The wild McKay correspondence for p^n-cyclic groups and quotient singularities

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Chapter 1

Introduction

The subject of this thesis is the wild McKay correspondence for cyclic groups of prime power order; especially, studying the $\boldsymbol{v}$*-function* and its integral. The $\boldsymbol{v}$-function is a key ingredient in the wild McKay correspondence. Furthermore, it can be considered as a common generalization of the *age* invariant in the tame McKay correspondence or of the *Artin conductor*, which is an important invariant in number theory (see [37] for details).

The McKay correspondence relates an algebraic invariant of a representation V of a finite group G with a geometric invariant of the associated quotient variety $X := V/G$. Depending on which type of invariant one considers, there are different approaches to the McKay correspondence. In characteristic zero, Batyrev [3] showed a typical one and then Denef–Loeser [13] refined it by using motivic integration. Yasuda [45] generalized their results to arbitrary characteristics, in particular, including the wild case—that is, the case where the given finite group G has order divisible by the characteristic of the base field. In what follows, we work over an algebraically closed field k of characteristic $p > 0$.

Theorem (The wild McKay correspondence [45, Corollary 16.3])**.** *Assume that a finite group G acts on an affine space $V = \mathbb{A}_k^d$ linearly and faithfully. Let $X := V/G$ and Δ be a $\mathbb{Q}$-Weil divisor on X such that the canonical morphism $V \to (X, \Delta)$ is crepant. (Note that if G has no pseudo-reflections, then $\Delta = 0$.) Then we have*

$$M_{\mathrm{st}}(X, \Delta) = \int_{G\text{-}\mathrm{Cov}(D)} \mathbb{L}^{d-\boldsymbol{v}}.$$

Here $M_{\mathrm{st}}(X, \Delta)$ denotes the stringy motive *of the pair (X, Δ), G-$\mathrm{Cov}(D)$ denotes the moduli space of G-covers of the formal disk $D := \operatorname{Spec} k[[t]]$, and $\boldsymbol{v}$ denotes the $\boldsymbol{v}$-function $\boldsymbol{v}\colon G\text{-}\mathrm{Cov}(D) \to \mathbb{Q}$ associated to the given G-action.*

To simplify, we consider only the case that G has no pseudo-reflections. We shall discuss the case that G has pseudo-reflections later, see Remark 2.4.2.

Since the stringy motive $M_{\mathrm{st}}(X)$ contains information on singularities of the quotient variety X, the wild McKay correspondence theorem allows us to study singularities by computing the $\boldsymbol{v}$-function $\boldsymbol{v}$ and its integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-\boldsymbol{v}}$. However, in spite of its importance, it is difficult to compute them in a general situation.

Example 1.0.1. For the following cases, we have the explicit formulae of the $\boldsymbol{v}$-functions.

(1) For the tame case (that is, $p \nmid \# G$), the $\boldsymbol{v}$-function is exactly the age invariant.

(2) For $G = \mathbb{Z}/p\mathbb{Z}$, Yasuda [42] gave it; the $\boldsymbol{v}$-function can be computed by the ramification filtration.

(3) For the permutation representations, Wood–Yasuda [37] gave it; the $\boldsymbol{v}$-function can be computed by the Artin conductor.

(4) For $G = \mathbb{Z}/p^n\mathbb{Z}$, Yasuda and the author [33] gave it; this is a topic of this thesis.

(5) For subgroups $G \subset SL(2,k)$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$, Yamamoto [41] gave it; this is an example which is not determined by the ramification filtration.

Let us consider the case $G = \mathbb{Z}/p^n\mathbb{Z}$, which is of our principal interest.

We find out that the $\boldsymbol{v}$-function in this case can be computed in terms of ramification jumps of the field extension corresponding to the given G-cover. Note that values $\boldsymbol{v}(E)$ at non connected G-covers E follow from that of connected G-covers, thus the case of connected G-covers is essential. Let E be a connected G-cover of $D = \operatorname{Spec} k[[t]]$ and let $L/k((t))$ be the corresponding G-extension. By the Artin–Schreier–Witt theory, the extension $L/k((t))$ is given by an equation $\wp(g_0, g_1, \dots, g_{n-1}) = (f_0, f_1, \dots, f_{n-1})$, where $(f_0, f_1, \dots, f_{n-1}) \in W_n[k((t))]$ is reduced. We can decompose the extension $L/k((t))$ into a tower of p-cyclic extensions

$$L = K_{n-1} \supset K_{n-2} \supset \cdots \supset K_{-1} = k((t)),$$

where $K_i = K_{i-1}(g_i)$. The key facts here are first that the value $\boldsymbol{v}(E)$ is expressed in terms of the ramification jumps of the extensions K_i/K_{i-1} (see Lemmas 3.2.3 and 3.2.11), and second that these ramification jumps are determined by the orders of f_i (see Corollary 3.2.13). As a consequence, we get the following.

Theorem 1.0.2 (Theorem 3.2.14)**.** *Assume that the given G-representation is indecomposable of dimension d. With the notation as above, we denote $j_m = -\operatorname{ord} f_m$.*

The i-th upper ramification jump u_i and the i-th lower ramification jump l_i are given by

$$u_{i-1} := \max\{p^{n-1-m} j_m \mid m = 0, 1, \dots, i-1\},$$
$$l_{i-1} := u_0 + (u_1 - u_0)p + \cdots + (u_i - u_{i-1})p^i,$$

respectively. Then we have

$$\boldsymbol{v}(E) = \sum_{\substack{0 \le i_0 + i_1 p + \cdots + i_{n-1} p^{n-1} < d, \\ 0 \le i_0, i_1, \dots, i_{n-1} < p}} \left\lceil \frac{i_0 p^{n-1} l_0 + i_1 p^{n-2} l_1 + \cdots + i_{n-1} l_{n-1}}{p^n} \right\rceil.$$

Since the $\boldsymbol{v}$-function is additive with respect to direct sum of G-representations (it is immediate from Definition 2.4.12; see also [37, Lemma 3.4]), the decomposable G-representation case follows from the theorem above. We remark that for each positive integer $d \le p^n$, there exits exactly one indecomposable G-representation of dimension d modulo isomorphisms; it corresponds to the Jordan block of size d with eigenvalue 1, see [11, p. 431, (64.2) Lemma] for example.

In the relation to the minimal model program, it is natural to ask: How can we determine representation-theoretically when a quotient variety V/G (with G an arbitrary finite group) is terminal, canonical, log terminal or log canonical? In characteristic zero, we have the Reid–Shepherd-Baron–Tai criterion (see [23, Theorem 3.21] for example); by studying the age invariant, we obtain a criterion for the quotient varieties being canonical, terminal, or not. In positive characteristics, as an application of the wild McKay correspondence theorem, Yasuda proved the following.

Proposition 1.0.3 ([45, Corollary 16.4])**.** *We follow the notation of the wild McKay correspondence theorem. Assume that G has no pseudo-reflections (that is, $\Delta = 0$).*

(1) We have

$$\mathrm{discrep}(\mathrm{center} \subset X_{\mathrm{sing}}; X) = d - 1 - \max\left\{\dim X_{\mathrm{sing}}, \dim \int_{G\text{-}\mathrm{Cov}(D) \setminus \{o\}} \mathbb{L}^{d-v}\right\},$$

where o denotes the point corresponding to the trivial G-cover.

(2) If the integral $\int_{G\text{-}\mathrm{Cov}(D)} \mathbb{L}^{d-v}$ converges, then X is log terminal. If X has a log resolution, then the converse is also true.

In the case $G = \mathbb{Z}/p^n\mathbb{Z}$, combining this proposition with the explicit formula of the $\boldsymbol{v}$-function (Theorem 1.0.2), the author [32] give a numerical criterion for the quotient variety $X = V/G$ being canonical or log canonical. According to the Artin–Schreier–Witt theory, the integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-\boldsymbol{v}}$ can be written as an infinite sum of the form

$$\int_{G\text{-Cov}(D)} \mathbb{L}^{d-\boldsymbol{v}} = \sum_{\boldsymbol{j}} [G\text{-Cov}(D;\boldsymbol{j})] \mathbb{L}^{d-\boldsymbol{v}|_{G\text{-Cov}(D;\boldsymbol{j})}}.$$

Here $G\text{-Cov}(D) = \coprod G\text{-Cov}(D;\boldsymbol{j})$ is a stratification such that the restriction $\boldsymbol{v}|_{G\text{-Cov}(D;\boldsymbol{j})}$ of the $\boldsymbol{v}$-function to each stratum $G\text{-Cov}(D;\boldsymbol{j})$ is constant. It is hard to compute the above infinite sum directly. However, by expressing the infinite sum in terms of upper ramification jumps, we show that the integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-\boldsymbol{v}}$ converges if and only if some linear function in upper ramification jumps tends to $-\infty$ (cf. proof of Theorem 4.2.7). For an indecomposable G-representation V of dimension d, we define the following invariants:

$$S_V^{(m)} := \sum_{\substack{0 \le i_0 + i_1 p + \cdots + i_{n-1} p^{n-1} < d, \\ 0 \le i_0, i_1, \ldots, i_{n-1} < p}} i_m \quad (0 \le m \le n-1),$$

$$D_V^{(m)} := p^{n-1} \left(S_V^{(m)} - (p-1) \sum_{l=m+1}^{n-1} p^{m-1} S_V^{(l)} \right) \quad (0 \le m \le n-1).$$

We generalize these invariants to the cases when V is decomposable in the way that they become additive with respect to direct sum of G-representations. Now we can give a criterion as follows.

Theorem 1.0.4 (Corollary 4.2.10)**.**

(1) $X = V/G$ is canonical if the strict inequalities

$$1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} < 0 \quad (0 \le m \le n-1)$$

hold. Furthermore, if X has a log resolution, then the converse is also true.

(2) X is log canonical if and only if the inequalities

$$1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} \le 0 \quad (0 \le m \le n-1)$$

hold.

Especially, if the given G-representation V is indecomposable, then we can compute the invariants $D_V^{(m)}$ explicitly and hence get more simple criterion.

Corollary 1.0.5 (Theorem 4.2.12). *Assume that V is an indecomposable G-representation of dimension d.*

(1) $X = V/G$ is canonical if $d \geq p + p^{n-1}$. Furthermore, if X has a log resolution, then the converse is also true.

(2) X is log canonical if and only if $d \geq p - 1 + p^{n-1}$.

We can generalize immediately these criteria for the case that G is a finite group whose p-Sylow subgroup is cyclic, see Theorem 4.2.17.

As mentioned as above, it is hard to compute the integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-v}$ for general exponent n. Let us focus on the case $n = 2$ to get precise evaluation of discrepancies. By computing $\dim \int_{G\text{-Cov}(D;\boldsymbol{j})} \mathbb{L}^{d-v}$, as an application of Proposition 1.0.3, we get a refinement of Corollary 1.0.5.

Corollary 1.0.6 (Theorem 5.2.4). *Assume that V is an indecomposable $G = \mathbb{Z}/p^2\mathbb{Z}$-representation of dimension d ($p + 1 < d \leq p^2$) (with this assumption, V has no pseudo-reflection and hence $V \to X := V/G$ is crepant). Then,*

$$X \text{ is } \begin{cases} \text{terminal,} \\ \text{canonical,} \\ \text{log canonical,} \\ \text{not log canonical} \end{cases} \quad \text{if and only if } \begin{cases} d \geq 2p + 1, \\ d \geq 2p, \\ d \geq 2p - 1, \\ d < 2p - 1. \end{cases}$$

We give more comments on singularities. From the viewpoint of the minimal model program, klt singularities form an important class of singularities. In characteristic zero, klt singularities are rational and hence Cohen–Macaulay ([14, 22]; see also [24, Theorem 5.22] and its references). In positive characteristics, there exist many examples which are klt but not Cohen–Macaulay; they are constructed by Yasuda [42, 44], Kovács [25], Cascini–Tanaka [9], Gongyo–Nakamura–Tanaka [15], Totaro [36], Arvidsson–Bernasconi–Lacini [1], and Bernasconi [4]. Since many quotient varieties by p-groups may not be Cohen–Macaulay (cf. Proposition 2.1.6), thus we can give more such examples by using Theorem 1.0.4 or by its corollaries.

Note that we work over an algebraically closed field of characteristic $p > 0$ throughout this thesis for the simplicity reason; it is straightforward to generalize our results to any field of characteristic p by base change.

The outline of this thesis is as follows. In Chapter 2, we recall basic facts and definitions of representation theory, motivic integration, and singularities to give the statement of the wild McKay correspondence theorem. In Chapter 3, by using the Artin–Schreier–Witt theory, we describe the moduli space G-Cov(D) of G-covers of $D = \operatorname{Spec} k[[t]]$ and then decompose it to strata G-Cov$(D; \boldsymbol{j})$.

After that, we see that the $\boldsymbol{v}$-function in our case is written in terms of valuations of Witt vectors and by upper/lower ramification jumps of G-extension; we get an explicit formula for the $\boldsymbol{v}$-function. In Chapter 4, we apply the formula to the integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-\boldsymbol{v}}$; we determine when the integral converges. As an application, we give some criteria for whether associated quotient variety V/G is canonical (resp. log canonical) or not. In Chapter 5, by restricting ourselves to the case $G = \mathbb{Z}/p^2\mathbb{Z}$, we get a refinement of our criteria proved in the previous chapter.

Notation and convention

Unless otherwise noted, we follow the following notation and convention. We work over an algebraically closed field k of characteristic $p > 0$. We denote by $K = k((t))$ the field of formal Laurent power series over k. By a k-variety, we mean it is an integral k-scheme of finite type. The set $\mathbb{N}$ of natural numbers is the set of positive integers (that is, $\mathbb{N}$ does not contain 0).

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Chapter 2

Preliminaries

2.1 Representation theory

In this section, we recall some basic facts of the representation theory. Let G be an arbitrary finite group and k an algebraically closed field of characteristic $p > 0$. A *G-representation* is a group homomorphism $\rho\colon G \to GL(V)$, where V is a k-vector space and $GL(V)$ is a group of k-linear automorphisms of V. Throughout this thesis, we consider only the case V has finite dimension. We simply say that V is a representation when the finite group G in question is clear from context. As in the case of characteristic zero, we can define some basic terminologies (for example, subrepresentation, faithful and so on).

Definition 2.1.1. Let V be a G-representation. We say that V is *indecomposable* if it is impossible to express V as a direct sum of two non trivial subrepresentations. We say that V is *decomposable* if V is not indecomposable.

Remark 2.1.2. Since we consider only finite dimensional vector spaces, every G-representation V can be expressed as a direct sum of indecomposable subrepresentations. See, for example, [11, p. 81, (14.2) Theorem].

Moreover, for $\mathbb{Z}/p^n\mathbb{Z}$-representations, we have the following.

Proposition 2.1.3 ([11, p. 431, (64.2) Lemma])**.** *For each positive integer $d \leq p^n$, there exists exactly one indecomposable $\mathbb{Z}/p^n\mathbb{Z}$-representation of dimension d modulo isomorphisms; it corresponds to the Jordan block of size d with eigenvalue* 1.

Remark 2.1.4. It is easy to see that a d-dimensional G-representation is faithful if and only if $p^{n-1} < d \leq p^n$.

A G-representation V defines a G-action on V. We denote by $k[V]$ the coordinate ring of V (note that k is infinite since it is algebraically closed). The G-action on V also defines a G-action on $k[V]$.

Definition 2.1.5. Let V be a G-representation.

(1) We say that $g \in G$ is a *pseudo-reflection* if the fixed subspace V^g has codimension 1.

(2) We say that $g \in G$ is a *bi-reflection* if $\mathrm{rank}(g - \mathrm{id}) \leq 2$.

It is well-known that the invariant ring $k[V]^G$ is Cohen–Macaulay if $p \nmid \#G$ (or k has characteristic zero), see [19, Section 2]. In the case $p \mid \#G$, the following holds.

Proposition 2.1.6 ([8, Theorem 9.2.2])**.** *Let G be a p-group and let V be a G-representation. If the invariant ring $k[V]^G$ is Cohen–Macaulay, then G is generated by bi-reflections.*

Corollary 2.1.7. *Let V be an indecomposable faithful $\mathbb{Z}/p^n\mathbb{Z}$-representation of dimension d. If $d > 3$, then the invariant ring $k[V]^{\mathbb{Z}/p^n\mathbb{Z}}$ is not Cohen–Macaulay.*

Proof. Note that a generator σ of $\mathbb{Z}/p^n\mathbb{Z}$ corresponds to the Jordan block of size d with eigenvalue 1, and hence $\mathrm{rank}(\sigma - \mathrm{id}) = d - 1$. By the contraposition of Proposition 2.1.6, we obtain the desired conclusion. □

The following is used later.

Proposition 2.1.8 ([8, Theorem 3.8.1])**.** *Let G be a p-group and let V be a G-representation. Then the invariant ring $k[V]^G$ is a unique factorization domain.*

2.2 Motivic integration

The wild McKay correspondence theorem is given in terms of motivic integration. In this section, we introduce the notion of motivic measure and integration. For general theory of motivic integration, refer [10] if necessary.

2.2.1 The Grothendieck ring of varieties

Motivic integration takes values in a variant of the Grothendieck ring of k-varieties. First, we recall the definition. We denote by $\mathbf{Var}_k$ the category of k-varieties.

Definition 2.2.1. The *Grothendieck ring of k-varieties* $K_0(\mathbf{Var}_k)$ is the quotient of the free abelian group $\bigoplus_{\mathbf{Var}_k} \mathbb{Z}[X]$ generated by isomorphic classes $[X]$ of k-varieties X by the following *scissor relation*: If $Z \subset X$ is a closed subvariety of X, then $[X] = [X \setminus Z] + [Z]$. The multiplication on $K_0(\mathbf{Var}_k)$ is given by $[X][Y] := [X \times Y]$.

It is easy to see that $K_0(\mathbf{Var}_k)$ is a commutative ring with unit. Its additive identity is $0 = [\emptyset]$ and multiplicative one is $1 = [\operatorname{Spec} k]$.

Let X be a k-variety and let $Z \subset X$ be a closed subvariety of X with the same underlying topological space. Since $X \setminus Z = \emptyset$, by the scissor relation, thus we have $[X] - [Z] = [X \setminus Z] = [\emptyset] = 0$ and hence $[X] = [Z]$. This shows that the class $[X]$ of a k-variety X is determined independently of the scheme structure. Therefore, for a closed subset $Z \subset X$ of a k-variety X, we can define the class $[Z]$ by considering its reduced induced closed subscheme structure. In particular, the classes $[C]$ of locally closed subsets C are well-defined.

Let X be a k-variety. By a *constructible subset* of X, we mean it is a disjoint union of finitely many locally closed subsets of X.

Lemma 2.2.2. *Let X be a k-variety and let $C = \bigsqcup_{i=1}^{n} C_i \subset X$ be a constructible subset (that is, each C_i is locally closed). We define $[C] := \sum_{i=1}^{n} [C_i]$. The class $[C]$ is well-defined.*

Proof. See, for example, [10, Chapter 2, Section 1.3]. □

We put $\mathbb{L} := [\mathbb{A}_k^1]$. Then we can write $[\mathbb{G}_m] = \mathbb{L} - 1$, where $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$.

2.2.2 Localization, completion, and modification

Definition 2.2.3. We define $\mathcal{M} := K_0(\mathbf{Var}_k)[\mathbb{L}^{-1}]$ to be the localization of the Grothendieck ring of k-varieties $K_0(\mathbf{Var}_k)$ by the class of affine line $\mathbb{L} = [\mathbb{A}_k^1]$. We call an element of the form $[X]\mathbb{L}^n \in \mathcal{M}$ with X a k-scheme of finite type and with n a integer an *effective element*.

Definition 2.2.4. The *dimension* of an effective element $[X]\mathbb{L}^n$ is defined by $\dim[X]\mathbb{L}^n := \dim X + n$.

Lemma 2.2.5. *The dimension* $\dim[X]\mathbb{L}^n$ *of an effective element* $[X]\mathbb{L}^n$ *is well-defined.*

Proof. It follows from the property of the *Euler–Poincaré polynomial* $P(X)$ of a k-variety X: For a k-variety X, we have $\deg P(X) = 2 \dim X$. See [10, Corollary 3.5.1]. □

The dimension gives rise to a natural filtration of the ring $\mathcal{M}$. For each integer $m \in \mathbb{Z}$, we define $F_m \subset \mathcal{M}$ to be the subgroup generated by effective elements $[X]\mathbb{L}^n$ of dimension $\leq -m$. It is easy to see that the family $\{F_m\}_m$ satisfies $F_m F_n \subset F_{n+m}$ and hence $\{\mathcal{M}/F_m\}$ is a projective system of abelian groups.

Definition 2.2.6. We define the *complete Grothendieck ring of k-varieties* $\hat{\mathcal{M}}$ by

$$\hat{\mathcal{M}} := \varprojlim_{m} \mathcal{M}/F_m,$$

which is complete with respect to the induced topology.

We call an image of an effective element in $\hat{\mathcal{M}}$ an *effective element* again. A series $\{x_n\}_{n\geq 0}$ of effective elements with $\dim x_n \to -\infty$ $(n \to \infty)$ converges to 0, and the infinite sum $\sum_{n\geq 0} x_n$ converges in the ring $\hat{\mathcal{M}}$.

Definition 2.2.7. Let I be a countable set and $x_i \in \hat{\mathcal{M}}$ $(i \in I)$ be effective elements indexed by I. We say that the sum $\sum_{i\in I} x_i$ *converges* if for every integer m, there are only finitely many x_i of dimension $\geq m$. If the sum $\sum_{i\in I} x_i$ does not converge, we say that it *diverges*; in this case, we denote $\sum_{i\in I} x_i = \infty$.

Definition 2.2.8. Let $\sum_{i\in I} x_i$ be a countably infinite sum of effective elements. We define its *dimension* by

$$\dim \sum_{i\in I} x_i := \sup_{i\in I} \dim x_i.$$

We say that the sum $\sum_{i\in I} x_i$ is *dimensionally bounded* if it has finite dimension.

We remark that being dimensionally bounded does not imply convergence; if given sum has infinitely many terms of the same dimension, then it diverges.

We give two variants of the Grothendieck ring of k-varieties. The one is needed for the McKay correspondence.

Definition 2.2.9. A morphism $f\colon Y \to X$ of schemes is called a *universal homeomorphism* if every scheme X' over X, the induced morphism $f_{X'}\colon Y \times_X X' \to X'$ is a homeomorphism. If this is the case, we say that Y is universal homeomorphic to X.

We consider the equivalence relation generated by the following: for a morphism $f\colon Y \to X$ of k-varieties and for a non negative integer $m \in \mathbb{Z}_{\geq 0}$, if every geometric fiber of f, say over an algebraically closed filed L, is universal homeomorphic to the quotient of $\mathbb{A}^m_L$ by some finite group action, then $[Y] = [X]\mathbb{L}^m$.

Definition 2.2.10. We define the ring $K_0'(\mathbf{Var}_k)$ to be the quotient of the Grothendieck ring $K_0(\mathbf{Var}_k)$ by the equivalence considered above. We denote by $\mathcal{M}' = K_0'(\mathbf{Var}_k)[\mathbb{L}^{-1}]$ the localization and by $\hat{\mathcal{M}}'$ the completion of it.

The other is needed to define the invariant called stringy motive for $\mathbb{Q}$-Gorenstein varieties (see Remark 2.4.4).

Definition 2.2.11. Let r be a positive integer. We define the ring $K_0(\mathbf{Var}_k)_r := K_0(\mathbf{Var}_k)[\mathbb{L}^{-1/r}]$ to be the localization of the Grothendieck ring $K_0(\mathbf{Var}_k)$ by the r-th root of $\mathbb{L}$. In the similar way, we also define the complete ring $\hat{\mathcal{M}}_r$ and $\hat{\mathcal{M}}'_r$.

2.2.3 Jet schemes and arc schemes

Let X be a k-variety of dimension d and let n be a non negative integer. By a *n-jet* of X, we mean it is a morphism $\operatorname{Spec} k[t]/(t^{n+1}) \to X$ of k-schemes. By an *arc* of X, we mean it is a morphism $\operatorname{Spec} k[[t]] \to X$ of k-schemes. We denote by **Set** the category of sets, by $\mathbf{Sch}_k$ the category of k-schemes, by $\mathbf{Aff}_k$ the category of affine k-schemes, and by $\mathbf{Alg}_k$ the category of k-algebra.

Proposition 2.2.12. *The following contravariant functor*

$$J_nX : \mathbf{Sch}_k \to \mathbf{Set},\ Z \mapsto \operatorname{Hom}_{\mathbf{Sch}_k}(Z \times_k \operatorname{Spec} k[t]/(t^{n+1}), X)$$

*is representable by a k-scheme of finite type; We also denote it by J_nX and call it the n-*jet scheme *of X. In particular, the set of k-valued points $(J_nX)(k)$ is the set of n-jets of X.*

Proof. It follows from the fact that the functor is a Weil restriction, see [6, p. 276] or [10, Proposition 2.1.3]. We can also prove by a concrete construction for affine schemes, and then glue them for general k-schemes (cf. [20, Section 2]). □

For non negative integers n and m with $n < m$, the natural homomorphism $k[t]/(t^{m+1}) \to k[t]/(t^{n+1})$ induces a morphism $\pi_{m,n} : J_mX \to J_nX$ of k-schemes. We call it the *truncation morphism*. The following is an important property of the truncation morphisms.

Lemma 2.2.13. *Let $U \subset X$ be an open subscheme of X. Then the n-jet scheme J_nU of U identifies with the open subscheme $(\pi_{n,0})^{-1}(U)$ of J_nX. Moreover, the truncation morphisms $\pi_{m,n}$ are affine.*

Proof. See [10, Proposition 2.2.3 and Corollary 2.2.4]. □

Definition 2.2.14. The truncation morphisms $\{\pi_{m,n} : J_mX \to J_nX\}_{n<m}$ form a projective system. We define the *arc scheme* $J_\infty X$ of X to be the projective limit of the jet schemes of X:

$$J_\infty X := \varprojlim J_nX.$$

We call the canonical morphism $\pi_n : J_\infty X \to J_nX$ the *truncation morphism* again.

In general, projective limits of schemes may not exist. However, the arc scheme $J_\infty X = \varprojlim J_nX$ exists since the truncation morphisms $\pi_{m,n}$ are affine, see [17, Proposition (8.2.3)].

Proposition 2.2.15. *The following contravariant functor*

$$\mathbf{Aff}_k \to \mathbf{Set},\ \operatorname{Spec} A \mapsto \operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} A[[t]], X)$$

is representable by the arc scheme $J_\infty X$.

Proof. For the case $X = \operatorname{Spec} R$ is affine, we have

$$\begin{aligned}(J_\infty X)(A) &= \varprojlim \operatorname{Hom}_{\mathbf{Alg}_k}(R, A[t]/(t^{n+1})) \\ &= \operatorname{Hom}_{\mathbf{Alg}_k}(R, A[[t]]) \\ &= \operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} A[[t]], X).\end{aligned}$$

For the general case, it is the consequence of Bhatt's theorem below. ◻

Theorem 2.2.16 ([5])**.** *Let A be a ring which is I-adically complete for some ideal $I \subset A$. For every scheme X, we have $X(A) \simeq \varprojlim X(A/I^n)$ via the natural map.*

2.2.4 Motivic measure and integration

Let X be a k-variety of dimension d.

Definition 2.2.17.

(1) We say that $C \subset J_\infty X$ is a *cylinder at level n* if the image $\pi_n(C) \subset J_n X$ is a constructible subset such that $C = \pi_n^{-1}(\pi_n(C))$. We simply say that C is a *cylinder* if it is a cylinder at some level n.

(2) We say that a cylinder $C \subset J_\infty X$ is *stable at level n* if C is a cylinder at level n and if $\pi_{m+1}(C) \to \pi_m(C)$ is a trivial $\mathbb{A}_k^d$-bundle for every $m \geq n$. We simply say that C is a *stable cylinder* if it is stable at some level n.

First, we define a measure μ_X of cylinders in the arc space $J_\infty X$ of X.

Definition 2.2.18. We set $J_\infty^{(n)}(X) := J_\infty X \setminus \pi_n^{-1}(J_\infty(X_{\mathrm{sing}}))$, where X_{sing} is the singular locus of X. The *motivic measure* of cylinders in $J_\infty X$ is the map

$$\mu_X \colon \{\text{cylinders in } J_\infty X\} \to \hat{\mathcal{M}}$$

defined as follows: For a stable cylinder C at level n, we define

$$\mu_X(C) := [\pi_n(C)]\mathbb{L}^{-nd}.$$

For a general cylinder C, we define

$$\mu_X(C) := \lim_{n\to\infty} \mu_X(C \cap J_\infty^{(n)} X).$$

Indeed, $C \cap J_\infty^{(n)} X$ is a stable cylinder and the limit $\lim_{n\to\infty} C \cap J_\infty^{(n)} X$ exists in $\hat{\mathcal{M}}$. See, for example, [12, Lemma 4.1 and 4.2], [30, Section 4.5], or [10, Chapter 6]. Since we consider stable cylinders C, thus the measure $\mu_X(C)$ is independent of the choice of level n.

Next, we define measurable subsets of $J_\infty X$ and motivic integral on it. We consider a semi-norm $\|\bullet\|$ on $\hat{\mathcal{M}}$ defined by

$$\|\bullet\|\colon \hat{\mathcal{M}} \to \mathbb{R}_{\geq 0},\ x \mapsto 2^{-n},$$

where $n = \max\{m \mid x \in F_m\}$ (for the definition of F_m, see Section 2.2.2). From the triangle inequality, the following holds: For cylinders $C, D \subset J_\infty X$, we have

$$\|\mu_X(C \cup D)\| \leq \max\{\|\mu_X(C)\|, \|\mu_X(D)\|\}.$$

For subsets $A, B \subset S$ of a set S, we denote by $A \triangle B \coloneqq (A \cup B) \setminus (A \cap B)$ the symmetric difference of them.

Definition 2.2.19. We say that $C \subset J_\infty X$ is *measurable* if for every positive real number $\epsilon > 0$, there exists a family $\{C_i(\epsilon)\}_{i \geq 0}$ of cylinders satisfying the following:

(1) $(C \triangle C_0(\epsilon)) \subset \bigcup_{i>0} C_i(\epsilon)$,

(2) for every $i > 0$, $\|\mu_X(C_i(\epsilon))\| < \epsilon$.

With notation as above, we define the *motivic measure* of a measurable subset $C \subset J_\infty X$ by

$$\mu_X(C) \coloneqq \lim_{\epsilon \to 0} \mu_X(C_0(\epsilon)).$$

Lemma 2.2.20. *In the definition above, the limit* $\lim_{\epsilon \to 0} \mu_X(C_0(\epsilon))$ *exists and is independent of the choice of the family* $\{C_i(\epsilon)\}_{i \geq 0}$.

Proof. See [2, Theorem 6.18] and [13, Theorem A.6]. □

Definition 2.2.21. Let $C \subset J_\infty X$ be a measurable subset and let $F\colon C \to \mathbb{Z} \cup \{\infty\}$ a function. We say that F is *measurable* if each fiber of F is measurable. Then we define the integral by

$$\int_C \mathbb{L}^F \coloneqq \sum_{n \in \mathbb{Z}} \mu_X(F^{-1}(n))\mathbb{L}^n \in \hat{\mathcal{M}} \cup \{\infty\}.$$

Note that $\int_C \mathbb{L}^F$ does not necessarily converge.

As integrands F, we often consider the following *order functions*.

Definition 2.2.22. Let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf. Then we define

$$\operatorname{ord} \mathcal{I} : J_\infty X \to \mathbb{Z} \cup \{\infty\},\ \gamma \mapsto \operatorname{length} k[[t]]/\gamma^{-1}\mathcal{I}.$$

For the closed subscheme $Z \subset X$ defined by $\mathcal{I}$, we also write $\operatorname{ord} Z = \operatorname{ord} \mathcal{I}$.

2.3 Singularities

We shall briefly recall basic notions concerning singularities, refer [23, Section 2.1] if necessary. Let X be a normal k-variety of dimension d. We denote by X_{sing} the singular locus of it. By a $\mathbb{Q}$-Weil divisor *over* X, we mean it is a $\mathbb{Q}$-Weil divisor on some normal k-variety Y with $f\colon Y \to X$ a proper birational morphism.

Let $f\colon Y \to X$ be a proper birational morphism such that Y is normal. When X is $\mathbb{Q}$-Gorenstein, we can define the *relative canonical divisor* K_f in the usual way, which is a $\mathbb{Q}$-Weil divisor with a support contained in the exceptional locus $\mathrm{Exc}(f)$. Then we can write $K_f = \sum_E a(E;X)E$. We call $a(E;X)$ the *discrepancy* of E with respect to X. For a prime divisor E on Y, the closure of $f(E) \subset X$ is called the *center* of E on X and denoted by $\mathrm{center}_X E$. We define the *discrepancy* of X by

$$\mathrm{discrep}(\mathrm{center} \subset X_{\mathrm{sing}}; X) \coloneqq \inf_E \left\{ a(E;X) \,\middle|\, \mathrm{center}_X E \subset X_{\mathrm{sing}} \right\},$$

where E runs through the set of all exceptional divisor over X (cf. [23, Definition 4.7.4]).

Definition 2.3.1. Let X be a normal $\mathbb{Q}$-Gorenstein k-variety. Then we say that

$$X \text{ is } \begin{cases} \textit{terminal}, \\ \textit{canonical}, \\ \textit{log terminal}, \\ \textit{log canonical}, \end{cases} \text{if } \mathrm{discrep}(\mathrm{center} \subset X_{\mathrm{sing}}; X) \begin{cases} > 0, \\ \geq 0, \\ > -1, \\ \geq -1. \end{cases}$$

Remark 2.3.2. It is well-known that we have either $\mathrm{discrep}(\mathrm{center} \subset X_{\mathrm{sing}}; X) \geq -1$ or $= -\infty$ (see [24, Corollary 2.31]). We also remark that for factorial k-varieties X, its discrepancy $\mathrm{discrep}(\mathrm{center} \subset X_{\mathrm{sing}}; X)$ are integers if not $-\infty$. Therefore, by Proposition 2.1.8, for $X = \mathbb{A}^d_k/(\mathbb{Z}/p^n\mathbb{Z})$, being canonical is equivalent to being log terminal.

We also need to consider log pairs as is usual in birational geometry. By a *log pair*, we mean it is the pair (X, Δ) of a normal $\mathbb{Q}$-Gorenstein k-variety X and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor Δ on X. Replacing K_f by $K_f - f^*\Delta$, we can consider the discrepancies $a(E; X, \Delta)$ and $\mathrm{discrep}(\mathrm{center} \subset X_{\mathrm{sing}}; X, \Delta)$. We say that a log pair (X, Δ) is *Kawamata log terminal* (*klt* for short) or *log canonical* (*lc* for shot), if $\mathrm{discrep}(\mathrm{center} \subset X_{\mathrm{sing}}; X, \Delta) > -1$ or $\mathrm{discrep}(\mathrm{center} \subset X_{\mathrm{sing}}; X, \Delta) \geq -1$ respectively.

A *log resolution* of a log pair (X, Δ) is a proper birational morphism $f\colon Y \to X$ such that Y is regular, the exceptional locus $\mathrm{Exc}(f)$ is a Weil divisor, and $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f^{-1}(\Delta)$ is simple normal crossing.

2.4 The wild McKay correspondence

Let G be a finite group and let $V = \mathbb{A}_k^d$ be a faithful G-representation. Yasuda proved the wild McKay correspondence theorem.

Theorem 2.4.1 (The wild McKay correspondence [45, Corollary 16.3])**.** *Assume that a finite group G acts on an affine space $V = \mathbb{A}_k^d$ linearly and faithfully. Let $X = V/G$ and let Δ be a $\mathbb{Q}$-Weil divisor on X such that the canonical morphism $V \longrightarrow (X, \Delta)$ is crepant. (Note that if G has no pseudo-reflections, then $\Delta = 0$.) Then we have*

$$M_{\mathrm{st}}(X, \Delta) = \int_{G\text{-}\mathrm{Cov}(D)} \mathbb{L}^{d-\boldsymbol{v}}.$$

Here $M_{\mathrm{st}}(X, \Delta)$ denotes the stringy motive of the pair (X, Δ), G-$\mathrm{Cov}(D)$ denotes the moduli space of G-covers of the formal disk $D := \operatorname{Spec} k[[t]]$, and $\boldsymbol{v}$ denotes the $\boldsymbol{v}$-function $\boldsymbol{v}\colon G\text{-}\mathrm{Cov}(D) \longrightarrow \mathbb{Q}$ associated to the given G-action.

In this section, we introduce notions such as stringy motives, G-covers, and $\boldsymbol{v}$-functions.

Remark 2.4.2. For each $\mathbb{Z}/p^n\mathbb{Z}$-representation, we determine when it has pseudo-reflections (see Corollary 4.1.4). Moreover, we show that if given faithful $\mathbb{Z}/p^n\mathbb{Z}$-representation has a pseudo-reflection, then the divisor Δ as in Theorem 2.4.1 on the quotient variety X is irreducible and has multiplicity $p - 1$. Hence, the pair (X, Δ) is not log canonical unless $p = 2$.

From a viewpoint of applying the wild McKay correspondence to study quotient singularities, we often add assumption that G has no pseudo-reflections for simplicity.

2.4.1 Stringy motives

Definition 2.4.3. Let X be a normal k-variety of dimension d. We assume that X is 1-Gorenstein, that is, the canonical sheaf ω_X of X is invertible. We define the *ω-Jacobian ideal $\mathcal{J}_X$* by

$$\mathcal{J}_X \omega_X = \mathrm{Im}\left(\bigwedge^d \Omega_{X/k} \longrightarrow \omega_X \right).$$

For a log pair (X, Δ) with X a 1-Gorenstein normal k-variety, we define the *stringy motive $M_{\mathrm{st}}(X, \Delta)$* by

$$M_{\mathrm{st}}(X, \Delta) := \int_{J_\infty X} \mathbb{L}^{\operatorname{ord}\Delta + \operatorname{ord}\mathcal{J}_X}.$$

When $\Delta = 0$, we write $M_{\mathrm{st}}(X) = M_{\mathrm{st}}(X, 0)$.

Remark 2.4.4. For an r-Gorenstein k-variety X, we can define the ω-Jacobian ideal $\mathcal{J}_X$ and hence the stringy motive $M_{\mathrm{st}}(X, \Delta)$ as an element of a modified complete Grothendieck ring of k-varieties $\hat{\mathcal{M}}_r$ if it converges. For details, see [10, Chapter 7, Section 3.2.4 and Definition 3.4.3]. However, for simplicity, we assume that X is 1-Gorenstein in this thesis. Indeed, in our situation when $G = \mathbb{Z}/p^n\mathbb{Z}$ acts on $\mathbb{A}_k^d$ linearly and faithfully, the quotient variety $X = \mathbb{A}_k^d/G$ is 1-Gorenstein (see Proposition 2.1.8).

Stringy motives contain information of the singularities. For example, we can express the stringy motive $M_{\mathrm{st}}(X)$ in terms of the discrepancies.

Theorem 2.4.5. *Let X be a log terminal k-variety of dimension d. Suppose that X has a log resolution $f\colon Y \to X$. We write the relative canonical divisor K_f as $K_f = \sum_{i\in I} a_i E_i$ (E_i are prime divisors and $a_i \in \mathbb{Q} \setminus \{0\}$). Then we have*

$$M_{\mathrm{st}}(X) = \sum_{J\subset I} [E_J^\circ] \prod_{j\in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1}, \text{ where } E_J^\circ := \left(\bigcap_{j\in J} E_j\right) \setminus \left(\bigcup_{j\notin J} E_j\right).$$

Proof. This is a consequence of the change of variables formula and the explicit computation of motivic integrals. See [42, Corollary 6.4]. □

Remark 2.4.6. In some works such as [10], the motivic measure of a stable cylinder C at level n is given by $\mu_X(C) = [\pi_n(C)]\mathbb{L}^{-(n+1)d}$. Therefore, the explicit formula of $M_{\mathrm{st}}(X)$ stated above is $\mathbb{L}^d$ times different from one stated in [10, Chapter 7, Proposition 3.4.4].

Proposition 2.4.7 ([42, Proposition 6.6]). *Let X be a normal $\mathbb{Q}$-Gorenstein k-variety. If the stringy motive $M_{\mathrm{st}}(X)$ converges, then X is log terminal. If X has a log resolution, then the converse is also true.*

In addition, we can get more precise information of singularities. Let X be a normal 1-Gorenstein k-variety of dimension d.

Definition 2.4.8. For a measurable subset $U \subset J_\infty X$, we define

$$\lambda(U) := \dim\left(\int_U \mathbb{L}^{\operatorname{ord} \mathcal{J}_X}\right),$$

provided that the integral converges. We say that a measurable subset $U \subset J_\infty X$ is *small* if the integral $\int_U \mathbb{L}^{\operatorname{ord} \mathcal{J}_X}$ converges.

Theorem 2.4.9 ([44, Proposition 2.1]). *Let $\{C_r\}_r$ be a countable collection of small measurable subset $C_r \subset \pi_0^{-1}(X_{\mathrm{sing}})$, where $\pi_0\colon J_\infty X \to J_0 X = X$ denotes the truncation morphism. Suppose that $\pi_0^{-1}(X_{\mathrm{sing}})$ and $\bigcup_r C_r$ coincide outside a measure zero subset. Then we have*

$$\operatorname{discrep}(\text{center} \subset X_{\mathrm{sing}}; X) = d - 1 - \sup_r \lambda(C_r).$$

2.4.2 v-function and its integral

Let G be a finite group and let V be a G-representation of dimension d. We give the definition of the $\boldsymbol{v}$-function $\boldsymbol{v}_V : G\text{-Cov}(D) \to \mathbb{Q}$ associated to V.

By an *étale G-cover* $E^* \to D^* = \operatorname{Spec} K$ of D^*, we mean it is a finite étale morphism of degree $\#G$ endowed with a G-action such that $E^*/G = D^*$. Algebraically, for some G-étale K-algebra M, which ia a finite K-algebra of degree $\#G$ endowed with a G-action such that $M^G \simeq K$, we can write $E^* = \operatorname{Spec} M$. Moreover, there exists a field extension L/K and a positive integer c such that $M = L^c$ and hence we can write

$$\operatorname{Spec} M = \overbrace{\operatorname{Spec} L \sqcup \cdots \sqcup \operatorname{Spec} L}^{c}.$$

By a *G-cover* $E \to D$ of D, we mean it is the normalization E of D in an étale G-cover $E^* \to D^*$. We denote by $G\text{-Cov}(D^*)$ (resp. $G\text{-Cov}(D)$) the set of all étale G-covers of D^* (resp. D). Since there is a one-to-one correspondence between $G\text{-Cov}(D^*)$ and $G\text{-Cov}(D)$, we sometimes identify them.

For an arbitrary finite group G, the moduli space $G\text{-Cov}(D)$ of G-covers of D is considered in [35]. However, for using motivic integration when $G = \mathbb{Z}/p^n\mathbb{Z}$, we can use varieties described in Section 3.1. See also Remark 3.1.12 and Remark 3.1.15.

Let E be a G-cover of D and let $\mathcal{O}_E$ be its coordinate ring. There are two G-actions on $\mathcal{O}_E^{\oplus d}$. First, by definition, G acts on E and hence it extends diagonally to a G-action on $\mathcal{O}_E^{\oplus d}$. Second, considering the scalar extension $k \subset \mathcal{O}_E$, we get

$$G \to GL(V) = GL(d, k) \to GL(d, \mathcal{O}_E).$$

Definition 2.4.10. We define the *tuning module* $\Xi_E \subset \mathcal{O}_E^{\oplus d}$ of E to be the subset of elements on which the two actions defined above coincide.

Lemma 2.4.11 ([43, Proposition 6.3],[37, Lemma 3.2])**.** *The tuning module Ξ_E is a free $k[[t]]$-module of rank d.*

Definition 2.4.12 ([39])**.** We define the *$\boldsymbol{v}$-function* $\boldsymbol{v}_V : G\text{-Cov}(D) \to \mathbb{Q}$ associated to V as follows. Let E^* be the étale G-cover of D^* corresponding to given G-cover E and let $\operatorname{Spec} L$ be a connected component of E. We denote by f_L the inertia degree of $L/k((t))$. Let $x_i = (x_{ij})_{1\le j\le d} \in \mathcal{O}_E^{\oplus d}$ be a $k[[t]]$-basis of Ξ_E. Then we define

$$\begin{aligned} \boldsymbol{v}_V(E) &:= \frac{f_L}{\#G} \operatorname{length} \frac{\mathcal{O}_E}{\big(\det(x_{ij})\big)} \\ &= \frac{f_L}{\#G} \operatorname{length} \frac{\mathcal{O}_E^{\oplus d}}{\mathcal{O}_E \cdot \Xi_E}. \end{aligned}$$

If E is connected and v denotes the normalized valuation on $\mathcal{O}_E$, then we have

$$\boldsymbol{v}_V(E) = \frac{f_L}{\#G} v\big(\det(x_{ij})\big).$$

Furthermore, in the case $G = \mathbb{Z}/p^n\mathbb{Z}$, we have either $f_L = 1$ or the length of the module is zero.

The $\boldsymbol{v}$-function $\boldsymbol{v}_V$ depends on the given G-representation V. We omit the subscript V referring to the representation when it is clear from context. By abuse of notation, we write $\Xi_{E^*} = \Xi_E$ and $\boldsymbol{v}(E^*) = \boldsymbol{v}(E)$ because E is the normalization of D in E^*.

For the precise definition of the integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-\boldsymbol{v}}$ with G arbitrary finite group, see [45, Section 14]. In the case $G = \mathbb{Z}/p^n\mathbb{Z}$, the integral has a more explicit presentation as we see in Section 4.1; we can consider (4.1) as its definition.

Chapter 3

The v-function in terms of ramification jumps

3.1 The moduli space of G-covers

In this section, we discuss the moduli space $G\text{-Cov}(D)$ of G-covers of the formal disk $D := \operatorname{Spec} k[[t]]$, which is the domain of the $\boldsymbol{v}$-function.

3.1.1 The Artin–Schreier–Witt theory

We describe the moduli space $G\text{-Cov}(D)$ by using the Artin–Schreier–Witt theory. First, let us recall basic facts from the theory of Witt vectors.

Let A be a commutative ring with unit. As a set, we define $W[A] := \prod_{\mathbb{N}} A$ as the product of countable copies of A. The elements of $W[A]$ are called *Witt vectors.*

Definition 3.1.1. For non negative integers $n \geq 0$, we define

$$W_n(X_0, X_1, \ldots, X_n) := \sum_{i=0}^{n} p^i X_i^{p^{n-i}} \in \mathbb{Z}[X_0, X_1, \ldots, X_n].$$

Proposition 3.1.2. *For each non negative integer $n \geq 0$ and polynomial $\Phi \in \mathbb{Z}[X, Y]$, there exist polynomials $\phi_i \in \mathbb{Z}[X_0, X_1, \ldots, X_i, Y_0, Y_1, \ldots, Y_i]$ ($i = 0, 1, \ldots, n$) such that they satisfy*

$$W_n(\phi_0, \phi_1, \ldots, \phi_n) = \Phi(W_n(X_0, X_1, \ldots, X_n), W_n(Y_0, Y_1, \ldots, Y_n)),$$

and they are uniquely determined.

Proof. See, for example, [31, Chapter II, Section 6, Theorem 6]. □

We denote by S_i (resp. P_i) the polynomials ϕ_i associated by proposition above with the polynomials $\Phi(X, Y) = X + Y$ (resp. $\Phi(X, Y) = XY$). For Witt vectors $\boldsymbol{a} = (a_0, a_1, \dots)$ and $\boldsymbol{b} = (b_0, b_1, \dots)$, we define binary operations as follows.

(addition) $\boldsymbol{a} + \boldsymbol{b} \coloneqq (S_0(a_0, b_0), S_1(a_0, a_1, b_0, b_1), \dots)$,

(multiplication) $\boldsymbol{a} \cdot \boldsymbol{b} \coloneqq (P_0(a_0, b_0), P_1(a_0, a_1, b_0, b_1), \dots)$.

Proposition 3.1.3.

(1) The addition and multiplication defined above make $W[A]$ a commutative ring with unit. Its additive identity is $\mathbf{0} = (0, 0, \dots)$ and multiplicative one is $\mathbf{1} = (1, 0, \dots)$. The ring $W[A]$ is called the ring of Witt vectors with coefficients in A.

(2) For each non negative integer m, the subset

$$I_m \coloneqq \{(a_0, a_1, \dots) \in W[A] \mid a_0 = a_1 = \cdots = a_{m-1} = 0\}$$

is an ideal of $W[A]$. The quotient ring $W_m[A] \coloneqq W[A]/I_m$ is called the ring of Witt vectors of length m.

From now on, we consider the case $A = K$. We introduce important morphisms. One is the *Frobenius morphism*

$$\mathrm{Frob}\colon W_m[K] \longrightarrow W_m[K], (a_0, a_1, \dots) \mapsto (a_0^p, a_1^p, \dots).$$

We denote by $\wp \coloneqq \mathrm{Frob} - \mathrm{id}$ the *Artin–Schreier morphism.* The other is the *Verschiebung morphism*

$$V\colon W_m[K] \longrightarrow W_{m+1}[K], (a_0, a_1, \dots) \mapsto (0, a_0, a_1, \dots).$$

Remark 3.1.4. The Frobenius morphism Frob is a endomorphism of the ring. The Artin–Schreier morphism $\wp$ and the Verschiebung morphism V are homomorphisms of additive groups. Moreover, the Verschiebung morphism V commutes with the Artin–Schreier morphism $\wp$. We also remark that for each $l \geq 1$ the equality

$$(a_0, a_1, \dots, a_{l-1}, a_l, \dots) = (a_0, a_1, \dots a_{l-1}, 0, \dots) + (0, \dots, 0, a_l, a_{l+1}, \dots)$$

holds. We can easily check those properties.

We denote by K^{sep} the separable closure of K and by K_{p^n} the maximal abelian extension of exponent p^n over K. As sets, we can describe

$$\begin{aligned} G\text{-Cov}(D^*) &= \operatorname{Hom}_{\text{cont}}(\operatorname{Gal}(K^{\text{sep}}/K), \mathbb{Z}/p^n\mathbb{Z}) \\ &= \operatorname{Hom}_{\text{cont}}(\operatorname{Gal}(K_{p^n}/K), \mathbb{Z}/p^n\mathbb{Z}) \\ &= \operatorname{Hom}_{\text{cont}}(\operatorname{Gal}(K_{p^n}/K), \tfrac{1}{p^n}\mathbb{Z}/\mathbb{Z}) \\ &\overset{(\heartsuit)}{=} \operatorname{Hom}_{\text{cont}}(\operatorname{Gal}(K_{p^n}/K), \mathbb{Q}/\mathbb{Z}) \\ &\overset{(\clubsuit)}{=} W_n[K]/\wp(W_n[K]). \end{aligned}$$

Since every element of $\operatorname{Gal}(K_{p^n}/K)$ has order dividing p^n, thus the image of any continuous morphisms $\operatorname{Gal}(K_{p^n}/K) \to \mathbb{Q}/\mathbb{Z}$ is contained in $(1/p^n)\mathbb{Z}/\mathbb{Z}$ and hence the equality ($\heartsuit$) holds (see [27, pp. 340–341] for details). The equality ($\clubsuit$) is a consequence of [27, Theorem 6.1.9]. More explicitly, we can see the following (see, for example, [26, Chapter VI, Exercise 50]).

Proposition 3.1.5. *For a Galois extension L/K, it is p^n-cyclic if and only if there exists a Witt vector $\boldsymbol{f} = (f_0, f_1, \ldots, f_{n-1}) \in W_n[K]$ with $f_0 \notin \wp(K)$ such that $L = K(g_0, g_1, \ldots, g_{n-1})$ where the Witt vector $\boldsymbol{g} = (g_0, g_1, \ldots, g_{n-1})$ is a root of an equation $\wp(\boldsymbol{g}) = \boldsymbol{f}$. Moreover, a generator σ of the Galois group* $\operatorname{Gal}(L/K)$ *is given by $\sigma(\boldsymbol{g}) = \boldsymbol{g} + \boldsymbol{1}$. Here we define $\sigma(\boldsymbol{g}) := (\sigma(g_0), \sigma(g_1), \ldots, \sigma(g_{n-1}))$.*

Remark 3.1.6. Let E^* be the étale G-cover of D^* corresponding to a class of a Witt vector $\boldsymbol{f} = (f_0, f_1, \ldots, f_{n-1}) \in W_n[K]$. It is clear that E^* is connected if and only if its coordinate ring is a field. Proposition 3.1.5 shows that E^* is connected if and only if $f_0 \notin \wp(K)$.

Next, we find good representations of elements of $W_n[K]/\wp(W_n[K])$.

Definition 3.1.7. We put $\mathbb{N}' := \{j \in \mathbb{Z} \mid j > 0, p \nmid j\}$. Laurent polynomials of the form

$$\sum_{i \in \mathbb{N}'} g^{(-i)} t^{-i} \in k[t^{-1}] \subset K$$

are called *representative polynomials.* We denote by RP_k the set of representative polynomials.

Remark 3.1.8. More generally, a Witt vector $\boldsymbol{g} = (g_l)_l \in W_n[K]$ is called *reduced* (or *standard form*) if $p \nmid v_K(g_l)$ and $v_K(g_l) < 0$ for every l, where v_K denotes the normalized valuation on k.

Lemma 3.1.9. *For each Witt vector $\boldsymbol{f} \in W_n[K]$, there exists a unique $\boldsymbol{g} \in W_n[K]$ such that each component g_i of $\boldsymbol{g}$ is a representative polynomial and $\boldsymbol{f} - \boldsymbol{g} \in \wp(W_n[K])$. We call a Witt vector $\boldsymbol{g} = (g_0, g_1, \ldots, g_{n-1})$ consisting of representative polynomials a* representative Witt vector.

We prove Lemma 3.1.9 by induction on n. The case $n = 1$ is proved in [42, Lemma 2.3], for example. However, we give the proof below.

Lemma 3.1.10. *We have* $\wp(k[[t]]) = k[[t]]$.

Proof. It is obvious that $\wp(k[[t]]) \subset k[[t]]$. For the converse, let $g = \sum_i g^{(i)} t^i \in k[[t]]$. Let us choose inductively the coefficients $f^{(i)}$ of f so that $\wp(f) = g$. Since k is algebraically closed, thus $\wp(k) = k$ and hence we can choose $f^{(0)} \in \wp^{-1}(g^{(0)})$. If we have chosen $f^{(0)}, f^{(1)}, \dots, f^{(i-1)}$ such that $\wp(f) \equiv g \pmod{t^i}$, then we can set

$$f^{(i)} := \begin{cases} -g^{(i)} & \text{if } p \nmid i, \\ (f^{(i/p)})^p - g^{(i)} & \text{if } p \mid i. \end{cases}$$

This shows that $\wp(f) = g$. □

Proof of Lemma 3.1.9. First, we consider the case $n = 1$. We denote by $f^{(i)}$ the coefficient of t^i in f. From the previous lemma, we may assume that $f^{(i)} = 0$ for $i > 0$. Let pi $(i > 0)$ be the largest multiple of p satisfying $f^{(-pi)} \neq 0$ if any. Replacing f with $f - \wp((f^{(-pi)})^{1/p} t^i)$, we get $f^{(-pi)} = 0$ without changing $f^{(i)}$ for $i < -pi$. Iterating this procedure, we obtain a desired polynomial g. For the uniqueness of g, assume that $g' \in k[t^{-1}]$ satisfies the same condition. If $g \neq g'$, then $p \nmid \mathrm{ord}(g - g')$. However, from the conditions of g and g', we have $g - g' \in \wp(K)$ and hence $p \mid \mathrm{ord}(g - g')$. This is a contradiction.

Next, we consider the case $n \geq 2$. Let us denote by f_l the l-th component of given Witt vector $\boldsymbol{f}$. We choose $h_0 \in K$ satisfying $g_0 = f_0 + \wp(h_0)$, where g_0 is the unique representative polynomial. In the ring $W_n[K]$ of Witt vectors of length n, we have

$$\begin{aligned}(f_0, \dots) + \wp(h_0, \dots) &= (f_0 + \wp(h_0), \dots) \\ &= (g_0, \dots).\end{aligned} \tag{3.1}$$

Without loss of generality, we may assume that $\boldsymbol{f} = (g_0, f_1, \dots)$. From the induction hypothesis, there exists $g_1, g_2, \dots, g_{n-1}$ uniquely such that each g_l is a representative polynomial and

$$(f_1, \dots, f_{n-1}) \equiv (g_1, \dots, g_{n-1}) \pmod{\wp(W_{n-1}[K])}.$$

holds. Since the Verschiebung morphism commutes with $\wp$, thus we have

$$(0, f_1, \dots, f_{n-1}) \equiv (0, g_1, \dots, g_{n-1}) \pmod{\wp(W_n[K])}.$$

Then

$$\begin{aligned}(g_0, f_1, \dots, f_{n-1}) &= (g_0, 0, \dots, 0) + (0, f_1, \dots, f_{n-1}) \\ &\equiv (g_0, 0, \dots, 0) + (0, g_1, \dots, g_{n-1}) \pmod{\wp(W_n[K])} \\ &= (g_0, g_1, \dots, g_{n-1}).\end{aligned}$$

The first and last equality follows from the property of the Verschiebung morphism. Therefore, we have proved the existence of $\boldsymbol{g}$.

And finally, we show the uniqueness. (3.1) shows that the first entry g_0 is uniquely determined. Suppose that $(g_0, g_1, \ldots, g_{n-1})$ and $(g_0, g_1', \ldots, g_{n-1}')$ satisfy the same condition. Then we have

$$(0, f_1, \ldots, f_{n-1}) \equiv (0, g_1, \ldots, g_{n-1}) \equiv (0, g_1', \ldots, g_{n-1}') \pmod{\wp(W_n[K])}.$$

Again from the induction hypothesis, this shows that $g_1, \ldots, g_{n-1}$ are uniquely determined. □

Corollary 3.1.11. *We have a one-to-one correspondence*

$$G\text{-Cov}(D^*) \leftrightarrow G\text{-Cov}(D) \leftrightarrow (\mathrm{RP}_k)^n.$$

Remark 3.1.12. Corollary 3.1.11 shows that G-Cov(D^*) is identified with the k-point set of the ind-scheme $\mathbb{A}_k^\infty := \varinjlim_{n\in\mathbb{N}} \mathbb{A}_k^n$, where the transition map $\mathbb{A}_k^n \to \mathbb{A}_k^{n+1}$ is the standard closed embedding. In fact, the coarse moduli space of G-Cov(D^*) is the inductive perfection (that is, it is the inductive limit with respect to Frobenius morphisms) of this space $\mathbb{A}_k^\infty$, see [18]. To get the fine moduli stack, we further need to take the product of it with the stack BG, see [35].

3.1.2 Stratification and parameterization

In what follows, we follow the convention that $\mathrm{ord}\, 0 := \infty$. For a Witt vector $\boldsymbol{f} = (f_l)_l \in W_n[K]$, we denote $\mathrm{ord}\, \boldsymbol{f} := (\mathrm{ord}\, f_l)_l \in (\mathbb{Z} \cup \{\infty\})^n$. For a G-cover E^* corresponding to the representative Witt vector $\boldsymbol{f} \in W_n[K]$, we denote $\mathrm{ord}\, E^* := \mathrm{ord}\, \boldsymbol{f}$.

Definition 3.1.13. For an n-tuple $\boldsymbol{j} = (j_l)_l \in (\mathbb{N}' \cup \{-\infty\})^n$, we put $-\boldsymbol{j} = (-j_l)_l$. We define

$$G\text{-Cov}(D^*; \boldsymbol{j}) := \{E^* \in G\text{-Cov}(D^*) \mid \mathrm{ord}\, E^* = -\boldsymbol{j}\},$$
$$\mathrm{RP}_{k,\boldsymbol{j}} := \prod_{l=0}^{n-1} \{f \in \mathrm{RP}_k \mid \mathrm{ord}\, f = -j_l\}$$

For the case $\boldsymbol{j} = (j_0)$, we write RP_{k,j_0} in stead of $\mathrm{RP}_{k,(j_0)}$.

When $n = 1$, we have the following one-to-one correspondences (see [42, 10, Proposition 2.11])

$$G\text{-Cov}(D^*; j) \leftrightarrow \mathrm{RP}_{k,j} \leftrightarrow k^\times \times k^{j-1-\lfloor j/p \rfloor}.$$

Here $\lfloor \bullet \rfloor$ denotes the floor function, which assigns a real number a to the greatest integer $\lfloor a \rfloor$ less than or equal to a. We remark that the space G-Cov$(D^*; -\infty)$ is a point.

Proposition 3.1.14. *For $\boldsymbol{j} = (j_l)_l \in (\mathbb{N}' \cup \{-\infty\})^n$, we have one-to-one correspondences*

$$G\text{-Cov}(D^*; \boldsymbol{j}) \leftrightarrow \mathrm{RP}_{k,\boldsymbol{j}} \leftrightarrow \prod_{j_l \neq -\infty} \left(k^\times \times k^{j_l - 1 - \lfloor j_l/p \rfloor}\right).$$

We now regard $k^\times \times k^n$ as the variety $\mathbb{G}_{m,k} \times \mathbb{A}_k^n$. Then the correspondence above gives a structure of k-variety to G-Cov$(D^*; \boldsymbol{j})$. Thus, G-Cov$(D^*; \boldsymbol{j})$ can be thought of as an infinite-dimensional space admitting the stratification

$$G\text{-Cov}(D^*) = \coprod_{\boldsymbol{j}} G\text{-Cov}(D^*; \boldsymbol{j})$$

into countable finite-dimensional strata.

Remark 3.1.15. Varieties $\mathbb{G}_{m,k} \times \mathbb{A}_k^n$ are neither fine or coarse moduli spaces of G-covers (see Remark 3.1.12). However, we can construct families of G-covers over these spaces in a similar way as in [42, Section 2.4] and get morphisms from these spaces to the corresponding fine moduli stacks which are bijective on geometric points. Thus, as justified in [34], we can use the above varieties as our parameter spaces of G-covers in our context of motivic integration.

3.1.3 Description of G-actions on G-covers

Let $\boldsymbol{f} = (f_l)_l \in \mathrm{RP}_{k,\boldsymbol{j}}$ be a representative Witt vector of order $\mathrm{ord}\,\boldsymbol{f} = -\boldsymbol{j}$ and let $\boldsymbol{g} = (g_l)_l$ be a root of $\wp(\boldsymbol{g}) = \boldsymbol{f}$. We assume that the extension $L = K(\boldsymbol{g})$ is a G-extension of K and that a generator σ of G acts on L by $\sigma(\boldsymbol{g}) = \boldsymbol{g} + \mathbf{1}$. We can decompose the extension L/K into a tower of p-cyclic extensions

$$L = K_{n-1} \supset K_{n-2} \supset \cdots \supset K_0 \supset K_{-1} = K$$

where $K_i = K_{i-1}(g_i)$. Indeed, $\sigma^{p^i}|_{K_i}$ fixes K_{i-1} and its order is p. For each extension K_i/K_{i-1}, the i-th component g_i of $\boldsymbol{g}$ is a root of an equation

$$g_i^p - g_i + (\text{polynomial in } g_0, g_1, \ldots, g_{i-1}) = f_i.$$

We denote by v_{K_i} the normalized valuation on K_i. For each index i, by Lemma 3.1.9, there exists an $h_i \in K_{i-1}$ such that $\tilde{f}_i = (g_i + h_i)^p - (g_i + h_i)$, $p \nmid v_{K_{i-1}}(\tilde{f}_i)$, and $v_{K_{i-1}}(\tilde{f}_i) < 0$. We set $\tilde{g}_i := g_i + h_i$ and $\tilde{\boldsymbol{g}} := (\tilde{g}_l)_l$. Since $\tilde{g}_l^{i_l}$ $(0 \le i_l < p)$ form a basis of K_l/K_{l-1}, thus the products $\tilde{g}_0^{i_0}\tilde{g}_1^{i_1}\cdots\tilde{g}_{n-1}^{i_{n-1}}$ $(0 \le i_0, i_1, \ldots, i_{n-1} < p)$ form a basis of L/K.

Notation 3.1.16. For a k-algebra M endowed with a G-action, we denote $\delta := \sigma - \mathrm{id}_M$ a k-linear operator. For $d \in \mathbb{Z}_{\ge 0}$, we write $M^{\delta^d = 0} := \mathrm{Ker}(\delta^d\colon M \to M)$.

For an n-tuple $I = (i_0, i_1, \ldots, i_{n-1}) \in \{0, 1, \ldots, p-1\}^n$, we use a multi-index notation $\tilde{\boldsymbol{g}}^I := \tilde{g}_0^{i_0}\tilde{g}_1^{i_1}\cdots\tilde{g}_{n-1}^{i_{n-1}}$. We remark that to give an n-tuple $I = (i_l)_l$ is equivalent to give an integer $a_I = \sum_{l=0}^{n-1} i_l p^l$.

Proposition 3.1.17. *For each integer a_I with $1 \le a_I < p^n$ and for each $h \in K$, we have $\delta^{a_I}(\tilde{\boldsymbol{g}}^I h) \in k^\times \cdot h$ and $\delta^{a_I+1}(\tilde{\boldsymbol{g}}^I h) = 0$. Therefore, for each integer d with $0 \le d \le p^n$, we have*

$$L^{\delta^d=0} = \bigoplus_{a_I=0}^{d-1} K \cdot \tilde{\boldsymbol{g}}^I.$$

Proof. We prove the proposition by induction on n. The case $n = 1$ is just [42, Lemma 2.15]; we prove this case by induction on i_0 ($1 \le i_0 < p$) as follows. For $i_0 = 1$, since $\sigma(g_0) = g_0 + 1$, thus we have $\delta(gh) = (\sigma(g) - g)h = h$ and $\delta^2(gh) = \delta(h) = 0$. For $i_0 \ge 2$, we have

$$\sigma(g_0^{i_0} h) = (g_0 + 1)^{i_0} h = (g_0^{i_0} + i_0 g_0^{i_0-1} + \cdots + i_0 g_0 + 1)h$$

and hence

$$\delta(g_0^{i_0} h) = (i_0 g_0^{i_0-1} + \cdots + i_0 g_0 + 1).$$

Applying δ^{i_0-1} and δ^{i_0} to this, we obtain the claim.

Let us consider the case $n \ge 2$. By direct computation, we get $\delta^{p^m} = \sigma^{p^m} - \mathrm{id}$ for $0 \le m \le n$. The Artin–Schreier–Witt theory says that σ^{p^m} fixes the subfield $K_{m-1} = K(\tilde{g}_0, \tilde{g}_1, \ldots, \tilde{g}_{m-1})$ and that $\sigma^{p^m}(\tilde{g}_m) = \tilde{g}_m + 1$. Furthermore, δ^{p^m} is not only k-linear but also K_{m-1}-linear. For $1 \le i_m < p$,

$$\begin{aligned} \delta^{p^m}(\tilde{g}_m^{i_m}) &= (\tilde{g}_m + 1)^{i_m} - \tilde{g}_m^{i_m} \\ &= i_m \tilde{g}_m^{i_m-1} + \cdots + i_m \tilde{g}_m + 1. \end{aligned}$$

Applying $(\delta^{p^m})^{i_m-1}$ to this, by the induction on i_m, we get

$$(\delta^{p^m})^{i_m}(\tilde{g}_m^{i_m}) = i_m \cdot (\delta^{p^m})^{i_m-1}(\tilde{g}_m^{i_m-1})$$

and hence $(\delta^{p^m})^{i_m}(\tilde{g}_m^{i_m}) = i_m!$. Therefore, we have

$$\begin{aligned} \delta^{i_0+i_1p+\cdots+i_np^n}\left(\tilde{g}_0^{i_0}\tilde{g}_1^{i_1}\cdots\tilde{g}_n^{i_n} h\right) &= \delta^{i_0+\cdots+i_{n-1}p^{n-1}}\left(\left(\delta^{p^n}\right)^{i_n}\left(\tilde{g}_0^{i_0}\cdots\tilde{g}_{n-1}^{i_{n-1}}\tilde{g}_n^{i_n} h\right)\right) \\ &= \delta^{i_0+\cdots+i_{n-1}p^{n-1}}\left(\tilde{g}_0^{i_0}\cdots\tilde{g}_{n-1}^{i_{n-1}} \cdot i_n!\, h\right), \end{aligned}$$

and the first assertion follows from the induction on n.

It is clear that $L^{\delta^{p^n}=0} = L = \bigoplus_{a_I=0}^{p^n-1} K \cdot \tilde{\boldsymbol{g}}^I$ holds. Assume $x = \sum_I x_I \tilde{\boldsymbol{g}}^I \in L^{\delta^{d-1}=0}$. Since $L^{\delta^{d-1}=0} \subset L^{\delta^d=0}$, thus we have $x_I = 0$ for $a_I \ge d$ by the induction on d. From the first assertion, we have $\delta^{d-1}(x) = \delta^{d-1}(x_J \tilde{\boldsymbol{g}}^J) = 0$, where J is the index satisfying $a_J = d - 1$. Again from the first assertion, this shows $x_J = 0$. Therefore, we have $L^{\delta^{d-1}=0} \subset \bigoplus_{a_I=0}^{d-2} K \cdot \tilde{\boldsymbol{g}}^I$. The converse also follows from the first assertion. □

Corollary 3.1.18. *We denote by $\mathcal{O}_L$ the integer ring of L and by v_L the normalized valuation on L. For an n-tuple $I = (i_0, i_1, \ldots, i_{n-1}) \in \{0, 1, \ldots, p-1\}^n$, we put $n_I := \lceil -v_L(\tilde{\boldsymbol{g}}^I)/p^n \rceil$. Here $\lceil \bullet \rceil$ denotes the ceiling function, which assigns a real number a to the least integer $\lceil a \rceil$ greater than or equal to a. Then we have*

$$\mathcal{O}_L = \prod_{v_L(\tilde{\boldsymbol{g}}^I t^n) \geq 0} k \cdot \tilde{\boldsymbol{g}}^I t^n = \bigoplus_I k[[t]] \cdot \tilde{\boldsymbol{g}}^I t^{n_I}.$$

Moreover, for each integer d with $0 \leq d \leq p^n$, we have

$$\mathcal{O}_L^{\delta^d=0} = \prod_{\substack{v_L(\tilde{\boldsymbol{g}}^I t^n) \leq 0, \\ 0 \leq a_I < d}} k \cdot \tilde{\boldsymbol{g}}^I t^n = \bigoplus_{0 \leq a_I < d} k[[t]] \cdot \tilde{\boldsymbol{g}}^{a_I} t^{n_I}.$$

Proof. By definition, $v_{K_l}(\tilde{g}_l^{i_l})$ takes distinct values modulo p when i_l runs from 0 to $p-1$. Therefore, $v_L(\tilde{\boldsymbol{g}}^I)$ takes distinct values modulo p^n when a_I runs from 0 to $p^n - 1$. This proves the first assertion. The second assertion follows from Proposition 3.1.17 and the first assertion. □

3.2 v-functions of p^n-cyclic representation

We follow the notation as in Section 3.1.3.

3.2.1 The indecomposable case

Let V be an indecomposable G-representation of dimension d and let E be a connected G-cover of $D = \operatorname{Spec} k[[t]]$. In this section, we give a formula of the value $\boldsymbol{v}_V(E)$ of the $\boldsymbol{v}$-function associated to V at E.

Remark 3.2.1. Since $\boldsymbol{v}_{V \oplus W} = \boldsymbol{v}_V + \boldsymbol{v}_W$ holds, thus the case of indecomposable representations is essential.

Remark 3.2.2. In the case that the given G-cover E is not connected, by definition, connected components of E are isomorphic to each other. Let E' be a connected component of E and let G' be its stabilizer subgroup. Since $G = \mathbb{Z}/p^n\mathbb{Z}$ is abelian, the subgroup G' is uniquely determined (that is, it is independent of the choice of the component E'). In fact, every components are isomorphic as G'-covers. Then we have

$$\boldsymbol{v}_V(E) = \boldsymbol{v}_{V'}(E'),$$

where V' is the restriction of V to G'.

We denote by $k[\boldsymbol{x}] = k[x_1, x_2, \dots, x_d]$ the coordinate ring of the affine space V. We choose the coordinates so that the chosen generator σ of G acts by

$$x_i \mapsto \begin{cases} x_i + x_{i+1} & \text{if } i \neq d, \\ x_d & \text{if } i = d. \end{cases}$$

It amounts to taking the Jordan standard form of σ. We have $d \leq p^n$, since the order of a Jordan block of size m with eigenvalue 1 is the greatest power of p does not exceeding m. Let $E^* = \operatorname{Spec} L$ be an étale G-cover of $D^* = \operatorname{Spec} K$, where L/K is a G-extension. We denote by $\mathcal{O}_L$ the integer ring of L. Then the tuning module Ξ_{E^*} of E^* is written as

$$\begin{aligned} \Xi_{E^*} &= \left\{ (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathcal{O}_L^{\oplus d} \;\middle|\; \sigma(\alpha_i) = \alpha_i + \alpha_{i+1} (i < d), \sigma(\alpha_d) = \alpha_d \right\} \\ &= \left\{ (\alpha, \delta(\alpha), \dots, \delta^{d-1}(\alpha)) \in \mathcal{O}_L^{\oplus d} \;\middle|\; \alpha \in \mathcal{O}_L^{\delta^d = 0} \right\}. \end{aligned}$$

Corollary 3.1.18 gives us a $k[[t]]$-basis of $\mathcal{O}_L^{\delta^d=0}$.

Lemma 3.2.3. *With the notation of Section 3.1.3, we have*

$$\boldsymbol{v}_V(E^*) = \sum_{\substack{0 \leq i_0 + i_1 p + \cdots + i_{n-1} p^{n-1} < d, \\ 0 \leq i_0, i_1, \dots, i_{n-1} < p}} \left\lceil - \frac{i_0 v_L(\tilde{g}_0) + i_1 v_L(\tilde{g}_1) + \cdots + i_{n-1} v_L(\tilde{g}_{n-1})}{p^n} \right\rceil.$$

Proof. Let n_I be an integer as in Corollary 3.1.18. By Proposition 3.1.17, we find that the matrix $\left(\delta^m(\tilde{\boldsymbol{g}}^I t^{n_I})\right)_{I,m}$ is a triangular and that the diagonal components $\delta^{a_I}(\tilde{\boldsymbol{g}}^I t^{n_I})$ are of the form $h t^{n_I}$ $(0 \neq h \in k)$. Then

$$\begin{aligned} \boldsymbol{v}_V(E^*) &= \frac{1}{\# G} v_L\Big(\det\left(\delta^m(\tilde{\boldsymbol{g}}^I t^{n_I})\right)_{I,m}\Big) \\ &= \frac{1}{p^n} \sum_{0 \leq a_I < d} v_L(t^{n_I}) \\ &= \sum_{0 \leq a_I < d} n_I, \end{aligned}$$

which is the desired conclusion. □

3.2.2 Ramification jumps

We next determine the values $v_L(\tilde{g}_l)$ by studying ramification of L/K. We begin with recalling the notions of lower and upper ramification groups. The basic reference here is [31]. For a while, we follow the notation below: Let K be a complete discrete valuation field with the perfect residue field of characteristic $p > 0$, and let L/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$. We denote by $\mathcal{O}_L$ the valuation ring of L, by $\mathfrak{p}_L$ the prime ideal of $\mathcal{O}_L$, and by v_L the normalized valuation on L.

Definition 3.2.4. For each integer $i \geq -1$, we set

$$\begin{aligned} G_i &:= \left\{\gamma \in G \,\middle|\, \gamma \text{ acts trivially on } \mathcal{O}_L/\mathfrak{p}_L^{i+1}\right\} \\ &= \{\gamma \in G \mid v_L(\gamma(a) - a) \geq i + 1 \text{ for all } a \in \mathcal{O}_L\} \end{aligned}$$

and call it the i-th *lower ramification group* of L/K. For the equality of two definition, see [31, Chapter IV, Section 1, Lemma 1].

We remark that the lower ramification groups form a descending sequence $\{G_i\}_i$ of normal subgroups of G, and $G_i = \{1\}$ for sufficiently large i.

Let us next define upper ramification groups. For $t \in \mathbb{R}_{\geq -1}$, we put

$$\begin{aligned} G_t &:= G_{\lceil t \rceil}, \\ (G_0 : G_t) &:= 1 \quad (t < 0). \end{aligned}$$

Then we define the *Hasse–Herbrand function* $\varphi = \varphi_{L/K} \colon \mathbb{R}_{\geq -1} \longrightarrow \mathbb{R}_{\geq -1}$ by

$$\varphi_{L/K}(u) := \int_0^u \frac{dt}{(G_0 : G_t)}.$$

This function $\varphi_{L/K}$ is strictly increasing and a self-homeomorphism of $\mathbb{R}_{\geq -1}$. We denote $\psi = \psi_{L/K} := \varphi_{L/K}^{-1}$.

Definition 3.2.5. For $u \geq -1$, we call $G^u := G_{\psi_{L/K}(u)}$ the u-th *upper ramification group* of L/K.

As with the lower ramification groups, the upper ramification groups form a descending sequence $\{G^u\}_u$ of normal subgroups of G, and $G^u = \{1\}$ for sufficiently large u.

Proposition 3.2.6. *For (normal) subgroups of G, the following hold.*

(1) Let $H \subset G$ be a subgroup. For each integer $i \geq -1$, we have $H_i = G_i \cap H$.

(2) Let $H \subset H$ be a normal subgroup. For each real number $u \geq -1$, we have $(G/H)^u = G^u H/H$.

Here the filtration $\{H_i\}_i$ on H is induced from the H-extension derived from L/K.

Proof. See, for example, [31, Chapter IV, Proposition 2 and 14]. □

Definition 3.2.7. We say that l is a *lower ramification jump* of L/K if $G_l \neq G_{l+1}$. Also, we say that u is an *upper ramification jump* of L/K if $G^u \neq G^{u+\epsilon}$ for all $\epsilon > 0$.

We now restrict ourselves to the case of our principal interest where $G = \mathbb{Z}/p^n\mathbb{Z}$ and $K = k((t))$. From [31, p. 67,Corollary 3], each graded piece G_i/G_{i+1} is either 1 or $\mathbb{Z}/p^n\mathbb{Z}$. Therefore, there are exactly n lower ramification jumps and hence there are exactly n upper ramification jumps.

By definition, we can write

$$\psi(u) = \int_0^u (G^0 : G^w) dw.$$

Let $u_0 < u_1 < \cdots < u_{n-1}$ be the upper ramification jumps. Then, for any real number u with $u_{i-1} < u \leq u_i$, we have $(G^0 : G^u) = p^i$. We remark that $u_0 \geq 0$ because the residue field k is algebraically closed. Therefore, we get

$$\begin{aligned}\psi(u_i) &= \int_0^{u_i} (G^0 : G^w) dw \\ &= \int_0^{u_0} + \int_{u_0}^{u_1} + \cdots + \int_{u_{i-1}}^{u_i} (G^0 : G^w) dw \\ &= u_0 + (u_1 - u_0)p + \cdots + (u_i - u_{i-1})p^i. \end{aligned} \tag{3.2}$$

Note that $\psi(u_i)$ are the lower ramification jumps of L/K by definition. In particular, when $G = \mathbb{Z}/p\mathbb{Z}$, the unique lower ramification jumps is equal to the unique upper ramification jump; we call it simply the *ramification jump*.

With the notation as in Section 3.1.3, the following is immediate from Proposition 3.2.6.

Lemma 3.2.8. *The highest lower ramification jump of K_i/K is equal to the ramification jump of K_i/K_{i-1}.*

Lemma 3.2.9. *The highest upper jump of K_m/K is equal to the $(m+1)$-th upper jump u_m of L/K.*

Proof. Note that the Galois group of K_m/K is the quotient of $G = \mathbb{Z}/p^n\mathbb{Z}$ by the subgroup $p^{n-m-1}\mathbb{Z}/p^n\mathbb{Z}$. From Proposition 3.2.6, the upper ramification jump of K_m/K are exactly $u_0, u_1, \ldots, u_m$, those jumps from a subgroup of G to another both of which contain $p^{n-m-1}\mathbb{Z}/p^n\mathbb{Z}$. The highest one among them is u_m. □

Proposition 3.2.10. *The $(i+1)$-th upper ramification jump of L/K is equal to the ramification jump of K_i/K_{i-1}.*

Proof. The $(i+1)$-th upper ramification jump u_i is equal to the highest upper ramification jump of K_i/K, which is equal to the ramification jump of K_i/K_{i-1}. □

Lemma 3.2.11. *The ramification jump of K_i/K_{i-1} is equal to $v_{K_i}(\tilde{g}_i)$.*

Proof. First, we remark that $p \nmid v_{K_{i-1}}(\tilde{f}_i)$ and $v_{K_{i-1}}(\tilde{f}_i) < 0$. Let $j = -v_{K_{i-1}}(\tilde{f}_i)$ and $j = pq - r$ ($q, r \in \mathbb{Z}$, $0 \le r < p$). Since $\tilde{g}_i$ is a root of the equation $\tilde{g}_i^p - \tilde{g}_i = \tilde{f}_i$, thus we have

$$v_{K_i}(\tilde{g}_i) = -j = -qp + r.$$

We can take an integer $l \in \{1, 2, \dots, p-1\}$ so that $lr = pc + 1$ for some non negative integer $c \in \mathbb{Z}_{>0}$. Then we have

$$\begin{aligned} v_{K_i}(t^{lq-c}\tilde{g}_i^l) &= p(lq - c) - lj \\ &= p(lq - c) - l(qp - r) \\ &= 1, \end{aligned}$$

and hence $s := t^{lq-c}\tilde{g}_i^l$ is a uniformizer of K_i.

By direct computation, we have

$$\begin{aligned} \sigma(s) &= t^{lq-c}(\tilde{g}_i + 1)^l \\ &= t^{lq-c}\tilde{g}_i^l + lt^{lq-c}\tilde{g}_i^{l-1} + \text{higher degree terms} \end{aligned}$$

and hence

$$\sigma(s) - s = lt^{lq-c}\tilde{g}_i^{l-1} + \text{higher degree terms}.$$

Therefore

$$\begin{aligned} v_{K_i}(\sigma(s) - s) &= p(lq - c) + (l - 1)(-qp + r) \\ &= j + 1, \end{aligned}$$

which completes the proof. □

From Lemma 3.2.3, $\boldsymbol{v}_V(\operatorname{Spec} L)$ is expressed in terms of valuations of $\tilde{g}_i$'s, which are in turn related to upper ramification jumps of L/K by the above results. To determine $\boldsymbol{v}_V(\operatorname{Spec} L)$, we now compute the upper ramification jumps in terms of the corresponding representative Witt vectors.

Theorem 3.2.12. *Let L/K be a G-extension given by an equation $\wp(\boldsymbol{g}) = \boldsymbol{f}$, where $\boldsymbol{f}$ is reduced. Then, the highest upper ramification jump is given by*

$$\max\left\{-p^{n-1-i}v_K(f_i) \;\middle|\; i = 0, 1, \dots, n-1\right\}.$$

Here we follow the convention that $v_K(0) = \infty$.

Proof. For an integer m, we define

$$W_n^{(m)}[K] := \left\{(f_0, f_1, \dots, f_{n-1}) \,\middle|\, p^{n-i-1} v_K(f_i) \geq m\right\}.$$

From [7, p. 26, Corollary], for $\boldsymbol{f} \in W_n^{(-m)}[K] \setminus W_n^{(1-m)}[K]$, the corresponding extension L/K has Artin conductor $m+1$ (for the character $\chi\colon G \to \mathbb{C}$ of any faithful irreducible G-representation over $\mathbb{C}$). From [31, Chapter VI, Proposition 5], the highest upper ramification jump is m. □

This theorem together with Lemma 3.2.9 shows the following corollary:

Corollary 3.2.13. *In the same situation as above, the upper ramification jumps of L/K are given by*

$$\begin{aligned} -v_K(f_0) \leq \max\{-p v_K(f_0), -v_K(f_1)\} \leq \cdots \\ \cdots \leq \max\{-p^{n-1-i} v_K(f_i) \mid i = 0, 1, \dots, n-1\}. \end{aligned}$$

We obtain the following as a conclusion of this section.

Theorem 3.2.14. *Let $E^* = \operatorname{Spec} L$ be a connected étale G-cover of $D^* = \operatorname{Spec} K$. Assume that the G-extension L/K is defined by an equation $\wp(\boldsymbol{g}) = \boldsymbol{f}$, where $\boldsymbol{f} \in \mathrm{RP}_k{}^n$ is a representative Witt vector of order $-\boldsymbol{j}$ ($\boldsymbol{j} \in (\mathbb{N}' \cup \{-\infty\})^n$) with $j_0 \neq -\infty$. For $1 \leq i \leq n$, put*

$$\begin{aligned} u_{i-1} &= \max\{p^{n-1-m} j_m \mid m = 0, 1, \dots, i-1\}, \\ l_{i-1} &= u_0 + (u_1 - u_0)p + \cdots + (u_i - u_{i-1})p^i. \end{aligned}$$

Then

$$\boldsymbol{v}(E^*) = \sum_{\substack{0 \leq i_0 + i_1 p + \cdots + i_{n-1} p^{n-1} < d, \\ 0 \leq i_0, i_1, \dots, i_{n-1} < p}} \left\lceil \frac{i_0 p^{n-1} l_0 + i_1 p^{n-2} l_1 + \cdots + i_{n-1} l_{n-1}}{p^n} \right\rceil.$$

Proof. By Corollary 3.2.13, u_i is the $(i+1)$-th upper ramification jump of L/K. We can conclude that l_i is the $(i+1)$-th lower ramification jump of L/K by (3.2), and hence l_i is the lower ramification jump of K_i/K_{i-1} by Proposition 3.2.10. Lemma 3.2.11 shows that $v_L(\tilde{g}_i) = -p^{n-1-i} l_i$. By substituting them to the formula in Lemma 3.2.3, we get the formula desired. □

Remark 3.2.15. The previous theorem in particular shows that the function $\boldsymbol{v}$ is constant on each G-$\mathrm{Cov}(D; \boldsymbol{j})$.

Chapter 4

Convergence of the integral and singularities

4.1 The p^n-cyclic wild McKay correspondence

We repeat the wild McKay correspondence theorem.

Theorem ([45, Corollary 16.3]). *Let G be an arbitrary finite group. Assume that G acts on $V = \mathbb{A}_k^d$ linearly and faithfully. Put $X := V/G$ and let Δ be the $\mathbb{Q}$-Weil divisor on X such that $V \longrightarrow (X, \Delta)$ is crepant. Then we have*

$$M_{\mathrm{st}}(X, D) = \int_{G\text{-}\mathrm{Cov}(D)} \mathbb{L}^{d - v_V}.$$

We consider the case $G = \mathbb{Z}/p^n\mathbb{Z}$, which is our principal interest. In this case, we can describe the measure on $G\text{-}\mathrm{Cov}(D) = \coprod_{\boldsymbol{j}} G\text{-}\mathrm{Cov}(D; \boldsymbol{j})$ explicitly as follows. For a constructible subset $C \subset G\text{-}\mathrm{Cov}(D; \boldsymbol{j})$, then the *measure* ν is given by

$$\nu(C) := [C] \in \hat{\mathcal{M}},$$

where $\hat{\mathcal{M}}$ is the complete Grothendieck ring of k-varieties (see Section 2.2.2 if necessary). For a function $F\colon\ G\text{-}\mathrm{Cov}(D) \longrightarrow \mathbb{Q}$ which is constant on each stratum $G\text{-}\mathrm{Cov}(D; \boldsymbol{j})$, we put $F(\boldsymbol{j}) = F(G\text{-}\mathrm{Cov}(D; \boldsymbol{j}))$. Then we can write

$$\begin{aligned} \int_{G\text{-}\mathrm{Cov}(D)} \mathbb{L}^F &= \sum_{r \in \mathbb{Q}} \nu(F^{-1}(r)) \mathbb{L}^r && (4.1) \\ &= \sum_{\boldsymbol{j}} \nu(G\text{-}\mathrm{Cov}(D; \boldsymbol{j})) \mathbb{L}^{F(\boldsymbol{j})}. \end{aligned}$$

By putting $\boldsymbol{j} = (j_0, j_1, \ldots, j_{n-1}) \in (\mathbb{N}' \cup \{-\infty\})^n$, we have

$$\nu(G\text{-Cov}(D; \boldsymbol{j})) = \prod_{j_l \neq -\infty} (\mathbb{L} - 1)\mathbb{L}^{j_l - 1 - \lfloor j_l/p \rfloor},$$

and hence

$$\int_{G\text{-Cov}(D)} \mathbb{L}^F = \sum_{\boldsymbol{j}} \left(\prod_{j_l \neq -\infty} (\mathbb{L} - 1)\mathbb{L}^{j_l - 1 - \lfloor j_l/p \rfloor} \right) \mathbb{L}^{F(\boldsymbol{j})}.$$

It is well-known that if the given G-action has no pseudo-reflection, then the canonical morphism $V \to V/G$ is crepant. If the action is indecomposable, we can check easily whether G has pseudo-reflections or not as follows.

Lemma 4.1.1. *Let A be a square matrix of size d. We denote by $C_m(\lambda)$ the number of the Jordan block of size m with eigenvalue λ in the Jordan standard form of A. Then we have*

$$C_m(\lambda) = \operatorname{rank}(A - \lambda I)^{m-1} - 2 \operatorname{rank}(A - \lambda I)^m + \operatorname{rank}(A - \lambda I)^{m+1},$$

where I denotes the identity matrix.

Proof. We may assume that A is a Jordan block say with eigenvalue λ'. If $\lambda' \neq \lambda$, the formula is obvious. Let us assume $\lambda' = \lambda$. Then $A - \lambda I$ is nilpotent and $\operatorname{rank}(A - \lambda I)^m = \max\{0, d - m\}$. By direct computation, we obtain the formula. □

Lemma 4.1.2. *Let J be a Jordan block of size d with eigenvalue 1 over k. We write $d = qp + r$ ($0 \leq r < p$). Then the Jordan standard form of J^p has r Jordan blocks of size $q + 1$ and $p - r$ Jordan blocks of size q; in particular, it has exactly p blocks.*

Proof. Since $\operatorname{char} k = p$, it is easy to see that $\operatorname{rank}(J^p - I)^m = \operatorname{rank}(J - I)^{pm} = \max\{0, d - pm\}$. Especially, we have $\operatorname{rank}(J^p - I)^{q-1} = d - p(q - 1) = p + r$, $\operatorname{rank}(J^p - I)^q = d - pq = r$ and $\operatorname{rank}(J^p - I)^{q+1} = \operatorname{rank}(J^p - I)^{q+2} = 0$. Therefore, by Lemma 4.1.1, we get $C_{q+1}(1) = r$ and $C_q(1) = p - r$. The equality $r(q + 1) + (p - r)q = qp + r = d$ completes the proof. □

Lemma 4.1.3. *Let J_d be the Jordan block of size d with eigenvalue 1 ($1 \leq d \leq p^n$). For $1 \leq m < n$, $J_d^{p^m}$ is a pseudo-reflection if and only if $d = p^m + 1$.*

Proof. We shall prove by induction on m. When $m = 1$, the claim follows immediately from Lemma 4.1.2. Let $m > 1$. We write $d = qp + r$ ($0 \leq r < p$).

Then

$$\begin{aligned} J_d^{p^m} &= \left(J_d^p\right)^{p^{m-1}} \\ &\equiv \left(J_{q+1}^{\oplus r} \oplus J_q^{\oplus p-r}\right)^{p^{m-1}} \\ &= \left(J_{q+1}^{p^{m-1}}\right)^{\oplus r} \oplus \left(J_q^{p^{m-1}}\right)^{\oplus p-r}, \end{aligned}$$

where $\equiv$ denotes the similarity equivalence. The matrix $J_d^{p^m}$ is a pseudo-reflection if and only if one of the following holds:

(1) $J_{q+1}^{p^{m-1}}$ is a pseudo-reflection, $r = 1$ and $J_q^{p^{m-1}} = 1$,

(2) $J_q^{p^{m-1}}$ is a pseudo-reflection, $p - r = 1$ and $J_{q+1}^{p^{m-1}} = 1$.

In the latter case (2), by the induction hypothesis, we get $q = p^{m-1} + 1$, which contradicts the equality $J_{q+1}^{p^{m-1}} = 1$. In the former case (1), we get $q+1 = p^{m-1}+1$ and hence $d = p^m + 1$. Conversely, it is obvious that $J_{p^m+1}^{p^m}$ is a pseudo-reflection. □

Corollary 4.1.4. *Let J be a matrix of the Jordan normal form with a unique eigenvalue* 1.

(1) For a given integer $m \geq 0$, the matrix J^{p^m} is a pseudo-reflection if and only if J has one Jordan block of size $p^m + 1$ and all the other blocks have size $\leq p^m$.

(2) Let p^n be the order of J. The group $\langle J \rangle \cong \mathbb{Z}/p^n\mathbb{Z}$ contains a pseudo-refection if and only if J has one Jordan block of size $p^{n-1} + 1$ and all the other blocks have size $\leq p^{n-1}$. Moreover, if this is the case, the pseudo-reflections in the group are $J^{ip^{n-1}}$, $1 \leq i \leq p - 1$.

Proof. (1). The "if" part immediately follows from the last lemma. If there are at least two blocks say A and B of size $> p^m$, then neither A^{p^m} or B^{p^m} are the identity matrix. This shows that J^{p^m} is not a pseudo-reflection. If there is no block of size $> p^m$, then $J^{p^m} = 1$, which is not a pseudo-reflection. Thus, for J^{p^m} being a pseudo-reflection, J needs to have one and only one block of size $> p^m$ whose p^m-th power is a pseudo-reflection. Again from Lemma 4.1.3, this block needs to have size $p^m + 1$.

(2). For the group having pseudo-reflections, the matrix J needs to be of the form as in (1) for some m. Because of the order, we have $m = n - 1$. Conversely, if J is of this form for $m = n - 1$, then the group contains the pseudo-reflection $J^{p^{n-1}}$. Thus the first assertion of (2) holds. To show the second assertion, we first note that when two elements A, B of $\langle J \rangle$ generate the same subgroup, then A is a pseudo-reflection if and only if B is a pseudo-reflection. Therefore we only need to consider the p-powers J^{p^m}. The only pseudo-reflection among them is the one for $m = n - 1$. The second assertion follows. □

Proposition 4.1.5. *Suppose that $G = \mathbb{Z}/p^n\mathbb{Z}$ acts on $\mathbb{A}_k^d$ linearly and effectively and that there exists a pseudo-reflection. Let $H \subset \mathbb{A}_k^d$ be the hyperplane fixed by a pseudo-reflection in G (this hyperplane is independent of the pseudo-reflection from Corollary 4.1.4). Let $\overline{H}$ be the image of H in the quotient variety $\mathbb{A}_k^d/G$ with the reduced structure. Then the map $\mathbb{A}_k^d \to (\mathbb{A}_k^d/G, (p-1)\overline{H})$ is crepant.*

Proof. Let $V := \mathbb{A}_k^d$ and $X := V/G$. Let $v \in H$ be a general k-point whose stabilizer subgroup $S \subset G$ has order p and let $x \in \overline{H}$ be its image. To compute the right coefficient of the boundary divisor on X, it is enough to consider the morphism $\operatorname{Spec} \hat{\mathcal{O}}_{V,v} \to \operatorname{Spec} \hat{\mathcal{O}}_{X,x}$ between the formal neighborhoods of v and x. This morphism is isomorphic to the one similarly defined for the quotient morphism $V \to V/S$ associated to the induced action of $S = \mathbb{Z}/p\mathbb{Z}$ on V with pseudo-reflections. In this case, we know from [42] that the coefficient of the boundary divisor is $p - 1$. This shows the proposition. □

4.2 Convergence of stringy motives

In this section, we determine when the integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-v}$ converges.

4.2.1 Integrals over connected G-covers

First, we consider the integral $\int_{G\text{-Cov}^0(D)} \mathbb{L}^{d-\boldsymbol{v}_V}$ over $G\text{-Cov}^0(D)$, where $G\text{-Cov}^0(D) = \coprod_{j_0 \neq -\infty} G\text{-Cov}(D; \boldsymbol{j})$ denotes the set of connected G-covers of the formal disk $D = \operatorname{Spec} k[[t]]$. As we see in Theorem 3.2.14, for a connected G-cover E, the value $\boldsymbol{v}_V(E)$ is determined by the upper ramification jumps of the corresponding G-extension L/K. By abuse of notation, let us consider $\boldsymbol{v}_V$ as a function in variables $\boldsymbol{u} = (u_0, u_1, \dots, u_{n-1})$. The following is well known (see, for example, [28, Lemma 3.5]).

Lemma 4.2.1. *Let $\boldsymbol{u} = (u_0, u_1, \dots, u_{n-1})$ be an increasing sequence of positive integers of length n. Then $\boldsymbol{u}$ occurs as the set of upper ramification jumps of a G-extension of K if and only if the following conditions hold:*

(1) $p \nmid u_0$, and

(2) for $1 \leq i \leq n-1$, either

(2.a) $u_i = pu_{i-1}$ or

(2.b) *both* $u_i > pu_{i-1}$ *and* $p \nmid u_i$.

Definition 4.2.2. We denote by $\mathcal{U}$ the set of increasing sequences of positive integers of length n satisfying the conditions of Lemma 4.2.1. For $\boldsymbol{u} = (u_0, u_1, \dots, u_{n-1})$, set

$$\mathcal{J}(\boldsymbol{u}) := \left\{ \boldsymbol{j} = (j_0, j_1, \dots, j_{n-1}) \,\middle|\, u_m = \max\{p^{m-1-i} j_i \mid i = 0, 1, \dots, m-1\} \right\}.$$

Then we obtain

$$\int_{G\text{-Cov}^0(D)} \mathbb{L}^{d-v_V} = \sum_{\boldsymbol{u}\in\mathcal{U}} \left(\sum_{\boldsymbol{j}\in\mathcal{J}(\boldsymbol{u})} [G\text{-Cov}(D; \boldsymbol{j})] \right) \mathbb{L}^{d-v_V(\boldsymbol{u})}.$$

In addition, by definition, we have

$$\dim \sum_{\boldsymbol{j}\in\mathcal{J}(\boldsymbol{j})} [G\text{-Cov}(D; \boldsymbol{j})] = d + u_0 - \lfloor u_0/p \rfloor + u_1 - \lfloor u_1/p \rfloor + \cdots + u_{n-1} - \lfloor u_{n-1}/p \rfloor.$$

Therefore, it is enough to study when the function

$$u_0 - \lfloor u_0/p \rfloor + u_1 - \lfloor u_1/p \rfloor + \cdots + u_{n-1} - \lfloor u_{n-1}/p \rfloor - v_V(u_0, u_1, \dots, u_{n-1}) \quad (4.2)$$

in variables $u_0, u_1, \dots, u_{n-1}$ tends to $-\infty$ as $u_0 + u_1 + \cdots + u_{n-1}$ tends to ∞ and when the function is bounded below.

Let us define some invariants in order to study the function v_V.

Definition 4.2.3. For a positive integer d ($d \le p^n$) and a non-negative integer m ($m \le n-1$), we define

$$S_d^{(m)} := \sum_{\substack{0 \le i_0 + i_1 p + \cdots i_{n-1} p^{n-1} < d, \\ 0 \le i_0, i_1, \dots, i_{n-1} < p}} i_m.$$

When an indecomposable G-representation V of dimension d is given, we also write $S_V^{(m)} = S_d^{(m)}$.

Namely, $S_d^{(m)}$ is the sum of the $(m+1)$-th digits of the integers $0, 1, \dots, d-1$ in base-p notation. We can write them explicitly as follows.

Lemma 4.2.4. *Put* $d = d_0 + d_1 p + \cdots + d_{n-1} p^{n-1}$ ($0 \le d_m < p$ *for* $m = 0, 1, \dots, n-2$; $0 \le d_{n-1}$). *Then the equality*

$$S_d^{(m)} = p^m S^{(0)}_{d_m + \cdots + d_{n-1} p^{n-1-m}} + \sum_{l=0}^{m-1} p^l d_l d_m \quad (m > 0)$$

holds. In addition, we have

$$S_d^{(0)} = (d_1 + d_2 p + \cdots d_{n-1} p^{n-2}) \frac{p(p-1)}{2} + \frac{d_0(d_0 - 1)}{2}.$$

Proof. Let $q = d_1 + d_2 p + \cdots + d_{n-1}p^{n-2}$. By definition, we have

$$\begin{aligned}
S_d^{(m)} &= \sum_{\substack{0 \le i_0 + i_1 p + \cdots i_{n-1}p^{n-1} < d, \\ 0 \le i_0, i_1, \ldots, i_{n-1} < p}} i_m \\
&= \sum_{i_0=0}^{p-1} \sum_{\substack{0 \le i_1 + i_2 p + \cdots i_{n-1}p^{n-2} < q, \\ 0 \le i_1, i_1, \ldots, i_{n-1} < p}} i_m + \sum_{i_0=0}^{d_0-1} \sum_{i_1 + i_2 p + \cdots + i_{n-1}p^{n-2} = q} i_m \\
&= \sum_{i_0=0}^{p-1} \sum_{\substack{0 \le i_1 + i_2 p + \cdots i_{n-1}p^{n-2} < q, \\ 0 \le i_1, i_1, \ldots, i_{n-1} < p}} i_m + \sum_{i_0=0}^{d_0-1} d_m \\
&= p \sum_{\substack{0 \le i_0 + i_1 p + \cdots i_{n-2}p^{n-2} < q, \\ 0 \le i_0, i_1, \ldots, i_{n-2} < p}} i_{m-1} + d_0 d_m \\
&= p S_q^{(m-1)} + d_0 d_m.
\end{aligned}$$

Inductively, we get

$$\begin{aligned}
S_d^{(m)} &= p(p S^{(m-2)}_{d_2 + d_3 p + \cdots + d_{n-1}p^{n-3}} + d_1 d_m) + d_0 d_m \\
&\cdots \\
&= p^m S^{(0)}_{d_m + d_{m+1}p + \cdots + d_{n-1}p^{n-1-m}} + \sum_{l=0}^{m-1} p^l d_l d_m,
\end{aligned}$$

which is the first equality. Similarly, we have

$$\begin{aligned}
S_d^{(0)} &= \sum_{\substack{0 \le i_0 + i_1 p + \cdots i_{n-1}p^{n-1} < d, \\ 0 \le i_0, i_1, \ldots, i_{n-1} < p}} i_0 \\
&= \sum_{i_0=0}^{p-1} \sum_{\substack{0 \le i_1 + i_2 p + \cdots + i_{n-1}p^{n-2} < q, \\ 0 \le i_1, i_2, \ldots, i_{n-1} < p}} i_0 + \sum_{i_0=0}^{d_0-1} i_0 \\
&= q \sum_{i_0}^{p-1} i_0 + \frac{d_0(d_0 - 1)}{2} \\
&= q \frac{p(p-1)}{2} + \frac{d_0(d_0 - 1)}{2},
\end{aligned}$$

and hence we obtain the second equality. □

Definition 4.2.5. Let V be an indecomposable G-representation of dimension d. We define

$$D_V^{(m)} := p^{n-1}\left(S_V^{(m)} - (p-1)\sum_{l=m+1}^{n-1} p^{m-l} S_V^{(l)}\right).$$

For decomposable G-representations, we define the invariants $S_V^{(m)}$ and $D_V^{(m)}$ in the way that they become additive for direct sums; that is, for a decomposable G-representation $V = \bigoplus_\lambda V_\lambda$ (each V_λ is indecomposable), we define $S_V^{(m)} := \sum_\lambda S_{V_\lambda}^{(m)}$ and $D_V^{(m)} := \sum_\lambda D_{V_\lambda}^{(m)}$ respectively.

Lemma 4.2.6. *For integers q_m and r_m ($m = 0, 1, \ldots, n-1$), we have*

$$\boldsymbol{v}_V(q_0p^n + r_0, q_1p^n + r_1, \ldots, q_{n-1}p^n + r_{n-1}) = \sum_{m=0}^{n-1} D_V^{(m)} q_m + \boldsymbol{v}_V(r_0, r_1, \ldots, r_{n-1}).$$

Proof. Since the function $\boldsymbol{v}_V$ and the invariants $D_V^{(m)}$ are additive with respect to direct sum of G-representations, we may assume that V is indecomposable of dimension d.

By a direct computing we obtain from Theorem 3.2.14 the equality

$$\begin{aligned}
&\boldsymbol{v}_V(q_0p^n + r_0, u_1p^n + r_1, \ldots, q_{n-1}p^n + r_{n-1}) \\
&= \sum_{\substack{0 \le i_0 + i_1 p + \cdots i_{n-1}p^{n-1} < d, \\ 0 \le i_0, i_1, \ldots, i_{n-1} < p}} \sum_{m=0}^{n-1} i_m p^{n-1-m}(q_0 + (q_1 - q_0)p + \cdots + (q_m - q_{m-1})p^m) \\
&\qquad + \boldsymbol{v}_V(r_0, r_1, \ldots, r_{n-1}).
\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
&\sum_{m=0}^{n-1} i_m p^{n-1-m}(q_0 + (q_1 - q_0)p + \cdots + (q_m - q_{m-1})p^m) \\
=&p^{n-1} i_0 q_0 + p^{n-2} i_1(-(p-1)q_0 + pq_1) + \cdots \\
&+ i_{n-1}(-(p-1)q_0 - p(p-1)q_1 - \cdots - p^{n-2}(p-1)q_{n-2} + p^{n-1}q_{n-1}) \\
=&p^{n-1}(i_0 - (p-1)(p^{-1}i_1 + \cdots + p^{-n+1}i_{n-1}))q_0 \\
&+ p^{n-1}(i_1 - (p-1)(p^{-1}i_2 + \cdots + p^{-n+2}i_{n-1}))q_1 + \cdots \\
&+ p^{n-1} i_{n-1} q_{n-1} \\
=&\left(i_0 p^{n-1} - \sum_{l=1}^{n-1} i_l p^{n-1-l}(p-1) \right) q_0 \\
&+ \left(i_1 p^{n-1} - \sum_{l=2}^{n-1} i_l p^{n-l}(p-1) \right) q_1 + \cdots \\
&+ i_{n-1} p^{n-1} \cdot q_{n-1},
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\sum_{\substack{0 \le i_0 + i_2 p + \cdots i_{n-1} p^{n-1} < d, \\ 0 \le i_0, i_1, \ldots, i_{n-1} < p}} \sum_{m=0}^{n-1} i_m p^{n-1-m}(q_0 + (q_1 - q_0)p + \cdots + (q_m - q_{m-1})p^m) \\
&= D_V^{(0)} q_0 + D_V^{(1)} q_1 + \cdots + D_V^{(n-1)} q_{n-1}
\end{aligned}
$$

The above equalities show the lemma. □

We state the following as a conclusion of this section.

Theorem 4.2.7. *Let V be a faithful G-representation of dimension d (not necessarily indecomposable). The integral $\int_{G\text{-Cov}^0(D)} \mathbb{L}^{d-v_V}$ on the space $G\text{-Cov}^0(D)$ of the connected G-covers converges if and only if the strict inequalities*

$$
1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} < 0 \quad (m = 0, 1, \ldots, n-1)
$$

hold. If the inequalities ≤ 0 hold, then the integral $\int_{G\text{-Cov}^0(D)} \mathbb{L}^{d-v}$ is dimensionally bounded.

Proof. It is obvious that the integral $\int_{G\text{-Cov}^0(D)} \mathbb{L}^{d-v_V}$ converges if and only if (4.2) tends to $-\infty$ as the all variables u_m increase.

From Lemma 4.2.6, we have

$$\begin{aligned}
&u_0 - \lfloor u_0/p \rfloor + u_1 - \lfloor u_1/p \rfloor + \cdots + u_{n-1} - \lfloor u_{n-1}/p \rfloor - v_V(u_0, u_1, \ldots, u_{n-1}) \\
&\equiv_{\mathrm{bdd}} u_0 - u_0/p + u_1 - u_1/p + \cdots + u_{n-1} - u_{n-1}/p - \sum_{m=0}^{n-1} D_V^{(m)} u_m/p^n \\
&= \sum_{m=0}^{n-1} \left(1 - \frac{1}{p} - \frac{D_V^{(m)}}{p^n}\right) u_m =: f(\boldsymbol{u}),
\end{aligned}$$

where $\equiv_{\mathrm{bdd}}$ means equivalence modulo bounded functions. What we want to study is the limit of the function $f(\boldsymbol{u})$. Thus we consider $f(\boldsymbol{u})$ as a function defined on

$$\tilde{\mathcal{U}} := \{\boldsymbol{u} = (u_0, u_1, \ldots, u_{n-1}) \in \mathbb{R}^n \mid u_0 \geq 1, u_i \geq p u_{i-1}\ (i = 1, 2, \ldots, n-1)\}$$

instead of $\mathcal{U}$. For $t \in \mathbb{R}_{\geq 1}$, let $\tilde{\mathcal{U}}_t := \tilde{\mathcal{U}} \cap \{u_{n-1} = t\}$ be the intersection of the polyhedron $\tilde{\mathcal{U}}$ and the hyperplane $u_{n-1} = t$. Assume $t \geq p^{n-1}$ so that $\tilde{\mathcal{U}}_t$ becomes non empty. Obviously, $\tilde{\mathcal{U}}_t$ is bounded, that is, it is a polytope. Since f is a linear function, thus the maximum value of $f|_{\tilde{\mathcal{U}}_t}$ is attained at the one of its vertices

$$(1, p \ldots, p^{n-2}, t), (1, p \ldots, p^{n-3}, t/p, t), \ldots, (t/p^{n-1}, \ldots, t/p, t) \in \tilde{\mathcal{U}}_t.$$

By substituting, we have

$$\begin{aligned}
f(\overbrace{1, p \ldots, p^{m-1}}^{m}, t/p^{n-1-m}, \ldots, t/p, t) &\equiv_{\mathrm{bdd}} \sum_{l=m}^{n-1} \left(1 - \frac{1}{p} - \frac{D_V^{(l)}}{p^n}\right) \frac{t}{p^{n-1-l}} \\
&= \left(1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}}\right) t.
\end{aligned}$$

Therefore, this shows that the function $f(\boldsymbol{u})$ tends to $-\infty$ if and only if the coefficients $1 - 1/p^{n-m} - \sum_{l=m}^{n-1} D_V^{(l)}/p^{2n-1-l}$ are all negative. □

4.2.2 Criteria of the convergence and singularities

We consider the integral $\int_{G\text{-}\mathrm{Cov}(D)} \mathbb{L}^{d-v}$ over the moduli space G-Cov(D) of all G-covers with the results in the previous section. An aim of this section is to prove the following.

Theorem 4.2.8. *Let V be a faithful G-representation of dimension d. The integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-v_V}$ converges if and only if the strict inequalities*

$$1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} < 0 \quad (m = 0, 1, \ldots, n-1)$$

hold. If the inequalities ≤ 0 hold, then the integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-v}$ is dimensionally bounded.

Lemma 4.2.9. *Let V be an effective G-representation of dimension d and let W be the restriction of V to $H = \mathbb{Z}/p^{n-1}\mathbb{Z} \subset G$. Then, for $m = 1, 2, \ldots, n-1$, we have $D_V^{(m)} = pD_W^{(m-1)}$.*

Proof. Since the invariants $D_V^{(m)}$ are additive with respect to direct sum, thus we may assume that V is indecomposable. Moreover, by Lemma 4.1.2, we obtain

$$W \simeq W_{q+1}^{\oplus r} \oplus W_q^{\oplus p-r},$$

where $d = r+qp$ $(0 \leq r < p)$ and W_e denotes the indecomposable G-representation of dimension e. Put $d = d_0 + d_1 p + \cdots d_{n-1}p^{n-1}$ $(0 \leq d_m < p$ for $m = 0, 1, \ldots, n-2)$. Note that $r = d_0$ and $q = d_1 + d_2 p + \ldots + d_{n-1}p^{n-2}$. By definition, we have

$$\begin{aligned} S_{q+1}^{(m-1)} &= \sum_{\substack{0 \leq i_0 + i_1 p + \cdots + i_{n-2}p^{n-2} < q+1, \\ 0 \leq i_0, i_1, \ldots, i_{n-2} < p}} i_{m-1} \\ &= \sum_{\substack{0 \leq i_0 + i_1 p + \cdots + i_{n-2}p^{n-2} < q, \\ 0 \leq i_0, i_1, \ldots, i_{n-2} < p}} i_{m-1} + \sum_{\substack{0 \leq i_0 + i_1 p + \cdots + i_{n-2}p^{n-2} = q+1, \\ 0 \leq i_0, i_1, \ldots, i_{n-2} < p}} i_{m-1} \\ &= \sum_{\substack{0 \leq i_0 + i_1 p + \cdots + i_{n-2}p^{n-2} < q, \\ 0 \leq i_0, i_1, \ldots, i_{n-2} < p}} i_{m-1} + d_m \\ &= S_q^{(m-1)} + d_m, \end{aligned}$$

and hence

$$rS_{q+1}^{(m-1)} + (p-r)S_q^{(m-1)} = pS_q^{(m-1)} + rd_m.$$

Therefore, combining $S_d^{(m)} = pS_q^{(m-1)} + d_0 d_m$ (see the proof of Lemma 4.2.9), we obtain

$$S_d^{(m)} = rS_{q+1}^{(m-1)} + (p-r)S_q^{(m-1)}.$$

Now the lemma follows from the definition of $D_V^{(m)}$. □

Proof of Theorem 4.2.8. We prove by induction on n. The case $n = 1$ is just [42, Proposition 6.9]; the proof is following. By direct computing, we have

$$\begin{aligned}\int_{G\text{-Cov}(D)} \mathbb{L}^{d-v} &= \sum_j [G\text{-Cov}(D; j)]\mathbb{L}^{d-v(j)} \\ &= \sum_j (\mathbb{L}-1)\mathbb{L}^{j-1-\lfloor j/p \rfloor} \cdot \mathbb{L}^{d-v(j)} \\ &= (\mathbb{L}-1)\mathbb{L}^{d-1} \sum_j \mathbb{L}^{j-1-\lfloor j/p \rfloor},\end{aligned}$$

putting $j = qp + r$ $(0 < q < p)$, we have $v(qp + r) = D_V^{(0)} q + v(r)$ and hence

$$= (\mathbb{L}-1)\mathbb{L}^{d-1} \sum_{r=1}^{p-1} \sum_q \mathbb{L}^{(p-1-D_V^{(0)})q+r-v(r)}.$$

Therefore, we get that the integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-v}$ converges if and only if the inequality $p - 1 - D_V^{(0)} < 0$ holds.

Let $n \geq 2$ and let $H = \mathbb{Z}/p^{n-1}\mathbb{Z}$ be the subgroup of G of index p and W the restriction of V to H. By definition, a non connected G-cover has p components which are stabilized under H. This means that non connected G-covers are also H-covers. Let us divide the integral as follows:

$$\int_{G\text{-Cov}(D)} \mathbb{L}^{d-v_V} = \int_{H\text{-Cov}(D)} \mathbb{L}^{d-v_W} + \int_{G\text{-Cov}^0(D)} \mathbb{L}^{d-v_V},$$

where $G\text{-Cov}^0(D)$ is the moduli space of connected G-covers of D. Note that the necessary and sufficient condition on convergence of $\int_{H\text{-Cov}(D)} \mathbb{L}^{d-v_W}$ is given by the induction hypothesis, and one of $\int_{G\text{-Cov}^0(D)} \mathbb{L}^{d-v_V}$ is by Theorem 4.2.7. From Lemma 4.2.9, we have

$$\begin{aligned}1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} &= 1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{pD_W^{(l-1)}}{p^{2n-1-l}} \\ &= 1 - \frac{1}{p^{(n-1)-(m-1)}} - \sum_{l=m-1}^{n-2} \frac{D_W^{(l)}}{p^{2(n-1)-1-l}}\end{aligned}$$

for $m = 1, 2, \ldots, n-1$. Therefore, the convergence of the integral $\int_{G\text{-Cov}^0(D)} \mathbb{L}^{d-v_V}$ implies that of the integral $\int_{H\text{-Cov}(D)} \mathbb{L}^{d-v_W}$, and hence the proof is completed. □

Corollary 4.2.10. *Let V be a faithful G-representation (not necessarily indecomposable) and let $X := V/G$ be the quotient variety.*

(1) X is canonical if the strict inequalities $1 - 1/p^{n-m} - \sum_{l=m}^{n-1} D_V^{(l)}/p^{2n-1-l} < 0$ ($m = 0, 1, \dots, n-1$) hold. Furthermore, if there is a log resolution of X, then the converse is also true.

(2) X is log canonical if and only if the inequalities $1-1/p^{n-m}-\sum_{l=m}^{n-1} D_V^{(l)}/p^{2n-1-l} \leq 0$ ($m = 0, 1, \dots, n-1$) hold.

Proof. (1). If the strict inequalities hold, then the integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-v}$ and hence the stringy motive $M_{\mathrm{st}}(X)$ converges. From Proposition 2.4.7, we obtain the claim.

(2). Holding the inequalities is equivalent to that the integral $\int_{G\text{-Cov}(D)} \mathbb{L}^{d-v_V}$ is dimensionally bounded. Hence, from Theorem 5.0.1, we obtain the desired conclusion. □

Example 4.2.11. As an example, that is not covered by previous works, consider the case $p = 2$, $n = 3$ and the indecomposable G-representation V_5 of dimension 5. By direct computing, we get

$$S_5^{(0)} = 2, S_5^{(1)} = 2, S_5^{(2)} = 1,$$
$$D_{V_5}^{(0)} = 3, D_{V_5}^{(1)} = 6, D_{V_5}^{(2)} = 4.$$

We consider a decomposable G-representation $V := V_5^{\oplus m}$ ($m \geq 1$). Note that V is not Cohen–Macaulay. Using $D_V^{(l)} = mD_{V_5}^{(l)}$, we easily see whether the inequalities in Corollary 4.2.10 hold or not. Therefore, we obtain that V/G is log canonical if $m \geq 1$ and that V/G is canonical if $m \geq 2$.

4.2.3 Indecomposable cases

We give more precise estimation for the indecomposable cases.

Theorem 4.2.12. *Assume that V is a faithful indecomposable G-representation of dimension d which has no pseudo-reflections. Let $X := V/G$ be the quotient variety.*

(1) X is canonical if $d \geq p + p^{n-1}$. Furthermore, if there is X has a log resolution, then the converse is also true.

(2) X is log canonical if and only if $d \geq p - 1 + p^{n-1}$.

Lemma 4.2.13. *We consider the invariants $D_V^{(m)}$ as functions in variable d. Then the sum $\sum_{l=m}^{n-1} D_V^{(l)}/p^{2n-1-l}$ is strictly increasing.*

Proof. By definition, we have

$$\begin{aligned}\sum_{l=m}^{n-1} p^l D_V^{(l)} &= \sum_{l=m}^{n-1} p^l \left(-(p-1) \sum_{j=l+1}^{n-1} p^{l-j} S_d^{(j)} + S_d^{(l)} \right) \\ &= p^{n-1} \Big(p^m S_d^{(m)} + (-(p-1)p^{m-(m+1)} + p^{m+1}) S_d^{(m+1)} + \\ &\quad \vdots \\ &\quad + (-(p-1)(p^{m-(n-1)} + p^{(m+1)-(n-1)} + \cdots + p^{(n-2)-(n-1)}) + p^{n-1}) S_d^{(n-1)} \Big) \\ &= p^{n-1} \sum_{l=m}^{n-1} \left(-(p-1) \sum_{j=m}^{l-1} p^{j-l} + p^l \right) S_d^{(l)} \\ &= p^{n-1} \sum_{l=m}^{n-1} (p^{m-l} - 1 + p^l) S_d^{(l)} \\ &\geq p^{n-1} \cdot p^{n-1} S_d^{(n-1)} = p^{n-1} D_V^{(n-1)}.\end{aligned}$$

Since $D_V^{(n-1)} = p^{n-1} S_d^{(n-1)}$ is strictly increasing with respect to d, thus we get the desired conclusion. □

Remark 4.2.14. From the proof above, $\sum_{l=m}^{n-1} p^l D_V^{(l)}$ is monotone decreasing with respect to m. Since $D_V^{(n-1)}$ is non-negative, thus $D_V^{(l)}$ are all non negative.

Proof of Theorem 4.2.12. The case $n = 1$ is proved in [44]; we can easily see when the inequalities in Corollary 4.2.10 hold by using $S_d^{(0)} = d(d-1)/2$. We may assume that $n \geq 2$.

(2). We consider the case $d = p - 1 + p^{n-1}$. By direct computation, we obtain

$$S_d^{(0)} = \frac{p^{n-1}(p-1)}{2} + \frac{(p-1)(p-2)}{2},$$

$$S_d^{(m)} = \begin{cases} p^{n-1}(p-1)/2 & \text{if } 0 < m < n-1, \\ p-1 & \text{if } m = n-1, \end{cases}$$

and hence

$$D_V^{(n-1)} = p^{n-1}(p-1),$$

$$D_V^{(n-2)} = p^{n-1}(p-1)\left(\frac{p^{n-1}}{2} - \frac{p-1}{p} \right).$$

For $m = n - 1$, we have

$$1 - \frac{1}{p^{n-(n-1)}} - \frac{D_V^{(n-1)}}{p^{2n-1-(n-1)}} = 1 - \frac{1}{p} - \frac{p-1}{p} = 0.$$

Since $D_V^{(m)}$ are strictly increasing with respect to d, from Corollary 4.2.10 (2), thus X is not log canonical when $d < p - 1 + p^{n-1}$.

On the other hand, for $m = 0, 1, \dots, n-2$, we have

$$1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} < 1 - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} \\ \leq 1 - \left(\frac{D_V^{(n-2)}}{p^{n+1}} + \frac{D_V^{(n)}}{p^n} \right).$$

Thus we obtain

$$1 - \left(\frac{D_V^{(n-2)}}{p^{n+1}} + \frac{D_V^{n-2}}{p^{n+1}} \right) = 1 - \frac{p-1}{p^2} \left(\frac{p^{n-1}}{2} - \frac{p-1}{p} \right) - \frac{p-1}{p} \\ = \frac{1}{2p^3} \left(2p^3 - (p-1)(p^n - 2(p-1)) - 2p^2(p-1) \right) \\ = \frac{2 - p(p-1)(p^{n-1} - 4)}{2p^3}.$$

Therefore, we see that the inequalities in Corollary 4.2.10 (2) hold when $d \geq p - 1 + p^{n-1}$. Hence the quotient $X = V/G$ is log canonical except when $(p, n) = (2, 3)$. We remark that if $(p, n) = (2, 3)$, the G-representation V has pseudo-reflections (see Corollary 4.1.4 (2) for details).

(1). We consider the case $d = p + p^{n-1}$. We similarly have

$$1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} < 1 - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} \leq 1 - \frac{D_V^{(n-1)}}{p^n}.$$

By direct computing, we obtain

$$S_d^{(n-1)} = p, D_V^{(n-1)} = p^n,$$

and hence

$$1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} < 1 - \frac{p^n}{p^n} = 0.$$

Therefore, then the quotient $X = V/G$ is canonical if $d \geq p + p^{n-1}$. □

Example 4.2.15. Let V be a regular representation of $\mathbb{Z}/4\mathbb{Z}$ over the field k of characteristic 2. The invariant ring $k[V]^G$ is a first example which is a unique factorization domain but not Cohen–Macaulay. In general, by Proposition 2.1.8, the invariant rings of p-groups are unique factorization domains. By Theorem 4.2.12, we get the quotient V/G is canonical.

Example 4.2.16. Let V be a d-dimensional indecomposable G-representation. Note that V/G is not Cohen–Macaulay if $d > 3$. Therefore, by Theorem 4.2.12, we get V/G is canonical but not Cohen–Macaulay for sufficiently large d.

4.2.4 Finite groups whose p-Sylow subgroup is cyclic

At the end of this chapter, we slightly generalize our criterion for finite groups whose p-Sylow subgroups are cyclic.

Theorem 4.2.17. *Let $\tilde{G}$ be a finite group whose p-Sylow subgroups is $G = \mathbb{Z}/p^n\mathbb{Z}$, and let $\tilde{V}$ be a faithful $\tilde{G}$-representation. Assume that $\tilde{G}$ has no pseudo-reflections. We denote by V the restriction of $\tilde{V}$ to G. (Note that V is equal to $\tilde{V}$ as a space.) Let $X := V/G$ and $\tilde{X} := \tilde{X}/\tilde{G}$ be the quotient varieties.*

(1) $\tilde{X}$ is lo terminal if the strict inequalities $1 - 1/p^{n-m} - \sum_{l=m}^{n-1} pD_V^{(l)}/p^{2n-1-l} < 0$ $(m = 0, 1, \ldots, n-1)$ hold.

(2) $\tilde{X}$ is log canonical if the inequalities $1 - 1/p^{n-m} - \sum_{l=m}^{n-1} pD_V^{(l)}/p^{2n-1-l} \leq 0$ $(m = 0, 1, \ldots, n-1)$ hold.

Proof. Let $\pi\colon X \longrightarrow \tilde{X}$ be the canonical projection. Since $\tilde{G}$ has no pseudo-reflections, thus the projections $\tilde{V} \longrightarrow \tilde{X}$ and $V \longrightarrow X$ are both étale in codimension one, and so is π (see [16, Proposition 17.7.7] for example).

(1). According to Corollary 4.2.10, X is canonical if the strict inequalities hold. Hence, from [40, Theorem 6.5], we see that $\tilde{X}$ is log terminal.

(2). According to Corollary 4.2.10 again, X is log canonical if and only if the inequalities hold. If $\tilde{X}$ is log canonical, from the contraposition of [40, Theorem 6.4], X is log canonical. Conversely, with similar proof as in [40, Theorem 6.5], we see that if X is log canonical then $\tilde{X}$ is log canonical. □

Remark 4.2.18. There are only finitely many indecomposable $\tilde{G}$-representation up to isomorphisms. Moreover, an explicit formula for the number of non isomorphic ones is given in [21].

Chapter 5

Stringy motives and discrepancies

We now consider an application of Theorem 2.4.9 for the case $G = \mathbb{Z}/p^n\mathbb{Z}$. We denote $X = V/G$ and $\delta(X) = \mathrm{discrep}(\mathrm{center} \subset X_{\mathrm{sing}}; X)$.

The quotient morphism $\mathbb{A}^d_k \to X$ and an arc $D = \operatorname{Spec} K \to X$ induce a G-cover of D, unless the arc maps into the branch locus of $\mathbb{A}^d_k \to X$; the last exceptional case occurs only for arcs in a measure zero subset of $J_\infty X$. For each $\boldsymbol{j} \in (\mathbb{N}' \cup \{-\infty\})^n$, let $M_{\boldsymbol{j}} \subset \pi_0^{-1}(X_{\mathrm{sing}})$ be the locus of arcs including a G-cover E with $\operatorname{ord} E = -\boldsymbol{j}$. Here $\pi_0 \colon J_\infty X \to J_0 X = X$ denotes the truncation morphism. The collection of $M_{\boldsymbol{j}}$ satisfies the condition of Theorem 2.4.9. Suppose that G has no pseudo-reflection. As a variant of Theorem 2.4.1, for $\boldsymbol{j} \neq (-\infty, -\infty, \dots, -\infty)$, we have

$$\int_{M_{\boldsymbol{j}}} \mathbb{L}^{\operatorname{ord} \mathcal{J}_X} = \int_{G\text{-}\mathrm{Cov}(D;\boldsymbol{j})} \mathbb{L}^{d - \boldsymbol{v}_V} = [G\text{-}\mathrm{Cov}(D;\boldsymbol{j})]\mathbb{L}^{d - \boldsymbol{v}_V(\boldsymbol{j})}$$

and

$$\lambda(M_{\boldsymbol{j}}) = \dim \nu(G\text{-}\mathrm{Cov}(D;\boldsymbol{j})) + d - \boldsymbol{v}_V(\boldsymbol{j}).$$

The case $\boldsymbol{j} = (-\infty, -\infty, \dots, -\infty)$ corresponds to the trivial G-cover $\coprod D \to D$. We have

$$\int_{M_{(-\infty,-\infty,\dots,-\infty)}} \mathbb{L}^{\operatorname{ord} \mathcal{J}_X} = [R/G] = [B],$$

where $R \subset \mathbb{A}^d_k$ and $B \subset X$ are the ramification and the branch loci of $\mathbb{A}^d_k \to X$ respectively. In particular,

$$\lambda(M_{(-\infty,-\infty,\dots,-\infty)}) = \dim R = \dim B.$$

These formulae for λ together with Theorem 2.4.9 (or the theorem stated below) enable us to estimate $\delta(X)$ in terms of the $\boldsymbol{v}$-function in theory.

Theorem 5.0.1 ([45, Corollary 16.4 (1)]). *Let G be an arbitrary finite group and let V be a G-representation of dimension d and let $X = V/G$ be the associated quotient variety. Assume that G has no pseudo-reflections. Then we have*

$$\delta(X) = d - 1 - \max\left\{\dim X_{\mathrm{sing}}, \dim \int_{G\text{-Cov}(D)\setminus\{o\}} \mathbb{L}^{d-\boldsymbol{v}_V}\right\},$$

where o denotes the point corresponding the trivial G-cover.

We shall carry it out in the case $G = \mathbb{Z}/p^2\mathbb{Z}$; computation in this case is already rather complicated.

5.1 Estimation of discrepancies for $G = \mathbb{Z}/p^2\mathbb{Z}$

We express given G-representation V as a direct sum $\bigoplus_{\alpha=1}^{a} V_{d_\alpha}$, where V_{d_α} denotes an indecomposable G-representation of dimension d_α.

To compute the value $\boldsymbol{v}_V(j_0, j_1) = \boldsymbol{v}_V(u_0, u_1)$ of the $\boldsymbol{v}$-function, we should decompose $(\mathbb{N}' \cup \{-\infty\})^2$ into four parts:

$$\{(-\infty, -\infty)\}, \{(-\infty, j) \mid j \in \mathbb{N}'\},$$
$$\{(j_0, j_1) \mid pj_0 > j_1\}, \{(j_0, j_1) \mid j_0 \neq -\infty, pj_0 < j_1\}.$$

Note that $\{(-\infty, -\infty)\}$ corresponds to the trivial G-cover. Hence, it is enough to consider the remaining three cases.

5.1.1 The $j_0 = -\infty$ part

This part $\coprod_j G\text{-Cov}(D; -\infty, j)$ corresponds to the G-covers of D which have p connected components. Each component is then an H-cover with $H = \mathbb{Z}/p\mathbb{Z} \subset G$ the subgroup of order p. Let E be a such G-cover and let E' be a connected component of it. We denote by W the restriction of V to H. Note that, by Lemma 4.1.2, we have

$$W \simeq \bigoplus_{\alpha}\left(W_{q_\alpha}^{\oplus r_\alpha} \oplus W_{q_\alpha}^{\oplus p - r_\alpha}\right),$$

where $d_\alpha = r_\alpha + q_\alpha p$ $(0 \leq r_\alpha < p)$ and W_e denotes the indecomposable H-representation of dimension e. Then we have

$$\boldsymbol{v}_V(E) = \boldsymbol{v}_W(E').$$

We also remark that, from the proof of Theorem 4.2.8, the integral $\int_{G\text{-Cov}(D;-\infty,j)} \mathbb{L}^{d-\boldsymbol{v}_W(j)}$ is dimensionally bounded if and only if $p - 1 - S_V^{(1)} \leq 0$ since $D_V^{(1)} = pD_W^{(0)} = pS_V^{(1)}$.

By direct computing, we get

$$\begin{aligned}\lambda(M_{(-\infty,j)}) &= \dim \nu(G\text{-Cov}(D;-\infty,j)) + d - \boldsymbol{v}_V(-\infty,j)\\ &= \dim \nu(H\text{-Cov}(D;j)) + d - \boldsymbol{v}_W(j)\\ &= \lambda(N_j),\end{aligned}$$

where N_j is defined in the same way as M_j for $\mathbb{A}_k^d/H$. From [44, Equation (3.1), (3.2)], if $p-1-S_V^{(1)} \leq 0$, then we have

$$\begin{aligned}\sup_{j_0=-\infty} \lambda(M_j) &= b + \max_{1\leq l_1<p}\{l_1 - \mathrm{sht}_W(l_1)\} \\ &= d - S_V^{(1)} + \max_{1\leq l_1<p}\{\mathrm{sht}_W(p-l_1)+l_1\},\end{aligned} \tag{5.1}$$

where b denotes the number of the indecomposable direct summands of the induced H-representation W and sht_W the function associated to W defined as follows. For an indecomposable representation W_e of dimension e, we define

$$\mathrm{sht}_{W_e}(l) := \sum_{i=1}^{e-1}\left\lfloor \frac{il}{p}\right\rfloor.$$

In general, for the case $W = \bigoplus_e W_e$, we define $\mathrm{sht}_W := \sum_e \mathrm{sht}_{W_e}$. As for the value of b, from Lemma 4.1.2, we have

$$b = \sum_{d_\alpha<p} d_\alpha + \sum_{d_\alpha\geq p} p.$$

Similarly, as for the value of $\mathrm{sht}_W(l)$, we have

$$\mathrm{sht}_W(l) = \sum_{\alpha=1}^{a}\left(r_\alpha \sum_{i=1}^{q_\alpha}\left\lfloor\frac{il}{p}\right\rfloor + (p-r_\alpha)\sum_{i=1}^{q_\alpha-1}\left\lfloor\frac{il}{p}\right\rfloor\right).$$

If $p-1-S_V^{(1)} > 0$, then we have $\sup_{j_0=-\infty}\lambda(M_j) = \infty$.

5.1.2 The $j_0 \neq -\infty$ part

Assume that j_0; this part corresponds to the connected G-covers. For $\boldsymbol{j} = (j_0, j_1)$ with $j_0 \neq -\infty$, we put $u_0 = j_0$ and $u_1 = \max\{pj_0, j_1\}$. Then we already know that

$$\boldsymbol{v}_V(j_0,j_1) = \boldsymbol{v}_V(u_0,u_1) = \sum_{\alpha=1}^{a}\sum_{\substack{0\leq i_0+i_1p<d_\alpha,\\ 0\leq i_0,i_1<p}}\left\lceil\frac{(i_0p - i_1(p-1))u_0 + i_1pu_1}{p^2}\right\rceil.$$

As functions in $\boldsymbol{j} = (j_0, j_1)$, we define

$$\boldsymbol{e}_V^{>}(j_0, j_1) = \boldsymbol{e}_V^{>}(j_0) \coloneqq \sum_{\alpha=1}^{a} \sum_{\substack{0 \le i_0 + i_1 p < d_\alpha, \\ 0 \le i_0, i_1 < p}} \left\lceil \frac{p i_0 j_0 + (p^2 - p + 1) i_1 j_0}{p^2} \right\rceil,$$

$$\boldsymbol{e}_V^{<}(j_0, j_1) \coloneqq \sum_{\alpha=1}^{a} \sum_{\substack{0 \le i_0 + i_1 p < d_\alpha, \\ 0 \le i_0, i_1 < p}} \left\lceil \frac{p i_0 j_0 + (-(p-1) j_0 + p j_1) i_1}{p^2} \right\rceil.$$

If $j_1 = -\infty$, then we have

$$G\text{-Cov}(D; j_0, -\infty) = \mathbb{G}_m \times \mathbb{A}^{j_0 - 1 - \lfloor j_0/p \rfloor}.$$

Since $\boldsymbol{v}_V(j_0, -\infty) = \boldsymbol{e}_V^{>}$ depends only on j_0, thus we get $\lambda(M_{j_0,\infty}) < \lambda(M_{j_0,j_1})$ for $-\infty < j_1 < p j_0$. Therefore, we may assume that $j_1 \neq -\infty$ to evaluate $\sup_{\boldsymbol{j}} \lambda(M_{\boldsymbol{j}})$. With this assumption, we write $j_i = l_i + m_i p + n_i p^2$ ($i = 1, 2$; $0 \le m_i < p$, $1 \le l_i < p$).

Firstly, we consider the case $p j_0 > j_1$. Since $p \nmid u_0$ and $u_1 = p u_0$, by Lemma 4.2.6, we have

$$\boldsymbol{e}_V^{>}(r + q p^2) = \left(D_V^{(0)} + p D_V^{(1)}\right) q + \boldsymbol{e}_V^{>}(r), \tag{5.2}$$

where $1 \le r < p$.

Lemma 5.1.1. *If* $p^3 - p - \left(D_V^{(0)} + p D_V^{(1)}\right) \le 0$, *then we have*

$$\sup_{p j_0 > j_1} \lambda(M_{\boldsymbol{j}}) = d + \max\left\{(p^2 - 1) m_0 + l_0 p - \boldsymbol{e}_V^{>}(m_0 p + l_0)\right\}. \tag{5.3}$$

Otherwise, we have $\sup_{p j_0 > j_1} \lambda(M_{\boldsymbol{j}}) = \infty$.

Proof. Let $p j_0 > j_1 \neq -\infty$. The maximum value of $\lambda(M_{\boldsymbol{j}})$ ($p j_0 > j_1$) is given when $j_1 = p j_0 - 1$, equivalently, $n_1 = n_0 p + m_0$, $m_1 = l_0 - 1$ and $l_1 = p - 1$. Therefore, we get

$$\lambda(M_{\boldsymbol{j}}) = d + \left(p^3 - p - \left(D_V^{(0)} + p D_V^{(1)}\right)\right) n_0 + (p^2 - 1) m_0 + l_0 p - \boldsymbol{e}_V^{>}(m_0 p + l_0).$$

If $p^3 - p - \left(D_V^{0} + p D_V^{(1)}\right) > 0$, then $\sup_{p j_0 > j_1} \lambda(M_{\boldsymbol{j}}) = \infty$. Otherwise, we have

$$\sup_{p j_0 > j_1} (\lambda(M_{\boldsymbol{j}})) = d + \max_{0 \le m_0, m_1 < p; 1 \le l_0, l_1 < p} \{(p^2 - 1) m_0 + l_0 p - \boldsymbol{e}_V^{>}(m_0 p + l_0)\},$$

which completes the proof. ◻

Secondly, we consider the case $pj_0 < j_1$. In this case, by Lemma 4.2.6, we have

$$\boldsymbol{e}_V^{<}(r_0 + q_0p^2, r_1 + q_1p^2) = D_V^{(0)}q_0 + D_V^{(1)}q_1 + \boldsymbol{e}_V^{<}(r_0, r_1), \tag{5.4}$$

where $1 \leq r_i < p$.

Lemma 5.1.2. *If* $p^2 - p - D_V^{(i)} \leq 0$ *(*$i = 0, 1$*), then we have*

$$\sup_{pj_0<j_1} \lambda(M_j) = d + \max_{\substack{0\leq m_0,m_1<p,\\ 1\leq l_0,l_1<p}} \left\{(p-1)m_0 + l_0 + (p-1)m_1 + l_1 - \boldsymbol{e}_V^{<}(m_0p + l_0, m_1p + l_1)\right\}. \tag{5.5}$$

Otherwise, we have $\sup_{pj_0<j_1} \lambda(M_j) = \infty$.

Proof. By direct computing, we get

$$\begin{aligned}\lambda(M_j) = d &+ (p^2 - p - D_V^{(0)})n_0 + (p-1)m_0 + l_0 \\ &+ (p^2 - p - D_V^{(1)})n_1 + (p-1)m_1 + l_1 - \boldsymbol{e}_V^{<}(m_0p + l_0, m_1p + l_1).\end{aligned}$$

Assuming that $p^2 - p - D_V^{(i)} \leq 0$ ($i = 0, 1$), we get the first assertion. It is obvious that $\sup_{pj_0<j_1} \lambda(M_j) = \infty$ if $p^2 - p - D_V^{(1)} > 0$. □

Theorem 5.1.3. *If* $p - 1 - S_V^{(1)} \leq 0$ *and* $p^3 - p - \left(D_V^{(1)} + pD_V^{(1)}\right) \leq 0$*, then we have* $\sup_j \lambda(M_j) < \infty$ *and*

$$\sup_j \lambda(M_j) = \max\left\{\sup_{j_0=-\infty} \lambda(M_j), \sup_{pj_0>j_1} \lambda(M_j), \sup_{pj_0<j_1} \lambda(M_j)\right\},$$

where the suprema on the right hand side are given by Eqs. (5.1), (5.3) *and* (5.5)*. Conversely, if* $p-1-S_V^{(1)} > 0$ *or if* $p^3-p-\left(D_V^{(0)} + pD_V^{(1)}\right) > 0$*, then* $\sup_j \lambda(M_j) = \infty$.

Proof. Since $D_V^{(1)} = pS_V^{(1)} = pD_W^{(0)}$, thus holding $p - 1 - S_V^{(1)} \leq 0$ is equivalent to holding $p^2 - p - D_V^{(1)} \leq 0$. Therefore, it is enough to show that $p^2 - p - D_V^{(0)} \leq 0$. Assume that $p^2-p-D_V^{(0)} < 0$. Then, Lemma 5.1.4 shows that the G-representation V is of the form $V_1^{\oplus x} \oplus V_3$ ($p = 2$). However, if this is the case, we have $D_V^{(0)} + pD_V^{(1)} = 5 < 6 = p^3 - p$, which contradicts to the assumption $p^3 - p - \left(D_V^{(0)} + pD_V^{(1)}\right) \leq 0$.

Consequently, if $p - 1 - S_V^{(1)} \leq 0$ and $p^3 - p - \left(D_V^{(0)} - pD_V^{(1)}\right) \leq 0$, then the suprema $\sup_{j_0=-\infty} \lambda(M_j)$, $\sup_{pj_0>j_1} \lambda(M_j)$, and $\sup_{pj_0<j_1} \lambda(M_j)$ are all finite and they are given by Eqs. (5.1), (5.3) and (5.5). The converse is obvious. □

Lemma 5.1.4. *Let* V *be a faithful* G*-representation. We have the following.*

(1) *The inequality* $D_V^{(0)} \geq p^2$ *holds if* $p \geq 5$.

(2) *The inequality* $D_V^{(0)} \geq p^2 - p$ *holds except if the* G*-representation* V *is of the form* $V_1^{\oplus x} \oplus V_3$ $(p = 2)$.

To prove this, we need the following.

Lemma 5.1.5. *For an indecomposable* $\mathbb{Z}/p^2\mathbb{Z}$*-representation* V *of dimension* d*, we have*

$$\frac{d(p-1)}{2} - \frac{p^2}{8} \leq S_d^{(0)} \leq \frac{d(p-1)}{2},$$
$$\frac{d(d-p)}{2p} \leq S_d^{(1)} \leq \frac{d(d-p)}{2p} + \frac{p}{8}.$$

Proof. We write $d = r + qp$ $(0 \leq r < p)$. By definition, we have

$$\begin{aligned} S_d^{(0)} &= \frac{(d-r)(p-1)}{2} + \frac{r(r-1)}{2} \\ &= \frac{d(p-1)}{2} + \frac{r(r-p)}{2}. \end{aligned}$$

From the inequality of arithmetic and geometric means,

$$\begin{aligned} 0 &\geq r(r-p)/2 \\ &\geq -(r + (r-p))^2/8 \\ &= -p^2/8. \end{aligned}$$

Therefore, we get

$$\frac{d(p-1)}{2} - \frac{p^2}{8} \leq S_d^{(0)} \leq \frac{d(p-1)}{2}.$$

Similarly, we have

$$\begin{aligned} S_d^{(1)} &= \frac{(d-r)\left(\frac{d-r}{p} - 1\right)}{2} + \frac{d-r}{p}r \\ &= \frac{d(d-p)}{2p} - \frac{r(r-p)}{2p}, \end{aligned}$$

which completes the proof. ☐

Proof of Lemma 5.1.4. (1). Without loss of generality, we may assume V is indecomposable of dimension d $(p + 1 \leq d \leq p^2)$. By Lemma 5.1.5, we get

$$\begin{aligned} D_V^{(0)} - p^2 &= pS_d^{(0)} - (p-1)S_d^{(1)} - p^2 \\ &\geq p\left(\frac{(p-1)d}{2} - \frac{p^2}{8}\right) - (p-1)\left(\frac{d(d-p)}{2p} + \frac{p}{8}\right) - p^2. \end{aligned}$$

We denote by $\Phi(d)$ the last expression above. Let us show that $\Phi(d) \geq 0$. By direct computation, we have

$$\Phi(d) = -\frac{p-1}{2p}d^2 + \frac{p^2-1}{2}d - \frac{p^3+9p^2-p}{8},$$

hence $\Phi(d)$ is upward-convex with d regarded as a real variable. It is enough to check the values $\Phi(p+1)$ and $\Phi(p^2)$ are both non negative. We get

$$\Phi(p+1) = \frac{p^2(3p(p-3)-7)+4}{8p}$$
$$\Phi(p^2) = \frac{p(3p(p-5)+2p+1)}{8}.$$

It is obvious that $\Phi(p+1) \geq 0$ and $\Phi(p^2) \geq 0$ if $p \geq 5$. Consequently, we have $D_V^{(0)} - p^2 \geq \Phi(d) \geq 0$.

(2). We may assume $p \leq 3$. Firstly, we consider the case that V is indecomposable of dimension $d \geq p+1$. When $p = 3$, d varies from 4 to 9. Checking each case by direct computation, we get

$$D_{V_4}^{(0)} = 7, \quad D_{V_5}^{(0)} = 8, \quad D_{V_6}^{(0)} = 12,$$
$$D_{V_7}^{(0)} = 8, \quad D_{V_8}^{(0)} = 7, \quad D_{V_9}^{(0))} = 9,$$

and hence $D_V^{(0)} \geq p^2 - p$ holds.

If $p = 2$, we have

$$D_{V_1}^{(0)} = 0, \quad D_{V_2}^{(0)} = 2, \quad D_{V_3}^{(0)} = 1, \quad D_{V_4}^{(0)} = 2,$$

and hence the proof is completed. □

Example 5.1.6. Let $p = 3$ and let V be the indecomposable $G = \mathbb{Z}/p^2\mathbb{Z}$-representation of dimension d ($p+1 < d \leq p^2$). Then, according to computations with Sage [29], we get Table 5.1.

5.2 Refinement of dimensional criterion for $G = \mathbb{Z}/p^2\mathbb{Z}$

We shall give lower bounds of $\boldsymbol{e}_V^{>}$ and $\boldsymbol{e}_V^{<}$ and apply them to estimate $\delta(X) = \mathrm{discrep}(\mathrm{center} \subset X_{\mathrm{sing}}; X)$ more precisely when the given G-representation is indecomposable of dimension d.

$d = \dim V$	$\sup_{j_0=-\infty} \lambda(M_j)$	$\sup_{pj_0>j_1} \lambda(M_j)$	$\sup_{pj_0<j_1} \lambda(M_j)$	$\delta(X)$
4	∞	∞	∞	
5	5	3	4	-1
6	5	2	3	0
7	4	1	4	2
8	4	0	4	3
9	4	-2	3	4

Table 5.1: discrepancies in characteristic 3

Lemma 5.2.1. *We have*

$$\boldsymbol{e}_V^{>}(j_0) \geq \frac{\left(D_V^{(0)} + pD_V^{(1)}\right)j_0}{p^2}, \quad \boldsymbol{e}_V^{<}(j_0, j_1) \geq \frac{D_V^{(0)} j_0 + D_V^{(1)} j_1}{p^2}.$$

Proof. It is immediate from Lemma 4.2.6, see also Eqs. (5.2) and (5.4). □

We can now give the upper and bound of $\sup \lambda(M_{\boldsymbol{j}})$. For the $j_0 = -\infty$ part, from [44, Theorem 1.2], we have the following bounds:

Lemma 5.2.2 ([44])**.** *We have*

$$\sup_{j_0=-\infty} \lambda(M_{\boldsymbol{j}}) \geq d + p - 1 - S_V^{(1)}. \tag{5.6}$$

Furthermore, if $S_V^{(1)} \geq p$*, then we have*

$$\sup_{j_0=-\infty} \lambda(M_{\boldsymbol{j}}) \leq d + 1 - \frac{2S_V^{(1)}}{p}. \tag{5.7}$$

Proposition 5.2.3. *Assume that* $S_V^{(1)} \geq p$*,* $D_V^{(0)} + pD_V^{(1)} \geq p^3$ *and* $D_V^{(0)} \geq p^2$*. Then, we have*

$$\sup \lambda(M_{\boldsymbol{j}}) \leq \max\left\{d + 1 - \frac{2S_V^{(1)}}{p}, d + p - \frac{D_V^{(0)} + pD_V^{(1)}}{p^2}, d + 2 - \frac{D_V^{(0)} + D_V^{(1)}}{p^2}\right\}.$$

Proof. We shall first consider $\sup_{pj_0>j_1} \lambda(M_{\boldsymbol{j}})$. By Lemma 5.2.1, we have

$$\begin{aligned}&(p^2-1)m_0 + l_0 p - \boldsymbol{e}_V^{>}(m_0 p + l_0)\\ &\qquad\qquad \leq \left(p^2 - 1 - \frac{D_V^{(0)} + pD_V^{(1)}}{p}\right)m_0 + \left(p - \frac{D_V^{(0)} + pD_V^{(1)}}{p^2}\right)l_0.\end{aligned}$$

Since we assume $D_V^{(0)} + pD_V^{(1)} \geq p^3$, thus the coefficients of m_0 and l_0 in the right hand side are non positive. Therefore, the right hand side attains the maximum at $m_0 = 0$ and $l_0 = 1$. From Lemma 5.1.1, we get

$$\sup_{pj_0 > j_1} \lambda(M_j) \leq d + p - \frac{D_V^{(0)} + pD_V^{(1)}}{p^2}. \tag{5.8}$$

Next, we shall consider $\sup_{pj_0<j_1} \lambda(M_j)$. By Lemma 5.2.1, we have

$$\begin{aligned}
&(p-1)m_0 + l_0 + (p-1)m_1 + l_1 - \boldsymbol{e}_V^{<}(m_0 p + l_0, m_1 p + l_1) \\
&\quad \leq \left(p - 1 - \frac{D_V^{(0)}}{p}\right) m_0 + \left(1 - \frac{D_V^{(0)}}{p^2}\right) l_0 + \left(p - 1 - \frac{D_V^{(1)}}{p}\right) m_1 + \left(1 - \frac{D_V^{(1)}}{p^2}\right) l_1.
\end{aligned}$$

Note that the assumption $S_V^{(1)} \geq p$ implies $D_V^{(1)} \geq p^2$. Since we have $D_V^{(0)} \geq p^2$ and $D_V^{(1)} \geq p^2$, thus the coefficients in the last expression are non positive. Therefore, the last expression takes the maximum at $m_0 = m_1 = 0$ and $l_0 = l_1 = 1$. From Lemma 5.1.2, we get

$$\sup_{pj_0 < j_1} \lambda(M_j) \leq d + 2 - \frac{D_V^{(0)} + D_V^{(1)}}{p^2}. \tag{5.9}$$

Combining (5.7–5.9), we get the claim. □

As a conclusion of this section, we get a refinement of Theorem 4.2.12.

Theorem 5.2.4. *Assume that $V = V_d$ is a faithful indecomposable G-representation of dimension d ($p + 1 < d \leq p^2$) (with this assumption, V has no pseudo-reflection and hence $V \to X := V/G$ is crepant). Then,*

$$X \text{ is } \begin{cases} \textit{terminal}, \\ \textit{canonical}, \\ \textit{log canonical}, \\ \textit{not log canonical} \end{cases} \quad \textit{if and only if} \begin{cases} d \geq 2p + 1, \\ d \geq 2p, \\ d \geq 2p - 1, \\ d < 2p - 1. \end{cases}$$

Proof. First, we consider the case $d < 2p - 1$. From the definition of $S_d^{(1)}$, we get $S_V^{(1)} < p - 1$. By Theorem 5.1.3, we get $\sup \lambda(M_j) = \infty$ and hence $\delta(X) = -\infty$.

Next, we consider the case $d = 2p - 1$. Since we also assume $d > p + 1$, thus we have $p \geq 3$. By direct computation, we have $S_d^{(1)} = p - 1$ and $D_V^{(0)} + pD_V^{(1)} =$

$2p^3 - 4p^2 + 3p - 1 > p^3 - p$ and hence $\delta(X) > -\infty$ by Theorem 5.1.3. We remark that by (5.6) we have

$$\sup \lambda(M_j) \geq \sup_{j_0=-\infty} \lambda(M_j) \geq d + p - 1 - S_d^{(1)}.$$

and hence $\delta(X) \leq -1$ by Theorem 2.4.9. Therefore, we get $\delta(X) = -1$.

Thirdly, we consider the case $d = 2p$. Then, we have

$$\begin{aligned} S_V^{(1)} &= p, \\ D_V^{(0)} + pD_V^{(1)} &= 2p^3 - 2p^2 + p \geq p^3 \geq p^3 - p + 1. \end{aligned}$$

and hence by Corollary 4.2.10, the quotient $X = V/G$ is canonical and hence $d(X) \geq 0$. On the other hand, by (5.6), we have

$$\sup \lambda(M_j) \geq \sup_{j_0=-\infty} \lambda(M_j) \geq d + p - 1 - S_V^{(1)} = 2p - 1,$$

and hence $\delta(X) \leq 0$. Thus we get $\delta(X) = 0$.

Finally, we consider the case $d \geq 2p + 1$. When $p = 3$, the assertion follows from Example 5.1.6. We assume that $p \geq 5$. We remark that we have

$$\begin{aligned} D_V^{(0)} + pD_V^{(1)} &= pS_d^{(0)} + (p^2 - p + 1)S_d^{(1)}, \\ D_V^{(0)} + D_V^{(1)} &= pS_d^{(0)} + S_d^{(1)}. \end{aligned}$$

Since $S_d^{(i)}$ $(i = 0, 1)$ are monotonically increasing function in d, so are $D_V^{(0)} + pD_V^{(1)}$ and $D_V^{(0)} + D_V^{(1)}$. From Proposition 5.2.3, it is enough to show

$$\max\left\{ d + 1 - \frac{2S_V^{(1)}}{p}, d + p - \frac{D_V^{(0)} + pD_V^{(1)}}{p^2}, d + 2 - \frac{D_V^{(0)} + D_V^{(1)}}{p^2} \right\} < d - 1 \quad (5.10)$$

in the case $d = 2p + 1$. In this case, we have

$$\begin{aligned} S_V^{(1)} &= p + 2, \\ D_V^{(0)} + pD_V^{(0)} &= 2p^3 - p + 2, \\ D_V^{(0)} + D_V^{(0)} &= p^3 - p^2 + p + 2, \end{aligned}$$

and hence

$$\begin{aligned}
d+1-\frac{2S_V^{(1)}}{p} &= d-1-\frac{4}{p},\\
d+p-\frac{D_V^{(0)}+pD_V^{(1)}}{p^2} &= d-1+1+p-\frac{2p^3-p+2}{p^2}\\
&= d-1+1-p-\frac{p-2}{p^2},\\
d+2-\frac{D_V^{(0)}+D_V^{(1)}}{p^2} &= d-1+3-\frac{p^3-p^2+p+2}{p^2}\\
&= d-1+4-p-\frac{p+2}{p^2}.
\end{aligned}$$

Thus (5.10) holds. □

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