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VANISHING THEOREMS FOR COHOMOLOGY GROUPS ASSOCIATED TO DISCRETE SUBGROUPS OF SEMISIMPLE LIE GROUPS

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Introduction. The aim of this paper is to prove two vanishing theorems for cohomology groups related to discrete uniform subgroups of semisimple Lie groups.

Let ρ be a representation of a real linear semisimple Lie group G and Γ a discrete subgroup of G such that $\Gamma \backslash G$ is compact. Assume that Γ contains no elements of finite order. In §1 we give a criterion in terms of the highest weight of ρ for the vanishing of $H^p(\Gamma, \rho)$, the p^{th} cohomology group of Γ with coefficient in ρ . This criterion is a generalisation of a theorem of Matsushima and Murakami [3].

In §2 we prove the following theorem (Corollary to Theorem 3). Let G be a complex semisimple Lie group without any simple component of rank 1. Then for any discrete subgroup Γ such that $\Gamma \backslash G$ is compact, the canonical complex structure on the space $\Gamma \backslash G$ is rigid. (This question whether these complex structures are rigid was raised by Professor Matsushima).

1. A vanishing theorem for the cohomology of discrete uniform subgroups

Let G be a connected real linear semisimple Lie group and Γ a discrete subgroup such that the quotient $\Gamma \backslash G$ is compact. Let \mathfrak{g}_0 be the Lie algebra of left-invariant vector-fields of G and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ a Cartan-decomposition of \mathfrak{g}_0 , \mathfrak{k}_0 being the algebra. Let K be the (compact) Lie subgroup corresponding to \mathfrak{k}_0 and $X = G/K$ the corresponding symmetric space. To every representation of G in a finite dimensional real (or complex) vector space F , Matsushima and Murakami [2] have associated certain cohomology groups: we follow their notation and denote these groups by $H^p(\Gamma, X, \rho)$. (In the case when Γ has no elements of finite order Γ acts freely on X and $H^p(\Gamma, X, \rho)$ is isomorphic to the p^{th} cohomology group of Γ with coefficients in the restriction ρ_Γ of ρ to Γ). In the same article, they prove moreover the following result (see in particular §6, §7). (Proposition 1 below).

The vectorfields in \mathfrak{g}_0 project under the natural map $G \rightarrow \Gamma \backslash G$ into vectorfields on $\Gamma \backslash G$. We will from now on identify \mathfrak{g}_0 with this algebra of vectorfields on $\Gamma \backslash G$. Let φ be the Killing form on \mathfrak{g}_0 and $\{X_i\}_{1 \leq i \leq N}$ and $\{X_\alpha\}_{N+1 \leq \alpha \leq n}$ be bases of \mathfrak{p}_0 and \mathfrak{k}_0 such that $\varphi(X_i, X_j) = \delta_{ij}$ and $\varphi(X_\alpha, X_\beta) = -\delta_{\alpha\beta}$. Let $A_0(\Gamma, X, \rho)$ be the vector space of C^∞ - p -forms η on $\Gamma \backslash G$ satisfying i) $i_X \eta = 0$ and ii) $\theta_X \eta = \rho(X) \eta$ for every $X \in \mathfrak{k}_0$ where i_X (resp θ_X) denotes interior derivation (resp. Lie derivation) of η with respect to the vectorfield X . Because of i) and ii) η is determined by its values $i_1 \cdots i_p = \eta(X_{i_1} \cdots X_{i_p})$. Finally, let Δ^p be the operator

$$\Delta^p: A_0^p(\Gamma, X, \rho) \rightarrow A_0^p(\Gamma, X, \rho)$$

defined by

$$\begin{aligned} \Delta^p \eta(X_{i_1} \cdots X_{i_p}) &= \sum_{k=1}^N (-X_k^2 + \rho(X_k)^2) \eta_{i_1 \cdots i_p} \\ &+ \sum_{k=1}^N \sum_{u=1}^p (-1)^{u-1} \{(-[X_{i_u}, X_k] + \rho([X_{i_u}, X_k]))\} \eta_{k i_1 \cdots \hat{i}_u \cdots i_p} \end{aligned}$$

With this notation, we have

Proposition 1. $H^p(\Gamma, X, \rho)$ is canonically isomorphic to the vector space $\{\eta \mid \eta \in A_0^p(\Gamma, X, \rho); \Delta^p \eta = 0\}$.

Again, following [2], we define two operators Δ_D^p and Δ_ρ^p as follows:

$$\begin{aligned} \Delta_D^p \eta(X_{i_1} \cdots X_{i_p}) &= -\sum_{k=1}^N X_k^2 \eta_{i_1 \cdots i_p} + \sum_{k=1}^N \sum_{u=1}^p (-1)^u [X_{i_u}, X_k] \eta_{k i_1 \cdots \hat{i}_u \cdots i_p} \\ \Delta_\rho^p \eta(X_{i_1} \cdots X_{i_p}) &= +\sum_{k=1}^N \rho(X_k)^2 \eta_{i_1 \cdots i_p} - \sum_{k=1}^N \sum_{u=1}^p (-1)^u \rho([X_{i_u}, X_k]) \eta_{k k i_1 \cdots \hat{i}_u \cdots i_p} \end{aligned}$$

Then $\Delta^p = \Delta_D^p + \Delta_\rho^p$. In §7 [2], it is moreover proved that

$$\sum_{i_1 < \cdots < i_p} \int_{\Gamma/G} \langle (\Delta_D^p \eta)_{i_1 \cdots i_p}, \eta_{i_1 \cdots i_p} \rangle_F \geq 0$$

where $\langle \cdot, \cdot \rangle_F$ is a positive definite scalar product on F for which $\rho(X)$ is (hermitian) symmetric (resp. skew-symmetric (hermitian)) for $X \in \mathfrak{p}_0$ (resp. \mathfrak{k}_0). It follows therefore that if $\Delta^p \eta = 0$,

$$\sum_{i_1 < \cdots < i_p} \int_{\Gamma/G} \langle (\Delta_\rho^p \eta)_{i_1 \cdots i_p}, \eta_{i_1 \cdots i_p} \rangle_F \geq 0$$

We obtain therefore

Proposition 2. If the quadratic form on the space of exterior p -forms on \mathfrak{p}_0 with values in F defined by

$$\eta \rightarrow \sum_{i_1 < \cdots < i_p} \langle (\Delta_\rho^p \eta)_{i_1 \cdots i_p}, \eta_{i_1 \cdots i_p} \rangle_F$$

is positive definite, then $H^p(\Gamma, X, \rho) = 0$.

In the main result of this section we give a sufficient criterion in terms of the “highest weight” of ρ with respect to a suitable Cartan-subalgebra of \mathfrak{g}_0 in order that Δ_ρ^p define a positive definite quadratic form.

Let \mathfrak{g} denote the complexification of \mathfrak{g}_0 and \mathfrak{k} and \mathfrak{p} those of \mathfrak{k}_0 and \mathfrak{p}_0 . We identify \mathfrak{k} and \mathfrak{p} with subspaces of \mathfrak{g} . Let $\mathfrak{h}_{\mathfrak{k}_0}$ be a Cartan-subalgebra of \mathfrak{k}_0 and \mathfrak{h}_0 a Cartan-subalgebra of \mathfrak{g}_0 such that $\mathfrak{h}_0 \supset \mathfrak{h}_{\mathfrak{k}_0}$. Let $\mathfrak{h}_{\mathfrak{p}_0} = \mathfrak{h}_0 \cap \mathfrak{p}_0$. Let \mathfrak{h} and $\mathfrak{h}_{\mathfrak{p}}$ denote respectively the complexifications of $\mathfrak{h}_{\mathfrak{k}_0}$, \mathfrak{h}_0 and $\mathfrak{h}_{\mathfrak{p}_0}$. Then \mathfrak{h} is a Cartan-subalgebra of \mathfrak{g} . Let Δ be the system of roots of \mathfrak{g} with respect to \mathfrak{h} . For $\alpha \in \Delta$ let $H_\alpha \in \mathfrak{h}$ be the unique element such that $\varphi(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. Then, it is well known that the real subspace $\mathfrak{h}^* = \sum_{\alpha \in \Delta} R H_\alpha$ of \mathfrak{g} spanned by the $\{H_\alpha\}_{\alpha \in \Delta}$ is the same as $i\mathfrak{h}_{\mathfrak{k}_0} \oplus \mathfrak{p}_0$. Moreover if θ is the extension to \mathfrak{g} to the Cartan involution θ_0 denoted by the Cartan-decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, then θ is an automorphism of \mathfrak{g} leaving \mathfrak{h} invariant. Hence θ acts on the dual of \mathfrak{h} and permutes the elements of Δ . The set Δ may then be decomposed as the disjoint union $A \cup B \cup C$ of three subsets A , B and C

where

$$A = \{\alpha \mid \alpha \in \Delta; \theta(\alpha) = \alpha; \theta(E_\alpha) = E_\alpha\}$$

$$B = \{\alpha \mid \alpha \in \Delta; \theta(\alpha) \neq \alpha\}$$

$$C = \{\alpha \mid \alpha \in \Delta; \theta(\alpha) = \alpha; \theta(E_\alpha) = -E_\alpha\}.$$

(In the sequel we sometimes write α^θ for $\theta(\alpha)$).

We introduce next a lexicographic order on the (real) dual of \mathfrak{h}^* as follows: let H_1, \dots, H_r be an orthonormal basis of \mathfrak{h}^* with respect to φ ($\varphi|_{\mathfrak{h}^*}$ is positive definite) chosen so that H_1, \dots, H_r form a basis of $i\mathfrak{h}_{\mathfrak{k}_0}$ and if the centre \mathfrak{c}_0 of \mathfrak{k}_0 is non-zero, of dimension r , then H_1, \dots, H_r belong to $i\mathfrak{c}_0$; for α, β in the (real) dual of \mathfrak{h}^* , $\alpha > \beta$ if the first non-vanishing difference $\alpha(H_i) - \beta(H_i)$ is greater than zero. Let Δ^+ be the system of positive roots with respect to this order and let $A^+ = A \cap \Delta^+$, $B^+ = B \cap \Delta^+$, $C^+ = C \cap \Delta^+$. Then θ leaves A^+ , B^+ and C^+ invariant. Let $\sum_1 = A^+ U \{\alpha \mid \alpha \in B^+; \theta(\alpha) > \alpha\}$ and $\sum_2 = C^+ U \{\alpha \mid \alpha \in B^+; \theta(\alpha) > \alpha\}$.

Theorem 1. *Let ρ denote a finite dimensional representation of G in a complex vector-space F , as also the induced representation of \mathfrak{g} . Let Λ_ρ be the highest weight of ρ with respect to the above defined Cartan-subalgebra and the order on the dual of \mathfrak{h}^* . Then if $\sum_\rho = \{\alpha \mid \alpha \in \sum_2, \varphi(\Lambda_\rho, \alpha) \neq 0\}$ contains more than q elements, then the Hermitian quadratic form Q_ρ defined by*

$$\eta \rightarrow \sum_{i_1 < \dots < i_p} \langle (\Delta_\rho^p \eta)_{i_1 \dots i_p}, \eta_{i_1 \dots i_p} \rangle_F$$

is positive definite for $p \leq q$. Hence $H^p(\Gamma, X, \rho) = 0$ for $1 \leq p \leq q$.

Before we proceed to the proof of the theorem, we will make a few preliminary simplifications:

Lemma 1. *Let E be the q^{th} exterior power of p and let α be the isomorphism onto $F \otimes E$ of the space of exterior q -forms on p with values in F defined by*

$$\eta \rightarrow \sum_{i_1 < \dots < i_q} \eta_{i_1 \dots i_q} \otimes (X_{i_1} \wedge \dots \wedge X_{i_q})$$

Then

$$T_p^q = 2\alpha \circ \Delta_p^q \circ \alpha^{-1} = 2(\rho \otimes 1)(c) + (1 \otimes \sigma)(c') - (\rho \otimes 1)(c') - (\rho \otimes \sigma)(c')$$

where

$$c = \sum_{i=1}^N X_i^2 - \sum_{\alpha=N+1}^n X_{\alpha}^2$$

and $c' = -\sum_{\alpha=N+1}^n X_{\alpha}^2$ are elements of the enveloping algebras of \mathfrak{g} and \mathfrak{k} and σ denotes the adjoint representation of \mathfrak{k} in E . Hence T_p^q is a symmetric endomorphism of $F \otimes E$ with respect to the scalar product

$$\begin{aligned} & \left\langle \sum_{i_1 < \dots < i_p} \eta_{i_1 \dots i_p} \otimes X_{i_1} \wedge \dots \wedge X_{i_p}, \sum_{j_1 < \dots < j_p} \eta_{j_1 \dots j_p} \otimes X_{j_1} \wedge \dots \wedge X_{j_p} \right\rangle \\ &= \sum_{i_1 < \dots < i_p} \langle \eta_{i_1 \dots i_p}, \eta_{i_1 \dots i_p} \rangle_F \end{aligned}$$

Proof. We have

$$(\Delta_p^q)_{i_1 \dots i_q} = \sum_{k=1}^N \rho(X_k)^2 \eta_{i_1 \dots i_q} + \sum_{k=1}^N \sum_{u=1}^q (-1)^{u-1} \rho([X_{i_u}, X_k]) \eta_{k i_1 \dots \hat{i}_u \dots i_q}$$

For every q -tuple $I_q = (i_1 < \dots < i_q)$, we write X_{I_q} for $X_{i_1} \wedge \dots \wedge X_{i_q}$. In this notation,

$$\alpha(\eta) = \sum_{I_q} \eta_{I_q} \otimes X_{I_q}$$

$$\begin{aligned} \frac{1}{2} T_p^q \alpha(\eta) &= \sum_{I_q} \left\{ \sum_{k=1}^N \rho(X_k)^2 \eta_{I_q} + \sum_{k=1}^N \sum_{u=1}^q (-1)^{u-1} \rho([X_{i_u}, X_k]) \eta_{k i_1 \dots \hat{i}_u \dots i_q} \right\} \otimes X_{I_q} \\ &= \sum_{I_q} \left\{ \sum_{k=1}^N \rho(X_k)^2 \eta_{I_q} + \sum_{J_q \Delta I_q = i_u j_v} (-1)^{u+v} \rho([X_{i_u}, X_{j_v}]) \eta_{J_q} \right\} \otimes X_{I_q} \\ &= \sum_{I_q} \left\{ \sum_{k=1}^N \rho(X_k)^2 \eta_{I_q} + \sum_{J_q \Delta I_q = i_u j_v} (-1)^{u+v} c_{i_u j_v}^{\alpha} \rho(X_{\alpha}) \eta_{J_q} \right\} \otimes X_{I_q} . \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma(X_{\alpha}) X_{J_q} &= \sum_{k=1}^n \sum_{u=1}^q (-1)^{v-1} c_{i_u j_v}^{\alpha} (X_k \wedge X_{j_1} \dots X_{j_v} \dots \wedge X_{j_q}) \\ &= \sum_{I_q \Delta J_q = j_v i_u} (-1)^{u+v} c_{j_v}^{i_u} X_{I_q} \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{2} T_p^q \alpha(\eta) &= \sum_{I_q} \sum_{k=1}^N \rho(X_k)^2 \eta_{I_q} \otimes X_{I_q} + \sum_{I_q} \rho(X_{\alpha}) \eta_{J_q} \otimes \sigma(X_{\alpha}) X_{J_q} \\ &= \left\{ \sum_{k=1}^N \rho(X_k)^2 \otimes 1 + \sum_{\alpha} \rho(X_{\alpha}) \otimes \sigma(X_{\alpha}) \right\} \alpha(\eta) \end{aligned}$$

Now the required result follows from the fact

$$\begin{aligned} 2\rho(X_\alpha) \otimes \sigma(X_\alpha) &= \{\rho(X_\alpha) \otimes 1 + 1 \otimes \sigma(X_\alpha)\}^2 - \rho(X_\alpha)^2 \otimes 1 - 1 \otimes \sigma(X_\alpha)^2 \\ &= (\rho \otimes \sigma)(X_\alpha)^2 - \rho(X_\alpha)^2 \otimes 1 - 1 \otimes \sigma(X_\alpha)^2 \end{aligned}$$

That T_ρ^q is a hermitian symmetric endomorphism follows from the facts that $\rho(X_i)$ and $\sigma(X_i)$ are hermitian symmetric while $\rho(X_\alpha)$ and $\sigma(X_\alpha)$ are skew-hermitian with respect to $\langle \cdot, \cdot \rangle_F$ and the extension to E of the Killing form on \mathfrak{p}_0 .

Lemma 2. a) *If Λ is the highest weight of an irreducible representation ρ of \mathfrak{g} induced by a representation ρ of G , then*

$$\rho(c) = \{\varphi(\Lambda, \Lambda) + \sum \varphi(\Lambda, \alpha)\}. \quad \text{Identity}$$

b) *when restricted to the (irreducible) K -subspace generated by the eigen-space corresponding to the highest weight Λ ,*

$$\rho(c') = \left\{ \frac{1}{4} \varphi(\Lambda + \Lambda^\theta, \Lambda + \Lambda^\theta) + \sum_{\alpha \in \Sigma_1} \varphi\left(\Lambda, \frac{\alpha + \alpha^\theta}{2}\right) \right\}. \quad \text{Identity.}$$

For a proof see [4]: Lemmas 4 and 16(c).

Lemma 3. *If Λ_1 and Λ_2 are the highest weights of two irreducible representations ρ_1 , ρ_2 of \mathfrak{g} , such that $\Lambda_1 - \Lambda_2$ is a non-negative linear combination of simple roots of \mathfrak{g} , then $\lambda_1 \geq \lambda_2$ where $\rho_k(c) = (\lambda_k. \text{ Identity})$ ($k=1, 2$). Equality can occur only if $\Lambda_1 = \Lambda_2$.*

The same conclusions hold for \mathfrak{k} and c' instead of \mathfrak{g} and c provided that Λ_1 and Λ_2 coincide on the center of \mathfrak{k} .

For the proof see Lemma 5 [4].

Proof of Theorem 1. We obtain the eigen-values of T_ρ^q as follows: Let

$$E = \sum_{\mu \in \mathfrak{M}} E_\mu \quad \text{and} \quad F = \sum_{\lambda \in L} F_\lambda \quad \text{and} \quad F_\lambda \otimes E_\mu = \sum_{\nu \in \mathfrak{M}_{\lambda\mu}} V_{\lambda\mu}^\nu$$

be the decomposition of E , F and $F_\lambda \otimes E_\mu$ into irreducible \mathfrak{k} -modules indexed by the highest weights (for the order defined by H_1, \dots, H_p on $i\mathfrak{h}_k$). Since ρ is an irreducible representation of \mathfrak{g} and c is a central element of $U(\mathfrak{g})$, $\rho(c)$ is a scalar operator. Similarly, since c' is central in $U(\mathfrak{k})$, $\rho(c') \otimes 1$, $1 \otimes \sigma(c')$ and $(\rho \otimes \sigma)(c')$ are scalars on F_λ , E , $F \otimes E_\lambda$ and $V_{\lambda\mu}^\nu$. Hence T_ρ^q acts as a scalar on each $V_{\lambda\mu}^\nu$. We denote the corresponding eigen-value by $a(\lambda, \mu, \nu)$. Among $V_{\lambda\mu}^\nu$ there is a unique irreducible component with highest weight $\nu = \lambda + \mu$ we denote the corresponding scalar $a(\lambda, \mu, \nu)$ by $a(\lambda, \mu)$ with this notation, we have

Assertion I. $a(\lambda, \mu, \nu) \geq a(\lambda, \mu)$; equality occurs only if $\nu = \lambda + \mu$.

Proof. We denote the representation in $V_{\lambda\mu}^\nu$ by $\rho_{\lambda\mu}^\nu$. Then since $(\rho \otimes 1)(c)$, $(\rho \otimes 1)(c')$ and $(1 \otimes \sigma)(c')$ all define the same scalar operator in $F_\lambda \otimes E_\mu$,

$$a(\lambda, \mu) + a(\lambda, \mu, \nu) = \rho_{\lambda\mu}^{\lambda+\mu}(c') - \rho_{\lambda\mu}^{\nu}(c')$$

(Here we have let $\rho_{\lambda\mu}^{\nu}(c')$ stand for the scalar). Now any weight in $F_{\lambda} \otimes E_{\mu}$ has the form $\lambda_1 + \mu_1$ where λ_1 and μ_1 are weights of F_{λ} and E_{μ} ; on the other hand $\lambda - \lambda_1$ and $\mu - \mu_1$ are non-negative linear combination of simple roots of k ; hence so is $(\lambda + \mu) - (\lambda_1 + \mu_1)$. It follows then from Lemma 3 that

$$a(\lambda, \mu) \geq a(\lambda, \mu, \nu)$$

Equality can occur only if $\lambda + \mu = \lambda_1 + \mu_1$ and there is only one component of $F_{\lambda} \otimes E_{\mu}$ with $\lambda + \mu$ as the highest weight. (Note that if \mathfrak{k} has a centre, then the central elements act as scalars on F_{λ} and E hence in all of $F_{\lambda} \otimes E_{\mu}$).

Assertion II. *Let f_{λ} be a highest weight vector of F such that $\|f_{\lambda}\|_F^2 = 1$. For $\alpha \in \Delta$, let E_{α} be a root vector of α . Suppose that $E_{\alpha_0}f_{\lambda} = 0$ for $\alpha \in A^+$. If there is an $\alpha_0 \in B^+$ with $E_{\alpha_0}f_{\lambda} \neq 0$, then $E_{\alpha_0}f_{\lambda} \in F_{\lambda_1}$ for some λ_1 and $a(\lambda, \mu) < a(\lambda_1, \mu_1)$*

Proof. Using the fact that θ is an involution, we have

$$\mathfrak{k} = \mathfrak{h}_{\mathfrak{k}} \oplus \sum_{\alpha \in A^+} \{CE_{\alpha} \oplus CE_{\alpha}\} \oplus \sum_{\substack{\alpha \in B^+ \\ \alpha > \alpha_0}} \{C(E_{\alpha} + E_{\alpha}\theta) \oplus C(E_{-\alpha} + E_{-\alpha}\theta)\}$$

and the order chosen on $\mathfrak{h}_{\mathfrak{k}}^* = i\mathfrak{h}_{\mathfrak{k}_0}$ has precisely $\{\alpha | \alpha \in A^+\}$ and $\left\{ \frac{\alpha + \alpha^{\theta}}{2} \mid \alpha \in B^+ \right\}$

as the positive roots. The roots of \mathfrak{k} are necessarily zero on the centre of \mathfrak{k} . It follows that the weights λ and $\lambda + \alpha_0$ (which is the weight corresponding to $E_{\alpha_0}f_{\lambda}$) have the same values on the centre. On the other hand, since $\lambda + \alpha_0$ and λ_1 are weights of the same irreducible representation of \mathfrak{k} , λ_1 and $\lambda + \alpha_0$ have the same values on the centre of \mathfrak{k} . It follows that $\lambda_1 = \lambda$ on the centre of \mathfrak{k} . Now $\lambda_1 - \lambda = \lambda_1 - (\lambda + \alpha_0) + \alpha_0$ and $\lambda_1 - (\lambda + \alpha_0)$ is a non-negative linear combination of simple roots. Hence $\lambda_1 - \lambda$ is a non-negative linear combination of simple roots and $\lambda_1 \neq \lambda$. A similar remark holds for $\lambda_1 + \mu$ and $\lambda + \mu$. It follows then from Lemma 3 above that

$$\rho_{\lambda}(c') < \rho_{\lambda_1}(c')$$

and

$$\rho_{\lambda\mu}^{\lambda+\mu}(c') < \rho_{\lambda_1\mu}^{\lambda_1+\mu}(c')$$

The operators $(\rho \otimes 1)(c)$ and $(1 \otimes \sigma)(c')$ on the other hand are scalars on the whole of $F \otimes E$. Hence from the expression for T_{ρ}^q , the Assertion follows.

Assertion III. *Suppose that $E_{\alpha}F_{\lambda} = 0$ for $\alpha \in A^+ \cup B^+$ but that there is an $\alpha_0 \in C^+$ such that $E_{\alpha_0}f_{\lambda} \neq 0$. Then $a(\lambda, \mu) > 0$.*

Proof. If $\{E_{\alpha}\}_{\alpha \in \Delta}$ are root vectors so chosen that $\varphi(E_{\alpha}, E_{-\alpha}) = 1$, then, it is well known that

$$c = \sum_{\alpha \in \Delta^+} E_\alpha E_{-\alpha} + \sum_{\alpha \in \Delta^+} E_{-\alpha} E_\alpha + \sum_{i=1}^1 H_i^2$$

It follows that

$$\rho(c)f_\lambda = \sum_{\alpha \in \Delta^+} \rho(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)f_\lambda + \sum_{i=1}^1 \rho(H_i)^2 f_\lambda$$

Using the facts, $E_\alpha f_\lambda = 0$ for $\alpha \in A^+ \cup B^+$ and that $[E_\alpha, E_{-\alpha}] = H_\alpha$, we have

$$\rho(c)f_\lambda = \sum_{\alpha \in A^+ \cup B^+} \lambda(H_\alpha)f_\lambda + \sum_{i=1}^p \lambda(H_i)^2 f_\lambda + \sum_{\alpha \in C^+} \rho(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)f_\lambda + \sum_{i=p+1}^1 \rho(H_i)^2 f_\lambda$$

Hence

$$\begin{aligned} \langle \rho(c)f_\lambda, f_\lambda \rangle_F &= \sum_{\alpha \in A^+ \cup B^+} \lambda(H_\alpha) + \sum_{i=1}^p \lambda(H_i)^2 + \sum_{\alpha \in C^+} \langle \rho(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)f_\lambda, f_\lambda \rangle \\ &\quad + \sum_{i=p+1}^1 \langle \rho(H_i)^2 f_\lambda, f_\lambda \rangle_F \end{aligned}$$

Now it is well known that F admits an orthogonal decomposition with respect to $\langle \cdot, \cdot \rangle_F$ into irreducible representations of the algebra $\mathfrak{g}' = CE_\alpha \oplus CE_{-\alpha} \oplus CH_\alpha$ for $\alpha \in C^+$ so that to prove that $\langle \rho(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)f_\lambda, f_\lambda \rangle \geq |\lambda(H_\alpha)|$ equality occurring only if $E_\alpha f_\lambda = 0$, we may assume that the \mathfrak{g}' -invariant subspace W spanned by f_λ is *irreducible* with respect to the three dimensional algebra. Now by Lemma 2,

$$\rho \left\{ E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha + \frac{H_\alpha^2}{\varphi(H_\alpha, H_\alpha)} \right\} f_\lambda = \left\{ \frac{(\lambda + k\alpha)(H_\alpha)^2}{\varphi(H_\alpha, H_\alpha)} + (\lambda + k\alpha)(H_\alpha) \right\} f_\lambda$$

where $\lambda + k\alpha$, $k \geq 0$ is the highest weight in W (of \mathfrak{g}'). Hence

$$\rho(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)f_\lambda = \frac{k\alpha(H_\alpha)^2}{\varphi(H_\alpha, H_\alpha)} + (\lambda + k\alpha)(H_\alpha)f_\lambda$$

so that

$$\langle \rho(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)f_\lambda, f_\lambda \rangle_F = (\lambda + k\alpha)(H_\alpha) + \frac{\alpha(H_\alpha)}{\varphi(H_\alpha, H_\alpha)} \geq |\lambda(H_\alpha)|$$

(It is well known that $(\lambda + k\alpha)(H_\alpha) \geq |\lambda(H_\alpha)|$ since $\lambda + k\alpha$ is the highest weight). Moreover equality occurs only if $k=0$; if $k=0$, however, λ is the highest weight so that $E_\alpha f_\lambda = 0$. We have thus shown that

$$\langle \rho(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)f_\lambda, f_\lambda \rangle \geq |\lambda(H_\alpha)|$$

equality occurring only if $E_\alpha f_\lambda = 0$. We have therefore,

$$\langle \rho(c)f_\lambda, f_\lambda \rangle \geq \sum_{\alpha \in A^+ \cup B^+} \lambda(H_\alpha) + \sum_{i=1}^p \lambda(H_i)^2 + \sum_{\alpha \in C^+} |\lambda(H_\alpha)| + \sum_{i=p+1}^1 \langle \rho(H_i)^2 f_\lambda, f_\lambda \rangle_F$$

equality occurring only if $E_\alpha f_\lambda = 0$ for all $\alpha \in C^+$. Moreover $S = \sum_{i=p+1}^1 \rho(H_i)^2$ is

a non-negative symmetric operator so that

$$\rho(c)f_\lambda, f_\lambda \geq \sum_{\alpha \in A^+ \cup B^+} |\lambda(H_\alpha)| + \sum \lambda(H_i)^2 + \langle Sf_\lambda, f_\lambda \rangle + \sum_{\alpha \in C^+} |\lambda(H_\alpha)|$$

with $S \geq 0$ (Note that for $\alpha \in A^+ \cup B^+$, $E_\alpha f_\lambda = 0$ so that $\lambda(H_\alpha) \geq 0$).

Using b) of Lemma 2, we have also

$$\begin{aligned} \rho(c') \otimes 1 \Big|_{F_\lambda \otimes E} &= \left\{ \sum_{i=1}^p \lambda(H_i)^2 + \sum_{\alpha \in \Sigma_1} \lambda(H_\alpha + H_{\alpha^\theta})/2 \right\}. \quad \text{Identity} \\ (\rho \otimes \sigma)(c') \Big|_{V_{\lambda, \mu}^{\lambda + \mu}} &= \sum_{i=1}^p (\lambda + \mu)(H_i)^2 + \sum_{\alpha \in \Sigma_1} (\lambda + \mu)(H_\alpha + H_{\alpha^\theta})/2. \quad \text{Identity} \end{aligned}$$

and

$$(1 \otimes \sigma)(c') \Big|_{F \otimes E_\mu} = \sum_{i=1}^p \mu(H_i)^2 + \sum_{\alpha \in \Sigma_1} \mu(H_\alpha + H_{\alpha^\theta})/2. \quad \text{Identity}$$

so that if $e_\mu \otimes E_\mu$ is a unit weight vector of weight μ ,

$$\begin{aligned} \langle T_p^q(f_\lambda \otimes e_\mu), f_\lambda \otimes e_\mu \rangle &\geq 2 \sum_{\substack{\alpha \in B^+ \\ \alpha > \alpha^\theta}} |\lambda(H_\alpha + H_{\alpha^\theta})/2| + 2 \sum_{\alpha \in C^+} \lambda(H_\alpha) \\ &\quad + 2 \langle S(f_\lambda), f_\lambda \rangle - 2 \sum_{i=1}^p \lambda(H_i) \mu(H_i) \end{aligned}$$

Now μ being a weight of σ_q it is the sum of q of the weights of the adjoint representation of k_0 in p_0 . Hence

$$\mu = \sum_{i=1}^q (\alpha_i + \alpha_2^\theta)/2$$

where all the α_i belong to Σ_2 . Hence

$$\langle T_p^q(f_\lambda \otimes e_\mu), f_\lambda \otimes e_\mu \rangle \geq 2 \sum_{\alpha \in \Sigma_2} \lambda(H_\alpha + H_{\alpha^\theta})/2 - 2 \sum_{i=1}^q \lambda(H_{\alpha_i} + H_{\alpha_i^\theta})/2$$

Here equality can occur only if $E_\alpha f_\lambda = 0$ for $\alpha \in \Delta^+$ and $\langle Sf_\lambda, f_\lambda \rangle = 0$. It follows therefore that $a(\lambda, \mu) > 0$ if there exists $\alpha_0 \in C^+$ with $E_{\alpha_0} f_\lambda \neq 0$.

In view of Assertions I, II and III, we see that T is positive definite if and only if $a(\lambda_0, \mu) > 0$ where λ_0 is the greatest of the dominant weights $\{\lambda \mid \lambda \in L\}$: this follows from the fact that $E_\alpha f_{\lambda_0} = 0$ for all $\alpha \in \Delta^+$ if and only if f_{λ_0} is the highest weight vector for ρ ; it follows that any weight of $\rho|_k$ is of the form $\lambda_0 - \sum m_i r(\alpha_i)$ where $m_i \geq 0$ and $r(\alpha_i)$ are the restriction of positive roots of \mathfrak{g} ; finally $r(\alpha_i) \neq 0$ hence greater than zero (see Lemma 16 (f) [4]).

Thus to complete the proof of the Theorem, we need only prove

Assertion IV. *If λ_0 is the restriction $r(\Lambda)$ of the highest weight Λ of ρ , then $a(\lambda_0, \mu) > 0$ for all $\mu \in M$ provided there are at least $(q+1)$ roots $\alpha \in \Sigma_2$ such that $\Lambda(H_\alpha + H_{\alpha^\theta}) > 0$.*

Proof. By evaluation on the highest weight $f_{\lambda_0} \otimes e_\mu$ we have (Lemma 2)

$$\begin{aligned} T_\rho(f_{\lambda_0} \otimes e_\mu) &= \{2 \sum_{\alpha \in \Sigma_2} \Lambda(H_\alpha + H_{\alpha^\theta})/2 + 2 \sum_{i=1}^p \Lambda(H_i)^2 - 2 \sum_{i=1}^p \Lambda(H_i)\mu(H_i)\} (f_{\lambda_0} \otimes e_\mu) \\ &= \{2 \sum_{\alpha \in \Sigma_2} \Lambda(H_\alpha + H_{\alpha^\theta})/2 - 2 \sum_{i=1}^q (H_{\alpha_i} + H_{\alpha_i^\theta})/2 + 2 \sum_{i=1}^q \Lambda(H_i)^2\} (f_{\lambda_0} \otimes e_\mu) \end{aligned}$$

where

$$\mu = r(\sum_{i=1}^q (\alpha_i + \alpha_i^\theta)/2). \quad \text{It follows that}$$

$$a(\lambda_0, \mu) > 0 \quad \text{under our hypothesis,}$$

since

$$\sum_{i=1}^p \Lambda(H_i)^2 \geq 0.$$

This completes the proof of the Theorem.

REMARK 1. Theorem 1 generalises Theorem 12.1 of [3] where only the case when G/K is hermitian symmetric, is considered. In fact, the present theorem is more general than Theorem 12.1 of [3] even in this case: $H^n(\Gamma, X, \rho)$ admits a type decomposition (see [3])

$$H^n(\Gamma, X, \rho) \simeq \bigoplus_{r+s=n} H^{rs}(\Gamma, X, \rho)$$

so that under the hypothesis of Theorem 1, we have

$$H^{rs}(\Gamma, X, \rho) = 0$$

for $r+s \leq q$. Theorem 12.1 of [3] is the special case $q = \dim G/K$. In section §2, we will give an interpretation of the groups $H^{rs}(\Gamma, X, \rho)$. In [4] all the representations for which T_ρ^1 is positive definite are determined.

REMARK 2. The author has checked in a number of *classical cases*, that if G is simple and non-compact and ρ is *any* nontrivial irreducible representation, then the number of elements in \sum_ρ is greater than or equal to the rank of the associated symmetric space.

2. Compact quotients of complex semisimple Lie groups

Let X be a complex manifold and $\tilde{X} \xrightarrow{\pi} X$ be the universal covering of X . Let Γ be the fundamental group of X acting fixed point free on \tilde{X} . Let ρ be a representation of Γ in a finite dimensional complex vector space. Let L_ρ denote the local system associated to ρ and W_ρ the holomorphic vector bundle associated to ρ . Let \underline{L}_ρ and \underline{W}_ρ denote respectively the sheaf of germs of sections of L_ρ and holomorphic sections of W_ρ . By the de Rham theorem, the cohomology groups $H^p(X, L_\rho)$ of X with coefficients in the local system L_ρ are the cohomology groups of the complex

$$A = \sum_p A^p(\Gamma, \tilde{X}, \rho)$$

defined as follows: $A^p(\Gamma, X, \rho)$ is the vector space of C^∞ -exterior p -forms η on X with values in F satisfying the condition

$$\eta(\gamma_*(t_1), \gamma_*(t_2), \dots, \gamma_*(t_p)) = \rho(\gamma)^{-1} \eta(t_1, \dots, t_p)$$

where t_1, \dots, t_p are tangent vectors to \tilde{X} and $\gamma_*(t)$ denotes the image by γ of the tangent vector t to X ; the boundary operator in the complex is the exterior differentiation of F -valued forms on \tilde{X} . The complex structure on X gives a decomposition of each of the space $A^p(\Gamma, \tilde{X}, \rho)$ as a direct sum $\sum_{r+s=p} A^{r,s}(\Gamma, \tilde{X}, \rho)$ according to the bidegree. Moreover $d=d'+d''$ where d' and d'' are of bidegree $(1, 0)$ and $(0, 1)$ respectively. This gives A a structure of a double complex. The term E_1^{pq} of the spectral sequence associated to this double complex is clearly the q^{th} cohomology of the complex

$$0 \rightarrow A^{p,0}(\Gamma, \tilde{X}, \rho) \rightarrow A^{p,1}(\Gamma, \tilde{X}, \rho) \rightarrow \dots \rightarrow A^{p,n}(\Gamma, X, \rho) \rightarrow 0$$

($n=\dim X$). Again, by the Dolbeault theorem, the q^{th} cohomology of this complex is $H^q(X, \underline{\Omega}^p \otimes \underline{W}_\rho)$ where $\underline{\Omega}^p$ is the holomorphic bundle of holomorphic p -forms, and $\underline{\Omega}^p \otimes \underline{W}$ is the sheaf of germs of holomorphic p -forms on X with coefficients in W . Moreover, the derivation d_1 in the term E_1 is clearly the map induced by the exterior differentiation

$$d: \underline{\Omega}^p \otimes \underline{W}_\rho \rightarrow \underline{\Omega}^{p+1} \otimes \underline{W}_\rho$$

(since we have $\underline{\Omega}^p \otimes \underline{W}_\rho \simeq \underline{\Omega}^p \otimes \underline{L}_\rho$, the operator d above makes sense: $\underline{\Omega}^p \otimes \underline{L}_\rho \rightarrow \underline{\Omega}^{p+1} \otimes \underline{L}_\rho$).

We have thus

Proposition 1. *There is a convergent spectral sequence $\{E_r^{pq}\}_{c \leq r \leq \infty}$ converging to $H^*(\Gamma, \tilde{X}, \rho)$ such that $E_1^{pq} = H^q(X, \underline{\Omega}^p \otimes \underline{W}_\rho)$ and d_1 is induced by the map $d: \underline{\Omega}^p \otimes \underline{W}_\rho \rightarrow \underline{\Omega}^{p+1} \otimes \underline{W}_\rho$.*

Now let $\tilde{X}=G$ be a simply connected complex Lie group and $\Gamma \subset G$ a discrete subgroup; then $X=\Gamma \backslash G$. Let \mathfrak{g} be the Lie algebra of left invariant vectorfields on G . (Then elements of \mathfrak{g} may be regarded as vectorfields on $\Gamma \backslash G$ as well). Let \mathfrak{g}^c denote the complexification of \mathfrak{g} . Then $\mathfrak{g}^c \simeq \mathfrak{u}_1 \oplus \mathfrak{u}_2$ where \mathfrak{u}_1 and \mathfrak{u}_2 are respectively the complex ideals of holomorphic and antiholomorphic left-invariant vectorfields. The natural projections $\mathfrak{g} \rightarrow \mathfrak{u}_1$ and $\mathfrak{g} \rightarrow \mathfrak{u}_2$ define isomorphisms of \mathfrak{g} on \mathfrak{u}_1 and \mathfrak{u}_2 respectively.

Suppose now that ρ is the restriction of a representation of G in a finite dimensional vector space F . In this special case we can compute the term E_2 as well.

In the first place, there is a canonical (holomorphic) isomorphism of the vector bundle W_ρ on X with the trivial bundle. In fact the vector bundle W_ρ is obtained as follows: the group Γ acts $G \times F$ by diagonal action:

$$\gamma(g, f) = (\gamma g, \rho(\gamma)f) \quad \text{for } \gamma \in \Gamma.$$

This is an (holomorphic) automorphism of the vector bundle $G \times F$ on itself covering the left translation by γ and hence this action defines a vector bundle on $\Gamma \backslash G$. Now let $\Phi: G \times F \rightarrow G \times F$ be the isomorphism

$$\Phi(g, f) = (g, \rho(g)^{-1}f)$$

Then

$$\Phi(\gamma g, \rho(\gamma)f) = (\gamma g, \rho(g)^{-1}f)$$

Hence Φ defines an isomorphism Φ_0 of W_p on the trivial bundle $X \times F$.

Now, for left-invariant holomorphic vectorfields Z_1, \dots, Z_{p+1} and a holomorphic p -form η with values in F ,

$$\begin{aligned} d\eta(Z_1, \dots, Z_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} Z_i \eta(Z_1, \dots, \hat{Z}_i, \dots, Z_{p+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \eta([Z_i, Z_j], Z_1 \cdots \hat{Z}_i \cdots \hat{Z}_j \cdots Z_{p+1}) \end{aligned}$$

It follows that

$$\begin{aligned} (\Phi d \Phi^{-1})(\eta)(Z_1, \dots, Z_{p+1})_{g_0} &= \sum_{i=1}^{p+1} (-1)^{i+1} \{ \rho(g_0)^{-1} Z_i \rho(g) \eta(Z_1, \dots, Z_i, \dots, Z_{p+1}) \}_{g_0} \\ &\quad + \sum_{i < j} (-1)^{i+j} \{ \rho(g_0)^{-1} ([Z_i, Z_j], Z_1 \cdots Z_i \cdots Z_j \cdots Z_{p+1}) \}_{g_0} \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(Z_i) \eta(Z_1 \cdots \hat{Z}_i \cdots Z_{p+1}) \\ &\quad + \sum_{i=1}^{p+1} (-1)^{i+1} Z_i \eta(Z_1 \cdots Z_i \cdots Z_{p+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \eta([Z_i, Z_j], Z_1 \cdots \hat{Z}_i \cdots \hat{Z}_j \cdots Z_{p+1}) \}_{g_0} \end{aligned}$$

(ρ has a natural extension to g^C hence to u_1)

It follows that if we identify germs of holomorphic W -valued forms on $\Gamma \backslash G$ with germs of holomorphic F -valued forms on $\Gamma \backslash G$ through the isomorphism Φ_0 , the operator d is transformed into the operator d_0 defined by

$$\begin{aligned} d_0 \eta(Z_1, \dots, Z_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} (Z_i + \rho(Z_i)) \eta(Z_1, \dots, \hat{Z}_i, \dots, Z_{p+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \eta([Z_i, Z_j], Z_1 \cdots \hat{Z}_i \cdots \hat{Z}_j \cdots Z_{p+1}) \dots \dots \dots \text{①} \end{aligned}$$

Now the map which associates to each W_p -valued holomorphic p -form η , the F -valued holomorphic form $\Phi_0(\eta)$ defined by

$$(\Phi_0 \eta)(Z_1, \dots, Z_p) = \Phi_0(\eta(Z_1, \dots, Z_p))$$

for every p -tuple (Z_1, \dots, Z_p) of projections of left invariant holomorphic vectorfields on G , defines an isomorphism Φ_p of the sheaf $\underline{\Omega}^p \otimes \underline{W}_p$ on the sheaf $\text{Hom}_C(\bigwedge^p \mathfrak{u}_1, \mathcal{O} \otimes F)$. Moreover clearly the diagram

$$\begin{array}{ccc}
 \Omega^p \otimes_{\mathcal{O}} W_{\rho} & \xrightarrow{\Phi_{\rho}} & \text{Hom}_C(\Lambda^p \mathfrak{u}_1, \mathcal{O} \otimes_{\mathcal{C}} F) \\
 \downarrow d & & \downarrow d_0 \\
 \Omega^{p+1} \otimes_{\mathcal{O}} W_{\rho} & \xrightarrow{\Phi_{\rho+1}} & \text{Hom}_C(\Lambda^{p+1} \mathfrak{u}_1, \mathcal{O} \otimes_{\mathcal{C}} F)
 \end{array}$$

where d_0 is defined by equation ① above, is commutative. Now \mathcal{O} is a sheaf of \mathfrak{u}_1 -modules: the map $f \rightarrow Zf$ for the projection on X of a left invariant holomorphic vectorfield Z on G defines a representation $\mathfrak{u}_1 (\simeq \mathfrak{g})$ in the Lie algebra of endomorphisms of \mathcal{O} . The stalks at a point $x \in X$ of the complex of sheaves

$$0 \rightarrow \mathcal{O} \otimes_{\mathcal{C}} F \rightarrow \text{Hom}_C(\mathfrak{u}_1, \mathcal{O} \otimes_{\mathcal{C}} F) \rightarrow \dots \rightarrow \text{Hom}(\Lambda^n \mathfrak{u}_1, \mathcal{O} \otimes_{\mathcal{C}} F) \rightarrow 0$$

from then clearly the standard complex of the Lie algebra \mathfrak{u} with values in $\mathcal{O}_x \otimes F$, where \mathcal{O}_x is the stalk at x of \mathcal{O} . Passing then to the q^{th} -cohomology groups of this sheaves, we see that, we obtain the standard complex

$$0 \rightarrow H^q(X, \mathcal{O}) \otimes_{\mathcal{C}} F \rightarrow \text{Hom}_C(\mathfrak{u}_1, H^q(X, \mathcal{O}) \otimes_{\mathcal{C}} F) \dots \text{Hom}_C(\Lambda^n \mathfrak{u}_1, H^q(X, \mathcal{O}) \otimes_{\mathcal{C}} F) \rightarrow 0$$

where $H^q(X, \mathcal{O})$ carries the \mathfrak{u}_1 -module structure defined by the action of \mathfrak{u}_1 on \mathcal{O} defined above and $H^q(X, \mathcal{O}) \otimes F$ is the tensor product of this representation and ρ .

Combining the preceding, with Proposition 1, we obtain

Theorem 2. *Let G be a connected complex Lie group and Γ a discrete subgroup. Let \mathcal{O} be the sheaf of germs of holomorphic functions on $X = \Gamma \backslash G$. Let ρ be a representation of G in a finite dimensional complex vector space F and L_{ρ} the associated local system. Then there is a convergent spectral sequence $\{E_r\}_{0 \leq r \leq \infty}$ converging to $H^*(X, L_{\rho})$ such that $E_2^{pq} = H^p(g, H^q(X, \mathcal{O}) \otimes F)$ where $H^q(X, \mathcal{O})$ and F are considered as g -modules as follows: a left-invariant vectorfield Y on G projects on X as a vectorfield whose 1-parameter group is a group of holomorphic automorphisms of X ; hence $f \rightarrow Xf$ defines an endomorphism of \mathcal{O} and hence a representation of g ; in F we have the representation ρ .*

Proof. The argument above is incomplete only in two details, under the isomorphism $g \xrightarrow{p_1} \mathfrak{u}_1$, we must show the following:

- i) If ρ^c is the extension to \mathfrak{g} of ρ , then $\rho^c \circ p_1$ and ρ are equivalent.
- ii) $Xf = p_1(X) \cdot f$

The former is a well known fact; the latter follows from the fact that if $p_2: \mathfrak{g} \rightarrow \mathfrak{u}_2$ is the projection onto antiholomorphic vectorfields, then, $p_2(X)f = 0$ for holomorphic f .

A corollary is the following

Theorem 3. *Let G be a connected complex semisimple Lie group and Γ a*

discrete subgroup such that $\Gamma \backslash G$ is compact. Then, $H^1(\Gamma \backslash G, \mathcal{O})$ where \mathcal{O} is the sheaf of germs of holomorphic functions on $\Gamma \backslash G$ vanishes provided that G has no 3-dimensional components.

Proof. Since $\Gamma \backslash G$ is compact $H^q(X, \mathcal{O})$ are finite dimensional so that, in view of the Whitehead Lemma for semisimple Lie algebras, we have, for any finite dimensional representation ρ of G in a vector space F , in the spectral sequence of Theorem 2

$$E_2^{10} = E_2^{20} = 0. \quad \text{On the other hand,} \\ E_\infty^{01} = E_3^{01}$$

is the homology of

$$0 \rightarrow E_2^{01} \rightarrow E_2^{20} = 0$$

Hence $E_\infty^{01} = E_2^{01} = H^0(g, H^1(X, \mathcal{O}) \otimes F)$. Now if $H^1(X, \mathcal{O}) \neq 0$, and if we choose F to be the dual of this module, then, $H^0(g, H^1(X, \mathcal{O}) \otimes F) \neq 0$. On the other hand since the spectral sequence converges to $H^*(X, L_\rho)$, this implies that $H^1(X, L_\rho) \neq 0$. But according to [1a] and [4] under the hypothesis of the theorem, viz., that G has no 3-dimensional components, $H^1(X, L_\rho) = 0$, a contradiction. Hence the theorem.

Corollary. *If $\Gamma \subset G$ is a discrete subgroup of a connected complex semisimple Lie group G such that $\Gamma \backslash G$ is compact, then the natural complex structure on $\Gamma \backslash G$ is locally rigid.*

Proof. $\Gamma \backslash G$ is holomorphically parallelisable. Hence the sheaf Θ of germs of holomorphic vectorfields is isomorphic to a direct sum of copies of \mathcal{O} . From Theorem 3, therefore, $H^1(\Gamma \backslash G, \Theta) = 0$. It is well known that this last implies that the complex structure is locally rigid.

REMARK. Reverting to the notation of §1, when $K \backslash G$ is hermitian symmetric, Matsushima and Murakami have given a type decomposition

$$H^q(\Gamma, X, \rho) \simeq \sum_{r+s=q} H^{rs}(\Gamma, X, \rho).$$

The groups $H^{rs}(\Gamma, X, \rho)$ have an interpretation in terms of the spectral sequence of Proposition 1 of this section. In fact, according to proposition 1, there is a spectral sequence converging to $H^*(\Gamma, X, \rho)$ with E_1^{pq} as $H^q(X, \underline{\Omega} \otimes \underline{W}_\rho)$. A simple calculation using Lemma 4.1 of [3] shows that E_2^{pq} is isomorphic to $H^{pq}(\Gamma, X, \rho)$ and that the spectral sequence degenerates from the E_2 stage onwards.

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