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A CHARACTERIZATION OF QF-ALGEBRAS

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We have defined a new class of rings in [3] which we call self mini-injective rings and we have noted that there exist artinian rings in the new class which are not quasi-Frobenius rings (briefly QF-rings).

We shall show in this note that if a ring R is an algebra over a field with finite dimension, then a self mini-injective algebra is a QF-algebra.

Throughout this note we assume a ring R contains the identity and every module is a unitary right R-module. We shall refer for the definitions of mini-injectives and the extending property, etc. to [3].

Let K be a field and R a K-algebra with finite dimension over K.

Theorem 1 (cf. [3], Theorems 13 and 14). *Let R be as above. Then the following conditions are equivalent.*

1) *R is self mini-injective as a right R-module.*

2) *R is self mini-injective as a left R-module.*

3) *Every projective right R-module has the extending property of direct decomposition of the socle.*

4) *Every projective left R-module has the extending property of direct decomposition of the socle.*

5) *R is a* QF-*algebra.*

Proof. R is self-injective as a left or right R-module if and only if R is a QF-algebra by [2]. In this case R is self-injective as both a right and left R-module by [1]. It is clear from [3], Theorem 3 and Proposition 8 that 1), 2) are equivalent to 3), 4), respectively. Hence, we may assume R is a basic algebra by [4] and [6].

1)→5). Let $R=\sum_{i=1}^{n}\oplus e_iR$ be the standard decomposition, namely $\{e_i\}$ is a set of mutually orthogonal primitive idempotents and $e_iR\not\approx e_{i'}R$ if $i\neq i'$. Since R is right self mini-injective, R is right QF-2 by [3], Proposition 8 and $S(e_iR)\not\approx S(e_{i'}R)$ for $i\neq i'$ by [3], Theorem 5, where $S(\ \ )$ means the socle. Now e_iR is uniform as a right R-module and so the injective envelope $E(e_iR)$ of e_iR is indecomposable. We put $M^*=\mathrm{Hom}_K(M,K)$ for a K-module M. Then $E(e_iR)^*$

is indecomposable and projective as a left R-module. Hence, $E(e_i/R)^* \approx Re_{i'}$ and $E(e_iR) \not\approx (Re_{i'})^*$. From the fact $E(e_iR) \approx E(e_jR)$ for $i \neq j$, a mapping $\pi: i \rightarrow i'$ is a permutation on $\{1,2,\cdots,n\}$. Accordingly, $\sum_{i=1}^{n} [E(e_iR): K] = \sum_{i=1}^{n} [Re_{\pi(i)}: K] = [R : K]$. Therefore, $E(R) = \sum^{n} \oplus E(e_iR) = R$.
The remaining part is clear.

In the above proof we have used only the facts that R is right QF-2 and $S(e_iR) \not\approx S(e_jR)$ if $i \neq j$. Hence, we have

Theorem 2. *Let R be a K-algebra as above. If R is right* QF-2 *and $S(eR) \not\approx S(e'R)$ if $eR \not\approx e'R$ then R is* QF, *where e and e' are primitive idempotents, where J is the Jacobson radical of R.*

Corollary. *Let R be the K-algebra as above. We assume R/J is a simple algebra. Then R is a* QF-*algebra if and only if R is a right* QF-2 *algebra.*

We note that the above facts are not true for right and left artinian rings (see [3], Example 2).

Next we shall consider a characterization of a right artinian and self mini-injective ring.

Theorem 3. *Let R be a right artinian ring. Then the following conditions are equivalent.*

1) *R is self mini-injective as a right R-module.*

2) *R satisfies*

i) *if $e_1R \not\approx e_2R$, any minimal right ideal in e_1R is not isomorphic to one in e_2R.*

ii) *there exists a minimal right ideal I contained in e_1J^k $(e_iJ^{k+1}=0)$ such that* $\mathrm{End}_R(I) = \{a \in \overline{e_1Re_1} \mid aI \subseteq I\}$, i.e. $\mathrm{End}_R(I)$ *is extended to* $\mathrm{End}_R(e_1R)$ *and $S(e_1R) = e_1Re_1I$ for each e_1, where the e_i is primitive idempotent and $S(\ )$ is socle and $\bar{R} = R/J$.*

Proof. 1)$\rightarrow$2). It is clear from [3], Theorem 5. 2)$\rightarrow$1). The second part of ii) implies that each minimal right ideal I' in e_1R is isomorphic to I. We assume $I \approx \overline{e_2R}$ and $I = xR$, $I' = x'R$. Then we may assume $xe_2 = x$ and $x'e_2 = x'$. We obtain from ii) that $x' = x'e_2 = \sum y_i x r_i$; $y_i \in e_1Re_1$, $r_i \in Re_2$. Now $xr_ie_2 = xe_2r_ie_2 = x\overline{e_2r_ie_2}$. Since a mapping $xz \rightarrow x\overline{e_2r_ie_2}z$ is an R-endomorphism of I, there exists an element $\bar{a}_i$ in $\overline{e_1Re_1}$ with $\bar{a}_ix = x\overline{e_2r_ie_2}$ from ii). Hence, $x' = (\sum y_i\bar{a}_i)x = \bar{b}x$, where $\bar{b} = \sum y_i\bar{a}_i$. We quote the proof of [3], Proposition 9. Since $\bar{b} \neq 0$, $x = \bar{b}^{-1}x'$. Put $g(x'z) = xz$; $z \in R$. Let f be any element in $\mathrm{Hom}_R(I, I')$. Then $gf \in \mathrm{End}_R(I)$. Hence, there exists $\bar{a}$ in $\overline{e_1Re_1}$ such that $gf(x) = \bar{a}x$ by ii). Therefore, $f(x) = g^{-1}(\bar{a}x) = \overline{ba}x$ and f is extended to an elemant in $\mathrm{End}_R(e_1R)$. We know similarly that $\mathrm{End}_R(I') = \bar{b}^{-1}\mathrm{End}_R(I)\bar{b} = \{\bar{c} \in \overline{e_1Re_1} \mid cI' \subseteq I'\}$.

Hence, I' satisfies ii). Let $h \in \mathrm{Hom}_R(I', R)$ and $R = \sum_{i=1}^{n} \oplus e_i R$. Let $\pi_i\colon R \to e_i R$ be the projection. If $\pi_i h \neq 0$, $e_i R \approx e_1 R$ by i). We assume $\pi_i h = h_i \neq 0$ for $i = 1, 2, \cdots, t$ and $h_j = 0$ for $j > t$. Since $e_1 R \approx e_i R$ for $i \leqslant t$, there exists $c_i \in e_i R e_i$ and $d_i \in e_i R e_1$ such that $c_i d_i = e_1$ and $d_i c_i = e_i$. Using d_i and c_i, we know as above that any element in $\mathrm{Hom}_R(I', h_i(I'))$ is extended to an element in $\mathrm{Hom}_R(e_1 R, e_i R)$ for $i \leqslant t$. Take $p_i \in R$ such that $p_i x' = h_i(x')$. Then $h(x') = \sum h_i(x') = (\sum p_i) x'$. Hence, R is right self mini-injective by [3], Theorem 2.

REMARK. The above three conditions in Theorem 3, 2) are independent.

Corollary 1. *Let R be a right artinian and right self mini-injective. Then R is a right* QF-2 *if and only if* $\mathrm{End}_R(I) = \overline{e_1 R e_1}$ *in ii) of Theorem* 3.

Corollary 2. *Let R be a right artinian ring and e a primitive idempotent. We assume that i) R is right* QF-2, ii) *any monomorphism of* $\overline{eRe}$ *into itself as a division ring is isomorphic for each e* (e.g. *algebraic extension of the prime field*) *and* iii) $S(eR) \not\approx S(e'R)$ *if* $eR \not\approx e'R$. *Then R is right self mini-injective.*

Proof. We may assume R is basic. Since $S(eR) \supset eJ^k \neq 0$ ($eJ^{k+1} = 0$), $S(eR) = eJ^k$ by i). Put $S(eR) = uR$. $eJeu \subset eJ^{k+1} = 0$ and so uR is a left $\overline{eRe}$-module. We assume $uR \approx \overline{e'R}$. Since R is basic, $\overline{e'R} = \overline{e'Re'}$. Hence, $\overline{ue'Re'} = uR$. Let $\bar{x}$ be in $\overline{eRe}$. Then $\bar{x}u = u\bar{y}$; $\bar{y} \in \overline{e'Re'}$. It is clear that the mapping $\bar{x} \to \bar{y}$ gives us a monomorphism of the division ring $\overline{eRe}$ into $\overline{e'Re'}$ as a division ring. Repeating this procedure, we can find a chain $e, e', \cdots, e^{(t)}$ of primitive idempotents. We know from iii) that $e^{(s)} = e$ for some s (cf. [3], the proof of Proposition 8). Hence, $\overline{eReu} = \overline{ue'Re'}$ by ii). Therefore, R is right self mini-injective by Theorem 3.

We do not know any example of a right QF-2 and right self mini-injective ring which is not QF.

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