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VANISHING THEOREMS FOR COHOMOLOGY GROUPS ASSOCIATED TO DISCRETE SUBGROUPS OF SEMISIMPLE LIE GROUPS

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Introduction. The aim of this paper is to prove two vanishing theorems for cohomology groups related to discrete uniform subgroups of semisimple Lie groups.

Let ρ be a representation of a real linear semisimple Lie group G and Γ a discrete subgroup of G such that $\Gamma \setminus G$ is compact. Assume that Γ contains no elements of finite order. In §1 we give a criterion in terms of the highest weight of ρ for the vanishing of $H^{p}(\Gamma, \rho)$, the p^{th} cohomology group of Γ with coefficient in ρ . This criterion is a generalisation of a theorem of Matsushima and Murakami [3].

In §2 we prove the following theorem (Corollary to Theorem 3). Let G be a complex semisimple Lie group without any simple component of rank 1. Then for any discrete subgroup Γ such that $\Gamma \setminus G$ is compact, the canonical complex structure on the space $\Gamma \setminus G$ is rigid. (This question whether these complex structures are rigid was raised by Professor Matsushima).

1. A vanishing theorem for the cohomology of discrete uniform subgroups

Let G be a connected real linear semisimple Lie group and Γ a discrete subgroup such that the quotient $\Gamma \setminus G$ is compact. Let \mathfrak{g}_0 be the Lie algebra of left-invariant vector-fields of G and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ a Cartan-decomposition of \mathfrak{g}_0 , \mathfrak{k}_0 being the algebra. Let K be the (compact) Lie subgroup corresponding to \mathfrak{k}_0 and X=G/K the corresponding symmetric space. To every representation of G in a finite dimensional real (or complex) vector space F, Matsushima and Murakami [2] have associated certain cohomology groups: we follow their notation and denote these groups by $H^p(\Gamma, X, \rho)$. (In the case when Γ has no elements of finite order Γ acts freely on X and $H^p(\Gamma, X, \rho)$ is isomorphic to the p^{th} cohomology group of Γ with coefficients in the restriction ρ_{Γ} of ρ to Γ). In the same article, they prove moreover the following result (see in particular $\S6, \S7$). (Proposition 1 below). The vectorfields in \mathfrak{g}_0 project under the natural map $G \to \Gamma \backslash G$ into vectorfields on $\Gamma \backslash G$. We will from now on identify \mathfrak{g}_0 with this algebra of vectorfields on $\Gamma \backslash G$. Let φ be the Killing form on \mathfrak{g}_0 and $\{X_i\}_{1 \leq i \leq N}$ and $\{X_{\alpha}\}_{N+1 \leq \alpha \leq n}$ be bases of \mathfrak{p}_0 and \mathfrak{k}_0 such that $\varphi(X_i, X_j) = \delta_{ij}$ and $\varphi(X_{\alpha}, X_{\beta}) = -\delta_{\alpha\beta}$. Let $A_0(\Gamma, X, \rho)$ be the vector space of C^{\sim} -p-forms η on $\Gamma \backslash G$ satisfying i) $i_X \eta = 0$ and ii) $\theta_X \eta = \rho(X)\eta$ for every $X \in \mathfrak{k}_0$ where i_X (resp θ_X) denotes interior derivation (resp. Lie derivation) of η with respect to the vectorfield X. Because of i) and ii) η is determined by its values $i_1 \cdots i_p = \eta(X_{i_1} \cdots X_{i_p})$. Finally, let Δ^p be the operator

$$\Delta^{p} \colon A^{p}_{0}(\Gamma, X, \rho) \to A^{p}_{0}(\Gamma, X, \rho)$$

defined by

$$\Delta^{p} \eta(X_{i_{1}} \cdots X_{i_{p}}) = \sum_{k=1}^{N} (-X_{k}^{2} + \rho(X_{k})^{2}) \eta_{i_{1} \cdots i_{p}}$$
$$+ \sum_{k=1}^{N} \sum_{u=1}^{p} (-1)^{u-1} \{ (-[X_{i_{u}}, X_{k}] + \rho([X_{i_{u}}, X_{k}])) \} \eta_{ki_{1} \cdots \hat{i}_{u} \cdots i_{p}} \}$$

With this notation, we have

Proposition 1. $H^{p}(\Gamma, X, \rho)$ is canonically isomorphic to the vector space $\{\eta | \eta \in A_{0}^{p}(\Gamma, X, \rho); \Delta^{p}\eta = 0\}.$

Again, following [2], we define two operators Δ_D^p and Δ_ρ^p as follows:

$$\Delta_{D}^{n}\eta(X_{i_{1}}\cdots X_{i_{p}}) = -\sum_{k=1}^{N} X_{k}^{2}\eta_{i_{1}\cdots i_{p}} + \sum_{k=1}^{N} \sum_{u=1}^{p} (-1)^{u}[X_{i_{u}}, X_{k}]\eta_{ki_{1}\cdots i_{u}\cdots i_{p}}$$
$$\Delta_{\rho}^{n}(X_{i_{1}}\cdots X_{i_{p}}) = +\sum_{k=1}^{n} \rho(X_{k})^{2}\eta_{i_{1}\cdots i_{p}} - \sum_{k=1}^{N} \sum_{u=1}^{p} (-1)^{u}\rho([X_{i_{u}}, X_{k}])\eta_{kki_{1}\cdots i_{u}\cdots i_{p}}$$

Then $\Delta^{p} = \Delta^{p}_{D} + \Delta^{p}_{\rho}$. In §7 [2], it is moreover proved that

$$\sum_{i_1 < \cdots < i_p} \int_{\Gamma/G} \langle (\Delta_D^p \eta)_{i_1 \cdots i_p}, \eta_{i_1 \cdots i_p} \rangle_F \ge 0$$

where \langle , \rangle_F is a positive definite scalar product on F for which $\rho(X)$ is (hermitian) symmetric (resp. skew-symmetric (hermitian)) for $X \in \mathfrak{p}_0$ (resp. \mathfrak{k}_0). It follows therefore that if $\Delta^p \eta = 0$,

$$\sum_{i_1 < \cdots < i_p} \int_{\Gamma/G} \langle (\Delta_{\rho}^{p} \eta)_{i_1 \cdots i_p}, \eta_{i_1 \cdots i_p} \rangle_F \ge 0$$

We obtain therefore

Proposition 2. If the quadratic form on the space of exterior p-forms on \mathfrak{p}_0 with values in F defined by

$$\eta \rightarrow \sum_{i_1 < \cdots < i_p} \langle (\Delta_{\rho}^{p} \eta)_{i_1 \cdots i_p}, \ \eta_{i_1 \cdots i_p} \rangle_F$$

is positive definite, then $H^{p}(\Gamma, X, \rho)=0$.

In the main result of this section we give a sufficient criterion in terms of the "highest weight" of ρ with respect to a suitable Cartan-subalgebra of g_0 in order that Δ_0^n define a positive definite quadratic form.

Let g denote the complexification of \mathfrak{g}_0 and \mathfrak{k} and \mathfrak{p} those of \mathfrak{k}_0 and \mathfrak{p}_0 . We identify \mathfrak{k} and \mathfrak{p} with subspaces of g. Let $\mathfrak{h}_{\mathfrak{k}_0}$ be a Cartan-subalgebra of \mathfrak{k}_0 and \mathfrak{h}_0 a Cartan-subalgebra of \mathfrak{g}_0 such that $\mathfrak{h}_0 \supset \mathfrak{h}_{\mathfrak{k}_0}$. Let $\mathfrak{h}_{\mathfrak{p}_0} = \mathfrak{h}_0 \cap \mathfrak{p}_0$. Let $\mathfrak{h}_{\mathfrak{k}}$ \mathfrak{h} and $\mathfrak{h}_{\mathfrak{p}}$ denote respectively the complexifications of $\mathfrak{h}_{\mathfrak{k}_0} \mathfrak{h}_0$ and $\mathfrak{h}_{\mathfrak{p}_0}$. Then \mathfrak{h} is a Cartansubalgebra of g. Let Δ be the system of roots of g with respect to \mathfrak{h} . For $\alpha \in \Delta$ let $H_{\alpha} \in \mathfrak{h}$ be the unique element such that $\varphi(H_{\alpha}, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. Then, it is well known that the real subspace $\mathfrak{h}^* = \sum_{\alpha \in \Delta} RH_{\alpha}$ of g spanned by the $\{H_{\alpha}\}_{\alpha \in \Delta}$ is the same as $\mathfrak{i}\mathfrak{h}_{\mathfrak{l}_0} \oplus \mathfrak{p}_0$. Moreover if θ is the extension to g to the Cartan involution θ_0 denfied by the Cartan-decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, then θ is an automorphism of g leaving \mathfrak{h} invariant. Hence θ acts on the dual of \mathfrak{h} and permutes the elements of Δ . The set Δ may then be decomposed as the disjoint union $A \cup B \cup C$ of three subsets A, B and C

where

$$A = \{ \alpha | \alpha \in \Delta; \ \theta(\alpha) = \alpha; \ \theta(E_{\alpha}) = E_{\alpha} \}$$
$$B = \{ \alpha | \alpha \in \Delta; \ \theta(\alpha) = \alpha \}$$
$$C = \{ \alpha | \alpha \in \Delta; \ \theta(\alpha) = \alpha; \ \theta(E_{\alpha}) = -E_{\alpha} \}.$$

(In the sequel we sometimes write α^{θ} for $\theta(\alpha)$).

We introduce next a lexicographic order on the (real) dual of \mathfrak{h}^* as follows: let H_1, \dots, H_I be an orthonormal basis of \mathfrak{h}^* with respect to $\varphi(\varphi|_{\mathfrak{h}^*}$ is positive definite) chosen so that H_1, \dots, H_I form a basis of $i\mathfrak{h}_{\mathfrak{l}_0}$ and if the centre \mathfrak{c}_0 of \mathfrak{k}_0 is non-zero, of dimension r, then H_1, \dots, H_r belong to $i\mathfrak{c}_0$; for α, β in the (real) dual of \mathfrak{h}^* , $\alpha > \beta$ if the first non-vanishing difference $\alpha(H_i) - \beta(H_i)$ is greater than zero. Let Δ^+ be the system of positive roots with respect to this order and let $A^+ = A \cap \Delta^+$, $B^+ = B \cap \Delta^{+1}$, $C = \cap C\Delta^+$. Then θ leaves A^+ , B^+ and C^+ invariant. Let $\sum_1 = A^+ U\{\alpha \mid \alpha \in B^+; \theta(\alpha) > \alpha\}$ and $\sum_2 = C^+ U\{\alpha \mid \alpha \in B^+; \theta(\alpha) > \alpha\}$.

Theorem 1. Let ρ denote a finite dimensional representation of G in a complex vector-space F, as also the induced representation of g. Let Λ_{ρ} be the highest weight of ρ with respect to the above defined Cartan-subalgebra and the order on the dual of \mathfrak{h}^* . Then if $\sum_{\rho} = \{\alpha \mid \alpha \in \sum_2, \varphi(\Lambda_{\rho}, \alpha) \neq 0\}$ contains more than q elements, then the Hermitian quadratic form Q_{ρ} defined by

$$\eta \to \sum_{i_1 < \cdots < i_p} \langle (\Delta_{\rho}^{p} \eta)_{i_1 \cdots i_p}, \eta_{i_1 \cdots i_p} \rangle_F$$

is positive definite for $p \le q$. Hence $H^{p}(\Gamma, X, \rho) = 0$ for $1 \le p \le q$.

Before we proceed to the proof of the theorem, we will make a few preliminary simplifications: M.S. RAGHUNATHAN

Lemma 1. Let E be the q^{th} exterior power of p and let α be the isomorphism onto $F \otimes E$ of the space of exterior q-forms on p with values in F defined by

$$\eta \to \sum_{i_1 < \cdots < iq} \eta_{i_1 \cdots i_q} \otimes (X_{i_1} \wedge \cdots \wedge X_{i_q})$$

Then

$$T^{\mathbf{q}}_{\rho} = 2\alpha \circ \Delta^{\mathbf{q}}_{\rho} \circ \alpha^{-1} = 2(\rho \otimes 1)(c) + (1 \otimes \sigma)(c') - (\rho \otimes 1)(c') - (\rho \otimes \sigma)(c')$$

where

$$c = \sum_{i=1}^{N} X_{i}^{2} - \sum_{\alpha = N+1}^{n} X_{\alpha}^{2}$$

and $c' = -\sum_{\alpha=N+1}^{n} X_{\alpha}^{2}$ are elements of the enveloping algebras of g and t and σ denotes the adjoint representation of t in E. Hence T_{ρ}^{q} is a symmetric endomorphism of $F \otimes E$ with respect to the scalar product

$$\langle \sum_{i_1 < \dots < i_p} \eta_{i_1 \dots i_p} \otimes X_{i_1} \wedge \dots \wedge X_{i_p}, \sum_{j_1 < \dots < j_p} \eta_{j_1 \dots j_p} \otimes X_{j_1} \wedge \dots \wedge X_{j_p} \rangle$$

$$= \sum_{i_1 < \dots < i_p} \langle \eta_{i_1 \dots i_p}, \eta_{i_1 \dots i_p} \rangle_F$$

Proof. We have

$$(\Delta_{\rho}^{q})_{i_{1}\cdots i_{q}} = \sum_{k=1}^{N} \rho(X_{k})^{2} \eta_{i_{1}\cdots i_{q}} + \sum_{k=1}^{N} \sum_{u=1}^{q} (-1)^{u-1} \rho([X_{i_{u}}, X_{k}]) \eta_{ki_{1}\cdots \hat{i}_{u}\cdots i_{q}}$$

For every q-tuple $I_q = (i_1 < \cdots < i_q)$, we write X_{Iq} for $X_{i_1} \land \cdots \land X_{iq}$. In this notation,

$$\begin{aligned} \alpha(\eta) &= \sum_{I_q} \eta_{I_q} \otimes X_{I_q} \\ \frac{1}{2} T^{q}{}_{\rho} \alpha(\eta) &= \sum_{I_q} \{ \sum_{k=1}^{N} \rho(X_k)^2 \eta_{I_q} + \sum_{k=1}^{N} \sum_{u=1}^{q} (-1)^{u-1} \rho([X_{i_u}, X_k]) \eta_{ki_1 \cdots \hat{i}_u \cdots i_q} \} \otimes X_{I_q} \\ &= \sum_{I_q} \{ \sum_{k=1}^{N} \rho(X_k)^2 \eta_{I_q} + \sum_{J_q \Delta I_q = i_u j_v} (-1)^{u+v} \rho([X_{i_u}, X_{j_v}]) \eta_{J_q} \} \otimes X_{I_q} \\ &= \sum_{I_q} \{ \sum_{k=1}^{N} \rho(X_k)^2 \eta_{I_q} + \sum_{J_q \Delta I_q = i_u j_v} (-1)^{u+v} c^{\omega}_{i_u j_v} \rho(X_{\omega}) \eta_{J_q} \} \otimes X_{I_q} \end{aligned}$$

On the other hand,

$$\sigma(X_{\alpha})X_{Jq} = \sum_{k=1}^{n} \sum_{u=1}^{q} (-1)^{v-1} c_{\alpha j_{v}}^{k} (X_{k} \wedge X_{j_{1}} \cdots X_{j_{v}} \cdots \wedge X_{j_{q}})$$
$$= \sum_{Iq \Delta Jq = j_{v}i_{u}} (-1)^{u+v} c_{j_{v}}^{i_{u}} X_{Iq}$$

It follows that

$$\frac{1}{2}T_{\rho}^{q}\alpha(\eta) = \sum_{Iq}\sum_{k=1}^{N}\rho(X_{k})^{2}\eta_{Iq}\otimes X_{Iq} + \sum_{Jq}\rho(X_{\omega})\eta_{Jq}\otimes\sigma(X_{\omega})X_{Jq}$$
$$= \{\sum_{k=1}^{N}\rho(X_{k})^{2}\otimes 1 + \sum_{j}\rho(X_{\omega})\otimes\sigma(X_{\omega})\}\alpha(\eta)\}$$

Now the required result follows from the fact

$$2\rho(X_{\alpha}) \otimes \sigma(X_{\alpha}) = \{\rho(X_{\alpha}) \otimes 1 + 1 \otimes \sigma(X_{\alpha})\}^{2} - \rho(X_{\alpha})^{2} \otimes 1 - 1 \otimes \sigma(X_{\alpha})^{2} \\ = (\rho \otimes \sigma)(X_{\alpha})^{2} - \rho(X_{\alpha})^{2} \otimes 1 - 1 \otimes \sigma(X_{\alpha})^{2}$$

That T^{q}_{ρ} is a hermitian symmetric endomorphism follows from the facts that $\rho(X_{i})$ and $\sigma(X_{i})$ are hermitian symmetric while $\rho(X_{\omega})$ and $\sigma(X_{\omega})$ are skew-hermitian with respect to \langle , \rangle_{F} and the extension to E of the Killing form on \mathfrak{p}_{0} .

Lemma 2. a) If Λ is the highest weight of an irreducible representation ρ of \mathfrak{g} induced by a representation ρ of G, then

$$\rho(c) = \{\varphi(\Lambda, \Lambda) + \sum \varphi(\Lambda, \alpha)\}.$$
 Identity

b) when restricted to the (irreducible) K-subspace generated by the eigen-space corresponding to the highest weight Λ ,

$$\rho(c') = \left\{ \frac{1}{4} \varphi(\Lambda + \Lambda^{\theta}, \Lambda + \Lambda^{\theta}) + \sum_{\alpha \in \Sigma_1} \varphi\left(\Lambda, \frac{\alpha + \alpha^{\theta}}{2}\right) \right\}.$$
 Identity.

For a proof see [4]: Lemmas 4 and 16(c).

Lemma 3. If Λ_1 and Λ_2 are the highest weights of two irreducible representations ρ_1 , ρ_2 of \mathfrak{g} , such that $\Lambda_1 - \Lambda_2$ is a non-negative linear combination of simple roots of \mathfrak{g} , then $\lambda_1 \geq \lambda_2$ where $\rho_k(c) = (\lambda_k$. Identity) (k=1, 2). Equality can occur only if $\Lambda_1 = \Lambda_2$.

The same conclusions hold for \mathfrak{k} and c' instead of \mathfrak{g} and c provided that Λ_1 and Λ_2 coincide on the center of \mathfrak{k} .

For the proof see Lemma 5 [4].

Proof of Theorem 1. We obtain the eigen-values of T_{ρ}^{q} as follows: Let

$$E = \sum_{\mu \in \mathcal{M}} E_{\mu}$$
 and $F = \sum_{\lambda \in \mathcal{L}} F_{\lambda}$ and $F_{\lambda} \otimes E_{\mu} = \sum_{\nu \in \mathcal{M}_{\lambda\mu}} V_{\lambda\mu}^{\nu}$

be the decomposition of E, F and $F_{\lambda} \otimes E_{\mu}$ into irreducible \mathfrak{k} -modules indexed by the highest weights (for the order defined by H_1, \dots, H_p on \mathfrak{ih}_k). Since ρ is an irreducible representation of \mathfrak{g} and c is a central element of $U(\mathfrak{g})$, $\rho(c)$ is a scalar operator. Similarly, since c' is central in $U(\mathfrak{k})$, $\rho(c') \otimes 1$, $1 \otimes \sigma(c')$ and $(\rho \otimes \sigma)(c')$ are scalars on $F_{\lambda} E$, $F \otimes E_{\lambda}$ and $V_{\lambda\mu}^{\nu}$. Hence $T_{\rho}^{\mathfrak{g}}$ acts as a scalar on each $V_{\lambda\mu}^{\nu}$. We denote the corresponding eigen-value by $a(\lambda, \mu, \nu)$. Among $V_{\lambda\mu}^{\nu}$ there is a unique irreducible component with highest weight $\nu = \lambda + \mu$ we denote the corresponding scalar $a(\lambda, \mu, \nu)$ by $a(\lambda, \mu)$ with this notation, we have

Assertion I. $a(\lambda, \mu, \nu) \ge a(\lambda, \mu)$; equality occurs only if $\nu = \lambda + \mu$.

Proof. We denote the representation in $V_{\lambda\mu}^{\nu}$ by $\rho_{\lambda\mu}^{\nu}$. Then since $(\rho \otimes 1)(c)$, $(\rho \otimes 1)(c')$ and $(1 \otimes \sigma)(c')$ all define the same scalar operator in $F_{\lambda} \otimes E_{\mu}$,

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$$a(\lambda, \mu) + a(\lambda, \mu, \nu) = \rho_{\lambda\mu}^{\lambda+\mu}(c') - \rho_{\lambda\mu}^{\nu}(c')$$

(Here we have let $\rho_{\lambda\mu}^{\nu}(c')$ stand for the scalar). Now any weight in $F_{\lambda} \otimes E_{\mu}$ has the form $\lambda_1 + \mu_1$ where λ_1 and μ_1 are weights of F_{λ} and E_{μ} ; on the other hand $\lambda - \lambda_1$ and $\mu - \mu_1$ are non-negative linear combination of simple roots of k; hence so is $(\lambda + \mu) - (\lambda_1 + \mu_1)$. It follows then from Lemma 3 that

$$a(\lambda, \mu) \ge a(\lambda, \mu, \nu)$$

Equality can occur only if $\lambda + \mu = \lambda_1 + \mu_1$ and there is only one component of $F_{\lambda} \otimes E_{\mu}$ with $\lambda + \mu$ as the highest weight. (Note that if \mathfrak{k} has a centre, then the central elements act as scalars on F_{λ} and E hence in all of $F_{\lambda} \otimes E_{\mu}$).

Assertion II. Let f_{λ} be a highest weight vector of F such that $||f_{\lambda}||_{F}^{2}=1$. For $\alpha \in \Delta$, let E_{α} be a root vector of α . Suppose that $E_{\alpha_{0}}f_{\lambda}=0$ for $\alpha \in A^{+}$. If there is an $\alpha_{0} \in B^{+}$ with $E_{\alpha_{0}}f_{\lambda} \neq 0$, then $E_{\alpha_{0}}f_{\lambda} \in F_{\lambda_{1}}$ for some λ_{1} and $a(\lambda, \mu) < a(\lambda_{1}, \mu_{1})$

Proof. Using the fact that θ is an involution, we have

$$\mathfrak{k} = \mathfrak{h}_{\mathfrak{k}} \oplus \sum_{\alpha \in \mathcal{A}^+} \left\{ CE_{\alpha} \oplus CE_{\alpha} \right\} \oplus \sum_{\substack{\alpha \in \mathcal{B}^+ \\ \alpha > \alpha \theta}} \left\{ C(E_{\alpha} + E_{\alpha}\theta) \oplus C(E_{-\alpha} + E_{-\alpha}\theta) \right\}$$

and the order chosen on $\mathfrak{h}_{\mathfrak{f}}^* = i\mathfrak{h}_{\mathfrak{f}_0}$ has precisely $\{\alpha \mid \alpha \in A^+\}$ and $\left\{\frac{\alpha + \alpha^{\theta}}{2} \mid \alpha \in B^+\right\}$

as the positive roots. The roots of \mathfrak{k} are necessarily zero on the centre of \mathfrak{k} . It follows that the weights λ and $\lambda + \alpha_0$ (which is the weight corresponding to $E_{\alpha_0}f_{\lambda}$) have the same values on the centre. On the other hand, since $\lambda + \alpha_0$ and λ_1 are weights of the same irreducible representation of \mathfrak{k} , λ_1 and $\lambda + \alpha_0$ have the same values on the centre of \mathfrak{k} . It follows that $\lambda_1 = \lambda$ on the centre of \mathfrak{k} . Now $\lambda_1 - \lambda = \lambda_1 - (\lambda + \alpha_0) + \alpha_0$ and $\lambda_1 - (\lambda + \alpha_0) + \alpha_0$ is a non-negative linear combination of simple roots. Hence $\lambda_1 - \lambda$ is a non-negative linear combination of simple roots and $\lambda_1 \neq \lambda$. A similar remark holds for $\lambda_1 + \mu$ and $\lambda + \mu$. It follows then from Lemma 3 above that

$$\rho_{\lambda}(c') < \rho_{\lambda_1}(c')$$

and

$$\rho_{\lambda\mu}^{\lambda+\mu}(c') < \rho_{\lambda\mu}^{\lambda+\mu}(c')$$

The operators $(\rho \otimes 1)$ (c) and $(1 \otimes \sigma)$ (c') on the other hand are scalars on the whole of $F \otimes E$. Hence from the expression for T^q_{ρ} , the Assertion follows.

Assertion III. Suppose that $E_{\alpha}F_{\lambda}=0$ for $\alpha \in A^+ \cup B^+$ but that there is an $\alpha_0 \in C^+$ such that $E_{\alpha_0}f_{\lambda} \neq 0$. Then $a(\lambda, \mu) > 0$.

Proof. If $\{E_{\alpha}\}_{\alpha \in \Delta}$ are root vectors so chosen that $\varphi(E_{\alpha}, E_{-\alpha})=1$, then, it is well known that

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 $c = \sum_{\alpha \in \Delta^+} E_{\alpha} E_{-\alpha} + \sum_{\alpha \in \Delta^+} E_{-\alpha} E_{\alpha} + \sum_{i=1}^1 H_i^2$

It follows that

$$\rho(c)f_{\lambda} = \sum_{\alpha \in \Delta^{+}} \rho(E_{\alpha}E_{-\alpha} + E_{-\alpha}E_{\alpha})f_{\lambda} + \sum_{i=1}^{1} \rho(H_{i})^{2}f$$

Using the facts, $E_{\omega}f_{\lambda}=0$ for $\alpha \in A^+ \cup B^+$ and that $[E_{\omega}, E_{-\omega}]=H_{\omega}$, we have

$$\rho(c)f_{\lambda} = \sum_{\boldsymbol{\omega} \in \mathcal{A}^+ \cup \mathcal{B}^+} \lambda(H_{\boldsymbol{\omega}})f_{\lambda} + \sum_{i=1}^{p} \wedge (H_{i})^2 f_{\lambda} + \sum_{\boldsymbol{\omega} \in C} \rho(E_{\boldsymbol{\omega}}E_{-\boldsymbol{\omega}} + E_{-\boldsymbol{\omega}}E_{\boldsymbol{\omega}})f_{\lambda} + \sum_{i=p+1}^{1} \rho(H_{i})^2 f_{\lambda}$$

Hence

$$egin{aligned} &\langle
ho(c) f_{\lambda}, f_{\lambda}
angle_{F} = \sum_{arphi \in \mathcal{A}^{\perp} \cup B^{+}} \lambda(H_{arphi}) + \sum_{i=1}^{p} \lambda(H_{i})^{2} + \sum_{arphi \in C^{+}} \langle
ho(E_{arphi} E_{-arphi} + E_{-arphi} E_{arphi}) f_{\lambda}, f_{\lambda}
angle \ &+ \sum_{i=p+1}^{1} \langle
ho(H_{i})^{2} f_{\lambda}, f_{\lambda}
angle_{F} \end{aligned}$$

Now it is well known that F admits an orthogonal decomposition with respect to \langle , \rangle_F into irreducible representations of the algebra $g' = CE_{\alpha} \oplus CE_{-\alpha} \oplus CH_{\alpha}$ for $\alpha \in C^+$ so that to prove that $\langle \rho(E_{\alpha}E_{-\alpha} + E_{-\alpha}E_{\alpha})f_{\lambda}, f_{\lambda} \rangle \geq |\lambda(H_{\alpha})|$ equality occurring only if $E_{\alpha}f_{\lambda}=0$, we may assume that the g'-invariant subspace Wspanned by f_{λ} is *irreducible* with respect to the three dimensional algebra. Now by Lemma 2,

$$\rho\left\{E_{\omega}E_{-\omega}+E_{-\omega}E_{\omega}+\frac{H_{\omega}^{2}}{\varphi(H_{\omega}H_{\omega})}\right\}f_{\lambda}=\left\{\frac{(\lambda+k\alpha)(H_{\omega})^{2}}{\varphi(H_{\omega},H_{\omega})}+(\lambda+k\alpha)(H_{\omega})\right\}f_{\lambda}$$

where $\lambda + k\alpha$, $k \ge 0$ is the highest weight in W (of g'). Hence

$$\rho(E_{\omega}E_{-\omega}+E_{-\omega}E_{\omega})f_{\lambda}=\frac{k\alpha(H_{\omega})^{2}}{\varphi(H_{\omega},H_{\omega})}+(\lambda+k\alpha)(H_{\omega})f_{\lambda}$$

so that

$$<\!
ho(E_{a}E_{-a}+E_{-a}E_{a})f_{\lambda},\,f_{\lambda}\!>_{F}=(\lambda+klpha)(H_{a})\!+\!rac{lpha(H_{a})}{arphi(H_{a},\,H_{a})}\!\geq\!|\,\lambda(H_{a})|$$

(It is well known that $(\lambda + k\alpha)(H_{\alpha}) \ge |\lambda(H_{\alpha})|$ since $\lambda + k\alpha$ is the highest weight). Moreover equality occurs only if k=0; if k=0, however, λ is the highest weight so that $E_{\alpha}f_{\lambda}=0$. We have thus shown that

$$<\!
ho(E_{a}E_{-a}\!+\!E_{-a}E_{a})\!f_{\lambda},\,f_{\lambda}\!>\!\geq\!|\,\lambda(H_{a})|$$

equality occurring only if $E_{\alpha}f_{\lambda}=0$. We have therefore,

$$\langle \rho(c)f_{\lambda}, f_{\lambda} \rangle \geq \sum_{\alpha \in \mathcal{A}^+ \cup B^+} \lambda(H_{\alpha}) + \sum_{i=1}^{j} \lambda(H_i)^2 + \sum_{\alpha \in \mathcal{C}^+}^{p} |\lambda(H_{\alpha})| + \sum_{i=p+1}^{1} \langle \rho(H_i)^2 f_{\lambda}, f_{\lambda} \rangle_F$$

equality occurring only if $E_{\alpha}f_{\lambda}=0$ for all $\alpha \in C^+$. Moreover $S=\sum_{i=\ell+1}^{1}\rho(H_i)^2$ is

a non-negative symmetric operator so that

$$\rho(c)f_{\lambda}, f_{\lambda} \geq \sum_{\alpha \in A^{+} \cup B^{+}} |\lambda(H_{\alpha})| + \sum \lambda(H_{i})^{2} + \langle Sf_{\lambda}, f_{\lambda} \rangle + \sum_{\alpha \in C^{+}} |\lambda(H_{\alpha})|$$

with $S \ge 0$ (Note that for $\alpha \in A^+ \cup B^+$, $E_{\alpha}f_{\lambda} = 0$ so that $\lambda(H_{\alpha}) \ge 0$). Using b) of Lemma 2, we have also

$$\rho(c') \otimes 1 \Big|_{F_{\lambda \otimes E}} = \{ \sum_{i=1}^{p} \lambda(H_i)^2 + \sum_{\alpha \in \Sigma_1} \lambda(H_{\alpha} + H_{\alpha} \theta)/2 \}. \quad \text{Identity}$$
$$(\rho \otimes \sigma)(c') \Big|_{V_{\lambda \mu}^{\lambda + \mu}} = \sum_{i=1}^{p} (\lambda + \mu)(H_i)^2 + \sum_{\alpha \in \Sigma_1} (\lambda + \mu)(H_{\alpha} + H_{\alpha} \theta)/2 . \quad \text{Identity}$$

and

$$(1 \otimes \sigma)(c') \Big|_{F \otimes E_{\mu}} = \sum_{i=1}^{p} \mu(H_i)^2 + \sum_{\sigma \in \Sigma_1} \mu(H_{\sigma} + H_{\sigma} \theta)/2$$
. Identity

so that if $e_{\mu} \otimes E_{\mu}$ is a unit weight vector of weight μ ,

$$\langle T^{q}_{\rho}(f_{\lambda} \otimes e_{\mu}), f_{\lambda} \otimes e_{\mu} \rangle \geq 2 \sum_{\substack{\alpha \in B^{+} \\ \alpha > \alpha \theta}} |\lambda(H_{\alpha} + H_{\alpha}\theta)/2| + 2 \sum_{\alpha \in C^{+}} \lambda(H_{\alpha})$$
$$+ 2 \langle S(f_{\lambda}), f_{\lambda} \rangle - 2 \sum_{i=1}^{p} \lambda(H_{i}) \mu(H_{i})$$

Now μ being a weight of σ_q it is the sum of q of the weights of the adjoint representation of k_0 in p_0 . Hence

$$\mu = \sum_{i=1}^{q} (\alpha_i + \alpha_2^{\theta})/2$$

where all the α_i belong to $\sum_{i=1}^{n} \alpha_i$. Hence

$$\langle T^{q}_{\rho}(f_{\lambda}\otimes e_{\mu}), f_{\lambda}\otimes e_{\mu}\rangle \geq 2\sum_{\alpha\in\Sigma_{2}}\lambda(H_{\alpha}+H_{\alpha}\theta)/2-2\sum_{i=1}^{q}\lambda(H_{\alpha_{i}}+H_{\alpha_{i}}\theta)/2$$

Here equality can occur only if $E_{\alpha}f_{\lambda}=0$ for $\alpha \in \Delta^+$ and $\langle Sf_{\lambda}, f_{\lambda} \rangle = 0$. It follows therefore that $a(\lambda, \mu) > 0$ if there exists $\alpha_0 \in C^+$ with $E_{\alpha_0}f_{\lambda} \neq 0$.

In view of Assertions I, II and III, we see that T is positive definite if and only if $a(\lambda_0, \mu) > 0$ where λ_0 is the greatest of the dominant weights $\{\lambda \mid \lambda \in L\}$: this follows from the fact that $E_{\alpha}f_{\lambda_0}=0$ for all $\alpha \in \Delta^+$ if and only if f_{λ_0} is the highest weight vector for ρ ; it follows that any weight of $\rho \mid_k$ is of the form $\lambda_0 - \sum m_i r(\alpha_i)$ where $m_i \ge 0$ and $r(\alpha_i)$ are the restriction of positive roots of g; finally $r(\alpha_i) \neq 0$ hence greater than zero (see Lemma 16 (f) [4]).

Thus to complete the proof of the Theorem, we need only prove

Assertion IV. If λ_0 is the restriction $r(\Lambda)$ of the highest weight Λ of ρ , then $a(\lambda_0, \mu) > 0$ for all $\mu \in M$ provided there are at least (q+1) roots $\alpha \in \sum_2$ such that $\Lambda(H_{\alpha}+H_{\alpha}\theta) > 0$.

Proof. By evaluation on the highest weight $f_{\lambda_0} \otimes e_{\mu}$ we have (Lemma 2)

$$\begin{split} T_{\rho}(f_{\lambda_{0}}\otimes e_{\mu}) &= \{2\sum_{\alpha\in\Sigma_{2}}\Lambda(H_{\alpha}+H_{\alpha}\theta)/2+2\sum_{i=1}^{p}\Lambda(H_{i})^{2}-2\sum_{i=1}^{p}\Lambda(H_{i})\mu(H_{i})\}(f_{\lambda_{0}}\otimes e_{\mu})\\ &= \{2\sum_{\alpha\in\Sigma_{2}}\Lambda(H_{\alpha}+H_{\alpha}\theta)/2-2\sum_{i=1}^{q}(H_{\alpha i}+H_{\alpha i}\theta)/2+2\sum_{i=1}^{q}\Lambda(H_{i})^{2}\}(f_{\lambda_{0}}\otimes e_{\mu})\\ \text{where} \qquad \mu &= r(\sum_{i=1}^{q}(\alpha_{i}+\alpha_{i}^{\theta})/2). \text{ It follows that}\\ &a(\lambda_{0}, \mu) > 0 \quad \text{under our hypothesis,}\\ \text{since} \qquad \sum_{i=1}^{p}\Lambda(H_{i})^{2} \ge 0 \,. \end{split}$$

since

This completes the proof of the Theorem.

REMARK 1. Theorem 1 generalises Theorem 12.1 of [3] where only the case when G/K is hermitian symmetric, is considered. In fact, the present theorem is more general than Theorem 12.1 of [3] even in this case: $H^{n}(\Gamma, X, \rho)$ admits a type decomposition (see [3])

$$H^{n}(\Gamma, X, \rho) \simeq \prod_{r+s=n} H^{rs}(\Gamma, X, \rho)$$

so that under the hypothesis of Theorem 1, we have

$$H^{rs}(\Gamma, X, \rho) = 0$$

for $r+s \le q$. Theorem 12.1 of [3] is the special case $q = \dim G/K$. In section §2, we will give an interpretation of the groups $H^{rs}(\Gamma, X, \rho)$. In [4] all the representations for which T^{1}_{ρ} is positive definite are determined.

REMARK 2. The author has checked in a number of classical cases, that if G is simple and non-compact and ρ is any nontrivial irreducible representation, then the number of elements in \sum_{ρ} is greater than or equal to the rank of the associated symmetric space.

Compact quotients of complex semisimple Lie groups 2.

Let X be a complex manifold and $\tilde{X} \xrightarrow{\pi} X$ be the universal covering of X. Let Γ be the fundamental group of X acting fixed point free on \tilde{X} . Let ρ be a representation of Γ in a finite dimensional complex vector space. Let L_{ρ} denote the local system associated to ρ and W_{ρ} the holomorphic vector bundle associated to ρ . Let \underline{L}_{ρ} and \underline{W}_{ρ} denote respectively the sheaf of germs of sections of L_{ρ} and holomorphic sections of W_{ρ} . By the de Rham theorem, the cohomology groups $H^{p}(X, L_{\rho})$ of X with coefficients in the local system L_{ρ} are the cohomology groups of the complex

$$A = \sum_{p} A^{p}(\Gamma, \tilde{X}, \rho)$$

defined as follows: $A^{p}(\Gamma, X, \rho)$ is the vector space of C^{∞} -exterior p-forms η on X with values in F satisfying the condition

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$$\eta(\gamma_{*}(t_{1}), \gamma_{*}(t_{2}), \cdots, \gamma_{*}(t_{p})) = \rho(\gamma)^{-1}\eta(t_{1}, \cdots, t_{p})$$

where t_1, \dots, t_p are tangent vectors to \tilde{X} and $\gamma_*(t)$ denotes the image by γ of the tangent vector t to X; the boundary operator in the complex is the exterior differentiation of F-valued forms on \tilde{X} . The complex structure on X gives a decomposition of each of the space $A^p(\Gamma, \tilde{X}, \rho)$ as a direct sum $\sum_{r+s=p} A^{rs}(\Gamma, \tilde{X}, \rho)$ according to the bidegree. Moreover d=d'+d'' where d' and d'' are of bidegree (1, 0) and (0, 1) respectively. This gives A a structure of a double complex. The term E_1^{pq} of the spectral sequence associated to this double complex is clearly the q^{th} cohomology of the complex

$$0 \to A^{p,0}(\Gamma, \tilde{X}, \rho) \to A^{p,1}(\Gamma, \tilde{X}, \rho) \to \dots \to A^{p,n}(\Gamma, X, \rho) \to 0$$

 $(n = \dim X)$. Again, by the Dolbeault theorem, the q^{th} cohomology of this complex is $H^q(X, \underline{\Omega}^p \otimes \underline{W}_p)$ where $\underline{\Omega}^p$ is the holomorphic bundle of holomorphic p-forms, and $\underline{\Omega}^p \otimes \underline{W}$ is the sheaf of germs of holomorphic p-forms on X with coefficients in W. Moreover, the derivation d_1 in the term E_1 is clearly the map induced by the exterior differentiation

$$d: \ \underline{\Omega}^{p} \bigotimes_{\mathcal{O}} \underline{W}_{\rho} \to \underline{\Omega}^{p+1} \bigotimes_{\mathcal{O}} \underline{W}_{\rho}$$

(since we have $\underline{\Omega}^{p} \bigotimes_{\mathcal{O}} \underline{W}_{p} \simeq \underline{\Omega}^{p} \bigotimes_{\mathcal{C}} \underline{L}_{p}$, the operator d above makes sence: $\underline{\Omega}^{p} \bigotimes_{\mathcal{O}} \underline{L}_{p} \rightarrow \underline{\Omega}^{p+1} \bigotimes_{\mathcal{C}} \underline{L}_{p}$).

We have thus

Proposition 1. There is a convergent spectral sequence $\{E_r^{pq}\}_{c\leq r\leq \infty}$ converging to $H^*(\Gamma, \tilde{X}, \rho)$ such that $E_1^{pq} = H^q(X, \underline{\Omega}^p \otimes \underline{W}_{\rho})$ and d_1 is induced by the map $d: \underline{\Omega}^p \otimes \underline{W}_{\rho} \to \underline{\Omega}^{p+1} \otimes \underline{W}_{\rho}$.

Now let $\tilde{X}=G$ be a simply connected complex Lie group and $\Gamma \subset G$ a discrete subgroup; then $X=\Gamma \setminus G$. Let g be the Lie algebra of left invariant vectorfields on G. (Then elements of g may be regarded as vectorfields on $\Gamma \setminus G$ as well). Let g^c denote the complexification of g. Then $g^c \simeq \mathfrak{u}_1 \oplus \mathfrak{u}_2$ where \mathfrak{u}_1 and \mathfrak{u}_2 are respectively the complex ideals of holomorphic and antiholomorphic left-invariant vectorfields. The natural projections $g \to \mathfrak{n}_1$ and $g \to \mathfrak{u}_2$ define isomorphisms of g on \mathfrak{u}_1 and \mathfrak{u}_2 respectively.

Suppose now that ρ is the restriction of a representation of G in a finite dimensional vector space F. In this special case we can compute the term E_2 as well.

In the first place, there is a canonical (holomorphic) isomorphism of the vector bundle W_{ρ} on X with the trivial bundle. In fact the vector bundle W_{ρ} is obtained as follows: the group Γ acts $G \times F$ by diagonal action:

VANISHING THEOREMS FOR COHOMOLOGY GROUPS

$$\gamma(g, f) = (\gamma g, \rho(\gamma) f) \quad \text{for } \gamma \in \Gamma$$

This is an (holomorphic) automorphism of the vector bundle $G \times F$ on itself covering the left translation by γ and hence this action defines a vector bundle on $\Gamma \setminus G$. Now let $\Phi: G \times F \to G \times F$ be the isomorphism

$$\Phi(g,f) = (g, \rho(g)^{-1}f)$$

Then

$$\Phi(\gamma g, \rho(\gamma)f) = (\gamma g, \rho(g)^{-1}f)$$

Hence Φ defines an isomorphism Φ_0 of W_ρ on the trivial bundle $X \times F$.

Now, for left-invariant holomorphic vectorfields Z_1, \dots, Z_{p+1} and a holomorphic p-form η with values in F,

$$d\eta(Z_1, \cdots, Z_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} Z_i \eta(Z_1, \cdots, \hat{Z}_i, \cdots, Z_{p+1}) \\ + \sum_{i < j} (-1)^{i+j} \eta([Z_i, Z_j], Z_1 \cdots \hat{Z}_i \cdots \hat{Z}_j \cdots Z_{p+1})$$

It follows that

$$\begin{split} (\Phi d \Phi^{-1})(\eta)(Z_1, \cdots, Z_{p+1})_{g_0} &= \sum_{i=1}^{p+1} (-1)^{i+1} \{ \rho(g_0)^{-1} Z_i \rho(g) \eta(Z_1, \cdots, Z_i, \cdots, Z_{p+1}) \}_{g_0} \\ &+ \sum_{i < j} (-1)^{i+j} \{ \rho(g_0)^{-1} ([Z_i, Z_j], Z_1 \cdots Z_i \cdots Z_j \cdots Z_{p+1}) \}_{g_0} \\ &= \{ \sum_{i=1}^{p+1} (-1)^{i+1} \rho(Z_i) \eta(Z_1 \cdots \hat{Z}_i \cdots Z_{p+1}) \\ &+ \sum_{i=1}^{p+1} (-1)^{i+1} Z_i \eta(Z_1 \cdots Z_i \cdots Z_{p+1}) \\ &+ \sum_{i < j} (-1)^{i+j} \eta([Z_i, Z_j], Z_1 \cdots \hat{Z}_i \cdots \hat{Z}_j \cdots Z_{p+1}) \}_{g_0} \end{split}$$

(ρ has a natural extension to g^c hence to u_1)

It follows that if we identify germs of holomorphic *W*-valued forms on $\Gamma \setminus G$ with germs of holomorphic *F*-valued forms on $\Gamma \setminus G$ through the isomorphism Φ_0 , the operator *d* is transformed into the operator d_0 defined by

$$\begin{aligned} d_{0}\eta(Z_{1},\cdots,Z_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} (Z_{i} + \rho(Z_{i}))\eta(Z_{1},\cdots,\hat{Z}_{i},\cdots,Z_{p+1}) \\ &+ \sum_{i < j} (-1)^{i+j} \eta([Z_{i},Z_{j}],Z_{1} \cdots \hat{Z}_{i} \cdots \hat{Z}_{j} \cdots Z_{p+1}) \cdots \cdots \cdots & \textcircled{1} \end{aligned}$$

Now the map which associates to each W_{ρ} -valued holomorphic *p*-form η , the *F*-valued holomorphic form $\Phi_0(\eta)$ defined by

$$(\Phi_0\eta)(Z_1,\cdots,Z_p)=\Phi_0(\eta(Z_1,\cdots,Z_p))$$

for every *p*-tuple (Z_1, \dots, Z_p) of projections of left invariant holomorphic vectorfields on *G*, defines an isomorphism Φ_p of the sheaf $\underline{\Omega}^p \bigotimes_{\mathcal{O}} \underline{W}_p$ on the sheaf Hom_{*C*} $(\bigwedge^p \mathfrak{u}_1, \mathcal{O} \bigotimes_{\mathcal{O}} F)$. Moreover clearly the diagram

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where d_0 is defined by equation (1) above, is commutative. Now \mathcal{O} is a sheaf of \mathfrak{u}_1 -modules: the map $f \longrightarrow Zf$ for the projection on X of a left invariant holomorphic vectorfield Z on G defines a representation $\mathfrak{u}_1(\simeq \mathfrak{g})$ in the Lie algebra of endomorphism of \mathcal{O} . The stalks at a point $x \in X$ of the complex of sheaves

$$0 \to \mathcal{O} \underset{c}{\otimes} F \to \operatorname{Hom}_{c}(\mathfrak{u}_{1}, \mathcal{O} \underset{c}{\otimes} F) \to \cdots \to \operatorname{Hom}(\Lambda^{n}\mathfrak{u}_{1}, \mathcal{O} \underset{c}{\otimes} F) \to 0$$

from then clearly the standard complex of the Lie algebra u with values in $\mathcal{O}_x \otimes F$, where \mathcal{O}_x is the stalk at x of \mathcal{O} . Passing then to the q^{th} -cohomology groups of this sheaves, we see that, we obtain the standard complex

$$0 \to H^{q}(X, \mathcal{O}) \underset{c}{\otimes} F \to \operatorname{Hom}_{c}(\mathfrak{u}_{1}, H^{q}(X, \mathcal{O}) \underset{c}{\otimes} F) \cdots \operatorname{Hom}_{c}(\Lambda^{n}\mathfrak{u}_{1}, H^{q}(X, \mathcal{O}) \underset{c}{\otimes} F) \to 0$$

where $H^{q}(X, \mathcal{O})$ carries the \mathfrak{u}_{1} -module structure defined by the action of \mathfrak{u}_{1} on \mathcal{O} defined above and $H^{q}(X, \mathcal{O}) \otimes F$ is the tensor product of this representation and ρ .

Combining the preceding, with Proposition 1, we obtain

Theorem 2. Let G be a connected complex Lie group and Γ a discrete subgroup. Let \mathcal{O} be the sheaf of germs of holomorphic functions on $X = \Gamma \setminus G$. Let ρ be a representation of G in a finite dimensional complex vector space F and L_{ρ} the associated local system. Then there is a convergent spectral sequence $\{E_r\}_{0 \le r \le \infty}$ converging to $H^*(X, L_{\rho})$ such that $E_2^{pq} = H^p(g, H^q(X, \mathcal{O}) \bigotimes_{\sigma} F)$ where $H^q(X, \mathcal{O})$

and F are considered as g-modules as follows: a left-invariant vectorfield Y on G projects on X as a vectorfield whose 1-parameter group is a group of holomorphic automorphisms of X; hence $f \longrightarrow Xf$ defines an endomorphism of \mathcal{O} and hence a representation of g; in F we have the representation ρ .

Proof. The argument above is incomplete only in two details, under the isomorphism $g \xrightarrow{p_1} u_{i_1}$, we must show the following:

i) If ρ^c is the extension to g of ρ , then $\rho^c \circ \rho_1$ and ρ are equivalent.

ii) $Xf = p_1(X) \cdot f$

The former is a well known fact; the latter follows from the fact that if $p_2: g \to u_2$ is the projection onto antiholomorphic vectorfields, then, $p_2(X) f=0$ for holomorphic f.

A corollary is the following

Theorem 3. Let G be a connected complex semisimple Lie group and Γ a

discrete subgroup such that $\Gamma \setminus G$ is compact. Then, $H^1(\Gamma \setminus G, \mathcal{O})$ where \mathcal{O} is the sheaf of germs of holomorphic functions on $\Gamma \setminus G$ vanishes provided that G has no 3-dimensional components.

Proof. Since $\Gamma \setminus G$ is compact $H^q(X, \mathcal{O})$ are finite dimensional so that, in view of the Whitehead Lemma for semisimple Lie algebras, we have, for any finite dimensional representation ρ of G in a vector space F, in the spectral sequence of Theorem 2

$$E_2^{10} = E_2^{20} = 0$$
. On the other hand,
 $E_{\infty}^{01} = E_3^{01}$

is the homology of

$$0 \rightarrow E_2^{01} \rightarrow E_2^{20} = 0$$

Hence $E_{\infty}^{01} = E_{2}^{01} = H^{0}(\mathfrak{g}, H^{1}(X, \mathcal{O}) \underset{\sigma}{\otimes} F)$. Now if $H^{1}(X, \mathcal{O}) \neq 0$, and if we choose F to be the dual of this module, then, $H^{0}(\mathfrak{g}, H^{1}(X, \mathcal{O}) \otimes F) \neq 0$. On the other hand since the spectral sequence converges to $H^{*}(X, L_{\rho})$, this implies that $H^{1}(X, L_{\rho}) \neq 0$. But according to [1a] and [4] under the hypothesis of the theorem, viz., that G has no 3-dimensional components, $H^{1}(X, L_{\rho})=0$, a contradiction. Hence the theorem.

Corollary. If $\Gamma \subset G$ is a discrete subgroup of a connected complex semisimple Lie group G such that $\Gamma \setminus G$ is compact, then the natural complex structure on $\Gamma \setminus G$ is locally rigid.

Proof. $\Gamma \setminus G$ is holomorphically parallelisable. Hence the sheaf Θ of germs of holomorphic vectorfields is isomorphic to a direct sum of copies of \mathcal{O} . From Theorem 3, therefore, $H^1(\Gamma \setminus G, \Theta) = 0$. It is well known that this last implies that the complex structure is locally rigid.

REMARK. Reverting to the notation of §1, when $K \setminus G$ is hermitian symmetric, Matsushima and Murakami have given a type decomposition

$$H^{q}(\Gamma, X, \rho) \simeq \sum_{r+s=q} H^{rs}(\Gamma, X, \rho)$$

The groups $H^{r_s}(\Gamma, X, \rho)$ have an interpretation in terms of the spectral sequence of Proposition 1 of this section. In fact, according to proposition 1, there is a spectral sequence converging to $H^*(\Gamma, X, \rho)$ with E_1^{pq} as $H^q(X, \underline{\Omega} \bigotimes_{\mathcal{O}} \underline{W}_{\rho})$. A simple calculation using Lemma 4.1 of [3] shows that E_2^{pq} is isomorphic to $H^{pq}(\Gamma, X, \rho)$ and that the spectral sequence degenerates from the E_2 stage onwards.

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