \quad \text{(by Lemma 2.4)}\\ &=\langle\lambda,\ |x_i^1-x_i^2|\rangle\end{aligned}$$

Here we used Lemma 2.1 and a simple fact that if $\mu_n\in\mathcal{P}(R)$ converges weakly to $\mu\in\mathcal{P}(R)$ as $n\to\infty$, and if $\langle\mu_n, x^2\rangle$ is bounded in $n\geq 1$, then $\langle\mu_n, |x|\rangle$ also converges to $\langle\mu, |x|\rangle$ as $n\to\infty$. Furthermore noting that each marginal of λ is in $\mathcal{I}_2(\theta)$ by Lemma 2.2, we see by (2.9) and the translation invariance of λ that

$$\begin{aligned}&=\langle\lambda,\ |\sum_j P_t(i,j)(x_j^1-x_j^2)|\rangle\\ &\leq\langle\lambda,\ |\sum_j P_t(i,j)x_j^1-\theta|\rangle+\langle\lambda,\ |\sum_j P_t(i,j)x_j^2-\theta|\rangle\\ &=\langle\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}T_s^*\mu\, ds,\ |\sum_j P_t(i,j)x_j-\theta|\rangle\\ &\qquad\qquad+\langle\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}T_s^*\nu\, ds,\ |\sum P_t(i,j)x_j-\theta|\rangle\\ &=0,\ \text{letting } t\to\infty,\ \text{(by Lemma 2.2).}\end{aligned}$$

We are now in position to complete the proof of Theorem 1.1.

1°. We first claim that for each $\theta\in R$, there exists a $\nu_\theta\in\mathcal{S}\cap\mathcal{I}_2(\theta)$ such that $\langle\nu_0, x_i\rangle=\theta$ and $\lim_{t\to\infty}T_t^*\mu=\nu_\theta$ for every $\mu\in\mathcal{I}_2(\theta)$, which proves (i).

Let ν_θ be a limit point of $\{\frac{1}{t}\int_0^t T_s^*\delta_\theta ds\}$ as $t\to\infty$, i.e. $\nu_\theta=\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}T_s^*\delta_\theta ds$ for some $\{t_n\}$ with $t_n\to\infty$. From Lemma 2.2 it follows that $\nu_\theta\in\mathcal{S}\cap\mathcal{I}_2(\theta)$ and $\langle\nu_\theta, x_i\rangle=\theta$ for $i\in Z^d$. Next, let $\mu\in\mathcal{I}_2(\theta)$. For $f\in C_0^1(L^2(\gamma))$,

$$\begin{aligned}&|\langle T_t^*\mu, f\rangle-\langle\nu_\theta, f\rangle|\\ &=|\bar{E}^{\mu\times\nu_\theta}(f(x^1(t))-f(x^2(t)))|\\ &\leq \text{Const.}\sum_{i\,:\,(\text{finite sum})}\bar{E}^{\mu\times\nu_\theta}(|x_i^1(t)-x_i^2(t)|)\\ &\to 0 \text{ as } t\to\infty \text{ by Lemma 2.5, which yields (i)}\end{aligned}$$

2°. Suppose $\nu_\theta = c\nu' + (1-c)\nu''$ with $0<c<1$ and $\nu', \nu'' \in \mathcal{S} \cap \mathcal{I}$. Since $\nu_\theta \in \mathcal{I}_2(\theta)$ by Lemma 2.2, it follows $\nu' \in \mathcal{I}_2(\theta)$ and $\nu'' \in \mathcal{I}_2(\theta)$ so that by the step 1°, $\nu' = \nu'' = \nu_\theta$ holds. Hence ν_θ is extremal both in $\mathcal{S} \cap \mathcal{I}$ and $\mathcal{S} \cap \mathcal{I}_1$. To prove the converse, suppose $\mu \in \mathcal{I}_1$ be ergodic w.r.t. the Z^d-translation group. For $N>0$. Let $\pi_N: L^2(\gamma) \to L^2(\gamma)$ be defined by

$$(\pi_N x)_i = (x_i \wedge N) \vee (-N) \quad \text{for } i \in Z^d.$$

Denote by μ_N the image measure of μ by π_N. Then μ_N also is ergodic. Moreover it is easy to check $\mu_N \in \mathcal{I}_2(\theta_N)$ with $\theta_N = \langle \mu_N, x_i \rangle$. (cf. Lemma 5.2 in [7]). In the same way to use the coupling process as in the proof of Lemma 2.4.

$$\int \mu(dx) \bar{E}^{x, \pi_N x}(|x_i^1(t) - x_i^2(t)|) \leq \langle \mu, |x_i - (\pi_N x)_i| \rangle ,$$

hence for every $f \in C_0^1((L^2(\gamma))$ we have a constant $C_f < 0$ such that

$$\begin{aligned} |\langle T_t^* \mu, f \rangle - \langle T_t^* \mu_N, f \rangle| &= \left| \int \mu(dx) \bar{E}^{x, \pi_N x}(f(x^1(t)) - f(x^2(t))) \right| \\ &\leq C_f \langle \mu, |x_i - (\pi_N x)_i| \rangle . \end{aligned}$$

By 1°, $\lim_{t\to\infty} T_t^* \mu_N = \nu_{\theta_N}$. Since $\theta_N \to \theta = \langle \mu, x_i \rangle$ and $\langle \mu, |x_i - (\pi_N x)_i| \rangle \to 0$ as $N \to \infty$, we obtain $\lim_{t\to\infty} T_t^* \mu = \nu_\theta$, from which (ii) and the first part of (iii) follow. The latter part of (iii) is immediate since $\mu \in \mathcal{I}_1$ is represented as a mixture of ergodic ones in $\mathcal{I}_1$.

Finally we give the proof of Remark 1, (ii)′ after Theorem 1.1.

Proof of Remark 1, (ii)′.

We may assume $\theta_0 = 0$. Let $\nu \in (\mathcal{S} \cap \mathcal{I})_{\text{ext}}$ with $\langle \nu, x_i \rangle = +\infty$. By the proof of Theorem 2.19, pp. 439 of [6] it holds that $\nu \geq \nu_\theta$ for every $\theta > 0$. (This part of the proof of [6] works in the present situation without any change.) Using Lemma 2.1 we have a $C>0$ satisfying

$$\langle \nu_\theta, |x_i|^2 \rangle \leq C\theta^2 \quad \text{for } \theta \geq 1. \tag{2.10}$$

From $\langle \nu_\theta, x_i \rangle = \theta$ it follows

$$\langle \nu_\theta, x_i I(x_i \geq \theta/2) \rangle \geq \theta/2 .$$

So, using (2.10) and Schwarz's inequality, we get $\nu_\theta(x_i \geq \theta/2) \geq 1/4C$, so that

$$\nu(x_i \geq \theta/2) \geq \nu_\theta(x_i \geq \theta/2) \geq 1/4C \quad \text{for } \theta \geq 1,$$

which yields a contradiction; $\nu(x_i = \infty) \geq 1/4C$. Thus $(\mathcal{S} \cap \mathcal{I})_{\text{ext}} = (\mathcal{S} \cap \mathcal{I}_1)_{\text{ext}}$ and the proof of (ii)′ is complete.

3. Proof of Theorem 1.2

First note that under the assumptions (1.12) and (1.13) for the solution $x(t)$ of the SDE (1.1), the rescaled process $y(t)=x(\kappa t)$ satisfies the following SDE:

$$dy_i(t)=\kappa \sum_{j\in Z^d} A_{ij}y_j(t)dt+y_i(t)dM_i(t) \quad (i\in Z^d), \tag{3.1}$$

where $\{M_i(t)\}_{i\in Z^d}$ are continuous square-integrable martingale such that their quadratic variation processess satisfy $M_i(0)=0$ for $i\in Z^d$, and

$$c\delta_{ij}dt\le d\langle M_i, M_j\rangle(t)\le C\delta_{ij}dt \quad \text{for } i,j\in Z^d \text{ and } t>0. \tag{3.2}$$

Notice that we are now considering a continuous time Markov process $(\xi_t, \boldsymbol{P}_i)$ taking values in Z^d, generated by the infinitesimal martic κA, instead of A. As in [12], the proof is based on the following Feynman-Kac's formula with respect to the process $(\xi_t, \boldsymbol{P}_i)$. To avoid confusion in the subsequent we will use the notation $(\Omega, \mathcal{F}, \mathcal{F}_t, P^B)$ for the probability space where $\{B_i(t)\}$ (or $\{M_i(t)\}$) are defined.

Lemma 3.1. *Let* $y(0)=\boldsymbol{\theta}$ *with* $\theta>0$. *Then*

$$y_i(t) = \theta \boldsymbol{E}(\exp(\sum_{j\in Z^d}\int_0^t I(\xi_{t-s}=j)dM_j(s)-\frac{1}{2}\sum_{j\in Z^d}\int_0^t I(\xi_{t-s}=j)d\langle M_j\rangle(s)))\,,$$

where $I(A)$ stands for the indicator function for an event A, i.e. $I(A)=1$ if A occurs and $I(A)=0$ otherwise.

Proof. It is easy to see that the equation (3.1) with $y(0)=\boldsymbol{\theta}$ is equivalent to the following integral equation which is uniquely solvable:

$$y_i(t) = \theta+\int_0^t \sum_{j\in Z^d} P_{t-s}(i,j)y_j(s)dM_j(s), \quad i\in Z^d. \tag{3.3}$$

Using Ito's formula we get

$$\begin{aligned} &\exp(\sum\int_0^t I(\xi_{t-s}=j)(dM_j(s)-\frac{1}{2}d\langle M_j\rangle(s)))-1 \\ &= \int_0^t \exp(\sum_j\int_0^t I(\xi_{t-s}=j)(dM_j(s)-\frac{1}{2}d\langle M_j\rangle(s)))\sum_k I(\xi_{t-r}=k)dM_k(r). \end{aligned} \tag{3.4}$$

Taking the expectation of (3.4) with respect to $\boldsymbol{P}_i$, and using Markov property of $(\xi_t, \boldsymbol{P}_i)$ together with a stochastic Fubini's theorem (which is easily justified), one can see that

$$z_i(t) = \theta \boldsymbol{E}_i(\exp(\sum_{j\in Z^d}\int_0^t I(\xi_{t-s}=j)dM_j(s)-\frac{1}{2}\sum_{j\in Z^d}\int_0^t I(\xi_{t-s}=j)d\langle M_j\rangle(s)))\,,$$

solves the equation (3.3). Therefore the proof of Lemma 3.1 is complete by the uniquencess of the equation (3.3) with $y(0)=\boldsymbol{\theta}$.

For simplicity we hereafter assume $A_{ii}=A_0=-1$, and it suffices to show the exponential decay for $y_0(t)$. Let N_t be the number of jump times of ξ_s between the time interval $[0, t]$. Clearly N_t is a Poisson process with parameter κ. In order to estimate the Feynman-Kac expression of $y_0(t)$ we approximate $(\xi_t, \boldsymbol{P}_0)$ be a discrete time process.

Lemma 3.2. *There is a discrete time stochasitc process* $\{\eta_m\}_{m\geq 0}$ *taking values in* Z^d *such that* $\eta_0=0$, *and*

(i) $|\eta_m-\eta_{m-1}|\leq R_0$ *for* $m\geq 1$,

(ii) $\int_0^t I(\xi_s\neq\eta_{[s]})\,ds\leq 2N_t$,

where $[\cdot]$ *stands for the Gauss symbol, namely,* $[s]$ *is the integer part of s.*

(iii) $\hat{N}_n\equiv\#\{1\leq m\leq n\,|\,\eta_m\neq\eta_{m-1}\}\leq N_n$ *for* $n\geq 1$.

Proof. Let $\tau_0=0$ and let $\tau_1, \tau_2, \cdots$ be successive jump times of the process ξ_t, and set $\zeta_m=\xi_{\tau_m}$ for $m\geq 0$. Clearly $|\zeta_m-\zeta_{m-1}|\leq R_\sigma$

Define a sequence $\alpha_m (m\geq 0)$ by

$$\alpha_0=0, \text{ and } \alpha_m=\min\{\alpha_{m-1}+1, N_m\} \text{ for } m\geq 1.$$

Then, as easily seen, $\{\alpha_m\}$ is a non-decreasing sequence of integers with $\alpha_m-\alpha_{m-1}\leq 1$, and

$$\#\{1\leq m\leq n\,|\,\alpha_m\neq N_m\}\leq N_n \quad \text{for } n\geq 1. \tag{3.5}$$

Setting $\eta_m=\zeta_{\alpha_m}$ for $m\geq 0$, we see (i) and (iii) immediately. For (ii), from (3.5) it sollows

$$\begin{aligned}\int_0^n I(\xi_s\neq\eta_{[s]})\,ds &\leq \sum_{m=0}^{n-1} I(N_{m+1}-N_m\geq 1)+\sum_{m=0}^{n-1}\int_m^{m+1} I(\xi_s\neq\eta_m, N_{m+1}=N_m)\,ds \\ &\leq 2N_n\,,\end{aligned}$$

because $\xi_m\neq\eta_m$ implies $N_m\neq\alpha_m$.

Let

$$I(t)=\boldsymbol{E}_0(\exp(\sum_j\int_0^t I(\xi_{t-s}=j)dM_j(s)))\,.$$

Notice that the expectation is taken with respect to $(\xi_t, \boldsymbol{P}_0)$ so that $I(t)$ is a stochastic process on the probablilty space $(\Omega, \mathcal{F}, \mathcal{F}_t, P^B)$. We now want to get an exponential estimate for the process $I(t)$. Note that the probability law of $\{\xi_t-\xi_{t-s}\}_{0\leq s\leq t}$ under $\boldsymbol{P}_0$ coincides with that of $((\xi_s)_{0\leq s\leq t}, \boldsymbol{P}_0)$, so it holds

$$I(t)=\boldsymbol{E}_0(\exp(\sum_j\int_0^t I(\xi_t-\xi_s=j)dM_j(s)))\,. \tag{3.6}$$

In particular, for a fixed $n\geq 1$, we apply Lemma 3.2 to the process $(\{\xi_n-\xi_{m-s}\}_{0\leq s\leq n}, \boldsymbol{P}_0)$, and denote by $\{\eta_m^n\}_{0\leq m\leq n}$ the resultant discrete time process. Then we have

Lemma 3.3. *For a fixed integer $n\geq 1$ there is a discrete time stochastic process $\{\eta_m^n\}_{0\leq m\leq n}$ taking values in Z^d such that $\eta_0^n=0$ and*
(i) $|\eta_m^n-\eta_{m-1}^n|\leq R_0$ *for* $1\leq m\leq n$,
(ii) $\int_0^n I(\xi_n-\xi_s\neq\eta_{[n-s]}^n)\,ds\leq 2N_n$.
(iii) $\hat{N}_n\equiv\#\{1\leq m\leq n\,|\,\eta_m^n-\eta_{m-1}^n\neq 0\}\leq N_n$.

For an integer $n\geq 1$, let

$$J_1(n)=\sup_{n-1\leq t\leq n}\boldsymbol{E}_0(\exp(2\sum_j\int_0^t I(\eta_{[n-s]}^n=j)dM_j(s)))\,, \tag{3.7}$$

$$J_2(n)=\sup_{n-1\leq t\leq n}\boldsymbol{E}_0(\exp(2\sum_j\int_0^t (I(\xi_n-\xi_s=j)-I(\eta_{[n-s]}^n=j))dM_j(s)))\,. \tag{3.8}$$

Lemma 3.4.

$$\sup_{n-1\leq t\leq n} I(t)\leq e^{\kappa}J_1(n)^{1/2}J_2(n)^{1/2}\,.$$

Proof. Let $n-1\leq t\leq n$. Since $[N_t\neq N_n]$ is independent of $(\xi_s)_{0\leq s\leq t}$ under $\boldsymbol{P}_0$, we see

$$\begin{aligned}I(t)\leq&\boldsymbol{E}_0(\exp(\sum_j\int_0^t I(\xi_n-\xi_s=j)dM_j(s)))\\&\qquad+\boldsymbol{E}_0(\exp(\sum_j\int_0^t I(\xi_t-\xi_s=j)dM_j(s));\,N_t\neq N_n)\\\leq&\boldsymbol{E}_0(\exp(\sum_j\int_0^t I(\xi_n-\xi_s=j)dM_j(s)))+(1-e^{-\kappa})I(t)\,,\end{aligned}$$

so using Schwarz's inequality, we get

$$\begin{aligned}I(t)&\leq e^{\kappa}\,\boldsymbol{E}_0(\exp(\sum_j\int_0^t I(\xi_n-\xi_s=j)dM_j(s)))\\&\leq e^{\kappa}J_1(n)^{1/2}J_2(n)^{1/2}\,.\end{aligned}$$

Lemma 3.5. *There is a constant $C>0$ independentof κ such that*

$$P^B(\limsup_{n\to\infty}\frac{1}{n}\log J_2(n)\leq C\kappa)=1\,.$$

Proof. Let

$$M(t)=2\sum_j\int_0^t(I(\xi_n-\xi_s=j)-I(\eta_{[n-s]}^n=j)dM_j(s)\,.$$

$M(t)$ is a continuous martingale with respect to P^B with quadratic variation process

$$\begin{aligned}\langle M\rangle(t) &= 4\int_0^t \sum_j |I(\xi_n-\xi_s=j)-I(\eta^n_{[n-s]}=j)|\, d\langle M_j\rangle(s)\\ &\leq 8C\int_0^n I(\xi_n-\xi_s\neq\eta^n_{[n-s]})ds\\ &\leq 16CN_n,\quad \text{by Lemma 3.3.}\end{aligned}$$

Since $M(t)=B(\langle M\rangle(t))$ for some standard Brownian motion $B(t)$, using a simple formula on Brownirn motion:

$$E(\exp(\sup_{0\leq t\leq T} B(t))) = 2E(\exp B(T); B(T)>0)\leq 2e^{T/2},$$

we obtain

$$\begin{aligned}&E^B(J_2(n))\\ &\leq \boldsymbol{E}_0(E^B(\exp(\sup_{0\leq t\leq n} M(t))))\\ &\leq 2\,\boldsymbol{E}_0(\exp(8CN_n))\\ &\leq 2\exp((e^{8C}-1)\kappa n),\end{aligned}$$

so by Chebyshev's inequality

$$P^B(J_2(n)\geq \exp 2e^{8C}\,\kappa n)\leq 2\exp-((e^{8C}+1)\kappa n).$$

Thereofre we complete the proof of Lemma 3.5 by using Borel-Cantelli's lemma.

To estimate the main term $J_1(n)$ we divide it into three terms. Let

$$J_{11}(n) = \sup_{n-1\leq t\leq n} \boldsymbol{E}_0(\exp(2\sum_j\int_0^t I(\eta^n_{[n-s]}=j)dM_j(s)); 0<\hat{N}_n<n),$$

$$J_{12}(n) = \sup_{n-1\leq t\leq n} \boldsymbol{E}_0(\exp(2\sum_j\int_0^t I(\eta^n_{[n-s]}=j)dM_j(s)); \hat{N}_n\geq n),$$

$$J_{13}(n) = \sup_{n-1\leq t\leq n} \boldsymbol{E}_0(\exp(2\sum_j\int_0^t I(\eta^n_{[n-s]}=j)dM_j(s)); \hat{N}_n=0),$$

The estimate of $J_{13}(n)$ is trivial, and for $J_{12}(n)$ we obtain by the same method as $J_2(n)$ that

$$E^B(J_{12}(n))\leq 2e^{2Cn}\,\boldsymbol{P}_0(N_n\geq n)\leq 2\exp((e^{2C}-1)\kappa n),$$

hence

$$P^B(\limsup_{n\to\infty}\frac{1}{n}\operatorname{long} J_{12}(n)\leq C\kappa) = 1 \quad \text{for some } C>0 \text{ independent of } \kappa.$$

Now we proceed to estimate $J_{11}(n)$. Let $\delta(r)$ be a positive function on

(0, 1), which is specified later. For $0\leq k\leq n$, let W_k^n be the totality of sample pahts of the discrete time process $\eta^n=(\eta_m^n)_{0<m\leq n}$ having just k jumps. Then the cardinality of W_k^n is trivially estimated by

$$\# W_k^n \leq \binom{n}{k} R_0^{kd}.$$

Then

$$P^B(J_{11}(n)\geq \sum_{0<k<n} \boldsymbol{P}_0(\hat{N}_n=k)\exp(\delta(\frac{k}{n})n))$$

$$\leq \sum_{k=1}^{n-1}\sum_{w\in W_k^n} P^B(\sup_{0\leq t\leq n} 2\sum_j \int_0^t I(w_{[n-s]}=j)dM_j(s)>\delta(\frac{k}{n})n)$$

$$\leq \sum_{m=1}^{n-2}\sum_{w\in W_m^n} \frac{2}{\sqrt{2\pi}}\int_{C\delta(k/n)\sqrt{n}}^{\infty} \exp(-\frac{r^2}{2})dr \quad (\text{for some } C>0),$$

$$\leq \text{Const.} \sum_{k=1}^{n-1}\binom{n}{k} R_0^{kd}\exp(-C\delta(\frac{k}{n})^2 n) \quad (\text{for some } C>0),$$

$$\leq \text{Const.} \sum_{k=1}^{n-1} \frac{1}{\sqrt{n}\sqrt{\frac{k}{n}(1-\frac{k}{n})}} \exp n(h(\frac{k}{n})+(d\log R_0)\frac{k}{n}-C\delta(\frac{k}{n})^2),$$

where $h(r)=-r\log r-(1-r)\log(1-r)$ for $0<r<1$. For the last inequality we used Striling's formula.

Next we choose a function $\delta(r)$ such as

$$C\delta(r)^2=2(h(r)+(d\log R_0)r),$$

hence

$$\text{(3.9)} \quad P^B(J_{11}(n)\geq \sum_{m=1}^{n-1}\boldsymbol{P}_0(\hat{N}_n=k)\exp(\delta\frac{k}{n})n))$$

$$\leq \sum_{k=2}^{n-1} \frac{C}{\sqrt{n}\sqrt{\frac{k}{n}(1-\frac{k}{n})}} \exp -n(h(\frac{k}{n})+c\frac{k}{n}) \quad (\text{for some } c>0 \text{ and } C>0)$$

$$\equiv C(n).$$

Since it is not hard to show that $(C(n))$ is summable, by Borel-Cantelli's lemma,

$$\text{(3.10)} \quad P^B(J_{11}(n)\leq \sum_{k=1}^{n-1}\boldsymbol{P}_0(\hat{N}_n=k)\exp(\delta(\frac{k}{n})n) \text{ for sufficiently large } n)=1.$$

Next we claim

$$\text{(3.11)} \quad \limsup_{n\to\infty}\frac{1}{n}\log\sum_{k=1}^{n-1}\boldsymbol{P}_0(\hat{N}_n=k)\exp(\delta(\frac{k}{n})n\equiv C(\kappa)\to 0 \text{ as } \kappa\to 0.$$

Note that $\delta(r)$ is non-decreasing in $(0, r_0)$ with some $0<r_0<1$, and set $\delta_0(r)=\delta(r\wedge r_0)$. Using $\hat{N}_n \leq N_n$, by Lemma 3.3 we have

$$\sum_{k=0}^{n-1} \boldsymbol{P}_0(\hat{N}_n=k) \exp(\delta(\frac{k}{n})n) \leq \boldsymbol{E}_0(\exp(\delta_0(\frac{N_n}{n})n))+e^{|\delta|n} \boldsymbol{P}_0(N_n \geq r_0 n)$$

where $|\delta|=\sup \delta(r)$.
By a classical large deviation result on Poisson process with parameter $\kappa>0$, we know

$$\lim_{n\to\infty} \frac{1}{n} \log \boldsymbol{E}_0(\exp(n\delta_0(\frac{N_n}{n}))) = \sup_{r>0} \{\delta_0(r)+r \log \frac{\kappa}{r}+r-\kappa\} \equiv C_1(\kappa).$$

(eg. see [4]).

On the other hand by using Chebyshev's inequality we have

$$e^{|\delta|n} \boldsymbol{P}_0(N_m \geq r_0 n) \leq \exp \kappa(\frac{|\delta|}{r_0}-1).$$

Also, by elementary calcuations one can check that there exist $c>0$ and $c_\varepsilon>0$ for every $\varepsilon>0$ such that

$$c(\log 1/\kappa)^{-1} \leq C_1(\kappa) \leq c_\varepsilon(\log 1/\kappa)^{-1+\varepsilon} \text{ as } \kappa\to 0,$$

hence from these two estimates, we obtain (3.11).

Now we can complete the proof of Theorem 1.2. Summarizing Lemma 3.4, 3.5, (3.10), and (3.11) we have

$$P^B(\limsup_{t\to\infty} \frac{1}{t} \log I(t) \leq C(\kappa))=1 \text{ with some } C(k)=O((\log 1/\kappa))\kappa^{-1+\varepsilon} \text{ as } \kappa\to 0$$

for every $\varepsilon>0$.
Also, by (3.2)

$$\sum_j \int_0^t I(\xi_{t-s}=j)d\langle M_j\rangle(s) \geq ct,$$

hence the sample path $y_0((t)$ satisfies $y_0(t) \leq \theta \exp(-ct) I(t)$, and after all we obtain

$$P^B(\limsup_{t\to\infty} \frac{1}{t} \log y_0(t) \leq C(\kappa)-\frac{c}{2}) = 1,$$

which yields the upper bound in Theorem 1.2.

The lower bound in Theorem 1.2 is not difficult. In fact, by (3.2) it is sufficient to show

$$\text{(3.12)} \qquad \liminf_{t\infty\to} \frac{1}{t} \log I(t) \geq \liminf_{n\to\infty} \frac{1}{n} \log (\inf_{n-1\leq t\leq n} I(t)) \geq 0.$$

Let $n-1\leq t\leq n$. Using (3.2) and Jensen's inequality we see

$$I(t)\geq \boldsymbol{P}_0(\exp(\sum_j\int_0^t I(\xi_n-\xi_s=j)dM_j(s);\ N_n=N_{n-1})$$
$$\geq \boldsymbol{P}_0(N_n=N_{n-1})\exp(\sum_j\int_0^t \boldsymbol{P}_0(\xi_n-\xi_s=j\mid N_n=N_{n-1})dM_j(s))$$

Note that $M(t)=\sum_j\int_0^t \boldsymbol{P}_0(\xi_n-\xi_s=j\mid N_n=N_{n-1})dM_j(s)$ is a square integrable continuous martingale with quadratic variation process $\langle M\rangle(t)$, which satisfies $\langle M\rangle(t)\leq Ct$ by the assumption (3.2), so that there is a standard Brownian motion $B_n(t)$ such that

$$\inf_{n-1\leq t\leq n} M(t)\geq \inf_{n-1\leq t\leq n} B_n(\langle M\rangle)(t))\geq \inf_{0\leq t\leq n} B_n(Ct)\,.$$

Using this together with a Gaussian estimate and Borel-Cantelli's lemma, it is easy to see that

$$\liminf_{n\to\infty}\frac{1}{n}\inf_{n-1\leq t\leq n} M(t)\geq \lim_{n\to\infty}\frac{1}{n}\inf_{0\leq t\leq n} B_n(Ct)=0,\ \ P^B\text{—a.s.}$$

Since $\boldsymbol{P}_0(N_n=N_{n-1})=e^{-\kappa}$, we obtain (3.12). Therefore the proof of Theorem 1.2 is complete.

Proof of Corollary 1.3.

Let $\mu\in\mathcal{I}_1$. We want to show $\lim_{t\to\infty} T_t^*\mu=\delta_0$. By Theorem 1.2 it holds $\lim_{t\to\infty} T_t^*\delta_\theta=\delta_0$ for $\theta>0$. Using the coupling process introduced in the section 2 and its monotonicity:

$P^{x,x\wedge\theta}(x_i^1(t)\geq x_i^2(t)$ holds for all $i\in Z^d)=1$ for $t>0$, where $x\wedge\theta=(x_i\wedge\theta)$ for $\theta>0$, we see that for $K>0$,

$$\langle T_t^*\mu,x_i\wedge K\rangle=\int\mu(dx)E^{x,x\wedge\theta}((x_i^2(t)+(x_i^1(t)-x_i^2(t)))\wedge K)$$
$$\leq\int\mu(dx)E^{x,x\wedge\theta}(x_i^2(t)\wedge K)+\int\mu(dx)E^{x,x\wedge\theta}((x_i^1(t)-x_i^2(t))\wedge K).$$
$$\leq E^\theta(x_i(t)\wedge K)+\int\mu(dx)(E^x(x_i(t))-E^{x\wedge\theta}(x_i(t)))$$
$$=\langle T_t^*\delta_\theta,x_i\wedge K\rangle+\int\mu(dx)(x_i-x_i\wedge\theta)\,.$$

Letting $t\to\infty$ and $\theta\to\infty$ in this order, we get

$$\lim_{t\to\infty}\langle T_t^*\mu,x_i\wedge K\rangle=0\quad\text{for } i\in Z^d \text{ and } K>0,$$

which implies $\lim_{t\to\infty} T_t^*\mu=\delta_\theta$, completng the proof of Corollary 1.3.

REMARK. In this section we actually treated a linear model (3.1) with martingales as its deiving random force. Accordingly we can prove exponential decay of sample paths for more general SDE with coefficients $a_i(x)$ in place of $a(x_i)$ of the SDE (1.1), which may depend on other coordinates variables $\{x_j\}$, but should be assumed some uniformity conditions.

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