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NOTE ON MICROBUNDLES*)

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In [2] Lashof and Rothenberg have defined the css-group O and the Kan complex PD, and shown a certain exact sequence of abelian groups (Theorem (4.2)) which is fundamental to the studies of the PL-microbundles and smoothing.

In the present note we shall define a css-group H for the topological microbundles parallel to the css-group PL for the PL-microbundles (§1), and show an analogous exact sequence of abelian groups (§4) which seems to have some meaning to the study of the topological microbundles (§2, §3).

Our method is quite analogous to that of Lashof and Rothenberg [2], and Milnor [3], and is based on Heller's theory [1].

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0. Preliminaries

a) Directed systems of css-complexes.

Let Σ be a partially ordered set, i.e. a set in which we have a transitive relation $<$ defined for some (but not necessary all) pairs of elements. Σ is called a directed set if every pair of elements has a common successor: given σ and τ in Σ there is an element ρ in Σ satisfying $\sigma<\rho$ and $\tau<\rho$.

In the present note all css-complexes are supposed to satisfy Kan's extension condition unless otherwise stated.

Suppose to each element σ of Σ is assigned a css-complex[1] K_σ (css-group G_σ) and to each pair of elements $\sigma<\tau$ of Σ there corresponds a css-map $h_{\sigma\tau}$ of K_σ into K_τ (css-homomorphism $h_{\sigma\tau}$ of G_σ into G_τ) such that if $\rho<\sigma<\tau$ then

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1) For the theory of css-complexes, see for example Heller [1], Moore [5], Puppe [6].

$$h_{\rho\tau} = h_{\sigma\tau}\circ h_{\rho\sigma}\,.$$

A system of css-complexes (css-groups) of this sort is called a *directed system of* css-*complexes* (css-*groups*).

Given a directed system of css-complexes (css-groups), we can define naturally a new css-complex (css-group) called the *limit* css-*complex* K (css-*group* G) of the directed system. We shall denote $K=\varinjlim K_\sigma$ ($G=\varinjlim G_\sigma$).

Lemma 1. *Let* $\{K_\sigma, h_{\sigma\tau};\ \sigma, \tau\in\Sigma\}$, $\{L_\sigma, h_\tau;\ \sigma, \tau\in\Sigma\}$ *be directed systems of* css-*complexes. Suppose to each element* σ *of* Σ *is assigned a* css-*map* φ_σ *of* K_σ *into* L_σ *such that to each pair of elements* $\sigma<\tau$ *of* Σ *the following diagram*

$$\begin{array}{ccc} K_\sigma & \xrightarrow{\varphi_\sigma} & L_\sigma \\ {\scriptstyle h_{\sigma\tau}}\downarrow & & \downarrow{\scriptstyle h'_{\sigma\tau}} \\ K_\tau & \xrightarrow{\varphi_\tau} & L_\tau \end{array}$$

is commutative. Then, there exists a css-*map* φ *of* $K=\varinjlim K_\sigma$ *into* $L=\varinjlim L_\sigma$ *which corresponds an element* $\{k_\sigma\}$ *of* K *with representative* k_σ *to* $\{\varphi_\sigma(k_\sigma)\}$ *of* L. *If* φ_σ *is injective for each* $\sigma\in\Sigma$, *then the* css-*map* φ *is also injective. If* $\{K_\sigma, h_{\sigma\tau};\ \sigma, \tau\in\Sigma\}$, $\{L_\sigma, h'_\tau;\ \sigma, \tau\in\Sigma\}$ *are directed systems of* css-*groups and each* φ_σ *is a* css-*homomorphism, then the* css-*map* φ *is also a* css-*homomorphism.*

Proof. We shall prove the second assertion. Let $k=\{k_\sigma\}$, $k'=\{k'_\tau\}$ be elements of K such that $\varphi(k)=\varphi(k')$. Then there exists a common successor ρ of σ and τ such that $h'_{\sigma\rho}(\varphi_\sigma(k_\sigma))=h'_{\tau\rho}(\varphi_\tau(k'_\tau))$. The following diagrams are commutative:

$$\begin{array}{ccc} K_\sigma & \xrightarrow{\varphi_\sigma} & L_\sigma \\ {\scriptstyle h_{\sigma\rho}}\downarrow & & \downarrow{\scriptstyle h'_{\sigma\rho}} \\ K_\rho & \xrightarrow[\varphi_\rho]{} & L_\rho \end{array} \qquad \begin{array}{ccc} K_\tau & \xrightarrow{\varphi_\tau} & L_\tau \\ \downarrow{\scriptstyle h_{\tau\rho}} & & \downarrow{\scriptstyle h'_{\tau\rho}} \\ K_\rho & \xrightarrow[\varphi_\rho]{} & L_\rho \end{array}$$

Thus we have

$$\varphi_\rho\circ h_{\sigma\rho}(k_\sigma) = \varphi_\rho\circ h_{\tau\rho}(k'_\tau)\,.$$

Since φ_ρ is injective, we have $h_{\sigma\rho}(k_\sigma)=h_{\tau\rho}(k'_\tau)$. Thus we have $\{k_\sigma\}=\{k'_\tau\}$.

Let $\{G_\sigma, h_{\sigma\tau};\ \sigma, \tau\in\Sigma\}$, $\{H_\sigma, h'_{\sigma\tau};\ \sigma, \tau\in\Sigma\}$ be directed systems of css-groups, and for each $\sigma\in\Sigma$ H_σ is a css-subgroup of G_σ. Then, corresponding to each σ css-complex G_σ/H_σ, we have naturally a directed system of css-complexes $\{G_\sigma/H_\sigma, \bar{h}_{\sigma\tau}; \sigma, \tau\in\Sigma\}$. Let $G=\varinjlim G_\sigma$, $H=\varinjlim G_\sigma$, By Lemma 1, we can consider H as a css-subgroup of G. So we have a css-complex G/H. Then we have

Lemma 2. *$\varinjlim G_\sigma/H_\sigma$ and G/H are css-equivalent, that is, there exists a bijective css-map between them.*

Proof. Let $K=\varinjlim G_\sigma/H_\sigma$. Define $\varphi_q: K^{(q)}\to(G/H)^{(q)}$ by $\varphi_q(g)=\{g_\sigma\}$ mod $H^{(q)}$, for $g=\{g_\sigma \bmod H_\sigma^{(q)}\}\in K^{(q)}$. This is independent of the representative of g.

Clearly φ_q is surjective.

Let g, $g'\in K^{(q)}$, $g=\{g_\sigma \bmod H_\sigma^{(q)}\}$, $g'=\{g'_\tau \bmod H_\tau^{(q)}\}$ and $\varphi_q(g)=\varphi_q(g')$. Then we have

$$\{g_\sigma\} \bmod H^{(q)} = \{g'_\tau\} \bmod H^{(q)},$$

that is, there exists a common successor ρ of σ and τ such that

$$(h_{\sigma\rho}(g_\sigma))^{-1}h_{\tau\rho}(g_\tau)\in H_\rho^{(q)}.$$

Namely

$$h_{\sigma\rho}(g_\sigma) \bmod H_\rho^{(q)} = h_{\tau\rho}(g_\tau) \bmod H_\rho^{(q)}.$$

Thus we have $g=g'$.

For a weakly monotone map $\lambda: \Delta_p\to\Delta_q$, we can easily see that the following diagram

$$\begin{array}{ccccccc}\varinjlim G_\rho^{(q)}/H_\sigma^{(q)} & = & K^{(q)} & \xrightarrow{\varphi_q} & (G/H)^{(q)} & = & G^{(q)}/H^{(q)} \\ & & \downarrow{\scriptstyle\lambda^\#} & & \downarrow{\scriptstyle\lambda^\#} & & \\ \varinjlim G_\sigma^{(p)}/H_\sigma^{(p)} & = & K^{(p)} & \xrightarrow{\varphi_p} & (G/H)^{(p)} & = & G^{(p)}/H^{(p)}\end{array}$$

is commutative. Thus $\varphi=\{\varphi_q\}: K\to G/H$ is a surjective css-map.

We shall sometimes identify two css-equivalent css-complexes.

b) Heller's U-functor.

We shall recall Heller's theory [1]. If Γ is a css-group, a *universal group for* Γ is a css-group Υ containing Γ as a css-subgroup and with all homotopy groups $\pi_q(\Upsilon)=0$. For any css-group Γ, there corresponds a css-group $U(\Gamma)$, which is universal for Γ. Moreover, U is a covariant functor on the category of css-groups and css-homomorphisms into itself.

Explicitly, the css-group $U(\Gamma)$ is constructed as follows. Le $U(\Gamma)^{(q)}$ be the set of all map σ of css-complex Δ_q into css-group Γ preserving dimension but not in general incidence. The incidence operations are defined by composition of maps

$$\Delta_p \xrightarrow{\lambda} \Delta_q \longrightarrow \Gamma$$

for a weakly monotone map λ. The group operation in $U(\Gamma)^{(q)}$ is defined by that in Γ : if $\tau \in \Delta_q$ and σ, $\sigma' \in U(\Gamma)^{(q)}$, then

$$(\sigma\sigma')(\tau) = \sigma(\tau)\sigma'(\tau) .$$

With these definitions it is clear that $U(\Gamma) = \bigcup_{q \geqq 0} U(\Gamma)^{(q)}$ is a css-group. Γ may be identified with the subgroup of $U(\Gamma)$ consisting of those simplices which are css-maps $\sigma : \Delta_q \to \Gamma$. We shall denote the identification by

$$\iota_\Gamma : \Gamma \to U(\Gamma) .$$

Let Γ, Γ' be css-groups and $\varphi : \Gamma \to \Gamma'$ be a css-homomorphism. Then the css-homomorphism

$$U(\varphi) : U(\Gamma) \to U(\Gamma')$$

is defined as follows. Let $\sigma \in U(\Gamma)^{(q)}$. We define $U(\varphi)(\sigma) \in U(\Gamma')^{(q)}$ to be $\varphi \circ \sigma$. Then $U(\varphi)$ is a dimension preserving map. For a weakly monotone map $\lambda : \Delta_p \to \Delta_q$, and $\tau \in U(\Gamma)^{(q)}$

$$\begin{aligned} \lambda^{\#} \circ U(\varphi)(\tau) &= \lambda^{\#}(\varphi \circ \tau) \\ &= (\varphi \circ \tau) \circ \lambda \\ &= \varphi \circ (\tau \circ \lambda) \\ &= \varphi \circ \lambda^{\#}(\tau) \\ &= U(\varphi) \circ \lambda^{\#}(\tau) . \end{aligned}$$

Thus $U(\varphi)$ is a css-map, and clearly css-homomorphism.

By the definition, if φ is a css-monomorphism, the $U(\varphi)$ is also a css-monomorphism.

Now let Γ, Γ' be css-groups. Then $\Gamma \times \Gamma'$ is also css-groups. Then we have

Lemma 3. *There exists a* css-*isomorphism* $\alpha : U(\Gamma) \times U(\Gamma') \to U(\Gamma \times \Gamma')$ *such that the following diagram*

$$\begin{array}{ccc} \Gamma\times\Gamma' & \xrightarrow{\iota_\Gamma\times\iota_{\Gamma'}} & U(\Gamma)\times U(\Gamma') \\ & \searrow^{\iota_{\Gamma\times\Gamma'}} & \wr\!\!\parallel\downarrow\alpha \\ & & U(\Gamma\times\Gamma') \end{array}$$

is commutative.

Proof. Define

$$\alpha_q : U(\Gamma)^{(q)}\times U(\Gamma')^{(q)} \to U(\Gamma\times\Gamma')^{(q)}$$
$$\alpha_q(\sigma,\sigma') = \tau,$$
$$\tau(\omega) = (\sigma(\omega),\sigma'(\omega)),\ \omega\in\Delta_q .$$

Then α_q is clearly an injective map.

Let $\lambda:\Delta_p\to\Delta_q$ be a weakly monotone map. Then the following diagram

$$\begin{array}{ccc} U(\Gamma)^{(q)}\times U(\Gamma')^{(q)} & \xrightarrow{\alpha_q} & U(\Gamma\times\Gamma')^{(q)} \\ \lambda^{\#}\downarrow & & \downarrow\lambda^{\#} \\ U(\Gamma)^{(p)}\times U(\Gamma')^{(p)} & \xrightarrow{\alpha_p} & U(\Gamma\times\Gamma')^{(p)} \end{array}$$

is commutative. Thus $\alpha=\{\alpha_q\}: U(\Gamma)\times U(\Gamma')\to U(\Gamma\times\Gamma')$ is an injective css-map.

Let $(\sigma,\sigma'),(\rho,\rho')\in U(\Gamma)^{(q)}\times U(\Gamma')^{(q)}$, and $\alpha_q(\sigma,\sigma')=\tau$, $\alpha_q(\rho,\rho')=\tau'$. Then $(\sigma,\sigma')(\rho,\rho')=(\sigma\rho,\sigma'\rho')$. Let $\alpha_q(\sigma\rho,\sigma'\rho')=\tau''$. Then we can prove easily

$$\tau''(\omega) = (\tau\tau')(\omega), \quad \text{for} \quad \omega\in\Delta_q .$$

Thus α is a css-monomorphism, Clearly α is surjective.

Then commutativity is easily seen.

Lemma 4. *Let $\{\Gamma_m, h_{m,n}; m, n\in Z\}$ be a directed system of* css-*groups and $\Gamma=\varinjlim\Gamma_m$. Then $\{U(\Gamma_m), U(h_{m,n}); m, n\in Z\}$ is also a directed system of* css-*groups, and*

$$\varinjlim U(\Gamma_m)\simeq U(\Gamma).$$

Proof. Define $\varphi:\varinjlim U(\Gamma_m)\to U(\Gamma)$ by $\varphi(\{\sigma^q_{(m)}\})=\iota_m\circ\sigma^q_{(m)}$, where $\iota_m: \Gamma_m\to\Gamma$ is the projection map and $\sigma^q_{(m)}:\Delta_q\to\Gamma_m$ is a representative of an element σ^q of $(\varinjlim U(\Gamma_m))^{(q)}$. Let $\sigma^q_{(n)}$ be another representative of σ^q: $\{\sigma^q_{(n)}\}=\{\sigma^q_{(m)}\}$. Then there exists an integer p such that $m, n\leqq p$, $h_{mp}\circ\sigma^q_{(m)}=h_{np}\circ\sigma^q_{(n)}$. Then

$$\begin{aligned}\iota_n\circ\sigma^q_{(n)} &= \iota_p\circ h_{np}\circ\sigma^q_{(n)}\\ &= \iota_p\circ h_{mp}\circ\sigma^q_{(m)}\\ &= \iota_m\circ\sigma^q_{(m)}\,.\end{aligned}$$

Thus the above definition has no ambiguity.

Clearly φ is an onto css-homomorphism.

Now we shall prove that φ is injective. Let $\varphi(\{\sigma^q_{(m)}\})=\varphi(\{\tau^q_{(n)}\})$. Then we have $\iota_m\circ\sigma^q_{(m)}=\iota_n\circ\tau^q_{(n)}$. Therefore, there exists an integer p such that $m,\ n\leqq p$ and $h_{mp}\circ\sigma^q_{(m)}=h_{np}\circ\tau^q_{(n)}$. Thus we have $\{\sigma^q_{(m)}\}=\{\tau^q_{(n)}\}$.

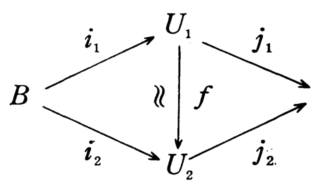
1. css-groups H_n, H

In this section we shall construct a css-group H_n for topological microbundles of dimension n. The construction of the css-group H_n is completely parallel to Milnor's construction [3] of the css-group PL_n for PL-microbundles of dimension n.

First we need to define the concept of an isomorphism-germ between topological microbundles. Let

$$\mathfrak{x}_\alpha : B \xrightarrow{i_\alpha} E_\alpha \xrightarrow{j_\alpha} B, \qquad \alpha=1,2$$

be two topological microbundles over B. Recall that $\mathfrak{x}_1$ and $\mathfrak{x}_2$ are *isomorphic* if there exist neighborhoods U_α of $i_\alpha(B)$ in E_α for $\alpha=1, 2$, and a homeomorphism $f: U_1\to U_2$ so that the diagram



is commutative.

DEFINITION. Two these homeomorphisms

$$\begin{aligned} f &: U_1 \to U_2\,,\\ f' &: U_1' \to U_2'\,,\end{aligned}$$

are said to define the same *isomorphism-germ* F from $\mathfrak{x}_1$ to $\mathfrak{x}_2$, if the two maps f, f' coincide on some sufficiently small neighborhood of $i_1(B)$. (Thus an isomorphism-germ

$$F: \mathfrak{x}_1 \to \mathfrak{x}_2$$

is an equivalence class of such homeomorphisms.)

Now consider the topological microbundle $g^*\mathfrak{x}_1$ and $g^*\mathfrak{x}_2$ induced by some continuous mapping $g: B' \to B$. Any isomorphism-germ $F: \mathfrak{x}_1 \to \mathfrak{x}_2$ clearly gives rise to an isomorphism-germ $g^*\mathfrak{x}_1 \to g^*\mathfrak{x}_2$. This induced isomorphism-germ will be denoted by g^*F.

For each integer $n \geqq 0$, we shall construct a css-group H_n as follows. Let Δ_k denote the standard ordered k-simplex. As usual let $\mathfrak{e}^n_{\Delta_k}$ denote the trivial topological microbundle

$$\mathfrak{e}^n_{\Delta_k}: \Delta_k \xrightarrow{\times 0} \Delta_k \times R^n \xrightarrow{p_1} \Delta_k .$$

DEFINITION. A k-simplex F of the css-complex H_n is an isomorphism-germ $F: \mathfrak{e}^n_{\Delta_k} \to \mathfrak{e}^n_{\Delta_k}$. The operation of composing isomorphism-germs makes the set $H^{(k)}_n$ of k-simplexes into a group. For each weakly monotone simplicial map $\lambda: \Delta_l \to \Delta_k$ define a homomorphism

$$\lambda^\#: H^{(k)}_n \to H^{(l)}_n$$

as follows. Let $\lambda^\#$ carry each isomorphism-germ F to the induced isomorphism-germ $\lambda^\# F$. Thus $H_n = \{H^{(k)}_n, \lambda^\#\}$ is a css-group.

We have a natural css-monomorphism

$$\iota_{r,s}: H_r \to H_s, \quad r \leqq s .$$

The family $\{H_r ; \iota_{r,s}\}$ is a directed system of css-groups. Define

$$H = \varinjlim H_n .$$

Then H is also a css-group.

We have a natural css-monomorphism

$$\mu_n: PL_n \to H_n ,$$

and the following diagram

$$(1) \qquad \begin{array}{ccc} PL_r & \xrightarrow{\mu_r} & H_r \\ {\scriptstyle \iota'_{r,s}} \downarrow & & \downarrow {\scriptstyle \iota_{r,s}} \\ PL_s & \xrightarrow{\mu_s} & H_s \end{array} \qquad (r \leqq s)$$

is commutative, where $\iota'_{r,s}: PL_r \to PL_s$ is a natural css-monomorphism. Therefore, by Lemma 1 we have a css-monomorphism

$$\mu: PL \to H .$$

Thus we can consider PL_n, PL as css-subgroup of H_n, H respectively. Then we can consider css-complexes H_n/PL_n, H/PL.

By the commutative diagram (1), we have a natural css-map

$$\omega_{r,s}: H_r/PL_r \to H_s/PL_s, \quad r \leqq s.$$

The family $\{H_r/PL_r; \omega_{r,s}\}$ is a directed system of css-complexes. By Lemma 2, we have an css-equivalence

$$H/PL = \varinjlim H_i/PL_i.$$

Let K be a css-complex not necessarily satisfying Kan's condition, L a css-complex. Then we shall denote by $[K, L]$ the css-homotopy classes of css-maps of K into L. As is remarked above, $[K, H_n]$, $[K, H]$, $[K, H_n/PL_n]$ and $[K H/PL]$, have meanings.

2. Kan complexes BPL_n, BPL; BH_n, BH

Since U is a covariant functor, to the css-monomorphism $\iota_{m,n}: H_m \to H_n$, $m \leqq n$, corresponds a css-monomorphism

$$U(\iota_{m,n}); \ U(H_m) \to U(H_n), \quad m \leqq n.$$

Then the family $\{U(H_m); U(\iota_{mn})\}$ is a directed system of css-groups. Define

$$U = \varinjlim U(H_n).$$

Then U is also a css-group, and by Lemma 4 U can be considered as $U(H)$, therefore, its all homotopy groups vanish.

Since U is a covariant functor, the following diagram

$$(2) \qquad \begin{array}{ccc} H_m & \xrightarrow{\nu_m} & U(H_m) \\ {\scriptstyle \iota_{m,n}}\downarrow & & \downarrow{\scriptstyle U(\iota_{m,n})} \\ H_n & \xrightarrow{\nu_n} & U(H_n) \end{array} \qquad (m \leqq n)$$

is commutative, where $\nu_m: H_m \to U(H_m)$ is the inclusion map ι_{H_m}. Therefore, by Lemma 1 we have a css-monomorphism

$$\nu: H \to U.$$

By Lemma 4 this css-monomorphism is nothing but the inclusion map $\iota_H: H \to U(H)$. Thus we can consider H as css-subgroup of U.

By the commutative diagram (2), we have a css-map

$$\iota_{m,n}: U(H_m)/H_m \to U(H_n)/H_n, \quad (m \leqq n).$$

The family $\{U(H_m)/H_m; \iota_{m,n}\}$ is a directed system of css-complexes. By Lemma 2, we have a css-equivalence

$$U/H = \varinjlim U(H_n)/H_n .$$

The css-group PL_n is a css-subgroup of H_n. Therefore, PL_n also can be considered as a css-subgroup of $U(H_n)$. The following diagram

$$\begin{array}{ccc} PL_m & \xrightarrow{\nu_m \circ \mu_m} & U(H_n) \\ {\scriptstyle \iota'_{m,n}} \downarrow & & \downarrow {\scriptstyle U(\iota_{m,n})} \\ PL_n & \xrightarrow{\nu_n \circ \mu_n} & U(H_n) \end{array} \qquad (m \leqq n)$$

is commutative. Therefore, we have a css-map

$$\iota_{m,n} : U(H_m)/PL_m \to U(H_n)/PL_n, \quad (m \leqq n) .$$

The family $\{U(H_m)/PL_m;\ \iota_{m\ n}\}$ is a directed system of css-complexes. By Lemma 2, we have a css-equivalence

$$U/PL = \varinjlim U(H_n)/PL_n .$$

Now the natural map

$$\pi_n : U(H_n) \to U(H_n)/H_n$$

can be considered as a H_n-bundle in Heller's sense (cf. Heller [1]). Namely, $U(H_n)/H_n$ is a classifiying css-complex of H_n-bundles. We shall denote $U(H_n)/H_n$ by BH_n, and U/H by BH. Similarly, we shall denote $U(H_n)/PL_n$ by BPL_n, and U/PL by BPL.

We shall denote the natural map $U/PL \to U/H$ by

$$\rho : BPL \to BH .$$

By Lemma 1 and 2, this css-map can be considered as the limit of css-maps $\rho_n : U(H_n)/PL_n \to U(H_n)/H_n$.

Let K be a locally finite simplicial complex. Choose some well-ordering for the vertices of K. Let $\tilde{K}$ be the css-complex consisting of all weakly monotone simplicial maps $f : \Delta_k \to K$, with $\lambda^{\#} : \tilde{K}^{(k)} \to K^{(l)}$ defined by $\lambda^{\#}(f) = f \circ \lambda$ for a weakly monotone map $\lambda : \Delta_l \to \Delta_k$.

Now consider a topological microbundle $\mathfrak{x}$ of dimension n over K.

DEFINITION. The H_n-bundle $\tilde{\mathfrak{x}} = (\tilde{E}, \pi, \tilde{K})$ *associated* with $\mathfrak{x}$ is constructed as follows. A k-simplex of the total css-complex $\tilde{E}$ consists of

1) a k-simplex $f \in \tilde{K}^{(k)}$, together with
2) an isomorphism-germ $F : e^n_{\Delta_k} \to f^* \mathfrak{x}$.

The function $\lambda^{\#} : \tilde{E}^{(k)} \to \tilde{E}^{(l)}$ are defined by the formula $\lambda^{\#}(f, F) = (f \circ \lambda, \lambda^* F)$. The right translation function

$$\tilde{E}\times H_n \rightarrow \tilde{E}$$

is just the operation of composing isomorphism-germs. Since this operation is free, it follows that $\tilde{E}$ is an H_n-bundle in Heller's sense and $\tilde{E}/H_n=\tilde{K}$.

Proposition 1. *Let K be a locally finite simplicial complex. Then the operation of assigning to each topological microbundle $\mathfrak{x}$ of dimension n over K its associated H_n-bundle $\mathfrak{x}$ sets up one to one correspondence between isomorphism classes of topological microbundles of dimension n over K and equivalence*[2] *classes of H_n-bundles over $\tilde{K}$.*

The proof in the case of PL-microbundles given in Milnor [3] applies without essential change. Details will be left to the readers.

By Heller's classification theorem (Heller [1], Theorem (10.1), we have

Proposition 2. *Let K be a* css-*complex. The equivalence classes of H_n-bundles X such that X/H_n is K, are in one to one correspondence with the* css-*homotopy classes $[K, BH_n]$ of* css-*maps $\alpha: K\rightarrow BH_n$.*

By Propositions 1 and 2, we have

Theorem 1. *Let K be a locally finite simplicial complex. Then the isomorphism classes of topological microbundle of dimension n over K are in one to one correspondence with the* css-*homotopy classes $[\tilde{K}, BH_n]$.*

3. Whitney sums

Let $H_k^{(p)}\ni\alpha$, $H_n^{(p)}\ni\beta$. The α and β are represented by following maps, respectively:

$$\Delta_p\times 0\subset U\subset\Delta_p\times R^k,$$
$$\Delta_p\times 0\subset V\subset\Delta_p\times R^n,$$
$$f: U\rightarrow\Delta_p\times R^k,$$
$$g: V\rightarrow\Delta_p\times R^n.$$

Define the *Whitney sum* $\alpha\oplus\beta\in H_{k+n}^{(p)}$ by the class represented by the following map:

$$\Delta_p\times 0\subset W\subset\Delta_p\times R^k\times R^n,$$
$$f\oplus g: W\rightarrow\Delta_p\times R^k\times R^n,$$
$$(f\oplus g)(x, u, v) = (x, p_2\circ f(x, u), p_2\circ g(x, v)),$$

2) By equivalence we say strong equivalence in Heller's sense (cf. Heller [1]).

where p_2 is the projection to the second factor. Then $\oplus$ is a css-map

$$\oplus : H_k \times H_n \to H_{k+n} .$$

By restruction, we get

$$\oplus : PL_k \times PL_n \to PL_{k+n} .$$

This css-map is defined in Lashof-Rothenberg [2].

Now we define css-map

$$\oplus : U(H_k) \times U(H_n) \to U(H_{n+k}) .$$

Let $(\sigma, \sigma') \in U(H_k)^{(q)} \times U(H_n)^{(q)}$. Define

$$\oplus(\sigma, \sigma') = \sigma'',$$
$$\sigma'' : \Delta_q \to H_{k+n} ,$$

by

$$\sigma''(\tau) = \sigma(\tau) \oplus \sigma'(\tau), \quad \tau \in \Delta_q .$$

For a weakly monotone map $\lambda : \Delta_p \to \Delta_q$,

$$\begin{aligned} \oplus (\lambda(\sigma), \lambda^{\#}(\sigma'))(\tau) &= \lambda^{\#}(\sigma)(\tau) \oplus \lambda^{\#}(\sigma')(\tau) \\ &= \sigma \circ \lambda(\tau) \oplus \sigma' \circ \lambda(\tau) \\ &= \sigma'' \circ \lambda(\tau) \\ &= \lambda^{\#}(\sigma'')(\tau) \\ &= \lambda^{\#} \circ \oplus (\sigma, \sigma')(\tau) . \end{aligned}$$

Thus the map $\oplus$ defined above is a css-map.

By the above definition the following diagram

$$\begin{array}{ccc} H_k \times H_n & \xrightarrow{\oplus} & H_{k+n} \\ {\scriptstyle \iota_{H_k} \times \iota_{H_n}} \downarrow & & \downarrow {\scriptstyle \iota_{H_{k+n}}} \\ U(H_k) \times U(H_n) & \xrightarrow{\oplus} & U(H_{k+n}) \end{array}$$

is commutative.

By left translation $PL_k \times PL_n$ acts on $H_k \times H_n$ and we have a commutative diagram

$$\begin{array}{ccc} (PL_k \times PL_n) \times (H_k \times H_n) & \to & H_k \times H_n \\ {\scriptstyle \oplus \times \oplus} \downarrow & & \downarrow {\scriptstyle \oplus} \\ PL_{k+n} \times H_{k+n} & \to & H_{k+n} . \end{array}$$

Thus the above map passes to the quotient

$$\oplus : H_k/PL_k \times H_n/PL_n \to H_{k+n}/PL_{k+n}\,.$$

Similarly we have

$$\oplus : U(H_k)/H_k \times U(H_n)/H_n \to U(H_{k+n})/H_{k+n},$$
$$\oplus : U(H_k)/PL_k \times U(H_n)/PL_n \to U(H_{k+n})/PL_{k+n}\,.$$

Let K be a css-complex, and

$$\alpha_k : K \to H_k,\ \alpha_n : K \to H_n$$

be css-maps. Then the above operation induces a map

$$\alpha_k \oplus \alpha_n : K \to H_{k+n}\,.$$

We note

$$(\alpha_k \oplus \alpha_n) \oplus \alpha_p = \alpha_k \oplus (\alpha_n \oplus \alpha_p)\,.$$

Thus we have

$$\oplus : [K, H_k] \times [K, H_n] \to [K, H_{k+n}]\,.$$

Similarly we have

$$\oplus : [K, U(H_k)] \times [K, U(H_n)] \to [K, U(H_{k+n})]\,,$$

and moreover

$$\oplus : [K, U(H_k)/H_k] \times [K, U(H_n)/H_n] \to [K, U(H_{k+n})/H_{k+n}],$$
$$\oplus : [K, U(H_k)/PL_k] \times [K, U(H_n/PL_n] \to [K, U(H_{k+n})/PL_{k+n}],$$
$$\oplus : [K, H_k/PL_k] \times [K, H_n/PL_n] \to [K, H_{k+n}/PL_{k+n}]\,.$$

Let A_n be one of the Kan complexes

$$PL_n,\ H_n,\ U(H_n),\ U(H_n)/H_n,\ U(H_n)/PL_n,\ H_n/PL_n\,,$$

and $\iota_{m,n}$ be one of the natural css-maps

$$\begin{array}{ll} PL_m \to PL_n, & H_m/PL_m \to H_n/PL_n, \\ H_m \to H_n, & U(H_m)/PL_m \to U(H_n)/PL_n, \\ U(H_m) \to U(H_n), & U(H_m)/H_m \to U(H_n)/H_n\,. \end{array}$$

Then the family $\{A_m ; \iota_{m,n}\}$ is a directed system of Kan complexes. Define $A = \varinjlim A_n$.

We shall call a css-complex K *finite*, if it has only a finite number of non-degenerate simplices. For any finite css-complex K an easy argument shows $[K, A] = \varinjlim [K, A_n]$. By the same argument as Lashof-

Rothenberg [2], §4, we have

Proposition 3. *Let K be a finite* css-*complex. We have the following commutative diagram:*

$$\begin{array}{ccc} [K, A_k]\times[K, A_n] & \xrightarrow{\oplus} & [K, A_{k+n}] \\ {\scriptstyle (\iota_{n,n+r})_*\times(\iota_{k,k+s})_*}\downarrow & & \downarrow{\scriptstyle (\iota_{k+n,k+n+r+s})_*} \\ [K, A_{k+r}]\times[K, A_{n+s}] & \xrightarrow{\oplus} & [K, A_{k+n+r+s}]. \end{array}$$

Consequently we have

$$\oplus : [K, A]\times[K, A] \to [K, A].$$

Proposition 4. *Let K be a finite* css-*complex. For $A=PL$, H the Whitney sum on $[K, A]$ is induced from group multipication. Further $[K, A]$ is an abelian group.*

Proposition 5. *Let K be a finite* css-*complex K. For $A=H/PL$, U/PL, U/H the Whitney sum induces on $[K, A]$ the structure of an associative abelian monoid with two sided identity.*

By Lemma 3, we have the following commutative diagram

$$\begin{array}{ccccc} & & & \nearrow^{\iota_{H_n}\times\iota_{H_n}} & U(H_k)\times U(H_n) \\ & & & & \wr\!\!\parallel\ \downarrow\alpha \\ PL_k\times PL_n & \xrightarrow{\mu_k\times\mu_n} & H_k\times H_n & \xrightarrow{\iota_{H_n\times H_n}} & U(H_k\times H_n) \\ \oplus\downarrow & & \oplus\downarrow & & \downarrow U(\oplus) \\ PL_{k+n} & \xrightarrow{\mu_{k+n}} & H_{k+n} & \xrightarrow{\iota_{H_{k+n}}} & U(H_{k+n}). \end{array}$$

Notice that the css-homomorphism $U(\oplus)\circ\alpha$ is nothing but the Whiteney sum

$$\oplus : U(H_k)\times U(H_n) \to U(H_{k+n})$$

defined above. Then we have

Proposition 6. *Let K be a* css-*complex. The following diagram*

$$\begin{array}{ccc} [K, H_k/PL_k]\times[K, H_n/PL_n] & \xrightarrow{\oplus} & [K, U_{k+n}/PL_{k+n}] \\ {\scriptstyle (\zeta_k)_*\times(\zeta_n)_k}\downarrow & & \downarrow{\scriptstyle (\zeta_{k+n})_*} \\ [K, U_k/PL_k]\times[K, U_n/PL_n] & \xrightarrow{\oplus} & [K, H_{k+n}/PL_{k+n}] \\ {\scriptstyle (\rho_k)_*\times(\rho_n)_*}\downarrow & & \downarrow{\scriptstyle (\rho_{k+n})_*} \\ [K, U_k/H_k]\times[K, U_n/H_n] & \xrightarrow{\oplus} & [K, U_{k+n}/H_{k+n}] \end{array}$$

is commutative, where $U_n=U(H_n)$.

Let K be a locally finite simplicial complex of finite dimention. Recall that the s-classes of topological microbundles over K form an abelian group $k_{\mathrm{Top}}(K)$ by Whitney sum (Milnor [4], §4).

The following theorem will give some meaning to the css-complex BH.

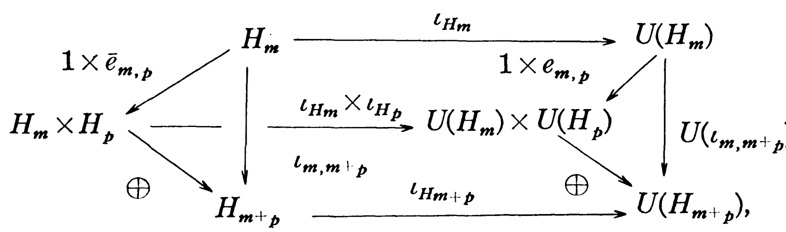
Theorem 2. *Let K be a finite simplicial complex. Then there exists an isomorphism of $k_{\mathrm{Top}}(K)$ onto $[\tilde{K}, BH]$ as semi-group.*

Proof. Let $(\mathfrak{x}) \in k_{\mathrm{Top}}(K)$, and the fibre dimension of $\mathfrak{x}$ be m. By Theorem 1, to $\mathfrak{x}$ corresponds a css-map $f: \tilde{K} \to BH_m = U(H_m)/H_m$. Let ι_m; $BH_m \to BH$ be the canonical inclusion map. To the s-class $(\mathfrak{x})$ we correspond the css-homotopy class $\{\iota_m \circ f\}$. We shall denote $\varphi((\mathfrak{x})) = \{\iota_m \circ f\}$.

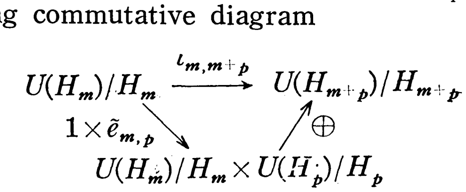
Now we shall prove that this class does not depends on the representative $\mathfrak{x}$ of the class $(\mathfrak{x}) \in k_{\mathrm{Top}}(K)$. Let $e_{m,p}^{(p)}: U(H_m)^{(q)} \to U(H_p)^{(q)}$ be the map which corresponds all elements of $U(H_m)^{(q)}$ to the unit of $U(H_p)^{(q)}$. Then

$$e_{m,p} = \{e_{m,p}^{(q)}\} : U(H_m) \to U(H_p)$$

is a css-homomorphism, and the following diagram



is commutative, where $\bar{e}_{m,p}$ is the restriction of $e_{m,p}$ over H_m. Thus we have the following commutative diagram



where $\tilde{e}_{m,p}: U(H_m)/H_m \to U(H_p)/H_p$ is the css-map induced for $e_{m,p}$: $U(H_n) \to U(H_p)$.

Let $(\mathfrak{x}) = (\mathfrak{y})$, and the fibre dimension of $\mathfrak{y}$ be n. Then there exist integers $0 \leqq p, q$ such that

$$(3) \qquad \mathfrak{x} \oplus \mathfrak{e}^p \sim \mathfrak{y} \oplus \mathfrak{e}^q .$$

Considering the definition of Whitney sums and the correspondence in

Theorem 1, we find that to the microbundle $\mathfrak{x}\oplus\mathfrak{e}^q$ corresponds the composite css-map

$$\tilde{K} \xrightarrow{f} U(H_m)/H_m \xrightarrow{1\times\tilde{e}_{m,p}} U(H_m)/H_m \times U(H_p)/H_p \xrightarrow{\oplus} U(H_{m+p})/H_{m+p}\,.$$

By the above commutative diagram, we obtain that to the microbundle $\mathfrak{x}\oplus\mathfrak{e}^p$ corresponds the css-map $\iota_{m,m+p}\circ f$. If we denote the css-map corresponding to the microbundle $\mathfrak{y}$ by $g:\tilde{K}\to BH_n$, then to the microbundle $\mathfrak{y}\oplus\mathfrak{e}^q$ corresponds the css-map $\iota_{n,n+q}\circ g$. By (3) $\iota_{m,m+q}\circ f$ and $\iota_{n,n+q}\circ g$ are css-homotopic each other. Thus we have $\{\iota_m\circ f\}=\{\iota_n\circ g\}$.

By Theorem 1, the above map φ is clearly surjective.

Let $\varphi((\mathfrak{x}))=\varphi((\mathfrak{y}))$ and to $\mathfrak{x}$ and $\mathfrak{y}$ correspond css-maps f and g, respectively. Then $\iota_m\circ f$ and $\iota_n\circ g$ are homotopic each other. Therefore, there exist integers p, $q\geqq 0$ such that $m+p=n+q$ and $\iota_{m,m+p}\circ f$ and $\iota_{n,n+q}\circ g$ are homotopic each other. So we have $(\mathfrak{x})=(\mathfrak{y})$.

Now ws shall show that φ is a homomorphism. Let $(\mathfrak{x})$, $(\mathfrak{y})\in k_{\mathrm{Top}}(K)$, and $\mathfrak{x}=(\tilde{E}, \pi, \tilde{K})$, $\mathfrak{y}=(\tilde{E}', \pi', \tilde{K})$ be associated H_m- and H_n-bundles to $\mathfrak{x}$ and $\mathfrak{y}$, and $f:\tilde{K}\to U(H_m)/H_m$, $g:\tilde{K}\to U(H_n)/H_n$ be css-maps corresponding to $\mathfrak{x}$ and $\mathfrak{y}$ respectively. We have the following commutative diagram;

$$\begin{array}{ccccc}
\tilde{K}\times\tilde{K} & \xrightarrow{f\times g} & U(H_m)/H_m\times U(H_n)/H_n & & \\
 & \searrow^{\bar{\alpha}\circ(f\times g)} & \downarrow{\scriptstyle\bar{\alpha}} & \searrow^{\oplus} & \\
 & & U(H_m\times H_n)/H_m\times H_n & \xrightarrow{\overline{U(\oplus)}} & U(H_{m+n})/H_{m+n},
\end{array}$$

where $\bar{\alpha}$ and $\overline{U(\oplus)}$ are the css-maps induced by α and $U(\oplus)$ respectively. Considering the correspondence in Theorem 1, we obtain that the H_{m+n}-bundle associated to $\mathfrak{x}\times\mathfrak{y}$ is induced by the css-map $\oplus\circ(f\times g)$. Let d: $K\to K\times K$, $\tilde{d}:\tilde{K}\times\tilde{K}$ be diagonal maps. As $\mathfrak{x}\oplus\mathfrak{y}=d^*(\mathfrak{x}\times\mathfrak{y})$, the H_{m+n}-bunble associated to $\mathfrak{x}\oplus\mathfrak{y}$ is induced by $\oplus\circ(f\times g)\circ d$. By Proposition 3, we obtain that φ is a homomorphism.

4. Exact sequence

Theorem 3. *For any finite* css-*complex* K, *the sequence*

$$[K, PL] \xrightarrow{\mu_*} [K, H] \xrightarrow{\lambda_*} [K, H/PL] \xrightarrow{\zeta_*} [K, BPL] \xrightarrow{\rho_*} [K, BH]$$

is an exact sequence of abelian groups.

Proof. That this is an exact sequence of base-pointed sets is the usual property of fibre spaces applied to the css-fibre spaces

$$\text{I} \quad PL \to H \to H/PL,$$
$$\text{II} \quad H/PL \to U/PL \to U/H,$$

noticing that I is fibration induced from II by inclusion $H/PL \to U/PL$.

That the maps are additive follows from definitions and from Proposition 6.

The fact that $[K, BPL]$ and $[K, BH]$ are abelian groups is known (Theorem 2). Now it only remains to show that $[K, H/PL]$ is actually a group, i.e. that inverses exist. Let $\alpha \in [K, H/PL]$. Then $\zeta_*(\alpha) \in [K, BPL]$ has an inverse $\nu \in [K, BPL]$. Since ρ_* is a group homomorphism, $\rho_*(\nu)=0$. Thus there is an $\alpha' \in [K, H/PL]$ with $\zeta_*(\alpha')=\nu$. Thus $\zeta_*(\alpha+\alpha')=0$, and there is a $\beta \in [K, H]$ with $\lambda_*(\beta)=\alpha+\alpha'$. Now β has an inverse $(-\beta)$ in $[K, H]$ so that $\lambda_*(-\beta)+(\alpha+\alpha')=\lambda_*(-\beta)+\lambda_*(\beta)=\lambda_*(-\beta+\beta)=0$. Thus $\alpha'+\lambda_*(-\beta)$ is an inverse to α.

Thus the theorem is proved.

Let O be the css-group defined in Lashof-Rothenberg [2]. Then O is a css-subgroup of H and BO can be considered as U/O. Let

$$\mu': \quad O \to H,$$
$$\lambda': \quad H \to H/O,$$
$$\zeta': \quad H/O \to U/O = BO,$$
$$\rho': \quad BO = U/O \to U/H = BH$$

be the naturally defined css-maps.

Then in quite a parallel way, we obtain the following

Theorem 4. *For any finite* css-*complex* K, *the sequence*

$$[K, O] \xrightarrow{\mu'_*} [K, H] \xrightarrow{\lambda'_*} [K, H/O] \xrightarrow{\zeta'_*} [K, BO] \xrightarrow{\rho'_*} [K, BH]$$

is an exact sequence of abelian groups.

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