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NOTE ON MICROBUNDLES*)

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In [2] Lashof and Rothenberg have defined the css-group O and the Kan complex PD, and shown a certain exact sequence of abelian groups (Theorem (4.2)) which is fundamental to the studies of the PL-microbundles and smoothing.

In the present note we shall define a css-group H for the topological microbundles parallel to the css-group PL for the PL-microbundles (§ 1), and show an analogous exact sequence of abelian groups (§ 4) which seems to have some meaning to the study of the topological microbundles (§ 2, § 3).

Our method is quite analogous to that of Lashof and Rothenberg [2], and Milnor [3], and is based on Heller's theory [1].

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0. Preliminaries

a) Directed systems of css-complexes.

Let Σ be a partially ordered set, i.e. a set in which we have a transitive relation $<$ defined for some (but not necessary all) pairs of elements. Σ is called a directed set if every pair of elements has a common successor: given σ and τ in Σ there is an element ρ in Σ satisfying $\sigma<\rho$ and $\tau<\rho$.

In the present note all css-complexes are supposed to satisfy Kan's extension condition unless otherwise stated.

Suppose to each element σ of Σ is assigned a css-complex[1] K_σ (css-group G_σ) and to each pair of elements $\sigma<\tau$ of Σ there corresponds a css-map $h_{\sigma\tau}$ of K_σ into K_τ (css-homomorphism $h_{\sigma\tau}$ of G_σ into G_τ) such that if $\rho<\sigma<\tau$ then

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1) For the theory of css-complexes, see for example Heller [1], Moore [5], Puppe [6].

$$h_{\rho\tau} = h_{\sigma\tau}\circ h_{\rho\sigma}\,.$$

A system of css-complexes (css-groups) of this sort is called a *directed system of* css-*complexes* (css-*groups*).

Given a directed system of css-complexes (css-groups), we can define naturally a new css-complex (css-group) called the *limit* css-*complex* K (css-*group* G) of the directed system. We shall denote $K=\varinjlim K_\sigma (G = \varinjlim G_\sigma)$.

Lemma 1. *Let* $\{K_\sigma, h_{\sigma\tau}; \sigma, \tau\in\Sigma\}$, $\{L_\sigma, h_\tau; \sigma, \tau\in\Sigma\}$ *be directed systems of* css-*complexes. Suppose to each element* σ *of* Σ *is assigned a* css-*map* φ_σ *of* K_σ *into* L_σ *such that to each pair of elements* $\sigma<\tau$ *of* Σ *the following diagram*

$$\begin{tikzcd} K_\sigma \arrow[r, "\varphi_\sigma"] \arrow[d, "h_{\sigma\tau}"'] & L_\sigma \arrow[d, "h'_{\sigma\tau}"] \\ K_\tau \arrow[r, "\varphi_\tau"] & L_\tau \end{tikzcd}$$

is commutative. Then, there exists a css-*map* φ *of* $K=\varinjlim K_\sigma$ *into* $L=\varinjlim L_\sigma$ *which corresponds an element* $\{k_\sigma\}$ *of* K *with representative* k_σ *to* $\{\varphi_\sigma(k_\sigma)\}$ *of* L. *If* φ_σ *is injective for each* $\sigma\in\Sigma$, *then the* css-*map* φ *is also injective. If* $\{K_\sigma, h_{\sigma\tau}; \sigma, \tau\in\Sigma\}$, $\{L_\sigma, h'_\tau; \sigma, \tau\in\Sigma\}$ *are directed systems of* css-*groups and each* φ_σ *is a* css-*homomorphism, then the* css-*map* φ *is also a* css-*homomorphism.*

Proof. We shall prove the second assertion. Let $k=\{k_\sigma\}$, $k'=\{k'_\tau\}$ be elements of K such that $\varphi(k)=\varphi(k')$. Then there exists a common successor ρ of σ and τ such that $h'_{\sigma\rho}(\varphi_\sigma(k_\sigma))=h'_{\tau\rho}(\varphi_\tau(k'_\tau))$. The following diagrams are commutative:

$$\begin{tikzcd} K_\sigma \arrow[r, "\varphi_\sigma"] \arrow[d, "h_{\sigma\rho}"'] & L_\sigma \arrow[d, "h'_{\sigma\rho}"] \\ K_\rho \arrow[r, "\varphi_\rho"'] & L_\rho \end{tikzcd} \qquad \begin{tikzcd} K_\tau \arrow[r, "\varphi_\tau"] \arrow[d, "h_{\tau\rho}"] & L_\tau \arrow[d, "h'_{\tau\rho}"] \\ K_\rho \arrow[r, "\varphi_\rho"'] & L_\rho \end{tikzcd}$$

Thus we have

$$\varphi_\rho\circ h_{\sigma\rho}(k_\sigma) = \varphi_\rho\circ h_{\tau\rho}(k'_\tau)\,.$$

Since φ_ρ is injective, we have $h_{\sigma\rho}(k_\sigma)=h_{\tau\rho}(k^\tau_l)$. Thus we have $\{k_\sigma\}=\{k'_\tau\}$.

Let $\{G_\sigma, h_{\sigma\tau}; \sigma, \tau\in\Sigma\}$, $\{H_\sigma, h'_{\sigma\tau}; \sigma, \tau\in\Sigma\}$ be directed systems of css-groups, and for each $\sigma\in\Sigma$ H_σ is a css-subgroup of G_σ. Then, corresponding to each σ css-complex G_σ/H_σ, we have naturally a directed system of css-complexes $\{G_\sigma/H_\sigma, \bar{h}_{\sigma\tau}; \sigma, \tau\in\Sigma\}$. Let $G=\varinjlim G_\sigma$, $H=\varinjlim G_\sigma$, By Lemma 1, we can consider H as a css-subgroup of G. So we have a css-complex G/H. Then we have

Lemma 2. $\varinjlim G_\sigma/H_\sigma$ *and* G/H *are* css-*equivalent, that is, there exists a bijective* css-*map between them.*

Proof. Let $K=\varinjlim G_\sigma/H_\sigma$. Define $\varphi_q: K^{(q)}\to(G/H)^{(q)}$ by $\varphi_q(g)=\{g_\sigma\}$ mod $H^{(q)}$, for $g=\{g_\sigma \bmod H_\sigma^{(q)}\}\in K^{(q)}$. This is independent of the representative of g.

Clearly φ_q is surjective.

Let g, $g'\in K^{(q)}$, $g=\{g_\sigma \bmod H_\sigma^{(q)}\}$, $g'=\{g'_\tau \bmod H_\tau^{(q)}\}$ and $\varphi_q(g)=\varphi_q(g')$. Then we have

$$\{g_\sigma\} \bmod H^{(q)} = \{g'_\tau\} \bmod H^{(q)},$$

that is, there exists a common successor ρ of σ and τ such that

$$(h_{\sigma\rho}(g_\sigma))^{-1}h_{\tau\rho}(g_\tau)\in H_\rho^{(q)}.$$

Namely

$$h_{\sigma\rho}(g_\sigma) \bmod H_\rho^{(q)} = h_{\tau\rho}(g_\tau) \bmod H_\rho^{(q)}.$$

Thus we have $g=g'$.

For a weakly monotone map $\lambda:\Delta_p\to\Delta_q$, we can easily see that the following diagram

$$\begin{array}{ccccc}
\varinjlim G_\rho^{(q)}/H_\sigma^{(q)} = K^{(q)} & \xrightarrow{\varphi_q} & (G/H)^{(q)} = G^{(q)}/H^{(q)} \\
\lambda^\# \downarrow & & \downarrow \lambda^\# \\
\varinjlim G_\sigma^{(p)}/H_\sigma^{(p)} = K^{(p)} & \xrightarrow{\varphi_p} & (G/H)^{(p)} = G^{(p)}/H^{(p)}
\end{array}$$

is commutative. Thus $\varphi=\{\varphi_q\}: K\to G/H$ is a surjective css-map.

We shall sometimes identify two css-equivalent css-complexes.

b) Heller's *U*-functor.

We shall recall Heller's theory [1]. If Γ is a css-group, a *universal group for* Γ is a css-group Υ containing Γ as a css-subgroup and with all homotopy groups $\pi_q(\Upsilon)=0$. For any css-group Γ, there corresponds a css-group $U(\Gamma)$, which is universal for Γ. Moreover, U is a covariant functor on the category of css-groups and css-homomorphisms into itself.

Explicitly, the css-group $U(\Gamma)$ is constructed as follows. Le $U(\Gamma)^{(q)}$ be the set of all map σ of css-complex Δ_q into css-group Γ preserving dimension but not in general incidence. The incidence operations are defined by composition of maps

$$\Delta_p \xrightarrow{\lambda} \Delta_q \longrightarrow \Gamma$$

for a weakly monotone map λ. The group operation in $U(\Gamma)^{(q)}$ is defined by that in Γ: if $\tau \in \Delta_q$ and $\sigma, \sigma' \in U(\Gamma)^{(q)}$, then

$$(\sigma\sigma')(\tau) = \sigma(\tau)\sigma'(\tau).$$

With these definitions it is clear that $U(\Gamma) = \bigcup_{q \geqq 0} U(\Gamma)^{(q)}$ is a css-group. Γ may be identified with the subgroup of $U(\Gamma)$ consisting of those simplices which are css-maps $\sigma : \Delta_q \to \Gamma$. We shall denote the identification by

$$\iota_\Gamma : \Gamma \to U(\Gamma).$$

Let Γ, Γ' be css-groups and $\varphi : \Gamma \to \Gamma'$ be a css-homomorphism. Then the css-homomorphism

$$U(\varphi) : U(\Gamma) \to U(\Gamma')$$

is defined as follows. Let $\sigma \in U(\Gamma)^{(q)}$. We define $U(\varphi)(\sigma) \in U(\Gamma')^{(q)}$ to be $\varphi \circ \sigma$. Then $U(\varphi)$ is a dimension preserving map. For a weakly monotone map $\lambda : \Delta_p \to \Delta_q$, and $\tau \in U(\Gamma)^{(q)}$

$$\begin{aligned} \lambda^{\#} \circ U(\varphi)(\tau) &= \lambda^{\#}(\varphi \circ \tau) \\ &= (\varphi \circ \tau) \circ \lambda \\ &= \varphi \circ (\tau \circ \lambda) \\ &= \varphi \circ \lambda^{\#}(\tau) \\ &= U(\varphi) \circ \lambda^{\#}(\tau). \end{aligned}$$

Thus $U(\varphi)$ is a css-map, and clearly css-homomorphism.

By the definition, if φ is a css-monomorphism, the $U(\varphi)$ is also a css-monomorphism.

Now let Γ, Γ' be css-groups. Then $\Gamma \times \Gamma'$ is also css-groups. Then we have

Lemma 3. *There exists a* css-*isomorphism* $\alpha : U(\Gamma) \times U(\Gamma') \to U(\Gamma \times \Gamma')$ *such that the following diagram*

$$\begin{tikzcd} \Gamma\times\Gamma' \arrow[r, "\iota_\Gamma\times\iota_{\Gamma'}"] \arrow[dr, "\iota_{\Gamma\times\Gamma'}"'] & U(\Gamma)\times U(\Gamma') \arrow[d, "\alpha", "\wr\|"'] \\ & U(\Gamma\times\Gamma') \end{tikzcd}$$

is commutative.

Proof. Define

$$\alpha_q : U(\Gamma)^{(q)}\times U(\Gamma')^{(q)} \to U(\Gamma\times\Gamma')^{(q)}$$
$$\alpha_q(\sigma, \sigma') = \tau,$$
$$\tau(\omega) = (\sigma(\omega), \sigma'(\omega)),\ \omega\in\Delta_q .$$

Then α_q is clearly an injective map.

Let $\lambda : \Delta_p\to\Delta_q$ be a weakly monotone map. Then the following diagram

$$\begin{tikzcd} U(\Gamma)^{(q)}\times U(\Gamma')^{(q)} \arrow[r, "\alpha_q"] \arrow[d, "\lambda^\#"'] & U(\Gamma\times\Gamma')^{(q)} \arrow[d, "\lambda^\#"] \\ U(\Gamma)^{(p)}\times U(\Gamma')^{(p)} \arrow[r, "\alpha_p"] & U(\Gamma\times\Gamma')^{(p)} \end{tikzcd}$$

is commutative. Thus $\alpha=\{\alpha_q\} : U(\Gamma)\times U(\Gamma')\to U(\Gamma\times\Gamma')$ is an injective css-map.

Let $(\sigma, \sigma'),(\rho, \rho')\in U(\Gamma)^{(q)}\times U(\Gamma')^{(q)}$, and $\alpha_q(\sigma, \sigma')=\tau$, $\alpha_q(\rho, \rho')=\tau'$. Then $(\sigma, \sigma')(\rho, \rho')=(\sigma\rho, \sigma'\rho')$. Let $\alpha_q(\sigma\rho, \sigma'\rho')=\tau''$. Then we can prove easily

$$\tau''(\omega) = (\tau\tau')(\omega), \quad \text{for} \quad \omega\in\Delta_q .$$

Thus α is a css-monomorphism, Clearly α is surjective.

Then commutativity is easily seen.

Lemma 4. *Let $\{\Gamma_m, h_{m,n} ; m, n\in Z\}$ be a directed system of* css-*groups and $\Gamma=\varinjlim\Gamma_m$. Then $\{U(\Gamma_m), U(h_{m,n}) ; m, n\in Z\}$ is also a directed system of* css-*groups, and*

$$\varinjlim U(\Gamma_m) \cong U(\Gamma) .$$

Proof. Define $\varphi : \varinjlim U(\Gamma_m)\to U(\Gamma)$ by $\varphi(\{\sigma^q_{(m)}\})=\iota_m\circ\sigma^q_{(m)}$, where ι_m : $\Gamma_m\to\Gamma$ is the projection map and $\sigma^q_{(m)} : \Delta_q\to\Gamma_m$ is a representative of an element σ^q of $(\varinjlim U(\Gamma_m))^{(q)}$. Let $\sigma^q_{(n)}$ be another representative of σ^q: $\{\sigma^q_{(n)}\} = \{\sigma^q_{(m)}\}$. Then there exists an integer p such that $m, n\leqq p$, $h_{mp}\circ\sigma^q_{(m)}=h_{np}\circ\sigma^q_{(n)}$. Then

$$\begin{aligned}\iota_n\circ\sigma^q_{(n)} &= \iota_p\circ h_{np}\circ\sigma^q_{(n)}\\ &= \iota_p\circ h_{mp}\circ\sigma^q_{(m)}\\ &= \iota_m\circ\sigma^q_{(m)}\,.\end{aligned}$$

Thus the above definition has no ambiguity.

Clearly φ is an onto css-homomorphism.

Now we shall prove that φ is injective. Let $\varphi(\{\sigma^q_{(m)}\})=\varphi(\{\tau^q_{(n)}\})$. Then we have $\iota_m\circ\sigma^q_{(m)}=\iota_n\circ\tau^q_{(n)}$. Therefore, there exists an integer p such that $m,\ n\leqq p$ and $h_{mp}\circ\sigma^q_{(m)}=h_{np}\circ\tau^q_{(n)}$. Thus we have $\{\sigma^q_{(m)}\}=\{\tau^q_{(n)}\}$.

1. css-groups H_n, H

In this section we shall construct a css-group H_n for topological microbundles of dimension n. The construction of the css-group H_n is completely parallel to Milnor's construction [3] of the css-group PL_n for PL-microbundles of dimension n.

First we need to define the concept of an isomorphism-germ between topological microbundles. Let

$$\mathfrak{x}_\alpha : B \xrightarrow{i_\alpha} E_\alpha \xrightarrow{j_\alpha} B, \qquad \alpha=1,2$$

be two topological microbundles over B. Recall that $\mathfrak{x}_1$ and $\mathfrak{x}_2$ are *isomorphic* if there exist neighborhoods U_α of $i_\alpha(B)$ in E_α for $\alpha=1, 2$, and a homeomorphism $f: U_1\to U_2$ so that the diagram

$$\begin{tikzcd} & U_1 \arrow[dd, "f", "\wr"'] \arrow[dr, "j_1"] & \\ B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\ & U_2 \arrow[ur, "j_2"'] & \end{tikzcd}$$

is commutative.

DEFINITION. Two these homeomorphisms

$$\begin{aligned} f &: U_1 \to U_2\,,\\ f' &: U_1' \to U_2'\,,\end{aligned}$$

are said to define the same *isomorphism-germ* F from $\mathfrak{x}_1$ to $\mathfrak{x}_2$, if the two maps f, f' coincide on some sufficiently small neighborhood of $i_1(B)$. (Thus an isomorphism-germ

$$F:\mathfrak{x}_1\to\mathfrak{x}_2$$

is an equivalence class of such homeomorphisms.)

Now consider the topological microbundle $g^*\mathfrak{x}_1$ and $g^*\mathfrak{x}_2$ induced by some continuous mapping $g: B' \to B$. Any isomorphism-germ $F: \mathfrak{x}_1 \to \mathfrak{x}_2$ clearly gives rise to an isomorphism-germ $g^*\mathfrak{x}_1 \to g^*\mathfrak{x}_2$. This induced isomorphism-germ will be denoted by g^*F.

For each integer $n \geqq 0$, we shall construct a css-group H_n as follows. Let Δ_k denote the standard ordered k-simplex. As usual let $\mathfrak{e}^n_{\Delta_k}$ denote the trivial topological microbundle

$$\mathfrak{e}^n_{\Delta_k}: \Delta_k \xrightarrow{\times 0} \Delta_k \times R^n \xrightarrow{p_1} \Delta_k .$$

DEFINITION. A k-simplex F of the css-complex H_n is an isomorphism-germ $F: \mathfrak{e}^n_{\Delta_k} \to \mathfrak{e}^n_{\Delta_k}$. The operation of composing isomorphism-germs makes the set $H^{(k)}_n$ of k-simplexes into a group. For each weakly monotone simplicial map $\lambda: \Delta_l \to \Delta_k$ define a homomorphism

$$\lambda^\#: H^{(k)}_n \to H^{(l)}_n$$

as follows. Let $\lambda^\#$ carry each isomorphism-germ F to the induced isomorphism-germ $\lambda^\# F$. Thus $H_n = \{H^{(k)}_n, \lambda^\#\}$ is a css-group.

We have a natural css-monomorphism

$$\iota_{r,s}: H_r \to H_s, \quad r \leqq s .$$

The family $\{H_r ; \iota_{r,s}\}$ is a directed system of css-groups. Define

$$H = \varinjlim H_n .$$

Then H is also a css-group.

We have a natural css-monomorphism

$$\mu_n: PL_n \to H_n ,$$

and the following diagram

$$(1) \qquad \begin{array}{ccc} PL_r & \xrightarrow{\mu_r} & H_r \\ {\scriptstyle \iota'_{r,s}}\downarrow & & \downarrow{\scriptstyle \iota_{r,s}} \\ PL_s & \xrightarrow{\mu_s} & H_s \end{array} \qquad (r \leqq s)$$

is commutative, where $\iota'_{r,s}: PL_r \to PL_s$ is a natural css-monomorphism. Therefore, by Lemma 1 we have a css-monomorphism

$$\mu: PL \to H .$$

Thus we can consider PL_n, PL as css-subgroup of H_n, H respectively. Then we can consider css-complexes H_n/PL_n, H/PL.

By the commutative diagram (1), we have a natural css-map

$$\omega_{r,s}: H_r/PL_r \to H_s/PL_s, \quad r \leqq s.$$

The family $\{H_r/PL_r\,;\ \omega_{r,s}\}$ is a directed system of css-complexes. By Lemma 2, we have an css-equivalence

$$H/PL = \varinjlim H_i/PL_i.$$

Let K be a css-complex not necessarily satisfying Kan's condition, L a css-complex. Then we shall denote by $[K, L]$ the css-homotopy classes of css-maps of K into L. As is remarked above, $[K, H_n]$, $[K, H]$, $[K, H_n/PL_n]$ and $[K\ H/PL]$, have meanings.

2. Kan complexes BPL_n, BPL; BH_n, BH

Since U is a covariant functor, to the css-monomorphism $\iota_{m,n}: H_m \to H_n$, $m \leqq n$, corresponds a css-monomorphism

$$U(\iota_{m,n})\,;\ U(H_m) \to U(H_n), \quad m \leqq n.$$

Then the family $\{U(H_m)\,;\ U(\iota_{mn})\}$ is a directed system of css-groups. Define

$$U = \varinjlim U(H_n).$$

Then U is also a css-group, and by Lemma 4 U can be considered as $U(H)$, therefore, its all homotopy groups vanish.

Since U is a covariant functor, the following diagram

$$(2) \qquad \begin{array}{ccc} H_m & \xrightarrow{\nu_m} & U(H_m) \\ {\scriptstyle \iota_{m,n}}\downarrow & & \downarrow{\scriptstyle U(\iota_{m,n})} \\ H_n & \xrightarrow{\nu_n} & U(H_n) \end{array} \qquad (m \leqq n)$$

is commutative, where $\nu_m: H_m \to U(H_m)$ is the inclusion map ι_{H_m}. Therefore, by Lemma 1 we have a css-monomorphism

$$\nu: H \to U.$$

By Lemma 4 this css-monomorphism is nothing but the inclusion map $\iota_H: H \to U(H)$. Thus we can consider H as css-subgroup of U.

By the commutative diagram (2), we have a css-map

$$\iota_{m,n}: U(H_m)/H_m \to U(H_n)/H_n, \quad (m \leqq n).$$

The family $\{U(H_m)/H_m\,;\ \iota_{m,n}\}$ is a directed system of css-complexes. By Lemma 2, we have a css-equivalence

$$U/H = \varinjlim U(H_n)/H_n .$$

The css-group PL_n is a css-subgroup of H_n. Therefore, PL_n also can be considered as a css-subgroup of $U(H_n)$. The following diagram

$$\begin{array}{ccc} PL_m & \xrightarrow{\nu_m \circ \mu_m} & U(H_n) \\ {\scriptstyle \iota'_{m,n}} \downarrow & & \downarrow {\scriptstyle U(\iota_{m,n})} \\ PL_n & \xrightarrow{\nu_n \circ \mu_n} & U(H_n) \end{array} \qquad (m \leqq n)$$

is commutative. Therefore, we have a css-map

$$\iota_{m,n} : U(H_m)/PL_m \to U(H_n)/PL_n , \quad (m \leqq n) .$$

The family $\{U(H_m)/PL_m ; \iota_{m\,n}\}$ is a directed system of css-complexes. By Lemma 2, we have a css-equivalence

$$U/PL = \varinjlim U(H_n)/PL_n .$$

Now the natural map

$$\pi_n : U(H_n) \to U(H_n)/H_n$$

can be considered as a H_n-bundle in Heller's sense (cf. Heller [1]). Namely, $U(H_n)/H_n$ is a classifiying css-complex of H_n-bundles. We shall denote $U(H_n)/H_n$ by BH_n, and U/H by BH. Similarly, we shall denote $U(H_n)/PL_n$ by BPL_n, and U/PL by BPL.

We shall denote the natural map $U/PL \to U/H$ by

$$\rho : BPL \to BH .$$

By Lemma 1 and 2, this css-map can be considered as the limit of css-maps $\rho_n : U(H_n)/PL_n \to U(H_n)/H_n$.

Let K be a locally finite simplicial complex. Choose some well-ordering for the vertices of K. Let $\tilde{K}$ be the css-complex consisting of all weakly monotone simplicial maps $f : \Delta_k \to K$, with $\lambda^{\#} : \tilde{K}^{(k)} \to K^{(l)}$ defined by $\lambda^{\#}(f) = f \circ \lambda$ for a weakly monotone map $\lambda : \Delta_l \to \Delta_k$.

Now consider a topological microbundle $\mathfrak{x}$ of dimension n over K.

DEFINITION. The H_n-bundle $\tilde{\mathfrak{x}} = (\tilde{E}, \pi, \tilde{K})$ *associated* with $\mathfrak{x}$ is constructed as follows. A k-simplex of the total css-complex $\tilde{E}$ consists of

1) a k-simplex $f \in \tilde{K}^{(k)}$, together with
2) an isomorphism-germ $F : e^n_{\Delta_k} \to f^* \mathfrak{x}$.

The function $\lambda^{\#} : \tilde{E}^{(k)} \to \tilde{E}^{(l)}$ are defined by the formula $\lambda^{\#}(f, F) = (f \circ \lambda, \lambda^* F)$. The right translation function

$$\tilde{E}\times H_n \to \tilde{E}$$

is just the operation of composing isomorphism-germs. Since this operation is free, it follows that $\tilde{E}$ is an H_n-bundle in Heller's sense and $\tilde{E}/H_n=\tilde{K}$.

Proposition 1. *Let K be a locally finite simplicial complex. Then the operation of assigning to each topological microbundle $\mathfrak{x}$ of dimension n over K its associated H_n-bundle $\mathfrak{x}$ sets up one to one correspondence between isomorphism classes of topological microbundles of dimension n over K and equivalence*[2] *classes of H_n-bundles over $\tilde{K}$.*

The proof in the case of PL-microbundles given in Milnor [3] applies without essential change. Details will be left to the readers.

By Heller's classification theorem (Heller [1], Theorem (10.1), we have

Proposition 2. *Let K be a* css-*-complex. The equivalence classes of H_n-bundles X such that X/H_n is K, are in one to one correspondence with the* css-*homotopy classes $[K, BH_n]$ of* css-*maps $\alpha: K\to BH_n$.*

By Propositions 1 and 2, we have

Theorem 1. *Let K be a locally finite simplicial complex. Then the isomorphism classes of topological microbundle of dimension n over K are in one to one correspondence with the* css-*homotopy classes $[\tilde{K}, BH_n]$.*

3. Whitney sums

Let $H_k^{(p)}\ni\alpha$, $H_n^{(p)}\ni\beta$. The α and β are represented by following maps, respectively:

$$\begin{aligned}
&\Delta_p\times 0\subset U\subset \Delta_p\times R^k,\\
&\Delta_p\times 0\subset V\subset \Delta_p\times R^n,\\
&f: U\to \Delta_p\times R^k,\\
&g: V\to \Delta_p\times R^n .
\end{aligned}$$

Define the *Whitney sum* $\alpha\oplus\beta\in H_{k+n}^{(p)}$ by the class represented by the following map:

$$\begin{aligned}
&\Delta_p\times 0\subset W\subset \Delta_p\times R^k\times R^n,\\
&f\oplus g: W\to\Delta_p\times R^k\times R^n,\\
&(f\oplus g)(x, u, v) = (x, p_2\circ f(x, u), p_2\circ g(x, v)),
\end{aligned}$$

2) By equivalence we say strong equivalence in Heller's sense (cf. Heller [1]).

where p_2 is the projection to the second factor. Then $\oplus$ is a css-map

$$\oplus : H_k \times H_n \to H_{k+n}\,.$$

By restruction, we get

$$\oplus : PL_k \times PL_n \to PL_{k+n}\,.$$

This css-map is defined in Lashof-Rothenberg [2].

Now we define css-map

$$\oplus : U(H_k) \times U(H_n) \to U(H_{n+k})\,.$$

Let $(\sigma, \sigma') \in U(H_k)^{(q)} \times U(H_n)^{(q)}$. Define

$$\oplus(\sigma, \sigma') = \sigma'',$$
$$\sigma'' : \Delta_q \to H_{k+n}\,,$$

by

$$\sigma''(\tau) = \sigma(\tau) \oplus \sigma'(\tau), \quad \tau \in \Delta_q\,.$$

For a weakly monotone map $\lambda : \Delta_p \to \Delta_q$,

$$\begin{aligned} \oplus\,(\lambda(\sigma), \lambda^{\#}(\sigma'))(\tau) &= \lambda^{\#}(\sigma)(\tau) \oplus \lambda^{\#}(\sigma')(\tau) \\ &= \sigma \circ \lambda(\tau) \oplus \sigma' \circ \lambda(\tau) \\ &= \sigma'' \circ \lambda(\tau) \\ &= \lambda^{\#}(\sigma'')(\tau) \\ &= \lambda^{\#} \circ \oplus\,(\sigma, \sigma')(\tau)\,. \end{aligned}$$

Thus the map $\oplus$ defined above is a css-map.

By the above definition the following diagram

$$\begin{tikzcd} H_k \times H_n \arrow[r, "\oplus"] \arrow[d, "\iota_{H_k} \times \iota_{H_n}"'] & H_{k+n} \arrow[d, "\iota_{H_{k+n}}"] \\ U(H_k) \times U(H_n) \arrow[r, "\oplus"] & U(H_{k+n}) \end{tikzcd}$$

is commutative.

By left translation $PL_k \times PL_n$ acts on $H_k \times H_n$ and we have a commutative diagram

$$\begin{tikzcd} (PL_k \times PL_n) \times (H_k \times H_n) \arrow[r] \arrow[d, "\oplus \times \oplus"'] & H_k \times H_n \arrow[d, "\oplus"] \\ PL_{k+n} \times H_{k+n} \arrow[r] & H_{k+n}\,. \end{tikzcd}$$

Thus the above map passes to the quotient

$$\oplus : H_k/PL_k \times H_n/PL_n \to H_{k+n}/PL_{k+n}.$$

Similarly we have

$$\oplus : U(H_k)/H_k \times U(H_n)/H_n \to U(H_{k+n})/H_{k+n},$$
$$\oplus : U(H_k)/PL_k \times U(H_n)/PL_n \to U(H_{k+n})/PL_{k+n}.$$

Let K be a css-complex, and

$$\alpha_k : K \to H_k, \ \alpha_n : K \to H_n$$

be css-maps. Then the above operation induces a map

$$\alpha_k \oplus \alpha_n : K \to H_{k+n}.$$

We note

$$(\alpha_k \oplus \alpha_n) \oplus \alpha_p = \alpha_k \oplus (\alpha_n \oplus \alpha_p).$$

Thus we have

$$\oplus : [K, H_k] \times [K, H_n] \to [K, H_{k+n}].$$

Similarly we have

$$\oplus : [K, U(H_k)] \times [K, U(H_n)] \to [K, U(H_{k+n})],$$

and moreover

$$\oplus : [K, U(H_k)/H_k] \times [K, U(H_n)/H_n] \to [K, U(H_{k+n})/H_{k+n}],$$
$$\oplus : [K, U(H_k)/PL_k] \times [K, U(H_n/PL_n] \to [K, U(H_{k+n})/PL_{k+n}],$$
$$\oplus : [K, H_k/PL_k] \times [K, H_n/PL_n] \to [K, H_{k+n}/PL_{k+n}].$$

Let A_n be one of the Kan complexes

$$PL_n, \ H_n, \ U(H_n), \ U(H_n)/H_n, \ U(H_n)/PL_n, \ H_n/PL_n,$$

and $\iota_{m,n}$ be one of the natural css-maps

$$\begin{array}{ll} PL_m \to PL_n, & H_m/PL_m \to H_n/PL_n, \\ H_m \to H_n, & U(H_m)/PL_m \to U(H_n)/PL_n, \\ U(H_m) \to U(H_n), & U(H_m)/H_m \to U(H_n)/H_n. \end{array}$$

Then the family $\{A_m; \iota_{m,n}\}$ is a directed system of Kan complexes. Define $A = \varinjlim A_n$.

We shall call a css-complex K *finite*, if it has only a finite number of non-degenerate simplices. For any finite css-complex K an easy argument shows $[K, A] = \varinjlim [K, A_n]$. By the same argument as Lashof-

Rothenberg [2], §4, we have

Proposition 3. *Let K be a finite* css-*complex. We have the following commutative diagram*:

$$\begin{tikzcd}
{[K, A_k] \times [K, A_n]} \arrow[r, "\oplus"] \arrow[d, "(\iota_{n,n+r})_* \times (\iota_{k,k+s})_*"'] & {[K, A_{k+n}]} \arrow[d, "(\iota_{k+n,k+n+r+s})_*"] \\
{[K, A_{k+r}] \times [K, A_{n+s}]} \arrow[r, "\oplus"] & {[K, A_{k+n+r+s}]}.
\end{tikzcd}$$

Consequently we have

$$\oplus : [K, A] \times [K, A] \to [K, A].$$

Proposition 4. *Let K be a finite* css-*complex. For $A = PL$, H the Whitney sum on $[K, A]$ is induced from group multipication. Further $[K, A]$ is an abelian group.*

Proposition 5. *Let K be a finite* css-*complex K. For $A = H/PL$, U/PL, U/H the Whitney sum induces on $[K, A]$ the structure of an associative abelian monoid with two sided identity.*

By Lemma 3, we have the following commutative diagram

$$\begin{tikzcd}
 & & U(H_k) \times U(H_n) \arrow[d, "\alpha", "\wr\|"'] \\
PL_k \times PL_n \arrow[r, "\mu_k \times \mu_n"] \arrow[d, "\oplus"'] & H_k \times H_n \arrow[r, "\iota_{H_n \times H_n}"] \arrow[d, "\oplus"'] \arrow[ur, "\iota_{H_n} \times \iota_{H_n}"] & U(H_k \times H_n) \arrow[d, "U(\oplus)"] \\
PL_{k+n} \arrow[r, "\mu_{k+n}"] & H_{k+n} \arrow[r, "\iota H_{k+n}"] & U(H_{k+n}).
\end{tikzcd}$$

Notice that the css-homomorphism $U(\oplus) \circ \alpha$ is nothing but the Whiteney sum

$$\oplus : U(H_k) \times U(H_n) \to U(H_{k+n})$$

defined above. Then we have

Proposition 6. *Let K be a* css-*complex. The following diagram*

$$\begin{tikzcd}
{[K, H_k/PL_k] \times [K, H_n/PL_n]} \arrow[r, "\oplus"] \arrow[d, "(\zeta_k)_* \times (\zeta_n)_k"'] & {[K, U_{k+n}/PL_{k+n}]} \arrow[d, "(\zeta_{k+n})_*"] \\
{[K, U_k/PL_k] \times [K, U_n/PL_n]} \arrow[r, "\oplus"] \arrow[d, "(\rho_k)_* \times (\rho_n)_*"'] & {[K, H_{k+n}/PL_{k+n}]} \arrow[d, "(\rho_{k+n})_*"] \\
{[K, U_k/H_k] \times [K, U_n/H_n]} \arrow[r, "\oplus"] & {[K, U_{k+n}/H_{k+n}]}
\end{tikzcd}$$

is commutative, where $U_n = U(H_n)$.

Let K be a locally finite simplicial complex of finite dimention. Recall that the s-classes of topological microbundles over K form an abelian group $k_{\mathrm{Top}}(K)$ by Whitney sum (Milnor [4], §4).

The following theorem will give some meaning to the css-complex BH.

Theorem 2. *Let K be a finite simplicial complex. Then there exists an isomorphism of $k_{\mathrm{Top}}(K)$ onto $[\tilde{K}, BH]$ as semi-group.*

Proof. Let $(\mathfrak{x})\in k_{\mathrm{Top}}(K)$, and the fibre dimension of $\mathfrak{x}$ be m. By Theorem 1, to $\mathfrak{x}$ corresponds a css-map $f: \tilde{K}\to BH_m = U(H_m)/H_m$. Let ι_m; $BH_m\to BH$ be the canonical inclusion map. To the s-class $(\mathfrak{x})$ we correspond the css-homotopy class $\{\iota_m\circ f\}$. We shall denote $\varphi((\mathfrak{x}))=\{\iota_m\circ f\}$.

Now we shall prove that this class does not depends on the representative $\mathfrak{x}$ of the class $(\mathfrak{x})\in k_{\mathrm{Top}}(K)$. Let $e_{m,p}^{(q)}: U(H_m)^{(q)}\to U(H_p)^{(q)}$ be the map which corresponds all elements of $U(H_m)^{(q)}$ to the unit of $U(H_p)^{(q)}$. Then

$$e_{m,p} = \{e_{m,p}^{(q)}\} : U(H_m) \to U(H_p)$$

is a css-homomorphism, and the following diagram

$$\begin{tikzcd}
 & H_m \arrow[dl, "1\times \bar{e}_{m,p}"'] \arrow[rr, "\iota_{H_m}"] \arrow[dd, "\iota_{m,m+p}"] & & U(H_m) \arrow[dl, "1\times e_{m,p}"'] \arrow[dd, "U(\iota_{m,m+p})"] \\
H_m\times H_p \arrow[rr, "\iota_{H_m}\times\iota_{H_p}"] \arrow[dr, "\oplus"'] & & U(H_m)\times U(H_p) \arrow[dr, "\oplus"'] & \\
 & H_{m+p} \arrow[rr, "\iota_{H_{m+p}}"] & & U(H_{m+p}),
\end{tikzcd}$$

is commutative, where $\bar{e}_{m,p}$ is the restriction of $e_{m,p}$ over H_m. Thus we have the following commutative diagram

$$\begin{tikzcd}
U(H_m)/H_m \arrow[rr, "\iota_{m,m+p}"] \arrow[dr, "1\times\tilde{e}_{m,p}"'] & & U(H_{m+p})/H_{m+p} \\
 & U(H_m)/H_m\times U(H_p)/H_p \arrow[ur, "\oplus"'] &
\end{tikzcd}$$

where $\tilde{e}_{m,p}: U(H_m)/H_m\to U(H_p)/H_p$ is the css-map induced for $e_{m,p}$: $U(H_n)\to U(H_p)$.

Let $(\mathfrak{x})=(\mathfrak{y})$, and the fibre dimension of $\mathfrak{y}$ be n. Then there exist integers $0\leqq p$, q such that

$$(3) \qquad \mathfrak{x}\oplus \mathfrak{e}^p \sim \mathfrak{y}\oplus\mathfrak{e}^q .$$

Considering the definition of Whitney sums and the correspondence in

Theorem 1, we find that to the microbundle $\mathfrak{x}\oplus\mathfrak{e}^q$ corresponds the composite css-map

$$\tilde{K} \xrightarrow{f} U(H_m)/H_m \xrightarrow{1\times\tilde{e}_{m,p}} U(H_m)/H_m \times U(H_p)/H_p$$
$$\xrightarrow{\oplus} U(H_{m+p})/H_{m+p}\,.$$

By the above commutative diagram, we obtain that to the microbundle $\mathfrak{x}\oplus\mathfrak{e}^p$ corresponds the css-map $\iota_{m,m+p}\circ f$. If we denote the css-map corresponding to the microbundle $\mathfrak{y}$ by $g:\tilde{K}\to BH_n$, then to the microbundle $\mathfrak{y}\oplus\mathfrak{e}^q$ corresponds the css-map $\iota_{n,n+q}\circ g$. By (3) $\iota_{m,m+q}\circ f$ and $\iota_{n,n+q}\circ g$ are css-homotopic each other. Thus we have $\{\iota_m\circ f\}=\{\iota_n\circ g\}$.

By Theorem 1, the above map φ is clearly surjective.

Let $\varphi((\mathfrak{x}))=\varphi((\mathfrak{y}))$ and to $\mathfrak{x}$ and $\mathfrak{y}$ correspond css-maps f and g, respectively. Then $\iota_m\circ f$ and $\iota_n\circ g$ are homotopic each other. Therefore, there exist integers p, $q\geqq 0$ such that $m+p=n+q$ and $\iota_{m,m+p}\circ f$ and $\iota_{n,n+q}\circ g$ are homotopic each other. So we have $(\mathfrak{x})=(\mathfrak{y})$.

Now ws shall show that φ is a homomorphism. Let $(\mathfrak{x})$, $(\mathfrak{y})\in k_{\mathrm{Top}}(K)$, and $\mathfrak{x}=(\tilde{E}, \pi, \tilde{K})$, $\mathfrak{y}=(\tilde{E}', \pi', \tilde{K})$ be associated H_m- and H_n-bundles to $\mathfrak{x}$ and $\mathfrak{y}$, and $f:\tilde{K}\to U(H_m)/H_m$, $g:\tilde{K}\to U(H_n)/H_n$ be css-maps corresponding to $\mathfrak{x}$ and $\mathfrak{y}$ respectively. We have the following commutative diagram;

$$\begin{tikzcd}
\tilde{K}\times\tilde{K} \arrow[r, "f\times g"] \arrow[dr, "\bar{\alpha}\circ(f\times g)"'] & U(H_m)/H_m\times U(H_n)/H_n \arrow[d, "\bar{\alpha}"] \arrow[dr, "\oplus"] & \\
& U(H_m\times H_n)/H_m\times H_n \arrow[r, "\overline{U(\oplus)}"] & U(H_{m+n})/H_{m+n},
\end{tikzcd}$$

where $\bar{\alpha}$ and $\overline{U(\oplus)}$ are the css-maps induced by α and $U(\oplus)$ respectively. Considering the correspondence in Theorem 1, we obtain that the H_{m+n}-bundle associated to $\mathfrak{x}\times\mathfrak{y}$ is induced by the css-map $\oplus\circ(f\times g)$. Let d: $K\to K\times K$, $\tilde{d}:\tilde{K}\times\tilde{K}$ be diagonal maps. As $\mathfrak{x}\oplus\mathfrak{y}=d^*(\mathfrak{x}\times\mathfrak{y})$, the H_{m+n}-bunble associated to $\mathfrak{x}\oplus\mathfrak{y}$ is induced by $\oplus\circ(f\times g)\circ d$. By Proposition 3, we obtain that φ is a homomorphism.

4. Exact sequence

Theorem 3. *For any finite* css-*complex K, the sequence*

$$[K, PL] \xrightarrow{\mu_*} [K, H] \xrightarrow{\lambda_*} [K, H/PL] \xrightarrow{\zeta_*} [K, BPL] \xrightarrow{\rho_*} [K, BH]$$

is an exact sequence of abelian groups.

Proof. That this is an exact sequence of base-pointed sets is the usual property of fibre spaces applied to the css-fibre spaces

$$\begin{array}{ll} \text{I} & PL \to H \to H/PL\,, \\ \text{II} & H/PL \to U/PL \to U/H, \end{array}$$

noticing that I is fibration induced from II by inclusion $H/PL \to U/PL$.

That the maps are additive follows from definitions and from Proposition 6.

The fact that $[K, BPL]$ and $[K, BH]$ are abelian groups is known (Theorem 2). Now it only remains to show that $[K, H/PL]$ is actually a group, i.e. that inverses exist. Let $\alpha \in [K, H/PL]$. Then $\zeta_*(\alpha) \in [K, BPL]$ has an inverse $\nu \in [K, BPL]$. Since ρ_* is a group homomorphism, $\rho_*(\nu)=0$. Thus there is an $\alpha' \in [K, H/PL]$ with $\zeta_*(\alpha')=\nu$. Thus $\zeta_*(\alpha+\alpha')=0$, and there is a $\beta \in [K, H]$ with $\lambda_*(\beta)=\alpha+\alpha'$. Now β has an inverse $(-\beta)$ in $[K, H]$ so that $\lambda_*(-\beta)+(\alpha+\alpha')=\lambda_*(-\beta)+\lambda_*(\beta)=\lambda_*(-\beta+\beta)=0$. Thus $\alpha'+\lambda_*(-\beta)$ is an inverse to α.

Thus the theorem is proved.

Let O be the css-group defined in Lashof-Rothenberg [2]. Then O is a css-subgroup of H and BO can be considered as U/O. Let

$$\begin{array}{ll} \mu': & O \to H, \\ \lambda': & H \to H/O, \\ \zeta': & H/O \to U/O = BO, \\ \rho': & BO = U/O \to U/H = BH \end{array}$$

be the naturally defined css-maps.

Then in quite a parallel way, we obtain the following

Theorem 4. *For any finite* css-*complex* K, *the sequence*

$$[K, O] \xrightarrow{\mu'_*} [K, H] \xrightarrow{\lambda'_*} [K, H/O] \xrightarrow{\zeta'_*} [K, BO] \xrightarrow{\rho'_*} [K, BH]$$

is an exact sequence of abelian groups.

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