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A CHARACTERIZATION OF QF-ALGEBRAS

MANABU HARADA

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We have defined a new class of rings in [3] which we call self mini-injective rings and we have noted that there exist artinian rings in the new class which are not quasi-Frobenius rings (briefly QF-rings).

We shall show in this note that if a ring $R$ is an algebra over a field with finite dimension, then a self mini-injective algebra is a QF-algebra.

Throughout this note we assume a ring $R$ contains the identity and every module is a unitary right $R$-module. We shall refer for the definitions of mini-injectives and the extending property, etc. to [3].

Let $K$ be a field and $R$ a $K$-algebra with finite dimension over $K$.

**Theorem 1** (cf. [3], Theorems 13 and 14). Let $R$ be as above. Then the following conditions are equivalent.

1) $R$ is self mini-injective as a right $R$-module.
2) $R$ is self mini-injective as a left $R$-module.
3) Every projective right $R$-module has the extending property of direct decomposition of the socle.
4) Every projective left $R$-module has the extending property of direct decomposition of the socle.
5) $R$ is a QF-algebra.

Proof. $R$ is self-injective as a left or right $R$-module if and only if $R$ is a QF-algebra by [2]. In this case $R$ is self-injective as both a right and left $R$-module by [1]. It is clear from [3], Theorem 3 and Proposition 8 that 1), 2) are equivalent to 3), 4), respectively. Hence, we may assume $R$ is a basic algebra by [4] and [6].

1) $\rightarrow$ 5). Let $R=\sum_{i=1}^{n} \oplus e_{i}R$ be the standard decomposition, namely $\{e_{i}\}$ is a set of mutually orthogonal primitive idempotents and $e_{i}R=e_{i}e_{i}R$ if $i \neq i'$. Since $R$ is right self mini-injective, $R$ is right QF-2 by [3], Proposition 8 and $S(e_{i}R)\neq S(e_{i}R)$ for $i \neq i'$ by [3], Theorem 5, where $S(\ )$ means the socle. Now $e_{i}R$ is uniform as a right $R$-module and so the injective envelope $E(e_{i}R)$ of $e_{i}R$ is indecomposable. We put $M^{*}=\text{Hom}_{K}(M,K)$ for a $K$-module $M$. Then $E(e_{i}R)^{*}$
is indecomposable and projective as a left \( R \)-module. Hence, \( E(e_i|R) \cong Re_i \)\(^R\) and \( E(e_i|R) \cong (Re_i)\* \). From the fact \( E(e_i|R) \cong E(e_j|R) \) for \( i \neq j \), a mapping \( \pi : i \to i' \) is a permutation on \( \{1, 2, \cdots, n\} \). Accordingly, \( \sum_{i=1}^{n} [E(e_i|R) : K] \sum_{i=1}^{n} [Re_i\pi(i) : K] = [R : K] \). Therefore, \( E(R) = \sum E(e_i|R) = R \).

The remaining part is clear.

In the above proof we have used only the facts that \( R \) is right QF-2 and \( S(e_i|R) \cong S(e_i|R) \) if \( i \neq j \). Hence, we have

**Theorem 2.** Let \( R \) be a \( K \)-algebra as above. If \( R \) is right QF-2 and \( S(e_i|R) \cong S(e_i'|R) \) if \( e_i \neq e_i' \) then \( R \) is QF, where \( e \) and \( e' \) are primitive idempotents, where \( J \) is the Jacobson radical of \( R \).

**Corollary.** Let \( R \) be the \( K \)-algebra as above. We assume \( R/J \) is a simple algebra. Then \( R \) is a QF-algebra if and only if \( R \) is a right QF-2 algebra.

We note that the above facts are not true for right and left artinian rings (see [3], Example 2).

Next we shall consider a characterization of a right artinian and self mini-injective ring.

**Theorem 3.** Let \( R \) be a right artinian ring. Then the following conditions are equivalent.

1) \( R \) is self mini-injective as a right \( R \)-module.
2) \( R \) satisfies
   i) if \( e_i R \cong e_i R \), any minimal right ideal in \( e_i R \) is not isomorphic to one in \( e_2 R \).
   ii) there exists a minimal right ideal \( I \) contained in \( e_i J^{i+1} = 0 \) such that \( \text{End}_R(I) = \{a \in e_i Re_i | a I \subseteq I\} \), i.e. \( \text{End}_R(I) \) is extended to \( \text{End}_R(e_i R) \) and \( S(e_i R) = e_i Re_i I \) for each \( e_i \), where the \( e_i \) is primitive idempotent and \( S(\cdot) \) is socle and \( R = R/J \).

**Proof.** 1) \( \to \) 2). It is clear from [3], Theorem 5. 2) \( \to \) 1). The second part of ii) implies that each minimal right ideal \( I' \) in \( e_i R \) is isomorphic to \( I \). We assume \( I \cong e_2 R \) and \( I=xR, I'=x' R \). Then we may assume \( xe_2=x \) and \( x'e_2=x' \). We obtain from ii) that \( x' = x'e_2 = \sum y_i x_r \), \( y_i \in e_i Re_i, r_i \in Re_2 \). Now \( x = x'e_2 = xe_2 e_2 - xe_2 e_2 \). Since a mapping \( xz \to xe_2 e_2 \) is an \( R \)-endomorphism of \( I \), there exists an element \( a_i \) in \( e_i Re_i \) with \( a_i x = xe_2 e_2 \). Since \( b \neq 0, x = b^{-1} x' \). Also, \( g(x') = xz, z \in R \). Let \( f \) be any element in \( \text{Hom}_R(I, I') \). Then \( gf \in \text{End}_R(I) \). Hence, there exists \( a_i \) in \( e_i Re_i \) such that \( gf(x) = ax \) by ii). Therefore, \( f(x) = g^{-1}(ax) = bax \) and \( f \) is extended to an element in \( \text{End}_R(e_i R) \). We know similarly that \( \text{End}_R(I') = b^{-1} \text{End}_R(I) b = \{\xi \in e_i Re_i | e_i I' \subseteq I'\} \).
Hence, $I'$ satisfies ii). Let $h \in \text{Hom}_R(I', R)$ and $R = \sum_{i=1}^{\infty} e_i R$. Let $\pi_i : R \to e_i R$ be the projection. If $\pi_i h = h_i \neq 0$ for $i = 1, 2, \ldots, t$ and $h_j = 0$ for $j > t$. Since $e_i R \cong e_i R$ for $i \leq t$, there exists $c_i \in e_i R$ and $d_i \in e_i R$ such that $c_i d_i = e_i$. Using $d_i$ and $c_i$, we know as above that any element in $\text{Hom}_R(I', h(I'))$ is extended to an element in $\text{Hom}_R(e_i R, e_i R)$ for $i \leq t$. Take $p_i \in R$ such that $p_i x' = h_i(x')$. Then $h(x') = \sum h_i(x') = (\sum p_i)x'$. Hence, $R$ is right self mini-injective by [3], Theorem 2.

REMARK. The above three conditions in Theorem 3, 2) are independent.

Corollary 1. Let $R$ be a right artinian and right self mini-injective. Then $R$ is a right QF-2 if and only if $\text{End}_R(I) = e_i R$, in ii) of Theorem 3.

Corollary 2. Let $R$ be a right artinian ring and $e$ a primitive idempotent. We assume that i) $R$ is right QF-2, ii) any monomorphism of $eR$ into itself as a division ring is isomorphic for each $e$ (e.g. algebraic extension of the prime field) and iii) $S(eR) \cong S(e'R)$ if $eR \cong e'R$. Then $R$ is right self mini-injective.

Proof. We may assume $R$ is basic. Since $S(eR) \supseteq eJ^k = 0 (eJ^{k+1} = 0)$, $S(eR) = eJ^k$ by i). Put $S(eR) = uR$. $eJ u \subseteq eJ^{k+1} = 0$ and so $uR$ is a left $eR$-module. We assume $uR \cong e'R$. Since $R$ is basic, $e'R = e'R$. Hence, $ue'R = uR$. Let $x$ be in $eR$. Then $xu = uy; y \in e'R$. It is clear that the mapping $x \mapsto y$ gives us a monomorphism of the division ring $eR$ into $e'R$ as a division ring. Repeating this procedure, we can find a chain $e, e', \ldots, e^{(t)}$ of primitive idempotents. We know from iii) that $e^{(s)} = e$ for some $s$ (cf. [3], the proof of Proposition 8). Hence, $eReu = e'R$ by ii). Therefore, $R$ is right self mini-injective by Theorem 3.

We do not know any example of a right QF-2 and right self mini-injective ring which is not QF.

References


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