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VANISHING THEOREMS FOR COHOMOLOGY GROUPS ASSOCIATED TO DISCRETE SUBGROUPS OF SEMISIMPLE LIE GROUPS

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Introduction. The aim of this paper is to prove two vanishing theorems for cohomology groups related to discrete uniform subgroups of semisimple Lie groups.

Let $\rho$ be a representation of a real linear semisimple Lie group $G$ and $\Gamma$ a discrete subgroup of $G$ such that $\Gamma \backslash G$ is compact. Assume that $\Gamma$ contains no elements of finite order. In §1 we give a criterion in terms of the highest weight of $\rho$ for the vanishing of $H^p(\Gamma, \rho)$, the $p^{th}$ cohomology group of $\Gamma$ with coefficient in $\rho$. This criterion is a generalisation of a theorem of Matsushima and Murakami [3].

In §2 we prove the following theorem (Corollary to Theorem 3). Let $G$ be a complex semisimple Lie group without any simple component of rank 1. Then for any discrete subgroup $\Gamma$ such that $\Gamma \backslash G$ is compact, the canonical complex structure on the space $\Gamma \backslash G$ is rigid. (This question whether these complex structures are rigid was raised by Professor Matsushima).

1. A vanishing theorem for the cohomology of discrete uniform subgroups

Let $G$ be a connected real linear semisimple Lie group and $\Gamma$ a discrete subgroup such that the quotient $\Gamma \backslash G$ is compact. Let $\mathfrak{g}_0$ be the Lie algebra of left-invariant vector-fields of $G$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ a Cartan-decomposition of $\mathfrak{g}_0$, $\mathfrak{k}_0$ being the algebra. Let $K$ be the (compact) Lie subgroup corresponding to $\mathfrak{k}_0$ and $X=G/K$ the corresponding symmetric space. To every representation of $G$ in a finite dimensional real (or complex) vector space $F$, Matsushima and Murakami [2] have associated certain cohomology groups: we follow their notation and denote these groups by $H^p(\Gamma, X, \rho)$. (In the case when $\Gamma$ has no elements of finite order $\Gamma$ acts freely on $X$ and $H^p(\Gamma, X, \rho)$ is isomorphic to the $p^{th}$ cohomology group of $\Gamma$ with coefficients in the restriction $\rho_{|\Gamma}$ of $\rho$ to $\Gamma$). In the same article, they prove moreover the following result (see in particular §6, §7). (Proposition 1 below).
The vectorfields in $g_0$ project under the natural map $G \to \Gamma \backslash G$ into vectorfields on $\Gamma \backslash G$. We will from now on identify $g_0$ with this algebra of vectorfields on $\Gamma \backslash G$. Let $\phi$ be the Killing form on $g_0$ and $\{X_i\}_{1 \leq i \leq N}$ and $\{X_\alpha\}_{N+1 \leq \alpha \leq \lambda}$ be bases of $p_0$ and $\mathfrak{t}_0$ such that $\phi(X_i, X_j) = \delta_{ij}$ and $\phi(X_\alpha, X_\beta) = -\delta_{\alpha \beta}$. Let $A_\rho(\Gamma, X, \rho)$ be the vector space of $C^\infty$-$p$-forms $\eta$ on $\Gamma \backslash G$ satisfying i) $i_X \eta = 0$ and ii) $\theta_X \eta = \rho(X)\eta$ for every $X \in \mathfrak{t}_0$ where $i_X$ (resp $\theta_X$) denotes interior derivation (resp. Lie derivation) of $\eta$ with respect to the vectorfield $X$. Because of i) and ii) $\eta$ is determined by its values $i_1 \cdots i_p = \eta(X_{i_1} \cdots X_{i_p})$. Finally, let $\Delta^p$ be the operator

$$\Delta^p : A^p_\rho(\Gamma, X, \rho) \to A^p_\rho(\Gamma, X, \rho)$$

defined by

$$\Delta^p \eta(X_{i_1} \cdots X_{i_p}) = \sum_{i=1}^N \left(-X_i^2 + \rho(X_i)\right) \eta_{i_1 \cdots i_p} + \sum_{i=1}^N \sum_{j=1}^p (-1)^{p+1} \left\{\left([-X_i, X_j] + \rho([X_i, X_j])\right)\eta_{ki_1 \cdots i_{u-1}ip} \right\}$$

With this notation, we have

**Proposition 1.** $H^p(\Gamma, X, \rho)$ is canonically isomorphic to the vector space $\{\eta | \eta \in A^p_\rho(\Gamma, X, \rho); \Delta^p \eta = 0\}$.

Again, following [2], we define two operators $\Delta_D^p$ and $\Delta_F^p$ as follows:

$$\Delta_D^p(X_{i_1} \cdots X_{i_p}) = -\sum_{i=1}^N X_i^2 \eta_{i_1 \cdots i_p} + \sum_{i=1}^N \sum_{j=1}^p (-1)^{p+1} \left\{\left([-X_i, X_j] + \rho([X_i, X_j])\right)\eta_{ki_1 \cdots i_{u-1}ip} \right\}$$

Then $\Delta^p = \Delta_D^p + \Delta_F^p$. In § 7 [2], it is moreover proved that

$$\sum_{i_1 < \cdots < i_p} \int_{\Gamma \backslash G} \langle \Delta_D^p \eta \rangle_{i_1 \cdots i_p}, \eta_{i_1 \cdots i_p} \rangle_F \geq 0$$

where $\langle , \rangle_F$ is a positive definite scalar product on $F$ for which $\rho(X)$ is (hermitian) symmetric (resp. skew-symmetric (hermitian)) for $X \in p_0$ (resp. $\mathfrak{t}_0$).

It follows therefore that if $\Delta^p \eta = 0$,

$$\sum_{i_1 < \cdots < i_p} \int_{\Gamma \backslash G} \langle \Delta_F^p \eta \rangle_{i_1 \cdots i_p}, \eta_{i_1 \cdots i_p} \rangle_F \geq 0$$

We obtain therefore

**Proposition 2.** If the quadratic form on the space of exterior $p$-forms on $p_0$ with values in $F$ defined by

$$\eta \to \sum_{i_1 < \cdots < i_p} \langle \Delta_F^p \eta \rangle_{i_1 \cdots i_p}, \eta_{i_1 \cdots i_p} \rangle_F$$

is positive definite, then $H^p(\Gamma, X, \rho) = 0$. 
In the main result of this section we give a sufficient criterion in terms of the "highest weight" of $\rho$ with respect to a suitable Cartan-subalgebra of $g_0$ in order that $\Delta^*_p$ define a positive definite quadratic form.

Let $g$ denote the complexification of $g_0$ and $\mathfrak{h}$ and $\mathfrak{p}$ those of $\mathfrak{h}_0$ and $\mathfrak{p}_0$. We identify $\mathfrak{h}$ and $\mathfrak{p}$ with subspaces of $g$. Let $\mathfrak{h}_0$ be a Cartan-subalgebra of $\mathfrak{h}_0$ and $\mathfrak{p}_0$ a Cartan-subalgebra of $g_0$ such that $\mathfrak{h}_0 \supset \mathfrak{h}_0$. Let $\mathfrak{h}_0 \cap \mathfrak{p}_0$. Let $\mathfrak{h}_0$ denote respectively the complexifications of $\mathfrak{h}_0$, $\mathfrak{h}_0$ and $\mathfrak{p}_0$. Then $\mathfrak{h}$ is a Cartan-subalgebra of $g$. Let $\Delta$ be the system of roots of $g$ with respect to $\mathfrak{h}$. For $\alpha \in \Delta$ let $H_\alpha \in \mathfrak{h}$ be the unique element such that $\varphi(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. Then, it is well known that the real subspace $\mathfrak{h}^* = \sum_{\alpha \in \Delta} RH_\alpha$ of $g$ spanned by the $\{H_\alpha\}_{\alpha \in \Delta}$ is the same as $\mathfrak{h}_0 \oplus \mathfrak{p}_0$. Moreover if $\theta$ is the extension to $g$ to the Cartan involution $\theta$ denfied by the Cartan-decomposition $g_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0$, then $\theta$ is an automorphism of $g$ leaving $\mathfrak{h}$ invariant. Hence $\theta$ acts on the dual of $\mathfrak{h}$ and permutes the elements of $\Delta$. The set $\Delta$ may then be decomposed as the disjoint union $A \cup B \cup C$ of three subsets $A$, $B$ and $C$

where

$$A = \{\alpha | \alpha \in \Delta; \theta(\alpha) = \alpha; \theta(E_\alpha) = E_\alpha\}$$

$$B = \{\alpha | \alpha \in \Delta; \theta(\alpha) \neq \alpha\}$$

$$C = \{\alpha | \alpha \in \Delta; \theta(\alpha) = \alpha; \theta(E_\alpha) = -E_\alpha\}.$$ 

(In the sequel we sometimes write $\alpha^\theta$ for $\theta(\alpha)$).

We introduce next a lexicographic order on the (real) dual of $\mathfrak{h}^*$ as follows: let $H_1, \cdots, H_r$ be an orthonormal basis of $\mathfrak{h}^*$ with respect to $\varphi$ ($\varphi | \mathfrak{h}^*$ is positive definite) chosen so that $H_1, \cdots, H_r$ form a basis of $i \mathfrak{h}_0$ and if the centre $c_0$ of $\mathfrak{t}_0$ is non-zero, of dimension $r$, then $H_1, \cdots, H_r$ belong to $i \mathfrak{c}$; for $\alpha, \beta$ in the (real) dual of $\mathfrak{h}^*$, $\alpha > \beta$ if the first non-vanishing difference $\alpha(H_i) - \beta(H_i)$ is greater than zero. Let $\Delta^+$ be the system of positive roots with respect to this order and let $A^+ = A \cap \Delta^+$, $B^+ = B \cap \Delta^+$, $C^+ = C \cap \Delta^+$. Then $\theta$ leaves $A^+$, $B^+$ and $C^+$ invariant. Let $\sum_i A^+ U \{\alpha | \alpha \in B^+; \theta(\alpha) > \alpha\}$ and $\sum_i C^+ U \{\alpha | \alpha \in B^+; \theta(\alpha) > \alpha\}$.

**Theorem 1.** Let $\rho$ denote a finite dimensional representation of $G$ in a complex vector-space $F$, as also the induced representation of $g$. Let $\Lambda_p$ be the highest weight of $\rho$ with respect to the above defined Cartan-subalgebra and the order on the dual of $\mathfrak{h}^*$. Then if $\sum_{\rho} = \{\alpha | \alpha \in \sum_{\rho}, \varphi(\Lambda_\rho, \alpha) = 0\}$ contains more than $q$ elements, then the Hermitian quadratic form $Q_\rho$ defined by

$$\eta \rightarrow \sum_{i_1 < \cdots < i_p} \langle (\Delta^*_p \eta)_{i_1 \cdots i_p}, \eta_{i_1 \cdots i_p} \rangle_F$$

is positive definite for $p \leq q$. Hence $H^p(\Gamma, X, \rho) = 0$ for $1 \leq p \leq q$.

Before we proceed to the proof of the theorem, we will make a few preliminary simplifications:
Lemma 1. Let \( E \) be the \( q \)th exterior power of \( p \) and let \( \alpha \) be the isomorphism onto \( F \otimes E \) of the space of exterior \( q \)-forms on \( p \) with values in \( F \) defined by

\[
\eta \rightarrow \sum_{i_1 < \cdots < i_q} \eta_{i_1 \cdots i_q} \otimes (X_{i_1} \wedge \cdots \wedge X_{i_q})
\]

Then

\[
T_q^\alpha = 2\alpha \circ \Delta_q \circ \alpha^{-1} = 2(\rho \otimes 1)(c) + (1 \otimes \sigma)(c') - (\rho \otimes 1)(c') - (\rho \otimes \sigma)(c')
\]

where

\[
c = \sum_{i=1}^p X_i \otimes \sum_{a=N+1}^{N+p} X_a^2
\]

and \( c' = - \sum_{a=N+1}^{N+p} X_a^2 \) are elements of the enveloping algebras of \( \mathfrak{g} \) and \( \mathfrak{k} \) and \( \sigma \) denotes the adjoint representation of \( \mathfrak{k} \) in \( E \). Hence \( T_q^\alpha \) is a symmetric endomorphism of \( F \otimes E \) with respect to the scalar product

\[
\langle \sum_{i_1 < \cdots < i_p} \eta_{i_1 \cdots i_p} \otimes X_{i_1} \wedge \cdots \wedge X_{i_p}, \sum_{j_1 < \cdots < j_p} \eta_{j_1 \cdots j_p} \otimes X_{j_1} \wedge \cdots \wedge X_{j_p} \rangle
\]

Proof. We have

\[
(\Delta_q^\alpha)^{i_1 \cdots i_q} = \sum_{k=1}^N \rho(X_k)\eta_{i_1 \cdots i_q} + \sum_{k=1}^N \sum_{a=N+1}^{N+p} (-1)^{a-1}\rho([X_{i_a}, X_k])\eta_{k i_1 \cdots i_a \cdots i_q}
\]

For every \( q \)-tuple \( I_a=(i_1<\cdots<i_q) \), we write \( X_{I_q} \) for \( X_{i_1} \wedge \cdots \wedge X_{i_q} \). In this notation,

\[
\alpha(\eta) = \sum_{I_q} \eta_{I_q} \otimes X_{I_q}
\]

\[
\frac{1}{2} T_q^\alpha(\eta) = \sum_{I_q} \sum_{k=1}^N \rho(X_k)\eta_{I_q} + \sum_{I_q} \sum_{a=N+1}^{N+p} (-1)^{a-1}\rho([X_{i_a}, X_k])\eta_{k i_1 \cdots i_a \cdots i_q} \otimes X_{I_q}
\]

\[
= \sum_{I_q} \sum_{k=1}^N \rho(X_k)\eta_{I_q} + \sum_{J_q \Delta I_q=i_1 u_1} (-1)^{a+p}\rho([X_{i_a}, X_{j_1}])\eta_{j_q} \otimes X_{I_q}
\]

On the other hand,

\[
\sigma(X_a)X_{J_q} = \sum_{k=1}^N \sum_{i=1}^N (-1)^{a-1} c_{j_k}^a (X_k \wedge X_{j_1} \cdots X_{j_{a-1}} \wedge X_{j_a})
\]

\[
= \sum_{I_q \Delta J_q=i_1 u_1} (-1)^{a+p} c_{i_1 u_1} X_{I_q}
\]

It follows that

\[
\frac{1}{2} T_q^\alpha(\eta) = \sum_{I_q} \sum_{k=1}^N \rho(X_k)\eta_{I_q} \otimes X_{I_q} + \sum_{I_q} \rho(X_a)\eta_{I_q} \otimes \sigma(X_a) \otimes X_{I_q}
\]

\[
= \{\sum_{I_q} \rho(X_k) \otimes 1 + \sum \rho(X_a) \otimes \sigma(X_a) \alpha(\eta) \}.
\]
Now the required result follows from the fact 
\[ 2\rho(X_a) \otimes \sigma(X_a) = (\rho(X_a) \otimes 1 + 1 \otimes \sigma(X_a))^2 - \rho(X_a) \otimes 1 - 1 \otimes \sigma(X_a)^2 \]
\[ = (\rho \otimes \sigma)(X_a)^2 - \rho(X_a) \otimes 1 - 1 \otimes \sigma(X_a)^2 \]
That \( T^\mu_\nu \) is a hermitian symmetric endomorphism follows from the facts that \( \rho(X_i) \) and \( \sigma(X_i) \) are hermitian symmetric while \( \rho(X_a) \) and \( \sigma(X_a) \) are skew-hermitian with respect to \( \langle , \rangle_F \) and the extension to \( E \) of the Killing form on \( p_0 \).

**Lemma 2.** a) If \( \Lambda \) is the highest weight of an irreducible representation \( \rho \) of \( g \) induced by a representation \( \rho \) of \( G \), then 
\[ \rho(c) = \{ \varphi(\Lambda, \Lambda) + \sum \varphi(\Lambda, \alpha) \}. \]
Identity
b) when restricted to the (irreducible) \( K \)-subspace generated by the eigen-space corresponding to the highest weight \( \Lambda \),
\[ \rho(c') = \left\{ \frac{1}{4} \varphi(\Lambda + \Lambda^0, \Lambda + \Lambda^0) + \sum_{\alpha \in \Sigma_1} \varphi\left( \Lambda, \frac{\alpha + \alpha^\theta}{2} \right) \right\}. \]
Identity.
For a proof see [4]: Lemmas 4 and 16(c).

**Lemma 3.** If \( \Lambda_1 \) and \( \Lambda_2 \) are the highest weights of two irreducible representations \( \rho_1, \rho_2 \) of \( g \), such that \( \Lambda_1 - \Lambda_2 \) is a non-negative linear combination of simple roots of \( g \), then \( \lambda_1 \geq \lambda_2 \) where \( \rho_{\mu}(c) = (\lambda_k) \). Identity \((k=1, 2) \). Equality can occur only if \( \Lambda_1 = \Lambda_2 \).

The same conclusions hold for \( \mathfrak{k} \) and \( c' \) instead of \( g \) and \( c \) provided that \( \Lambda_1 \) and \( \Lambda_2 \) coincide on the center of \( \mathfrak{k} \).
For the proof see Lemma 5 [4].

**Proof of Theorem 1.** We obtain the eigen-values of \( T^\mu_\nu \) as follows: Let 
\[ E = \sum_{\mu \in \mathfrak{M}} E_\mu \quad \text{and} \quad F = \sum_{\lambda \in \mathfrak{L}} F_\lambda \quad \text{and} \quad F_\lambda \otimes E_\mu = \sum_{\nu \in \mathfrak{M}_\lambda \mu} V^\nu_{\lambda \mu} \]
be the decomposition of \( E, F \) and \( F_\lambda \otimes E_\mu \) into irreducible \( \mathfrak{f} \)-modules indexed by the highest weights (for the order defined by \( H_1, \cdots, H_\mathfrak{g} \) on \( \mathfrak{g} \)). Since \( \rho \) is an irreducible representation of \( g \) and \( c \) is a central element of \( U(g) \), \( \rho(c) \) is a scalar operator. Similarly, since \( c' \) is central in \( U(\mathfrak{k}) \), \( \rho(c') \otimes 1, 1 \otimes \sigma(c') \) and \( (\rho \otimes \sigma)(c') \) are scalars on \( F_\lambda E, F \otimes E_\lambda \) and \( V^\nu_{\lambda \mu} \). Hence \( T^\mu_\nu \) acts as a scalar on each \( V^\nu_{\lambda \mu} \). We denote the corresponding eigen-value by \( a(\lambda, \mu, \nu) \). Among \( V^\nu_{\lambda \mu} \) there is a unique irreducible component with highest weight \( \nu = \lambda + \mu \) we denote the corresponding scalar \( a(\lambda, \mu, \nu) \) by \( a(\lambda, \mu) \) with this notation, we have

**Assertion I.** \( a(\lambda, \mu, \nu) \geq a(\lambda, \mu) \); equality occurs only if \( \nu = \lambda + \mu \).

Proof. We denote the representation in \( V^\nu_{\lambda \mu} \) by \( \rho^\nu_{\lambda \mu} \). Then since \( (\rho \otimes 1)(c), (\rho \otimes 1)(c') \) and \( (1 \otimes \sigma)(c') \) all define the same scalar operator in \( F_\lambda \otimes E_\mu \).
\[ a(\lambda, \mu) + a(\lambda, \mu, \nu) = \rho^{\lambda+\mu}_\alpha(c') - \rho^{\lambda+\mu}_\alpha(c') \]

(Here we have let \( \rho^{\lambda}_\alpha(c') \) stand for the scalar). Now any weight in \( F_\lambda \otimes E_\mu \) has the form \( \lambda + \mu_1 \) where \( \lambda_1 \) and \( \mu_1 \) are weights of \( F_\lambda \) and \( E_\mu \); on the other hand \( \lambda - \lambda_1 \) and \( \mu - \mu_1 \) are non-negative linear combinations of simple roots of \( k \); hence so is \( (\lambda+\mu) - (\lambda_1+\mu_1) \). It follows then from Lemma 3 that

\[ a(\lambda, \mu, \nu) = -a(\lambda, \mu, \nu) \]

Equality can occur only if \( \lambda + \mu = \lambda_1 + \mu_1 \) and there is only one component of \( F_\lambda \otimes E_\mu \) with \( \lambda + \mu \) as the highest weight. (Note that if \( \mathfrak{f} \) has a centre, then the central elements act as scalars on \( F_\lambda \) and \( E_\mu \) hence in all of \( F_\lambda \otimes E_\mu \)).

**Assertion II.** Let \( f_\lambda \) be a highest weight vector of \( F \) such that \( \| f_\lambda \|_F^2 = 1 \). For \( \alpha \in \Delta \), let \( E_\alpha \) be a root vector of \( \alpha \). Suppose that \( E_{\alpha_\theta} f_\lambda = 0 \) for \( \alpha \in A^+ \). If there is an \( \alpha_0 \in B^+ \) with \( E_{\alpha_\theta} f_\lambda = 0 \), then \( E_{\alpha_\theta} f_\lambda \in F_{\lambda_1} \) for some \( \lambda_1 \) and \( a(\lambda_1, \mu) < a(\lambda_1, \mu_1) \)

Proof. Using the fact that \( \theta \) is an involution, we have

\[ \mathfrak{f} = \mathfrak{h}_t \bigoplus \sum_{\alpha \in A^+} \{ CE_\alpha \otimes CE_\alpha \} \bigoplus \sum_{\alpha \in \Delta^\alpha} \{ C(E_\alpha + E_\alpha \theta) \bigoplus C(E_\alpha + E_\alpha \theta) \} \]

and the order chosen on \( \mathfrak{h}_t^* = i \mathfrak{h}_t \) has precisely \( \{ \alpha | \alpha \in A^+ \} \) and \( \{ \alpha + \alpha^\theta | \alpha \in B^+ \} \) as the positive roots. The roots of \( \mathfrak{f} \) are necessarily zero on the centre of \( \mathfrak{f} \). It follows that the weights \( \lambda \) and \( \lambda + \alpha_0 \) (which is the weight corresponding to \( E_{\alpha_\theta} f_\lambda \)) have the same values on the centre. On the other hand, since \( \lambda + \alpha_0 \) and \( \lambda_1 \) are weights of the same irreducible representation of \( \mathfrak{f} \), \( \lambda_1 \) and \( \lambda + \alpha_0 \) have the same values on the centre of \( \mathfrak{f} \). It follows that \( \lambda_1 = \lambda \) on the centre of \( \mathfrak{f} \). Now \( \lambda_1 - \lambda = (\lambda - \lambda_1) + \alpha_0 \) and \( \lambda_1 - (\lambda + \alpha_0) \) is a non-negative linear combination of simple roots. Hence \( \lambda_1 - \lambda \) is a non-negative linear combination of simple roots and \( \lambda_1 = \lambda \). A similar remark holds for \( \lambda_1 + \mu \) and \( \lambda + \mu \). It follows then from Lemma 3 above that

\[ \rho_\lambda(c') < \rho_{\lambda_1}(c') \]

and

\[ \rho^{\lambda+\mu}_\alpha(c') < \rho^{\lambda_1+\mu}_\alpha(c') \]

The operators \((\rho \otimes 1)(c)\) and \((1 \otimes \sigma)(c')\) on the other hand are scalars on the whole of \( F \otimes E \). Hence from the expression for \( T^g \), the Assertion follows.

**Assertion III.** Suppose that \( E_{\alpha} F_\lambda = 0 \) for \( \alpha \in A^+ \cup B^+ \) but that there is an \( \alpha_0 \in C^+ \) such that \( E_{\alpha_\theta} f_\lambda = 0 \). Then \( a(\lambda, \mu) > 0 \).

Proof. If \( \{ E_\alpha \}_{\alpha \in \Delta} \) are root vectors so chosen that \( \varphi(E_\alpha, E_{-\alpha}) = 1 \), then, it is well known that
It follows that

$$c = \sum_{a \in A^+} E_a E_{-a} + \sum_{a \in A^+} E_a E_{-a} + \sum_{i=1}^{\frac{1}{2}} H_i^2$$

Using the facts, $E_a f_\lambda = 0$ for $\alpha \in A^+ \cup B^+$ and that $[E_a, E_{-a}] = H_{a}$, we have

$$\rho(c) f_\lambda = \sum_{a \in A^+ \cup B^+} \rho(E_a E_{-a} + E_{-a} E_a) f_\lambda + \sum_{i=1}^{\frac{1}{2}} \rho(H_i) f_\lambda$$

Hence

$$\langle \rho(c) f_\lambda, f_\lambda \rangle_F = \sum_{a \in A^+ \cup B^+} \lambda(\alpha) + \sum_{i=1}^{\frac{1}{2}} \lambda(H_i) + \sum_{a \in C^+} \rho(E_a E_{-a} + E_{-a} E_a) f_\lambda + \sum_{i=1}^{\frac{1}{2}} \rho(H_i) f_\lambda$$

Now it is well known that $F$ admits an orthogonal decomposition with respect to $\langle \cdot, \cdot \rangle_F$ into irreducible representations of the algebra $g'=CE_a \oplus CE_{-a} \oplus CH_{a}$ for $\alpha \in C^+$ so that to prove that $\langle \rho(E_a E_{-a} + E_{-a} E_a) f_\lambda, f_\lambda \rangle \geq |\lambda(H_a)|$ equality occurring only if $E_a f_\lambda = 0$, we may assume that the $g'$-invariant subspace $W$ spanned by $f_\lambda$ is irreducible with respect to the three dimensional algebra. Now by Lemma 2,

$$\rho\left\{E_a E_{-a} + E_{-a} E_a + \frac{H_a^2}{\varphi(H_a H_a)}\right\} f_\lambda = \left\{\frac{\alpha(H_a)}{\varphi(H_a, H_a)} + (\lambda + k\alpha)(H_a)\right\} f_\lambda$$

where $\lambda + k\alpha$, $k \geq 0$ is the highest weight in $W$ (of $g'$). Hence

$$\rho(E_a E_{-a} + E_{-a} E_a) f_\lambda = \frac{k\alpha(H_a)^2}{\varphi(H_a, H_a)} + (\lambda + k\alpha)(H_a) f_\lambda$$

so that

$$\langle \rho(E_a E_{-a} + E_{-a} E_a) f_\lambda, f_\lambda \rangle_F = (\lambda + k\alpha)(H_a) + \frac{\alpha(H_a)}{\varphi(H_a, H_a)} \geq |\lambda(H_a)|$$

(It is well known that $(\lambda + k\alpha)(H_a) \geq |\lambda(H_a)|$ since $\lambda + k\alpha$ is the highest weight). Moreover equality occurs only if $k = 0$; if $k = 0$, however, $\lambda$ is the highest weight so that $E_a f_\lambda = 0$. We have thus shown that

$$\langle \rho(E_a E_{-a} + E_{-a} E_a) f_\lambda, f_\lambda \rangle \geq |\lambda(H_a)|$$

equality occurring only if $E_a f_\lambda = 0$. We have therefore,

$$\langle \rho(c) f_\lambda, f_\lambda \rangle \geq \sum_{a \in A^+ \cup B^+} \lambda(\alpha) + \sum_{i=1}^{\frac{1}{2}} \lambda(H_i) + \sum_{a \in C^+} |\lambda(H_a)| + \sum_{i=1}^{\frac{1}{2}} \rho(H_i)^2 f_\lambda, f_\lambda \rangle_F$$

equality occurring only if $E_a f_\lambda = 0$ for all $\alpha \in C^+$. Moreover $S = \sum_{i=\frac{1}{2}+1} \rho(H_i)^2$ is
a non-negative symmetric operator so that
\[ \rho(c)f_\lambda, f_\lambda \geq \sum_{\alpha \in A^0 \cup B^0} |\lambda(H_\alpha)| + \sum_{\alpha \in \Sigma^1} \lambda(H_\alpha) \chi + \langle Sf_\lambda, f_\lambda \rangle + \sum_{\alpha \in \Sigma^1} |\lambda(H_\alpha)| \]
with \( S \geq 0 \) (Note that for \( \alpha \in A^+ \cup B^+ \), \( E_\alpha f_\lambda = 0 \) so that \( \lambda(H_\alpha) \geq 0 \).

Using b) of Lemma 2, we have also
\[ \rho(c') \otimes 1 \bigg|_{F_\lambda \otimes F} = \left\{ \sum_{i=1}^\phi \lambda(H_i)^2 + \sum_{\alpha \in \Sigma^1} \lambda(H_\alpha + H_\alpha^\theta)/2 \right\} \text{ Identity} \]
\[ (\rho \otimes \sigma)(c') \bigg|_{F_\lambda \otimes F \mu} = \sum_{i=1}^\phi (\lambda + \mu)(H_i)^2 + \sum_{\alpha \in \Sigma^1} (\lambda + \mu)(H_\alpha + H_\alpha^\theta)/2 \text{ Identity} \]
and
\[ (1 \otimes \sigma)(c') \bigg|_{F_\lambda \otimes F \mu} = \sum_{i=1}^\phi \mu(H_i)^2 + \sum_{\alpha \in \Sigma^1} \mu(H_\alpha + H_\alpha^\theta)/2 \text{ Identity} \]
so that if \( e_\mu \otimes E_\mu \) is a unit weight vector of weight \( \mu \),
\[ \langle T^\phi(f_\lambda \otimes e_\mu), f_\lambda \otimes e_\mu \rangle \geq 2 \sum_{\alpha \in \Sigma^1} |\lambda(H_\alpha + H_\alpha^\theta)/2| + 2 \sum_{\alpha \in \Sigma^1} \lambda(H_\alpha) \]
\[ + 2 \langle Sf_\lambda, f_\lambda \rangle - 2 \sum_{i=1}^\phi \lambda(H_i) \mu(H_i) \]
Now \( \mu \) being a weight of \( \sigma_q \) it is the sum of \( q \) of the weights of the adjoint representation of \( k_0 \) in \( p_0 \). Hence
\[ \mu = \sum_{i=1}^\phi (\alpha_i + \alpha_\theta^2)/2 \]
where all the \( \alpha_i \) belong to \( \Sigma^1 \). Hence
\[ \langle T^\phi(f_\lambda \otimes e_\mu), f_\lambda \otimes e_\mu \rangle \geq 2 \sum_{\alpha \in \Sigma^1} \lambda(H_\alpha + H_\alpha^\theta)/2 - 2 \sum_{i=1}^\phi \lambda(H_\alpha + H_\alpha^\theta)/2 \]
Here equality can occur only if \( E_\alpha f_\lambda = 0 \) for \( \alpha \in \Delta^+ \) and \( \langle Sf_\lambda, f_\lambda \rangle = 0 \). It follows therefore that \( a(\lambda, \mu) > 0 \) if there exists \( \alpha_0 \in C^+ \) with \( E_\alpha f_\lambda = 0 \).

In view of Assertions I, II and III, we see that \( T \) is positive definite if and only if \( a(\lambda_0, \mu) > 0 \) where \( \lambda_0 \) is the greatest of the dominant weights \( \{\lambda | \lambda, \mu \in L\} \) : this follows from the fact that \( E_\alpha f_\lambda = 0 \) for all \( \alpha \in \Delta^+ \) if and only if \( f_\lambda \) is the highest weight vector for \( \rho \); it follows that any weight of \( \rho | \_k \) is of the form \( \lambda_0 - \sum m_i r(\alpha_i) \) where \( m_i \geq 0 \) and \( r(\alpha_i) \) are the restriction of positive roots of \( g \); finally \( r(\alpha_i) = 0 \) hence greater than zero (see Lemma 16 (f) [4]).

Thus to complete the proof of the Theorem, we need only prove

**Assertion IV. If \( \lambda_0 \) is the restriction \( r(\Lambda) \) of the highest weight \( \Lambda \) of \( \rho \), then \( a(\lambda_0, \mu) > 0 \) for all \( \mu \in M \) provided there are at least \( (q+1) \) roots \( \alpha \in \Sigma_\rho \) such that \( \Lambda(H_\alpha + H_\alpha^\theta) > 0 \).**

Proof. By evaluation on the highest weight \( f_\lambda \otimes e_\mu \) we have (Lemma 2)
\[ T_\rho(f_\alpha \otimes e_\mu) = \{ 2 \sum_{\alpha \in \Sigma_2} \Lambda(H_\alpha + H_\rho) / 2 + 2 \sum_{i=1}^p \Lambda(H_i)^2 - 2 \sum_{i=1}^p \Lambda(H_i) \mu(H_i) \} (f_\alpha \otimes e_\mu) \]

\[ = \{ 2 \sum_{\alpha \in \Sigma_2} \Lambda(H_\alpha + H_\rho) / 2 - 2 \sum_{i=1}^p (H_{a_i} + H_{a_2}) / 2 + 2 \sum_{i=1}^p \Lambda(H_i)^2 \} (f_\alpha \otimes e_\mu) \]

where

\[ \mu = r \left( \sum_{i=1}^p (\alpha_i + \alpha_i^*) / 2 \right). \]

It follows that

\[ a(\lambda_\alpha, \mu) > 0 \quad \text{under our hypothesis}, \]

since

\[ \sum_{i=1}^p \Lambda(H_i)^2 \geq 0. \]

This completes the proof of the Theorem.

**Remark 1.** Theorem 1 generalises Theorem 12.1 of [3] where only the case when \( G/K \) is hermitian symmetric, is considered. In fact, the present theorem is more general than Theorem 12.1 of [3] even in this case: \( H^*(\Gamma, X, \rho) \) admits a type decomposition (see [3])

\[ H^*(\Gamma, X, \rho) = \prod_{r+s=q} H^{rs}(\Gamma, X, \rho) \]

so that under the hypothesis of Theorem 1, we have

\[ H^{rs}(\Gamma, X, \rho) = 0 \]

for \( r+s \leq q \). Theorem 12.1 of [3] is the special case \( q = \dim G/K \). In section §2, we will give an interpretation of the groups \( H^{rs}(\Gamma, X, \rho) \). In [4] all the representations for which \( T_\rho \) is positive definite are determined.

**Remark 2.** The author has checked in a number of classical cases, that if \( G \) is simple and non-compact and \( \rho \) is any nontrivial irreducible representation, then the number of elements in \( \sum_\alpha \rho \) is greater than or equal to the rank of the associated symmetric space.

### 2. Compact quotients of complex semisimple Lie groups

Let \( X \) be a complex manifold and \( \tilde{X} \xrightarrow{\pi} X \) be the universal covering of \( X \). Let \( \Gamma \) be the fundamental group of \( X \) acting fixed point free on \( \tilde{X} \). Let \( \rho \) be a representation of \( \Gamma \) in a finite dimensional complex vector space. Let \( L_\rho \) denote the local system associated to \( \rho \) and \( W_\rho \) the holomorphic vector bundle associated to \( \rho \). Let \( L_\rho \) and \( W_\rho \) denote respectively the sheaf of germs of sections of \( L_\rho \) and holomorphic sections of \( W_\rho \). By the de Rham theorem, the cohomology groups \( H^p(X, L_\rho) \) of \( X \) with coefficients in the local system \( L_\rho \) are the cohomology groups of the complex

\[ A = \sum_{p} A^p(\Gamma, \tilde{X}, \rho) \]

defined as follows: \( A^p(\Gamma, X, \rho) \) is the vector space of \( C^\infty \)-exterior \( p \)-forms \( \eta \) on \( X \) with values in \( F \) satisfying the condition
where $t_1, \ldots, t_p$ are tangent vectors to $\tilde{X}$ and $\gamma_X(t)$ denotes the image by $\gamma$ of the tangent vector $t$ to $X$; the boundary operator in the complex is the exterior differentiation of $F$-valued forms on $\tilde{X}$. The complex structure on $X$ gives a decomposition of each of the space $A^p(\Gamma, \tilde{X}, \rho)$ as a direct sum $\sum_{r+s=p} A^{rs}(\Gamma, \tilde{X}, \rho)$ according to the bidegree. Moreover $d=d'+d''$ where $d'$ and $d''$ are of bidegree $(1,0)$ and $(0,1)$ respectively. This gives $A$ a structure of a double complex. The term $E_1^{pq}$ of the spectral sequence associated to this double complex is clearly the $q^{th}$ cohomology of the complex

$$0 \to A^p(\Gamma, \tilde{X}, \rho) \to A^{p+1}(\Gamma, \tilde{X}, \rho) \to \cdots \to A^{p+q}(\Gamma, X, \rho) \to 0$$

($n=\dim X$). Again, by the Dolbeault theorem, the $q^{th}$ cohomology of this complex is $H^q(X, \Omega^p \otimes W)$ where $\Omega^p$ is the holomorphic bundle of holomorphic $p$-forms, and $\Omega^p \otimes W$ is the sheaf of germs of holomorphic $p$-forms on $X$ with coefficients in $W$. Moreover, the derivation $d_1$ in the term $E_1$ is clearly the map induced by the exterior differentiation

$$d: \Omega^p \otimes W_\rho \to \Omega^{p+1} \otimes W_\rho$$

(since we have $\Omega^p \otimes W_\rho \simeq \Omega^p \otimes L_\rho$, the operator $d$ above makes sense: $\Omega^p \otimes L_\rho \to \Omega^{p+1} \otimes L_\rho$).

We have thus

**Proposition 1.** There is a convergent spectral sequence $\{E_\text{pq}^{rs}\}_{0 \leq r \leq m}$ converging to $H^k(\Gamma, \tilde{X}, \rho)$ such that $E_1^{pq} = H^q(X, \Omega^p \otimes W)$ and $d_1$ is induced by the map $d: \Omega^p \otimes W_\rho \to \Omega^{p+1} \otimes W_\rho$.

Now let $\tilde{X}=G$ be a simply connected complex Lie group and $\Gamma \subset G$ a discrete subgroup; then $X=G \backslash G$. Let $\mathfrak{g}$ be the Lie algebra of left invariant vectorfields on $G$. (Then elements of $\mathfrak{g}$ may be regarded as vectorfields on $G \backslash G$ as well). Let $\mathfrak{g}^c$ denote the complexification of $\mathfrak{g}$. Then $\mathfrak{g}^c \simeq \mathfrak{u}_1 \oplus \mathfrak{u}_2$ where $\mathfrak{u}_1$ and $\mathfrak{u}_2$ are respectively the complex ideals of holomorphic and antiholomorphic left-invariant vectorfields. The natural projections $\mathfrak{g} \to \mathfrak{u}_1$ and $\mathfrak{g} \to \mathfrak{u}_2$ define isomorphisms of $\mathfrak{g}$ on $\mathfrak{u}_1$ and $\mathfrak{u}_2$ respectively.

Suppose now that $\rho$ is the restriction of a representation of $G$ in a finite dimensional vector space $F$. In this special case we can compute the term $E_2$ as well.

In the first place, there is a canonical (holomorphic) isomorphism of the vector bundle $W_\rho$ on $X$ with the trivial bundle. In fact the vector bundle $W_\rho$ is obtained as follows: the group $\Gamma$ acts $G \times F$ by diagonal action:
\[ \gamma(g, f) = (\gamma g, \rho(\gamma)f) \quad \text{for } \gamma \in \Gamma. \]

This is an (holomorphic) automorphism of the vector bundle $G \times F$ on itself covering the left translation by $\gamma$ and hence this action defines a vector bundle on $\Gamma \setminus G$. Now let $\Phi \colon G \times F \to G \times F$ be the isomorphism

\[ \Phi(g, f) = (g, \rho(g)^{-1}f) \]

Then

\[ \Phi(\gamma g, \rho(\gamma)f) = (\gamma g, \rho(g)^{-1}f) \]

Hence $\Phi$ defines an isomorphism $\Phi_0$ of $W_\rho$ on the trivial bundle $X \times F$.

Now, for left-invariant holomorphic vectorfields $Z_1, \ldots, Z_{p+1}$ and a holomorphic $p$-form $\eta$ with values in $F$,

\[
d\eta(Z_1, \ldots, Z_{p+1}) = \sum_{i=1}^{p+1} (-1)^i \iota^i Z_i \eta(Z_1, \ldots, \hat{Z}_i, \ldots, Z_{p+1}) + \sum_{i<j} (-1)^{i+j} \eta([Z_i, Z_j], Z_i \cdots \hat{Z}_i \cdots Z_j \cdots Z_{p+1})
\]

It follows that

\[
(\Phi d\Phi^{-1})(\eta)(Z_1, \ldots, Z_{p+1})_{\mathfrak{g}_0} = \sum_{i=1}^{p+1} (-1)^i \iota^i \rho(g_0)^{-1} Z_i \rho(g) \eta(Z_1, \ldots, Z_i, \ldots, Z_{p+1})_{\mathfrak{g}_0} + \sum_{i<j} (-1)^{i+j} \left\{ \rho(g_0)^{-1}([Z_i, Z_j], Z_i \cdots \hat{Z}_i \cdots Z_j \cdots Z_{p+1}) \right\}_{\mathfrak{g}_0} = \sum_{i=1}^{p+1} (-1)^i \iota^i \rho(Z_i) \eta(Z_1, \cdots, Z_{p+1}) + \sum_{i<j} (-1)^{i+j} \eta([Z_i, Z_j], Z_i \cdots \hat{Z}_i \cdots Z_j \cdots Z_{p+1})_{\mathfrak{g}_0}.
\]

($\rho$ has a natural extension to $\mathfrak{g}^c$ hence to $\mathfrak{u}_\rho$)

It follows that if we identify germs of holomorphic $W$-valued forms on $\Gamma \setminus G$ with germs of holomorphic $F$-valued forms on $\Gamma \setminus G$ through the isomorphism $\Phi_0$, the operator $d$ is transformed into the operator $d_0$ defined by

\[
d_0 \eta(Z_1, \ldots, Z_{p+1}) = \sum_{i=1}^{p+1} (-1)^i \iota^i (Z_i + \rho(Z_i)) \eta(Z_1, \ldots, \hat{Z}_i, \ldots, Z_{p+1}) + \sum_{i<j} (-1)^{i+j} \eta([Z_i, Z_j], Z_i \cdots \hat{Z}_i \cdots Z_j \cdots Z_{p+1})
\]

Now the map which associates to each $W_\rho$-valued holomorphic $p$-form $\eta$, the $F$-valued holomorphic form $\Phi_\rho(\eta)$ defined by

\[ (\Phi_\rho \eta)(Z_1, \cdots, Z_p) = \Phi_\rho(\eta(Z_1, \cdots, Z_p)) \]

for every $p$-tuple $(Z_1, \cdots, Z_p)$ of projections of left invariant holomorphic vectorfields on $G$, defines an isomorphism $\Phi_\rho$ of the sheaf $\Omega^p \otimes W_\rho$ on the sheaf $\mathcal{C}(\bigwedge \mathfrak{u}_\rho, \mathcal{O} \otimes F)$. Moreover clearly the diagram
where \( d_0 \) is defined by equation 1 above, is commutative. Now \( \mathcal{O} \) is a sheaf of \( \mathfrak{u}_r \)-modules: the map \( f \rightarrow Zf \) for the projection on \( X \) of a left invariant holomorphic vectorfield \( Z \) on \( G \) defines a representation \( \mathfrak{u}_r(=\mathfrak{g}) \) in the Lie algebra of endomorphism of \( \mathcal{O} \). The stalks at a point \( x \in X \) of the complex of sheaves

\[
0 \rightarrow \mathcal{O} \otimes F \rightarrow \text{Hom}_c(\Lambda^1 \mathfrak{u}_r, \mathcal{O} \otimes F) \rightarrow \cdots \rightarrow \text{Hom}_c(\Lambda^n \mathfrak{u}_r, \mathcal{O} \otimes F) \rightarrow 0
\]

from then clearly the standard complex of the Lie algebra \( \mathfrak{u} \) with values in \( \mathcal{O} \otimes F \), where \( \mathcal{O}_x \) is the stalk at \( x \) of \( \mathcal{O} \). Passing then to the \( d^n \)-cohomology groups of this sheaves, we see that, we obtain the standard complex

\[
0 \rightarrow H^q(X, \mathcal{O}) \otimes F \rightarrow \text{Hom}_c(\mathfrak{u}_1, H^q(X, \mathcal{O}) \otimes F) \cdots \rightarrow \text{Hom}_c(\Lambda^n \mathfrak{u}_1, H^q(X, \mathcal{O}) \otimes F) \rightarrow 0
\]

where \( H^q(X, \mathcal{O}) \) carries the \( \mathfrak{u}_r \)-module structure defined by the action of \( \mathfrak{u}_r \) on \( \mathcal{O} \) defined above and \( H^q(X, \mathcal{O}) \otimes F \) is the tensor product of this representation and \( \rho \).

Combining the preceding, with Proposition 1, we obtain

**Theorem 2.** Let \( G \) be a connected complex Lie group and \( \Gamma \) a discrete subgroup. Let \( \mathcal{O} \) be the sheaf of germs of holomorphic functions on \( X=\Gamma \backslash G \). Let \( \rho \) be a representation of \( G \) in a finite dimensional complex vector space \( F \) and \( L_\rho \) the associated local system. Then there is a convergent spectral sequence \( \{E_r\}_{r \geq 0} \) converging to \( H^*(X, L_\rho) \) such that \( E^q_2=H^q(g, H^q(X, \mathcal{O}) \otimes F) \) where \( H^q(X, \mathcal{O}) \) and \( F \) are considered as \( g \)-modules as follows: a left-invariant vectorfield \( Y \) on \( G \) projects on \( X \) as a vectorfield whose 1-parameter group is a group of holomorphic automorphisms of \( X \); hence \( f \rightarrow Xf \) defines an endomorphism of \( \mathcal{O} \) and hence a representation of \( g \); in \( F \) we have the representation \( \rho \).

Proof. The argument above is incomplete only in two details, under the isomorphism \( \mathfrak{g} \rightarrow \mathfrak{u}_r \), we must show the following:

i) If \( \rho^c \) is the extension to \( \mathfrak{g} \) of \( \rho \), then \( \rho^c \circ p_1 \) and \( \rho \) are equivalent.

ii) \( Xf=p_1(X) \cdot f \)

The former is a well known fact; the latter follows from the fact that if \( p_2: \mathfrak{g} \rightarrow \mathfrak{u}_2 \) is the projection onto antiholomorphic vectorfields, then, \( p_2(X)f=0 \) for holomorphic \( f \).

A corollary is the following

**Theorem 3.** Let \( G \) be a connected complex semisimple Lie group and \( \Gamma \) a
discrete subgroup such that $\Gamma \backslash G$ is compact. Then, $H^q(\Gamma \backslash G, \mathcal{O})$ where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on $\Gamma \backslash G$ vanishes provided that $G$ has no 3-dimensional components.

Proof. Since $\Gamma \backslash G$ is compact $H^q(X, \mathcal{O})$ are finite dimensional so that, in view of the Whitehead Lemma for semisimple Lie algebras, we have, for any finite dimensional representation $\rho$ of $G$ in a vector space $F$, in the spectral sequence of Theorem 2

$$E_2^{10} = E_2^{20} = 0.$$  
$$E_\infty^{01} = E_3^{01}$$

is the homology of

$$0 \to E_2^{01} \to E_2^{20} = 0$$

Hence $E_\infty^{01} = E_2^{01} = H^0(\mathfrak{g}, H^1(X, \mathcal{O}) \otimes F)$. Now if $H^1(X, \mathcal{O}) \neq 0$, and if we choose $F$ to be the dual of this module, then, $H^0(\mathfrak{g}, H^1(X, \mathcal{O}) \otimes F) \neq 0$. On the other hand since the spectral sequence converges to $H^*(X, L_\rho)$, this implies that $H^1(X, L_\rho) \neq 0$. But according to [1a] and [4] under the hypothesis of the theorem, viz., that $G$ has no 3-dimensional components, $H^1(X, L_\rho) = 0$, a contradiction. Hence the theorem.

Corollary. If $\Gamma \subset G$ is a discrete subgroup of a connected complex semisimple Lie group $G$ such that $\Gamma \backslash G$ is compact, then the natural complex structure on $\Gamma \backslash G$ is locally rigid.

Proof. $\Gamma \backslash G$ is holomorphically parallelisable. Hence the sheaf $\Theta$ of germs of holomorphic vector fields is isomorphic to a direct sum of copies of $\mathcal{O}$. From Theorem 3, therefore, $H^1(\Gamma \backslash G, \Theta) = 0$. It is well known that this last implies that the complex structure is locally rigid.

Remark. Reverting to the notation of §1, when $K \backslash G$ is hermitian symmetric, Matsushima and Murakami have given a type decomposition

$$H^q(\Gamma, X, \rho) = \sum_{r+s=q} H^{rs}(\Gamma, X, \rho).$$

The groups $H^{rs}(\Gamma, X, \rho)$ have an interpretation in terms of the spectral sequence of Proposition 1 of this section. In fact, according to proposition 1, there is a spectral sequence converging to $H^*(\Gamma, X, \rho)$ with $E_2^{rs}$ as $H^q(\Gamma, X, \Theta \otimes W_\rho)$. A simple calculation using Lemma 4.1 of [3] shows that $E_2^{rs}$ is isomorphic to $H^p(\Gamma, X, \rho)$ and that the spectral sequence degenerates from the $E_2$ stage onwards.

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