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VANISHING THEOREMS FOR COHOMOLOGY GROUPS ASSOCIATED TO DISCRETE SUBGROUPS OF SEMISIMPLE LIE GROUPS

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Introduction. The aim of this paper is to prove two vanishing theorems for cohomology groups related to discrete uniform subgroups of semisimple Lie groups.

Let ρ be a representation of a real linear semisimple Lie group G and Γ a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Assume that Γ contains no elements of finite order. In §1 we give a criterion in terms of the highest weight of ρ for the vanishing of $H^p(\Gamma, \rho)$, the p^{th} cohomology group of Γ with coefficient in ρ. This criterion is a generalisation of a theorem of Matsushima and Murakami [3].

In §2 we prove the following theorem (Corollary to Theorem 3). Let G be a complex semisimple Lie group without any simple component of rank 1. Then for any discrete subgroup Γ such that $\Gamma\backslash G$ is compact, the canonical complex structure on the space $\Gamma\backslash G$ is rigid. (This question whether these complex structures are rigid was raised by Professor Matsushima).

1. A vanishing theorem for the cohomology of discrete uniform subgroups

Let G be a connected real linear semisimple Lie group and Γ a discrete subgroup such that the quotient $\Gamma\backslash G$ is compact. Let $\mathfrak{g}_0$ be the Lie algebra of left-invariant vector-fields of G and $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$ a Cartan-decomposition of $\mathfrak{g}_0$, $\mathfrak{k}_0$ being the algebra. Let K be the (compact) Lie subgroup corresponding to $\mathfrak{k}_0$ and $X=G/K$ the corresponding symmetric space. To every representation of G in a finite dimensional real (or complex) vector space F, Matsushima and Murakami [2] have associated certain cohomology groups: we follow their notation and denote these groups by $H^p(\Gamma, X, \rho)$. (In the case when Γ has no elements of finite order Γ acts freely on X and $H^p(\Gamma, X, \rho)$ is isomorphic to the p^{th} cohomology group of Γ with coefficients in the restriction ρ_Γ of ρ to Γ). In the same article, they prove moreover the following result (see in particular §6, §7). (Proposition 1 below).

The vectorfields in $\mathfrak{g}_0$ project under the natural map $G\to\Gamma\backslash G$ into vectorfields on $\Gamma\backslash G$. We will from now on identify $\mathfrak{g}_0$ with this algebra of vectorfields on $\Gamma\backslash G$. Let φ be the Killing form on $\mathfrak{g}_0$ and $\{X_i\}_{1\leq i\leq N}$ and $\{X_\alpha\}_{N+1\leq\alpha\leq n}$ be bases of $\mathfrak{p}_0$ and $\mathfrak{k}_0$ such that $\varphi(X_i, X_j)=\delta_{ij}$ and $\varphi(X_\alpha, X_\beta)=-\delta_{\alpha\beta}$. Let $A_0(\Gamma, X, \rho)$ be the vector space of C^∞-p-forms η on $\Gamma\backslash G$ satisfying i) $i_X\eta=0$ and ii) $\theta_X\eta=\rho(X)\eta$ for every $X\in\mathfrak{k}_0$ where i_X (resp θ_X) denotes interior derivation (resp. Lie derivation) of η with respect to the vectorfield X. Because of i) and ii) η is determined by its values $i_1\cdots i_p=\eta(X_{i_1}\cdots X_{i_p})$. Finally, let Δ^p be the operator

$$\Delta^p\colon A_0^p(\Gamma, X, \rho)\to A_0^p(\Gamma, X, \rho)$$

defined by

$$\Delta^p\eta(X_{i_1}\cdots X_{i_p})=\sum_{k=1}^{N}(-X_k^2+\rho(X_k)^2)\eta_{i_1\cdots i_p}$$
$$+\sum_{k=1}^{N}\sum_{u=1}^{p}(-1)^{u-1}\{(-[X_{i_u}, X_k]+\rho([X_{i_u}, X_k]))\}\eta_{ki_1\cdots\hat{i}_u\cdots i_p}$$

With this notation, we have

Proposition 1. *$H^p(\Gamma, X, \rho)$ is canonically isomorphic to the vector space* $\{\eta\,|\,\eta\in A_0^p(\Gamma, X, \rho);\ \Delta^p\eta=0\}$.

Again, following [2], we define two operators Δ_D^p and Δ_ρ^p as follows:

$$\Delta_D^p\eta(X_{i_1}\cdots X_{i_p})=-\sum_{k=1}^{N}X_k^2\eta_{i_1\cdots i_p}+\sum_{k=1}^{N}\sum_{u=1}^{p}(-1)^u[X_{i_u}, X_k]\eta_{ki_1\cdots\hat{i}_u\cdots i_p}$$

$$\Delta_\rho^p(X_{i_1}\cdots X_{i_p})=+\sum_{k=1}^{n}\rho(X_k)^2\eta_{i_1\cdots i_p}-\sum_{k=1}^{N}\sum_{u=1}^{p}(-1)^u\rho([X_{i_u}, X_k])\eta_{kki_1\cdots i_u\cdots i_p}$$

Then $\Delta^p=\Delta_D^p+\Delta_\rho^p$. In §7 [2], it is moreover proved that

$$\sum_{i_1<\cdots<i_p}\int_{\Gamma/G}\langle(\Delta_D^p\eta)_{i_1\cdots i_p}, \eta_{i_1\cdots i_p}\rangle_F\geq 0$$

where $\langle\ ,\ \rangle_F$ is a positive definite scalar product on F for which $\rho(X)$ is (hermitian) symmetric (resp. skew-symmetric (hermitian)) for $X\in\mathfrak{p}_0$ (resp. $\mathfrak{k}_0$). It follows therefore that if $\Delta^p\eta=0$,

$$\sum_{i_1<\cdots<i_p}\int_{\Gamma/G}\langle(\Delta_\rho^p\eta)_{i_1\cdots i_p}, \eta_{i_1\cdots i_p}\rangle_F\geq 0$$

We obtain therefore

Proposition 2. *If the quadratic form on the space of exterior p-forms on $\mathfrak{p}_0$ with values in F defined by*

$$\eta\to\sum_{i_1<\cdots<i_p}\langle(\Delta_\rho^p\eta)_{i_1\cdots 1_p}, \eta_{i_1\cdots i_p}\rangle_F$$

is positive definite, then $H^p(\Gamma, X, \rho)=0$.

In the main result of this section we give a sufficient criterion in terms of the "highest weight" of ρ with respect to a suitable Cartan-subalgebra of $\mathfrak{g}_0$ in order that Δ_ρ^η define a positive definite quadratic form.

Let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$ and $\mathfrak{k}$ and $\mathfrak{p}$ those of $\mathfrak{k}_0$ and $\mathfrak{p}_0$. We identify $\mathfrak{k}$ and $\mathfrak{p}$ with subspaces of $\mathfrak{g}$. Let $\mathfrak{h}_{\mathfrak{k}_0}$ be a Cartan-subalgebra of $\mathfrak{k}_0$ and $\mathfrak{h}_0$ a Cartan-subalgebra of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \supset \mathfrak{h}_{\mathfrak{k}_0}$. Let $\mathfrak{h}_{\mathfrak{p}_0} = \mathfrak{h}_0 \cap \mathfrak{p}_0$. Let $\mathfrak{h}_\mathfrak{k}$ $\mathfrak{h}$ and $\mathfrak{h}_\mathfrak{p}$ denote respectively the complexifications of $\mathfrak{h}_{\mathfrak{k}_0}$ $\mathfrak{h}_0$ amd $\mathfrak{h}_{\mathfrak{p}_0}$. Then $\mathfrak{h}$ is a Cartan-subalgebra of $\mathfrak{g}$. Let Δ be the system of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For $\alpha \in \Delta$ let $H_\alpha \in \mathfrak{h}$ be the unique element such that $\varphi(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. Then, it is well known that the real subspace $\mathfrak{h}^* = \sum_{\alpha \in \Delta} RH_\alpha$ of $\mathfrak{g}$ spanned by the $\{H_\alpha\}_{\alpha \in \Delta}$ is the same as $i\mathfrak{h}_{\mathfrak{k}_0} \oplus \mathfrak{p}_0$. Moreover if θ is the extension to $\mathfrak{g}$ to the Cartan involution θ_0 denfied by the Cartan-decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, then θ is an automorphism of $\mathfrak{g}$ leaving $\mathfrak{h}$ invariant. Hence θ acts on the dual of $\mathfrak{h}$ and permutes the elements of Δ. The set Δ may then be decomposed as the disjoint union $A \cup B \cup C$ of three subsets A, B and C

where
$$\begin{aligned} A &= \{\alpha \mid \alpha \in \Delta;\ \theta(\alpha) = \alpha;\ \theta(E_\alpha) = E_\alpha\} \\ B &= \{\alpha \mid \alpha \in \Delta;\ \theta(\alpha) \neq \alpha\} \\ C &= \{\alpha \mid \alpha \in \Delta;\ \theta(\alpha) = \alpha;\ \theta(E_\alpha) = -E_\alpha\}\ . \end{aligned}$$

(In the sequel we sometimes write α^θ for $\theta(\alpha)$).

We introduce next a lexicographic order on the (real) dual of $\mathfrak{h}^*$ as follows: let $H_1, \cdots, H_l$ be an orthonormal basis of $\mathfrak{h}^*$ with respect to φ ($\varphi|_{\mathfrak{h}^*}$ is positive definite) chosen so that $H_1, \cdots, H_l$ form a basis of $i\mathfrak{h}_{\mathfrak{k}_0}$ and if the centre $\mathfrak{c}_0$ of $\mathfrak{k}_0$ is non-zero, of dimension r, then $H_1, \cdots, H_r$ belong to $i\mathfrak{c}_0$; for α, β in the (real) dual of $\mathfrak{h}^*$, $\alpha > \beta$ if the first non-vanishing difference $\alpha(H_i) - \beta(H_i)$ is greater than zero. Let Δ^+ be the system of positive roots with respect to this order and let $A^+ = A \cap \Delta^+$, $B^+ = B \cap \Delta^{+1}$, $C = \cap C\Delta^+$. Then θ leaves A^+, B^+ and C^+ invariant. Let $\sum_1 = A^+ U\{\alpha \mid \alpha \in B^+;\ \theta(\alpha) > \alpha\}$ and $\sum_2 = C^+ U\{\alpha \mid \alpha \in B^+;\ \theta(\alpha) > \alpha\}$.

Theorem 1. *Let ρ denote a finite dimensional representation of G in a complex vector-space F, as also the induced representation of $\mathfrak{g}$. Let Λ_ρ be the highest weight of ρ with respect to the above defined Cartan-subalgebra and the order on the dual of $\mathfrak{h}^*$. Then if $\sum_\rho = \{\alpha \mid \alpha \in \sum_2,\ \varphi(\Lambda_\rho, \alpha) \neq 0\}$ contains more than q elements, then the Hermitian quadratic form Q_ρ defined by*

$$\eta \to \sum_{i_1 < \cdots < i_p} \langle (\Delta_\rho^\eta \eta)_{i_1 \cdots i_p},\ \eta_{i_1 \cdots i_p} \rangle_F$$

is positive definite for $p \leq q$. Hence $H^p(\Gamma, X, \rho) = 0$ for $1 \leq p \leq q$.

Before we proceed to the proof of the theorem, we will make a few preliminary simplifications:

Lemma 1. *Let E be the q^{th} exterior power of $\mathfrak{p}$ and let α be the isomorphism onto $F\otimes E$ of the space of exterior q-forms on $\mathfrak{p}$ with values in F defined by*

$$\eta \to \sum_{i_1<\cdots<i_q} \eta_{i_1\cdots i_q} \otimes (X_{i_1}\wedge\cdots\wedge X_{i_q})$$

Then

$$T^q_\rho = 2\alpha\circ\Delta^q_\rho\circ\alpha^{-1} = 2(\rho\otimes 1)(c)+(1\otimes\sigma)(c')-(\rho\otimes 1)(c')-(\rho\otimes\sigma)(c')$$

where

$$c = \sum_{i=1}^{N} X_i^2 - \sum_{\alpha=N+1}^{n} X_\alpha^2$$

and $c' = -\sum_{\alpha=N+1}^{n} X_\alpha^2$ are elements of the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{k}$ and σ denotes the adjoint representation of $\mathfrak{k}$ in E. Hence T^q_ρ is a symmetric endomorphism of $F\otimes E$ with respect to the scalar product

$$\left\langle \sum_{i_1<\cdots<i_p} \eta_{i_1\cdots i_p}\otimes X_{i_1}\wedge\cdots\wedge X_{i_p}, \sum_{j_1<\cdots<j_p} \eta_{j_1\cdots j_p}\otimes X_{j_1}\wedge\cdots\wedge X_{j_p}\right\rangle = \sum_{i_1<\cdots<i_p} \langle \eta_{i_1\cdots i_p}, \eta_{i_1\cdots i_p}\rangle_F$$

Proof. We have

$$(\Delta^q_\rho)_{i_1\cdots i_q} = \sum_{k=1}^{N} \rho(X_k)^2\eta_{i_1\cdots i_q} + \sum_{k=1}^{N}\sum_{u=1}^{q}(-1)^{u-1}\rho([X_{i_u}, X_k])\eta_{k i_1\cdots\hat{i}_u\cdots i_q}$$

For every q-tuple $I_q=(i_1<\cdots<i_q)$, we write X_{I_q} for $X_{i_1}\wedge\cdots\wedge X_{i_q}$. In this notation,

$$\alpha(\eta) = \sum_{I_q} \eta_{I_q}\otimes X_{I_q}$$

$$\begin{aligned}\frac{1}{2}T^q_\rho\alpha(\eta) &= \sum_{I_q}\Big\{\sum_{k=1}^{N}\rho(X_k)^2\eta_{I_q} + \sum_{k=1}^{N}\sum_{u=1}^{q}(-1)^{u-1}\rho([X_{i_u}, X_k])\eta_{k i_1\cdots\hat{i}_u\cdots i_q}\Big\}\otimes X_{I_q} \\ &= \sum_{I_q}\Big\{\sum_{k=1}^{N}\rho(X_k)^2\eta_{I_q} + \sum_{J_q\Delta I_q=i_u j_v}(-1)^{u+v}\rho([X_{i_u}, X_{j_v}])\eta_{J_q}\Big\}\otimes X_{I_q} \\ &= \sum_{I_q}\Big\{\sum_{k=1}^{N}\rho(X_k)^2\eta_{I_q} + \sum_{J_q\Delta I_q=i_u j_v}(-1)^{u+v}c^{\alpha}_{i_u j_v}\rho(X_\alpha)\eta_{J_q}\Big\}\otimes X_{I_q}\ .\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma(X_\alpha)X_{J_q} &= \sum_{k=1}^{n}\sum_{u=1}^{q}(-1)^{v-1}c^k_{\alpha j_v}(X_k\wedge X_{j_1}\cdots X_{j_v}\cdots\wedge X_{j_q}) \\ &= \sum_{I_q\Delta J_q=j_v i_u}(-1)^{u+v}c^{i_u}_{j_v}X_{I_q}\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}T^q_\rho\alpha(\eta) &= \sum_{I_q}\sum_{k=1}^{N}\rho(X_k)^2\eta_{I_q}\otimes X_{I_q} + \sum_{J_q}\rho(X_\alpha)\eta_{J_q}\otimes\sigma(X_\alpha)X_{J_q} \\ &= \Big\{\sum_{k=1}^{N}\rho(X_k)^2\otimes 1 + \sum\rho(X_\alpha)\otimes\sigma(X_\alpha)\Big\}\alpha(\eta)\Big\}\end{aligned}$$

Now the required result follows from the fact

$$\begin{aligned}2\rho(X_\alpha)\otimes\sigma(X_\alpha) &= \{\rho(X_\alpha)\otimes 1+1\otimes\sigma(X_\alpha)\}^2-\rho(X_\alpha)^2\otimes 1-1\otimes\sigma(X_\alpha)^2\\ &= (\rho\otimes\sigma)(X_\alpha)^2-\rho(X_\alpha)^2\otimes 1-1\otimes\sigma(X_\alpha)^2\end{aligned}$$

That T^q_ρ is a hermitian symmetric endomorphism follows from the facts that $\rho(X_i)$ and $\sigma(X_i)$ are hermitian symmetric while $\rho(X_\alpha)$ and $\sigma(X_\alpha)$ are skew-hermitian with respect to $\langle\ ,\ \rangle_F$ and the extension to E of the Killing form on $\mathfrak{p}_0$.

Lemma 2. a) *If Λ is the highest weight of an irreducible representation ρ of $\mathfrak{g}$ induced by a representation ρ of G, then*

$$\rho(c)=\{\varphi(\Lambda,\ \Lambda)+\sum\varphi(\Lambda,\ \alpha)\}.\quad \textit{Identity}$$

b) *when restricted to the (irreducible) K-subspace generated by the eigen-space corresponding to the highest weight Λ,*

$$\rho(c')=\left\{\frac{1}{4}\varphi(\Lambda+\Lambda^\theta,\ \Lambda+\Lambda^\theta)+\sum_{\alpha\in\Sigma_1}\varphi\left(\Lambda,\ \frac{\alpha+\alpha^\theta}{2}\right)\right\}.\quad \textit{Identity}.$$

For a proof see [4]: Lemmas 4 and 16(c).

Lemma 3. *If Λ_1 and Λ_2 are the highest weights of two irreducible representations ρ_1, ρ_2 of $\mathfrak{g}$, such that $\Lambda_1-\Lambda_2$ is a non-negative linear combination of simple roots of $\mathfrak{g}$, then $\lambda_1\geq\lambda_2$ where $\rho_k(c)=(\lambda_k.$ Identity) ($k=1, 2$). Equality can occur only if $\Lambda_1=\Lambda_2$.*

The same conclusions hold for $\mathfrak{k}$ and c' instead of $\mathfrak{g}$ and c provided that Λ_1 and Λ_2 coincide on the center of $\mathfrak{k}$.

For the proof see Lemma 5 [4].

Proof of Theorem 1. We obtain the eigen-values of T^q_ρ as follows: Let

$$E=\sum_{\mu\in M}E_\mu \quad\text{and}\quad F=\sum_{\lambda\in L}F_\lambda \quad\text{and}\quad F_\lambda\otimes E_\mu=\sum_{\nu\in M_{\lambda\mu}}V^\nu_{\lambda\mu}$$

be the decomposition of E, F and $F_\lambda\otimes E_\mu$ into irreducible $\mathfrak{k}$-modules indexed by the highest weights (for the order defined by $H_1,\cdots,H_p$ on $i\mathfrak{h}_k$). Since ρ is an irreducible representation of $\mathfrak{g}$ and c is a central element of $U(\mathfrak{g})$, $\rho(c)$ is a scalar operator. Similarly, since c' is central in $U(\mathfrak{k})$, $\rho(c')\otimes 1$, $1\otimes\sigma(c')$ and $(\rho\otimes\sigma)(c')$ are scalars on F_λ E, $F\otimes E_\lambda$ and $V^\nu_{\lambda\mu}$. Hence T^q_ρ acts as a scalar on each $V^\nu_{\lambda\mu}$. We denote the corresponding eigen-value by $a(\lambda,\ \mu,\ \nu)$. Among $V^\nu_{\lambda\mu}$ there is a unique irreducible component with highest weight $\nu=\lambda+\mu$ we denote the corresponding scalar $a(\lambda,\ \mu,\ \nu)$ by $a(\lambda,\ \mu)$ with this notation, we have

Assertion I. $a(\lambda,\ \mu,\ \nu)\geq a(\lambda,\ \mu)$; *equality occurs only if $\nu=\lambda+\mu$.*

Proof. We denote the representation in $V^\nu_{\lambda\mu}$ by $\rho^\nu_{\lambda\mu}$. Then since $(\rho\otimes 1)(c)$, $(\rho\otimes 1)(c')$ and $(1\otimes\sigma)(c')$ all define the same scalar operator in $F_\lambda\otimes E_\mu$,

$$a(\lambda,\ \mu)+a(\lambda,\ \mu,\ \nu) = \rho_{\lambda\mu}^{\lambda+\mu}(c')-\rho_{\lambda\mu}^{\nu}(c')$$

(Here we have let $\rho_{\lambda\mu}^{\nu}(c')$ stand for the scalar). Now any weight in $F_\lambda\otimes E_\mu$ has the form $\lambda_1+\mu_1$ where λ_1 and μ_1 are weights of F_λ and E_μ; on the other hand $\lambda-\lambda_1$ and $\mu-\mu_1$ are non-negative linear combination of simple roots of k; hence so is $(\lambda+\mu)-(\lambda_1+\mu_1)$. It follows then from Lemma 3 that

$$a(\lambda,\ \mu)\geq a(\lambda,\ \mu,\ \nu)$$

Equality can occur only if $\lambda+\mu=\lambda_1+\mu_1$ and there is only one component of $F_\lambda\otimes E_\mu$ with $\lambda+\mu$ as the highest weight. (Note that if $\mathfrak{k}$ has a centre, then the central elements act as scalars on F_λ and E hence in all of $F_\lambda\otimes E_\mu$).

Assertion II. *Let f_λ be a highest weight vector of F such that $\|f_\lambda\|_F^2=1$. For $\alpha\in\Delta$, let E_α be a root vector of α. Suppose that $E_{\alpha_0}f_\lambda=0$ for $\alpha\in A^+$. If there is an $\alpha_0\in B^+$ with $E_{\alpha_0}f_\lambda\neq 0$, then $E_{\alpha_0}f_\lambda\in F_{\lambda_1}$ for some λ_1 and $a(\lambda,\ \mu)<a(\lambda_1,\ \mu_1)$*

Proof. Using the fact that θ is an involution, we have

$$\mathfrak{k} = \mathfrak{h}_{\mathfrak{k}}\oplus \sum_{\alpha\in A^+}\{CE_\alpha\oplus CE_\alpha\}\oplus \sum_{\substack{\alpha\in B^+\\ \alpha>\alpha\theta}}\{C(E_\alpha+E_\alpha\theta)\oplus C(E_{-\alpha}+E_{-\alpha}\theta)\}$$

and the order chosen on $\mathfrak{h}_{\mathfrak{k}}^*=i\mathfrak{h}_{\mathfrak{k}_0}$ has precisely $\{\alpha\,|\,\alpha\in A^+\}$ and $\left\{\dfrac{\alpha+\alpha^\theta}{2}\middle|\alpha\in B^+\right\}$ as the positive roots. The roots of $\mathfrak{k}$ are necessarily zero on the centre of $\mathfrak{k}$. It follows that the weights λ and $\lambda+\alpha_0$ (which is the weight corresponding to $E_{\alpha_0}f_\lambda$) have the same values on the centre. On the other hand, since $\lambda+\alpha_0$ and λ_1 are weights of the same irreducible representation of $\mathfrak{k}$, λ_1 and $\lambda+\alpha_0$ have the same values on the centre of $\mathfrak{k}$. It follows that $\lambda_1=\lambda$ on the centre of $\mathfrak{k}$. Now $\lambda_1-\lambda=\lambda_1-(\lambda+\alpha_0)+\alpha_0$ and $\lambda_1-(\lambda+\alpha_0)$ is a non-negative linear combination of simple roots. Hence $\lambda_1-\lambda$ is a non-negative linear combination of simple roots and $\lambda_1\neq\lambda$. A similar remark holds for $\lambda_1+\mu$ and $\lambda+\mu$. It follows then from Lemma 3 above that

$$\rho_\lambda(c') < \rho_{\lambda_1}(c')$$

and

$$\rho_{\lambda\mu}^{\lambda+\mu}(c') < \rho_{\lambda_1\mu}^{\lambda_1+\mu}(c')$$

The operators $(\rho\otimes 1))$ (c) and $(1\otimes\sigma)$ (c') on the other hand are scalars on the whole of $F\otimes E$. Hence from the expression for T_ρ^q, the Assertion follows.

Assertion III. *Suppose that $E_\alpha F_\lambda=0$ for $\alpha\in A^+\cup B^+$ but that there is an $\alpha_0\in C^+$ such that $E_{\alpha_0}f_\lambda\neq 0$. Then $a(\lambda,\ \mu)>0$.*

Proof. If $\{E_\alpha\}_{\alpha\in\Delta}$ are root vectors so chosen that $\varphi(E_\alpha,\ E_{-\alpha})=1$, then, it is well known that

$$c=\sum_{\alpha\in\Delta^+}E_\alpha E_{-\alpha}+\sum_{\alpha\in\Delta^+}E_{-\alpha}E_\alpha+\sum_{i=1}^{l}H_i^2$$

It follows that

$$\rho(c)f_\lambda=\sum_{\alpha\in\Delta^+}\rho(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)f_\lambda+\sum_{i=1}^{l}\rho(H_i)^2f$$

Using the facts, $E_\alpha f_\lambda=0$ for $\alpha\in A^+\cup B^+$ and that $[E_\alpha, E_{-\alpha}]=H_\alpha$, we have

$$\rho(c)f_\lambda=\sum_{\alpha\in A^+\cup B^+}\lambda(H_\alpha)f_\lambda+\sum_{i=1}^{p}\wedge(H_i)^2f_\lambda+\sum_{\alpha\in C}\rho(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)f_\lambda+\sum_{i=p+1}^{l}\rho(H_i)^2f_\lambda$$

Hence

$$\langle\rho(c)f_\lambda, f_\lambda\rangle_F=\sum_{\alpha\in A^+\cup B^+}\lambda(H_\alpha)+\sum_{i=1}^{p}\lambda(H_i)^2+\sum_{\alpha\in C^+}\langle\rho(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)f_\lambda, f_\lambda\rangle$$
$$+\sum_{i=p+1}^{l}\langle\rho(H_i)^2f_\lambda, f_\lambda\rangle_F$$

Now it is well known that F admits an orthogonal decomposition with respect to $\langle\ ,\ \rangle_F$ into irreducible representations of the algebra $\mathfrak{g}'=\boldsymbol{C}E_\alpha\oplus\boldsymbol{C}E_{-\alpha}\oplus\boldsymbol{C}H_\alpha$ for $\alpha\in C^+$ so that to prove that $\langle\rho(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)f_\lambda, f_\lambda\rangle\geq|\lambda(H_\alpha)|$ *equality occurring only if* $E_\alpha f_\lambda=0$, we may assume that the $\mathfrak{g}'$-invariant subspace W spanned by f_λ is *irreducible* with respect to the three dimensional algebra. Now by Lemma 2,

$$\rho\left\{E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha+\frac{H_\alpha^2}{\varphi(H_\alpha H_\alpha)}\right\}f_\lambda=\left\{\frac{(\lambda+k\alpha)(H_\alpha)^2}{\varphi(H_\alpha, H_\alpha)}+(\lambda+k\alpha)(H_\alpha)\right\}f_\lambda$$

where $\lambda+k\alpha$, $k\geq 0$ is the highest weight in W (of $\mathfrak{g}'$). Hence

$$\rho(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)f_\lambda=\frac{k\alpha(H_\alpha)^2}{\varphi(H_\alpha, H_\alpha)}+(\lambda+k\alpha)(H_\alpha)f_\lambda$$

so that

$$\langle\rho(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)f_\lambda, f_\lambda\rangle_F=(\lambda+k\alpha)(H_\alpha)+\frac{\alpha(H_\alpha)}{\varphi(H_\alpha, H_\alpha)}\geq|\lambda(H_\alpha)|$$

(It is well known that $(\lambda+k\alpha)(H_\alpha)\geq|\lambda(H_\alpha)|$ since $\lambda+k\alpha$ is the highest weight). Moreover equality occurs only if $k=0$; if $k=0$, however, λ is the highest weight so that $E_\alpha f_\lambda=0$. We have thus shown that

$$\langle\rho(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)f_\lambda, f_\lambda\rangle\geq|\lambda(H_\alpha)|$$

equality occurring only if $E_\alpha f_\lambda=0$. We have therefore,

$$\langle\rho(c)f_\lambda, f_\lambda\rangle\geq\sum_{\alpha\in A^+\cup B^+}\lambda(H_\alpha)+\sum_{i=1}\lambda(H_i)^2+\sum_{\alpha\in C^+}^{p}|\lambda(H_\alpha)|+\sum_{i=p+1}^{l}\langle\rho(H_i)^2f_\lambda, f_\lambda\rangle_F$$

equality occurring only if $E_\alpha f_\lambda=0$ *for all* $\alpha\in C^+$. Moreover $S=\sum_{i=p+1}^{l}\rho(H_i)^2$ is a non-negative symmetric operator so that

$$\rho(c)f_\lambda, f_\lambda\rangle \geq \sum_{\alpha\in A^+\cup B^+} |\lambda(H_\alpha)| + \sum \lambda(H_i)^2 + \langle Sf_\lambda, f_\lambda\rangle + \sum_{\alpha\in C^+} |\lambda(H_\alpha)|$$

with $S\geq 0$ (Note that for $\alpha\in A^+\cup B^+$, $E_\alpha f_\lambda=0$ so that $\lambda(H_\alpha)\geq 0$).

Using b) of Lemma 2, we have also

$$\rho(c')\otimes 1\Big|_{F_\lambda\otimes E} = \{\sum_{i=1}^{p} \lambda(H_i)^2 + \sum_{\alpha\in\Sigma_1} \lambda(H_\alpha+H_{\alpha^\theta})/2\}. \quad \text{Identity}$$

$$(\rho\otimes\sigma)(c')\Big|_{V^{\lambda+\mu}_{\lambda\mu}} = \sum_{i=1}^{p} (\lambda+\mu)(H_i)^2 + \sum_{\alpha\in\Sigma_1} (\lambda+\mu)(H_\alpha+H_{\alpha^\theta})/2\,. \quad \text{Identity}$$

and

$$(1\otimes\sigma)(c')\Big|_{F\otimes E_\mu} = \sum_{i=1}^{p} \mu(H_i)^2 + \sum_{\alpha\in\Sigma_1} \mu(H_\alpha+H_{\alpha^\theta})/2\,. \quad \text{Identity}$$

so that if $e_\mu\otimes E_\mu$ is a unit weight vector of weight μ,

$$\langle T^q_\rho(f_\lambda\otimes e_\mu), f_\lambda\otimes e_\mu\rangle \geq 2\sum_{\substack{\alpha\in B^+\\ \alpha>\alpha\theta}} |\lambda(H_\alpha+H_{\alpha^\theta})/2| + 2\sum_{\alpha\in C^+} \lambda(H_\alpha)$$
$$+2\langle S(f_\lambda), f_\lambda\rangle - 2\sum_{i=1}^{p} \lambda(H_i)\mu(H_i)$$

Now μ being a weight of σ_q it is the sum of q of the weights of the adjoint representation of k_0 in p_0. Hence

$$\mu = \sum_{i=1}^{q} (\alpha_i + \alpha_2^\theta)/2$$

where all the α_i belong to $\sum_2$. Hence

$$\langle T^q_\rho(f_\lambda\otimes e_\mu), f_\lambda\otimes e_\mu\rangle \geq 2\sum_{\alpha\in\Sigma_2} \lambda(H_\alpha+H_\alpha\theta)/2 - 2\sum_{i=1}^{q} \lambda(H_{\alpha_i}+H_{\alpha_i^\theta})/2$$

Here equality can occur only if $E_\alpha f_\lambda=0$ for $\alpha\in\Delta^+$ and $\langle Sf_\lambda, f_\lambda\rangle=0$. It follows therefore that $a(\lambda, \mu)>0$ if there exists $\alpha_0\in C^+$ with $E_{\alpha_0} f_\lambda\neq 0$.

In view of Assertions I, II and III, we see that T is positive definite if and only if $a(\lambda_0, \mu)>0$ where λ_0 is the greatest of the dominant weights $\{\lambda\,|\,\lambda\in L\}$: this follows from the fact that $E_\alpha f_{\lambda_0}=0$ for all $\alpha\in\Delta^+$ if and only if f_{λ_0} is the highest weight vector for ρ; it follows that any weight of $\rho|_k$ is of the form $\lambda_0-\sum m_i r(\alpha_i)$ where $m_i\geq 0$ and $r(\alpha_i)$ are the restriction of positive roots of $\mathfrak{g}$; finally $r(\alpha_i)\neq 0$ hence greater than zero (see Lemma 16 (f) [4]).

Thus to complete the proof of the Theorem, we need only prove

Assertion IV. *If λ_0 is the restriction $r(\Lambda)$ of the highest weight Λ of ρ, then $a(\lambda_0, \mu)>0$ for all $\mu\in M$ provided there are at least $(q+1)$ roots $\alpha\in\sum_2$ such that $\Lambda(H_\alpha+H_\alpha\theta)>0$.*

Proof. By evaluation on the highest weight $f_{\lambda_0}\otimes e_\mu$ we have (Lemma 2)

$$T_\rho(f_{\lambda_0}\otimes e_\mu) = \{2\sum_{\alpha\in\Sigma_2}\Lambda(H_\alpha+H_{\alpha^\theta})/2+2\sum_{i=1}^{p}\Lambda(H_i)^2-2\sum_{i=1}^{p}\Lambda(H_i)\mu(H_i)\}(f_{\lambda_0}\otimes e_\mu)$$
$$=\{2\sum_{\alpha\in\Sigma_2}\Lambda(H_\alpha+H_{\alpha^\theta})/2-2\sum_{i=1}^{q}(H_{\alpha_i}+H_{\alpha_i}\theta)/2+2\sum_{i=1}^{q}\Lambda(H_i)^2\}(f_{\lambda_0}\otimes e_\mu)$$

where $\mu = r(\sum_{i=1}^{q}(\alpha_i+\alpha_i^\theta)/2)$. It follows that

$a(\lambda_0, \mu)>0$ under our hypothesis,

since $\sum_{i=1}^{p}\Lambda(H_i)^2\geq 0$.

This completes the proof of the Theorem.

REMARK 1. Theorem 1 generalises Theorem 12.1 of [3] where only the case when G/K is hermitian symmetric, is considered. In fact, the present theorem is more general than Theorem 12.1 of [3] even in this case: $H^n(\Gamma, X, \rho)$ admits a type decomposition (see [3])

$$H^n(\Gamma, X, \rho)\simeq \coprod_{r+s=n} H^{rs}(\Gamma, X, \rho)$$

so that under the hypothesis of Theorem 1, we have

$$H^{rs}(\Gamma, X, \rho)=0$$

for $r+s\leq q$. Theorem 12.1 of [3] is the special case $q=\dim G/K$. In section §2, we will give an interpretation of the groups $H^{rs}(\Gamma, X, \rho)$. In [4] all the representations for which T_ρ^1 is positive definite are determined.

REMARK 2. The author has checked in a number of *classical cases*, that if G is simple and non-compact and ρ is *any* nontrivial irreducible representation, then the number of elements in $\sum_\rho$ is greater than or equal to the rank of the associated symmetric space.

2. Compact quotients of complex semisimple Lie groups

Let X be a complex manifold and $\tilde{X}\xrightarrow{\pi}X$ be the universal covering of X. Let Γ be the fundamental group of X acting fixed point free on $\tilde{X}$. Let ρ be a representation of Γ in a finite dimensional complex vector space. Let L_ρ denote the local system associated to ρ and W_ρ the holomorphic vector bundle associated to ρ. Let $\underline{L}_\rho$ and $\underline{W}_\rho$ denote respectively the sheaf of germs of sections of L_ρ and holomorphic sections of W_ρ. By the de Rham theorem, the cohomology groups $H^p(X, L_\rho)$ of X with coefficients in the local system L_ρ are the cohomology groups of the complex

$$A=\sum_p A^p(\Gamma, \tilde{X}, \rho)$$

defined as follows: $A^p(\Gamma, X, \rho)$ is the vector space of C^∞-exterior p-forms η on X with values in F satisfying the condition

$$\eta(\gamma_*(t_1), \gamma_*(t_2), \cdots, \gamma_*(t_p)) = \rho(\gamma)^{-1}\eta(t_1, \cdots, t_p)$$

where $t_1, \cdots, t_p$ are tangent vectors to $\tilde{X}$ and $\gamma_*(t)$ denotes the image by γ of the tangent vector t to X; the boundary operator in the complex is the exterior differentiation of F-valued forms on $\tilde{X}$. The complex structure on X gives a decomposition of each of the space $A^p(\Gamma, \tilde{X}, \rho)$ as a direct sum $\sum_{r+s=p} A^{rs}(\Gamma, \tilde{X}, \rho)$ according to the bidegree. Moreover $d=d'+d''$ where d' and d'' are of bidegree (1, 0) and (0, 1) respectively. This gives A a structure of a double complex. The term E_1^{pq} of the spectral sequence associated to this double complex is clearly the q^{th} cohomology of the complex

$$0 \to A^{p,0}(\Gamma, \tilde{X}, \rho) \to A^{p,1}(\Gamma, \tilde{X}, \rho) \to \cdots \to A^{p,n}(\Gamma, X, \rho) \to 0$$

($n=\dim X$). Again, by the Dolbeault theorem, the q^{th} cohomology of this complex is $H^q(X, \underline{\Omega}^p \underset{\mathcal{O}}{\otimes} \underline{W}_\rho)$ where $\underline{\Omega}^p$ is the holomorphic bundle of holomorphic p-forms, and $\underline{\Omega}^p \underset{\mathcal{O}}{\otimes} \underline{W}$ is the sheaf of germs of holomorphic p-forms on X with coefficients in W. Moreover, the derivation d_1 in the term E_1 is clearly the map induced by the exterior differentiation

$$d\colon \underline{\Omega}^p \underset{\mathcal{O}}{\otimes} \underline{W}_\rho \to \underline{\Omega}^{p+1} \underset{\mathcal{O}}{\otimes} \underline{W}_\rho$$

(since we have $\underline{\Omega}^p \underset{\mathcal{O}}{\otimes} \underline{W}_\rho \simeq \underline{\Omega}^p \underset{C}{\otimes} \underline{L}_\rho$, the operator d above makes sence: $\underline{\Omega}^p \underset{C}{\otimes} \underline{L}_\rho \to \underline{\Omega}^{p+1} \underset{C}{\otimes} \underline{L}_\rho$).

We have thus

Proposition 1. *There is a convergent spectral sequence* $\{E_r^{pq}\}_{c\le r\le\infty}$ *converging to* $H^*(\Gamma, \tilde{X}, \rho)$ *such that* $E_1^{pq}=H^q(X, \underline{\Omega}^p \underset{\mathcal{O}}{\otimes} \underline{W}_\rho)$ *and* d_1 *is induced by the map* $d\colon \underline{\Omega}^p \underset{\mathcal{O}}{\otimes} \underline{W}_\rho \to \underline{\Omega}^{p+1} \otimes \underline{W}_\rho$.

Now let $\tilde{X}=G$ be a simply connected complex Lie group and $\Gamma\subset G$ a discrete subgroup; then $X=\Gamma\backslash G$. Let $\mathfrak{g}$ be the Lie algebra of left invariant vectorfields on G. (Then elements of $\mathfrak{g}$ may be regarded as vectorfields on $\Gamma\backslash G$ as well). Let $\mathfrak{g}^C$ denote the complexification of $\mathfrak{g}$. Then $\mathfrak{g}^C \simeq \mathfrak{n}_1 \oplus \mathfrak{n}_2$ where $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are respectively the complex ideals of holomorphic and antiholomorphic left-invariant vectorfields. The natural projections $\mathfrak{g} \to \mathfrak{n}_1$ and $\mathfrak{g} \to \mathfrak{n}_2$ define isomorphisms of $\mathfrak{g}$ on $\mathfrak{n}_1$ and $\mathfrak{n}_2$ respectively.

Suppose now that ρ is the restriction of a representation of G in a finite dimensional vector space F. In this special case we can compute the term E_2 as well.

In the first place, there is a canonical (holomorphic) isomorphism of the vector bundle W_ρ on X with the trivial bundle. In fact the vector bundle W_ρ is obtained as follows: the group Γ acts $G\times F$ by diagonal action:

$$\gamma(g, f) = (\gamma g, \rho(\gamma) f) \qquad \text{for } \gamma \in \Gamma .$$

This is an (holomorphic) automorphism of the vector bundle $G \times F$ on itself covering the left translation by γ and hence this action defines a vector bundle on $\Gamma \backslash G$. Now let Φ: $G \times F \to G \times F$ be the isomorphism

$$\Phi(g, f) = (g, \rho(g)^{-1} f)$$

Then

$$\Phi(\gamma g, \rho(\gamma) f) = (\gamma g, \rho(g)^{-1} f)$$

Hence Φ defines an isomorphism Φ_0 of W_ρ on the trivial bundle $X \times F$.

Now, for left-invariant holomorphic vectorfields $Z_1, \cdots, Z_{p+1}$ and a holomorphic p-form η with values in F,

$$\begin{aligned} d\eta(Z_1, \cdots, Z_{p+1}) = & \sum_{i=1}^{p+1} (-1)^{i+1} Z_i \eta(Z_1, \cdots, \hat{Z}_i, \cdots, Z_{p+1}) \\ & + \sum_{i<j} (-1)^{i+j} \eta([Z_i, Z_j], Z_1 \cdots \hat{Z}_i \cdots \hat{Z}_j \cdots Z_{p+1}) \end{aligned}$$

It follows that

$$\begin{aligned} (\Phi d \Phi^{-1})(\eta)(Z_1, \cdots, Z_{p+1})_{g_0} = & \sum_{i=1}^{p+1} (-1)^{i+1} \{\rho(g_0)^{-1} Z_i \rho(g) \eta(Z_1, \cdots, Z_i, \cdots, Z_{p+1})\}_{g_0} \\ & + \sum_{i<j} (-1)^{i+j} \{\rho(g_0)^{-1}([Z_i, Z_j], Z_1 \cdots Z_i \cdots Z_j \cdots Z_{p+1})\}_{g_0} \\ = & \{\sum_{i=1}^{p+1} (-1)^{i+1} \rho(Z_i) \eta(Z_1 \cdots \hat{Z}_i \cdots Z_{p+1}) \\ & + \sum_{i=1}^{p+1} (-1)^{i+1} Z_i \eta(Z_1 \cdots Z_i \cdots Z_{p+1}) \\ & + \sum_{i<j} (-1)^{i+j} \eta([Z_i, Z_j], Z_1 \cdots \hat{Z}_i \cdots \hat{Z}_j \cdots Z_{p+1})\}_{g_0} \end{aligned}$$

(ρ has a natural extension to g^C hence to u_1)
It follows that if we identify germs of holomorphic W-valued forms on $\Gamma \backslash G$ with germs of holomorphic F-valued forms on $\Gamma \backslash G$ through the isomorphism Φ_0, the operator d is transformed into the operator d_0 defined by

$$\begin{aligned} d_0 \eta(Z_1, \cdots, Z_{p+1}) = & \sum_{i=1}^{p+1} (-1)^{i+1} (Z_i + \rho(Z_i)) \eta(Z_1, \cdots, \hat{Z}_i, \cdots, Z_{p+1}) \\ & + \sum_{i<j} (-1)^{i+j} \eta([Z_i, Z_j], Z_1 \cdots \hat{Z}_i \cdots \hat{Z}_j \cdots Z_{p+1}) \cdots\cdots\cdots\cdots \text{(I)} \end{aligned}$$

Now the map which associates to each W_ρ-valued holomorphic p-form η, the F-valued holomorphic form $\Phi_0(\eta)$ defined by

$$(\Phi_0 \eta)(Z_1, \cdots, Z_p) = \Phi_0(\eta(Z_1, \cdots, Z_p))$$

for every p-tuple $(Z_1, \cdots, Z_p)$ of projections of left invariant holomorphic vectorfields on G, defines an isomorphism Φ_p of the sheaf $\underline{\Omega}^p \otimes_{\mathcal{O}} \underline{W}_\rho$ on the sheaf $\mathrm{Hom}_C(\overset{p}{\wedge} \mathfrak{u}_1, \mathcal{O} \otimes_C F)$. Moreover clearly the diagram

$$\begin{array}{ccc}
\underline{\Omega}^p \underset{\mathcal{O}}{\otimes} \underline{W}_\rho & \xrightarrow{\Phi_p} & \mathrm{Hom}_C(\Lambda^p \mathfrak{u}_1, \mathcal{O} \underset{C}{\otimes} F) \\
\downarrow d & & \downarrow d_0 \\
\underline{\Omega}^{p+1} \underset{\mathcal{O}}{\otimes} \underline{W}_\rho & \xrightarrow{\Phi_{p+1}} & \mathrm{Hom}_C(\Lambda^{p+1} \mathfrak{u}_1, \mathcal{O} \underset{C}{\otimes} F)
\end{array}$$

where d_0 is defined by equation (I) above, is commutative. Now $\mathcal{O}$ is a sheaf of $\mathfrak{u}_1$-modules: the map $f \longrightarrow Zf$ for the projection on X of a left invariant holomorphic vectorfield Z on G defines a representation $\mathfrak{u}_1 (\simeq \mathfrak{g})$ in the Lie algebra of endomorphism of $\mathcal{O}$. The stalks at a point $x \in X$ of the complex of sheaves

$$0 \to \mathcal{O} \underset{C}{\otimes} F \to \mathrm{Hom}_C(\mathfrak{u}_1, \mathcal{O} \underset{C}{\otimes} F) \to \cdots \to \mathrm{Hom}(\Lambda^n \mathfrak{u}_1, \mathcal{O} \underset{C}{\otimes} F) \to 0$$

from then clearly the standard complex of the Lie algebra u with values in $\mathcal{O}_x \otimes F$, where $\mathcal{O}_x$ is the stalk at x of $\mathcal{O}$. Passing then to the q^{th}-cohomology groups of this sheaves, we see that, we obtain the standard complex

$$0 \to H^q(X, \mathcal{O}) \underset{C}{\otimes} F \to \mathrm{Hom}_C(\mathfrak{u}_1, H^q(X, \mathcal{O}) \underset{C}{\otimes} F) \cdots \mathrm{Hom}_C(\Lambda^n \mathfrak{u}_1, H^q(X, \mathcal{O}) \underset{C}{\otimes} F) \to 0$$

where $H^q(X, \mathcal{O})$ carries the $\mathfrak{u}_1$-module structure defined by the action of $\mathfrak{u}_1$ on $\mathcal{O}$ defined above and $H^q(X, \mathcal{O}) \otimes F$ is the tensor product of this representation and ρ.

Combining the preceding, with Proposition 1, we obtain

Theorem 2. *Let G be a connected complex Lie group and Γ a discrete subgroup. Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on $X = \Gamma \backslash G$. Let ρ be a representation of G in a finite dimensional complex vector space F and L_ρ the associated local system. Then there is a convergent spectral sequence $\{E_r\}_{0 \le r \le \infty}$ converging to $H^*(X, L_\rho)$ such that $E_2^{pq} = H^p(g, H^q(X, \mathcal{O}) \underset{C}{\otimes} F)$ where $H^q(X, \mathcal{O})$ and F are considered as g-modules as follows: a left-invariant vectorfield Y on G projects on X as a vectorfield whose 1-parameter group is a group of holomorphic automorphisms of X; hence $f \longrightarrow Xf$ defines an endomorphism of $\mathcal{O}$ and hence a representation of g; in F we have the representation ρ.*

Proof. The argument above is incomplete only in two details, under the isomorphism $\mathfrak{g} \xrightarrow{p_1} \mathfrak{u}_1$, we must show the following:

i) If ρ^C is the extension to $\mathfrak{g}$ of ρ, then $\rho^C \circ p_1$ and ρ are equivalent.

ii) $Xf = p_1(X) \cdot f$

The former is a well known fact; the latter follows from the fact that if $p_2 \colon \mathfrak{g} \to \mathfrak{u}_2$ is the projection onto antiholomorphic vectorfields, then, $p_2(X) f = 0$ for holomorphic f.

A corollary is the following

Theorem 3. *Let G be a connected complex semisimple Lie group and Γ a* *discrete subgroup such that $\Gamma\backslash G$ is compact. Then, $H^1(\Gamma\backslash G, \mathcal{O})$ where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on $\Gamma\backslash G$ vanishes provided that G has no 3-dimensional components.*

Proof. Since $\Gamma\backslash G$ is compact $H^q(X, \mathcal{O})$ are finite dimensional so that, in view of the Whitehead Lemma for semisimple Lie algebras, we have, for any finite dimensional representation ρ of G in a vector space F, in the spectral sequence of Theorem 2

$$E_2^{10} = E_2^{20} = 0\,. \quad \text{On the other hand,}$$
$$E_\infty^{01} = E_3^{01}$$

is the homology of

$$0 \to E_2^{01} \to E_2^{20} = 0$$

Hence $E_\infty^{01} = E_2^{01} = H^0(\mathfrak{g}, H^1(X, \mathcal{O}) \otimes_C F)$. Now if $H^1(X, \mathcal{O}) \neq 0$, and if we choose F to be the dual of this module, then, $H^0(\mathfrak{g}, H^1(X, \mathcal{O}) \otimes F) \neq 0$. On the other hand since the spectral sequence converges to $H^*(X, L_\rho)$, this implies that $H^1(X, L_\rho) \neq 0$. But according to [1a] and [4] under the hypothesis of the theorem, viz., that G has no 3-dimensional components, $H^1(X, L_\rho)=0$, a contradiction. Hence the theorem.

Corollary. *If $\Gamma \subset G$ is a discrete subgroup of a connected complex semisimple Lie group G such that $\Gamma\backslash G$ is compact, then the natural complex structure on $\Gamma\backslash G$ is locally rigid.*

Proof. $\Gamma\backslash G$ is holomorphically parallelisable. Hence the sheaf Θ of germs of holomorphic vectorfields is isomorphic to a direct sum of copies of $\mathcal{O}$. From Theorem 3, therefore, $H^1(\Gamma\backslash G, \Theta)=0$. It is well known that this last implies that the complex structure is locally rigid.

REMARK. Reverting to the notation of § 1, when $K\backslash G$ is hermitian symmetric, Matsushima and Murakami have given a type decomposition

$$H^q(\Gamma, X, \rho) \simeq \sum_{r+s=q} H^{rs}(\Gamma, X, \rho)\,.$$

The groups $H^{rs}(\Gamma, X, \rho)$ have an interpretation in terms of the spectral sequence of Proposition 1 of this section. In fact, according to proposition 1, there is a spectral sequence converging to $H^*(\Gamma, X, \rho)$ with E_1^{pq} as $H^q(X, \underline{\Omega} \otimes_{\mathcal{O}} \underline{W}_\rho)$. A simple calculation using Lemma 4.1 of [3] shows that E_2^{pq} is isomorphic to $H^{pq}(\Gamma, X, \rho)$ and that the spectral sequence degenerates from the E_2 stage onwards.

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