Asymptotic behavior of solutions to nonlinear hyperbolic and dispersive equations with weakly dissipative structure

(弱い消散構造を伴う非線形双曲型及び分散型方程式の解の漸近挙動)

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Chapter 1

Introduction

This thesis is concerned with two types of nonlinear partial differential equations.

We first consider the large-time behavior of small solutions to the Cauchy problem for semilinear wave equations

$$\begin{cases} \partial_t^2 u - \Delta u = F(\partial u), & (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ u(0,x) = \varepsilon f(x), & \\ \partial_t u(0,x) = \varepsilon g(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.0.1}$$

where $u = u(t,x)$ is a $\mathbb{R}$-valued unknown function, $F(\partial u)$ is a nonlinearity, $\partial u = (\partial_t u, \partial_{x_1} u, \partial_{x_2} u)$. The initial data f, g are compactly-supported C^∞-functions and $\varepsilon > 0$ is a small parameter. To begin with, let us review the free wave equation $\partial_t^2 u - \Delta u = 0$ (i.e., $F \equiv 0$). It is well known that the following estimates hold for the free solution $u(t)$:

$$|u(t,x)| \le C\varepsilon(1+t)^{-1/2}, \qquad t \ge 0, \ x \in \mathbb{R}^2, \tag{1.0.2}$$

$$\|u(t)\|_E = \|u(0)\|_E, \qquad t \ge 0, \tag{1.0.3}$$

where the energy norm $\|\cdot\|_E$ is defined by

$$\|\phi(t)\|_E^2 = \frac{1}{2}\int_{\mathbb{R}^2} \sum_{a=0}^{2} |\partial_a \phi(t,x)|^2 \, dx.$$

Next, we consider (1.0.1) in the case $F(\partial u) = O(|\partial u|^p)$ near $\partial u = 0$, where $p > 1$. For the small amplitude solution, we may expect the nonlinearity can be treated as a perturbation and the solutions behave like free solutions if p is large enough. If we assume that (1.0.2) and (1.0.3) are valid in this case, we have

$$\int_0^\infty \|F(\partial u)(\tau)\|_E \, d\tau \le C\varepsilon^p \int_{\mathbb{R}} (1+\tau)^{-(p-1)/2} \, d\tau \le C\varepsilon^p$$

when $p > 3$. Therefore we may expect that there exists an unique global solution to (1.0.1) in nonlinear case when $p > 3$. Indeed, if $p > 3$, the small data global existence (which we refer as SDGE in what follows) holds for (1.0.1), that is, (1.0.1) admits a unique global C^∞-solution for suitably small ε. Moreover, there exists a solution u^+ to the free wave equation $\Box u^+ = 0$ such that

$$\lim_{t\to\infty} \|u(t) - u^+(t)\|_E = 0.$$

In other wards, the global solution $u(t)$ is asymptotically free in the sense of the energy. In contrast, for the case $p \leq 3$, the global existence does not hold in general even if ε is arbitrarily small. Furthermore, even if there exists the global solution to (1.0.1), it does not behave like free solutions in general. In the sense, $p = 3$ is one of the critical situations. Thus, we need some structural conditions to conclude that the SDGE holds and the solution is asymptotically free. One of the most famous conditions is so called *null condition* which was first introduced by Klainerman [48] and Christodoulou [6] for the quasilinear wave equations in three space dimensions, and developed by many researchers later. We will give a review on the detail of the null condition in Section 2.1 below. Recently, weaker conditions than the null condition are studied. The *Agemi-type condition* introduced in [39] is one of them. This condition includes the dissipative structure such as cubic nonlinear damping $\Box v = -(\partial_t v)^3$ in addition to the cubic null condition. Under the Agemi-type condition, the SDGE holds for (1.0.1). However, there are a lot of unsolved problems in the asymptotic behavior of solutions under the Agemi-type condition.

The second equations which we are interested in are the nonlinear Schrödinger equations. Let us consider the Cauchy problem

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\partial_x^2 u = N(u, \partial_x u), & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{1.0.4}$$

where $i = \sqrt{-1}$, $u = u(t,x)$ is a $\mathbb{C}$-valued unknown function and the nonlinear term $N(u, \partial_x u)$ is a cubic homogeneous polynomial in $(u, \overline{u}, \partial_x u, \overline{\partial_x u})$ with complex coefficients. The initial data $\varphi(x)$ is suitably small, smooth and decay fast as $|x| \to \infty$. For the free Schrödinger equation $i\partial_t u + \frac{1}{2}\partial_x^2 u = 0$, the properties of the solutions

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq C(1+t)^{-1/2}, \qquad \|u(t)\|_{L^2(\mathbb{R})} = \|u(0)\|_{L^2(\mathbb{R})},$$

for $t \geq 0$ is well-known. By these properties and the similar argument as above, the cubic nonlinearities for the nonlinear Schrödinger equations in one space dimension cause a situation similar to two-dimensional cubic semilinear wave equations. Therefore it would be natural to expect that the structure corresponding to semilinear wave equations may exist for nonlinear

Schrödinger equations. From this point of view, the condition introduced in [51] can be regarded as an NLS-analog of the Agemi-type condition. This condition guarantees the SDGE for (1.0.4), but, as is the case with the Agemi-type condition, the asymptotics for the global solutions are not well understood.

The purpose of this thesis is two-fold: The first one is to develop the understanding for asymptotic behavior of solutions to (1.0.1) under the Agemi-type condition. In particular, we focus on the energy of the solutions and make clear whether the energy decay occurs likewise the nonlinear damping or another kind of phenomenon occurs. The second is to consider these analogs in the Schrödinger case (1.0.4) with a suitable dissipative condition.

This thesis is organized as follows. In Chapter 2, we consider the asymptotic behavior of the solution u to (1.0.1) under the Agemi-type condition. For the single case, we prove the energy decay occurs unless the null condition and we give an upper bound estimate for $\|u(t)\|_E$. We also study a two-component system of semilinear wave equations with cubic nonlinearity satisfying the Agemi-type condition. We show that small amplitude solutions of this system behave like free solutions as $t \to +\infty$. Furthermore, we give a criterion for large time non-decay of the energy for small amplitude solutions in terms of the radiation fields associated with the initial data. This chapter is based on [63], [64] and [62]. Chapter 3 deals with the Cauchy problem for (1.0.4). Under a suitable weakly dissipative condition on the nonlinearity, we show that the small data solution has a logarithmic time decay in L^2 for the single case. For a two-component system case, we show each component of the solutions are asymptotically free in the large time and that the scattering states have a non-trivial restricted condition. We also provide criteria for large time decay or non-decay in L^2 of the small amplitude solutions in terms of the Fourier transforms of the initial data. This part is based on [53], [54] and [55].

Before closing this chapter, we introduce some notations and function spaces. We denote by $C_0^\infty(\mathbb{R}^d)$ the set of compactly-supported C^∞ functions on $\mathbb{R}^d$. For $1 \le p \le \infty$, we denote the Lebesgue space on $\mathbb{R}^d$ by $L^p(\mathbb{R}^d)$ and its norm by $\|\cdot\|_{L^p(\mathbb{R}^d)}$. For $m \in \mathbb{N}$ and $1 \le p \le \infty$, we denote by $W^{m,p}(\mathbb{R}^d)$ the $L^p(\mathbb{R}^d)$-based Sobolev space of order m

$$W^{m,p}(\mathbb{R}^d) := \{f \in L^p(\mathbb{R}^d)\ ;\ \partial_x^\alpha f \in L^p(\mathbb{R}^d),\ \alpha \in \mathbb{Z}_+^d,\ |\alpha| \le m\}$$

equipped with the norm

$$\|\phi\|_{W^{m,p}(\mathbb{R}^d)} := \sum_{|\alpha|\le m} \|\partial_x^\alpha f\|_{L^p(\mathbb{R}^d)},$$

where $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We write $H^m(\mathbb{R}^d) = W^{m,2}(\mathbb{R}^d)$. For $m,\ s \in \mathbb{Z}_+$, we denote by $H^{k,m}(\mathbb{R}^d)$ the ($L^2(\mathbb{R}^d)$-based) weighted Sobolev space

$$H^{m,s}(\mathbb{R}^d) := \{f \in H^m(\mathbb{R}^d)\ ;\ \langle\cdot\rangle^s f \in H^m(\mathbb{R}^d)\}$$

equipped with the norm

$$\|f\|_{H^{m,s}(\mathbb{R}^d)} := \|\langle\,\cdot\,\rangle^s f\|_{H^m(\mathbb{R}^d)},$$

where $\langle z \rangle = \sqrt{1+|z|^2}$. We will occasionally omit "$(\mathbb{R}^d)$" if it causes no confusion. The Fourier transform of ϕ is defined by

$$\big(\mathcal{F}\phi\big)(\xi) = \hat{\phi}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \phi(x)\, dx, \qquad \xi \in \mathbb{R},$$

and the inverse Fourier transform of ϕ is defined by

$$\big(\mathcal{F}^{-1}\phi\big)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \phi(\xi)\, d\xi, \qquad x \in \mathbb{R}.$$

We denote several positive constants by the same letter C, which may be different from one line to another.

Chapter 2

Asymptotic behavior of solutions to semilinear wave equations with weakly dissipative structure

2.1 Introduction and results

This chapter is based on the joint works [63] with Hideaki Sunagawa, [64] with Hideaki Sunagawa and Hiroki Terashita, and the author's work [62]. We consider large-time asymptotic behavior of the solution $u = (u_j(t,x))_{1\le j\le N}$ to the Cauchy problem

$$\Box u_j = F_j(\partial u), \qquad (t,x) \in (0,\infty)\times\mathbb{R}^2, \tag{2.1.1}$$

with the initial condition

$$u_j(0,x) = \varepsilon f_j(x), \quad \partial_t u_j(0,x) = \varepsilon g_j(x), \qquad x\in\mathbb{R}^2, \tag{2.1.2}$$

for $1 \le j \le N$, where $\Box = \partial_t^2 - \Delta = \partial_0^2 - (\partial_1^2 + \partial_2^2)$, $\partial_0 = \partial_t = \partial/\partial t$, $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$, $\varepsilon > 0$ is a small parameter and f_j, $g_j \in C_0^\infty(\mathbb{R}^2)$. We suppose that the nonlinearity $F = (F_j)_{1\le j\le N}$ is a $\mathbb{R}^N$-valued C^∞-function given by

$$F_j(\partial u) = F_j^{\mathrm{q}}(\partial u) + F_j^{\mathrm{c}}(\partial u) + O(|\partial u|^4)$$

near $\partial u = 0$, where the quadratic homogeneous part $F_j^{\mathrm{q}}(\partial u)$ and the cubic homogeneous part $F_j^{\mathrm{c}}(\partial u)$ are given by

$$F_j^{\mathrm{q}}(\partial u) = \sum_{k,l=1}^{N} \sum_{a,b,c=0}^{2} B_{jkl}^{ab} (\partial_a u_k)(\partial_b u_l),$$

$$F_j^{\mathrm{c}}(\partial u) = \sum_{k,l,m=1}^{N} \sum_{a,b,c=0}^{2} C_{jklm}^{abc} (\partial_a u_k)(\partial_b u_l)(\partial_c u_m),$$

with some real constants B_{jkl}^{ab} and C_{jklm}^{abc}, respectively.

To explain the backgrounds, we consider (2.1.1)–(2.1.2) in $\mathbb{R}^d$ and assume that F vanishes of order $p \geq 2$ in a neighborhood of $0 \in \mathbb{R}^{N\times(1+d)}$ for a while. If $p > 1 + 2/(d-1)$, it is well known that the SDGE holds. Moreover, the solution behaves like a solution to the free wave equation as $t \to \infty$. On the other hands, if $p \leq 1+2/(d-1)$, global existence fails to hold in general even when $\varepsilon > 0$ is arbitrarily small. (see [29], [8], etc). In this sense, the power $p_{\mathrm{c}}(d) := 1 + 2/(d-1)$ is a critical exponent for nonlinear perturbation. Note that $p_{\mathrm{c}}(2) = 3$ and $p_{\mathrm{c}}(3) = 2$. On the other hand, the small data global existence can hold for some class of nonlinearity of the critical power. One of the most famous example is the so called *null condition*, which has been originally introduced by Christodoulou[6] and Klainerman[48] in three dimensional case. Its counterparts for two dimensional case are developed later by several authors (see [8], [24], [33], [2] etc.). In what follows $\mathbb{S}^1$ stands for the unit circle in $\mathbb{R}^2$. We say that the quadratic (resp. cubic) null condition is satisfied if and only if $F^{\mathrm{q,red}}(\omega, Y)$ (resp. $F^{\mathrm{c,red}}(\omega, Y)$) vanishes identically on $\mathbb{S}^1 \times \mathbb{R}^N$, where $F^{\mathrm{q,red}} = (F_j^{\mathrm{q,red}})_{1\leq j\leq N}$, $F^{\mathrm{c,red}} = (F_j^{\mathrm{c,red}})_{1\leq j\leq N}$ are defined by

$$F_j^{\mathrm{q,red}}(\omega, Y) = \sum_{k,l,m=1}^{N} \sum_{a,b=0}^{2} B_{jkl}^{ab} \omega_a \omega_b Y_k Y_l,$$

$$F_j^{\mathrm{c,red}}(\omega, Y) = \sum_{k,l,m=1}^{N} \sum_{a,b,c=0}^{2} C_{jklm}^{abc} \omega_a \omega_b \omega_c Y_k Y_l Y_m$$

for $Y = (Y_j)_{1\leq j\leq N} \in \mathbb{R}^N$ and $\omega = (\omega_1, \omega_2) \in \mathbb{S}^1$ with the convention $\omega_0 = -1$. When $d = 2$, if both the quadratic null condition and the cubic null condition are satisfied, then the SDGE holds for (2.1.1)–(2.1.2). Moreover the global solution u under the null condition is asymptotically free in the sense of the energy. Note that we need only the quadratic null condition to conclude the same in three space dimensions case. We also remark that if only the quadratic null condition is assumed, it is shown by Godin [8] that the following estimate for the lifespan T_ε holds for the single case ($N = 1$):

$$\liminf_{\varepsilon\to+0} \varepsilon^2 \log T_\varepsilon \geq \frac{1}{\displaystyle\sup_{(\sigma,\omega)\in\mathbb{R}\times\mathbb{S}^1} \Big[-F^{\mathrm{c,red}}(\omega, 1)(\partial_\sigma \mathcal{F}_0[f, g](\sigma, \omega))^2\Big]}$$

with the convention $1/0 = +\infty$, where $\mathcal{F}_0[f, g]$ is the radiation field associated with the initial data. To be more specific, $\mathcal{F}_0[f, g] =: \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}$ is defined by

$$\mathcal{F}_0[f, g](\sigma, \omega) := -\partial_\sigma \mathcal{R}_2[f](\sigma, \omega) + \mathcal{R}_2[g](\sigma, \omega), \tag{2.1.3}$$

where

$$\mathcal{R}_2[\phi](\sigma, \omega) := \frac{1}{2\sqrt{2}\pi} \int_\sigma^\infty \frac{\mathcal{R}[\phi](s, \omega)}{\sqrt{s - \sigma}}\, ds, \quad \mathcal{R}[\phi](s, \omega) := \int_{y\cdot\omega=s} \phi(y)\, dS_y,$$

for $\phi \in C_0^\infty(\mathbb{R}^2)$. More information on the detailed lifespan estimates and the related topics can be found in [1], [3], [7], [22], [23], [25], [30], [31], [34], [66], [70], etc., and the references cited therein.

Recently, a lot of efforts have been made for the study on weaker structural conditions than the null condition mentioned above which ensure the small data global existence (see e.g., [56], [57], [58], [59], [2], [4], [5], [26], [49], [36], [40], [38], [39], [34], [35], [20], [21], [26], etc). It should be emphasized that the situation becomes much more complicated because long-range nonlinear effects must be taken into account. In [39], the following condition has been introduced:

(Ag) There exists an $N \times N$-matrix valued continuous function $\mathcal{A} = \mathcal{A}(\omega)$ on $\mathbb{S}^1$, which is a positive-definite symmetric matrix for each $\omega \in \mathbb{S}^1$, such that

$$Y \cdot \mathcal{A}(\omega) F^{\mathrm{c,red}}(\omega, Y) \geq 0, \quad (\omega, Y) \in \mathbb{S}^1 \times \mathbb{R}^N,$$

where the symbol $\cdot$ denotes the standard inner product in $\mathbb{R}^N$.

After the partial results [49], [26], [40], it has been shown in [39] that the quadratic null condition and (**Ag**) imply the small data global existence for (2.1.1)–(2.1.2) in two space dimensions. (see also [38] for the quadratic nonlinearities in three space dimensions case). It is obvious that (**Ag**) is weaker than the cubic null condition. We note that this condition is motivated by works of Rentaro Agemi in the late 1990's. He tried to find a structural condition which covers not only the standard null condition but also the wave equations with cubic nonlinear damping. Therefore it would be fair to call this the *Agemi-type condition.* As for the asymptotic behavior of the global solutions under (**Ag**), many interesting problems seem left unsolved. To the author's knowledge, only the following two cases ($\mathbf{Ag}_+$) and ($\mathbf{Ag}_0$) are well-understood:

($\mathbf{Ag}_+$) There exist an $\mathcal{A}(\omega)$ as in (**Ag**) and a positive constant C such that

$$Y \cdot \mathcal{A}(\omega) F^{\mathrm{c,red}}(\omega, Y) \geq C|Y|^4, \quad (\omega, Y) \in \mathbb{S}^1 \times \mathbb{R}^N.$$

Under (Ag_+), the total energy $\|u(t)\|_E$ decays like $O((\log t)^{-1/4+\delta})$ as $t \to +\infty$, where $\delta > 0$ can be arbitrarily small. See [39] for the detail. A typical example of $F^{\mathrm{c}}(\partial u)$ satisfying ($\mathbf{Ag}_+$) is the cubic damping term $-(\partial_t u)^3$. Note also that the energy decay of this kind never occur under the cubic null condition unless $f = g \equiv 0$ (see e.g., Chapter 9 in [34] for the detail). Therefore it will be fair to say that ($\mathbf{Ag}_+$) yields dissipative structure.

($\mathbf{Ag}_0$) There exists an $\mathcal{A}(\omega)$ as in **(Ag)** such that

$$Y \cdot \mathcal{A}(\omega) F^{\mathrm{c,red}}(\omega, Y) = 0, \quad (\omega, Y) \in \mathbb{S}^1 \times \mathbb{R}^N.$$

Note that ($\mathbf{Ag}_0$) is stronger than (**Ag**) if F is cubic (while it is equivalent to (**Ag**) in the quadratic case). Roughly speaking, it holds under ($\mathbf{Ag}_0$) that

$$\partial u(t,x) \sim |x|^{-1/2} \hat{\omega}(x) V(t; |x| - t, x/|x|)$$

as $t \to \infty$, where $\hat{\omega}(x) = (-1, x_1/|x|, x_2/|x|)$, and $V(t;\sigma,\omega)$ solves

$$\partial_t V = \frac{1}{t} Q(\omega, V) V$$

with a suitable skew-symmetric matrix Q depending on (ω, V). In contrast to ($\mathbf{Ag}_+$), decay of the total energy never occurs under ($\mathbf{Ag}_0$) except for the trivial solution. Typical example satisfying ($\mathbf{Ag}_0$) is

$$\begin{cases} \Box u_1 = -(\partial_t u_1)^2 \partial_t u_2, \\ \Box u_2 = (\partial_t u_1)^3. \end{cases}$$

For more details on ($\mathbf{Ag}_0$), see [38], [35] and Chapter 10 in [34].

However, there is a gap between (**Ag**) and ($\mathbf{Ag}_+$), ($\mathbf{Ag}_0$). Now we come to a following question naturally: *If the quadratic null condition and* (**Ag**) *is satisfied but the cubic null condition,* ($\mathbf{Ag}_+$) *and* ($\mathbf{Ag}_0$) *are violated, how does the solution to* (2.1.1)–(2.1.2) *behave as* $t \to +\infty$*? In particular, does the energy decay occur?* To the authors' knowledge, there are no previous works which address this question. The aim of this chapter is to give answers to this question.

Let us start with the single case ($N = 1$) of (2.1.1)-(2.1.2), that is,

$$\begin{cases} \Box u = F(\partial u), & (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ u(0,x) = \varepsilon f(x), & x \in \mathbb{R}^2, \\ \partial_t u(0,x) = \varepsilon g(x), & \end{cases} \tag{2.1.4}$$

where $u = u(t,x)$ is $\mathbb{R}$-valued unknown function and $f,\, g \in C_0^\infty(\mathbb{R}^2)$. In this case, cubic part of the nonlinearity $F^{\mathrm{c}}(\partial u)$ and its reduced form $F^{\mathrm{c,red}}(\omega, Y)$ can be written as the forms

$$F^{\mathrm{c}}(\partial u) = \sum_{a,b,c=0}^{2} C^{abc}(\partial_a u)(\partial_b u)(\partial_c u)$$

and

$$F^{\mathrm{c,red}}(\omega, Y) = \sum_{a,b,c=0}^{2} C^{abc}\omega_a\omega_b\omega_c Y^3.$$

We put $P(\omega) = F^{\mathrm{c,red}}(\omega, 1)$, then $(\mathbf{Ag})$ is equivalent to $P(\omega) \geq 0$ on $\mathbb{S}^1$ and $(\mathbf{Ag}_+)$ is equivalent to $P(\omega) > 0$ on $\mathbb{S}^1$. We also note that $(\mathbf{Ag}_0)$ holds if and only if the cubic null condition is satisfied. Therefore, for the single case, we are interested in the case in which $(\mathbf{Ag})$ is satisfied but the cubic null condition and $(\mathbf{Ag}_+)$ are violated.

The first result, which concerns the single case, is as follows.

Theorem 2.1.1. *Let $N = 1$. Assume that quadratic null condition and $(\mathbf{Ag})$ are satisfied but the cubic null condition is violated. For the global solution u to (2.1.4), there exist positive constants C and λ such that*

$$\|u(t)\|_E \leq \frac{C\varepsilon}{(1+\varepsilon^2\log(t+2))^\lambda}$$

for $t \geq 0$, provided that ε is sufficiently small.

Remark 2.1.1. We give some examples of $F_{\mathrm{c}}(\partial u)$ which satisfy $(\mathbf{Ag})$ but violate the cubic null condition and $(\mathbf{Ag}_+)$:

$$-(\partial_1 u)^2\partial_t u, \qquad -(\partial_1 u)^2(\partial_t u + \partial_2 u), \qquad -(\partial_t u + \partial_2 u)^3.$$

The corresponding $P(\omega)$'s are ω_1^2, $\omega_1^2(1-\omega_2)$, $(1-\omega_2)^3$, respectively. We will give more precise estimates of λ for these three cases in Subsection 2.4.4, below.

Theorem 2.1.1 says that the energy decay occurs under $(\mathbf{Ag})$ for the single case unless the cubic null condition. It may be natural to ask what happens in the system case. We next address this point. Let us consider the following two-component system:

$$\begin{cases} \Box u_1 = -(\partial_t u_2)^2\partial_t u_1, \\ \Box u_2 = -(\partial_t u_1)^2\partial_t u_2, \end{cases} \quad (t,x) \in (0,\infty)\times\mathbb{R}^2, \tag{2.1.5}$$

The system (2.1.5) satisfies $(\mathbf{Ag})$ with $\mathcal{A}(\omega)$ being 2×2 identity matrix. Indeed we have $Y \cdot F^{\mathrm{c,red}}(\omega, Y) = 2Y_1^2Y_2^2$. This expression tells us that both

$(\mathbf{Ag}_+)$ and $(\mathbf{Ag}_0)$ are violated. Note also that the system (2.1.5) possesses two conservation laws

$$\frac{d}{dt}\left(\|u_1(t)\|_E^2 + \|u_2(t)\|_E^2\right) = -2\int_{\mathbb{R}^2} \left(\partial_t u_1(t,x)\right)^2 \left(\partial_t u_2(t,x)\right)^2 dx \qquad (2.1.6)$$

and

$$\frac{d}{dt}\Big(\|u_1(t)\|_E^2 - \|u_2(t)\|_E^2\Big) = 0. \qquad (2.1.7)$$

However, these are not enough to say something about the large-time asymptotics for $u(t)$, and this is not trivial at all. Our next aim is to clarify the asymptotic behavior of the solution $u(t)$ to (2.1.5)–(2.1.2). The second result is as follows.

Theorem 2.1.2. *Assume that f, $g \in C_0^\infty(\mathbb{R}^2)$ and ε is suitably small. Then the global solution $u(t)$ to* (2.1.5)*–*(2.1.2) *is asymptotically free.*

Remark 2.1.2. If we consider the case $(f_1, g_1) = (f_2, g_2)$, the system (2.1.5) can be reduced to the single euation $\Box v = -(\partial_t v)^3$. Therefore we can adapt the result of [40], [39] (or Theorem 2.1.1) to see that the total energy $\|u(t)\|_E$ decays like $O((\log t)^{-1/4+\delta})$ as $t \to +\infty$.

We note that the total energy decay stated in Remark 2.1.2 is an exceptional case. Indeed, it follows from the conservation law (2.1.7) that at least one component u_1 or u_2 tends to a non-trivial free solution if $\|u_1(0)\|_E \neq \|u_2(0)\|_E$. It is far from obvious whether both u_1^+ and u_2^+ do not vanish in a certain case. We reveal a criterion for the energy non-decay in the terms of the radiation fields associated with the initial data as follows. This is our third result.

Theorem 2.1.3. *Let $\mathcal{F}_j(\sigma,\omega) = \mathcal{F}_0[f_j, g_j](\sigma,\omega)$ for $j = 1, 2$, where $\mathcal{F}_0$ is defined by* (2.1.3)*. Suppose that there exist (σ^*, ω^*), $(\sigma_*, \omega_*) \in \mathbb{R}\times\mathbb{S}^1$ satisfying*

$$|\partial_\sigma \mathcal{F}_1(\sigma^*, \omega^*)| > |\partial_\sigma \mathcal{F}_2(\sigma^*, \omega^*)| \qquad (2.1.8)$$

and

$$|\partial_\sigma \mathcal{F}_1(\sigma_*, \omega_*)| < |\partial_\sigma \mathcal{F}_2(\sigma_*, \omega_*)|,$$

respectively. Then we have $\lim_{t\to+\infty} \|u_1(t)\|_E > 0$ and $\lim_{t\to+\infty} \|u_2(t)\|_E > 0$ for suitably small ε.

Remark 2.1.3. From Theorem 2.1.3, we can construct the solution $u = (u_1, u_2)$ to (2.1.5)–(2.1.2) with energy of each component does not decay. Consequently, if we choose a suitable (f, g), both $u_1(t)$ and $u_2(t)$ can behave like non-trivial free solutions as $t \to +\infty$.

Remark 2.1.4. Our proof of Theorems 2.1.2 and 2.1.3 below do not rely on the conservation laws (2.1.6) and (2.1.7) at all. For example, the same proof is valid for the system

$$\begin{cases} \Box u_1 = -|\nabla_x u_2|^2 \partial_t u_1, \\ \Box u_2 = -|\nabla_x u_1|^2 \partial_t u_2, \end{cases}$$

or more generally, any cubic satisfying the quadratic or cubic null conditions can be added to the right-hand side of it.

Remark 2.1.5. Theorems 2.1.2 and 2.1.3 concern only the forward Cauchy problem (i.e., for $t > 0$). For the backward Cauchy problem, we can construct a blowing-up solution with arbitrary small $\varepsilon > 0$ and a suitable choice of f, g based on the idea of [8]. This should be contrasted with the behavior of solutions under $(\mathbf{Ag}_0)$.

2.2 Preliminaries

In this section, we collect several notations and estimates which will be used in the subsequent sections.

We define $S := t\partial_t + x_1\partial_1 + x_2\partial_2$, $L_1 := t\partial_1 + x_1\partial_t$, $L_2 := t\partial_2 + x_2\partial_t$, $\Omega := x_1\partial_2 - x_2\partial_1$, and we set $\Gamma = (\Gamma_j)_{0\le j\le 6} = (S, L_1, L_2, \Omega, \partial_0, \partial_1, \partial_2)$. For a multi-index $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_6) \in \mathbb{Z}_+^7$, we write $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_6$ and $\Gamma^\alpha = \Gamma_0^{\alpha_0}\Gamma_1^{\alpha_1}\cdots\Gamma_6^{\alpha_6}$. We define $|\cdot|_s$, $\|\cdot\|_s$ by

$$|\phi(t,x)|_s = \sum_{|\alpha|\le s} |\Gamma^\alpha \phi(t,x)|, \qquad \|\phi(t,\cdot)\|_s = \sum_{|\alpha|\le s} \|\Gamma^\alpha \phi(t,\cdot)\|_{L^2(\mathbb{R}^2)},$$

respectively. For $x \in \mathbb{R}^2\backslash\{0\}$, we write $r := |x|$, $\omega = (\omega_1, \omega_2) := x/|x|$, $\omega^\perp = (\omega_1^\perp, \omega_2^\perp) := (-\omega_2, \omega_1)$, $\partial_r := \omega_1\partial_1 + \omega_2\partial_2$, and $\partial_\pm := \partial_t \pm \partial_r$. Following relations will play an important roll in the reduction argument of Section 2.3:

$$\partial_+\partial_-(r^{1/2}\phi) = r^{1/2}\Box\phi + \frac{1}{4r^{3/2}}(4\Omega^2 + 1)\phi, \tag{2.2.1}$$

$$(t + r)(\partial_j - \omega_j\partial_r) = \omega_j^\perp(\Omega + \omega_1 L_2 - \omega_2 L_1), \qquad j = 1, 2, \tag{2.2.2}$$

$$(t + r)\partial_+ = S + \omega_1 L_1 + \omega_2 L_2, \tag{2.2.3}$$

and $\partial_+ + \partial_- = 2\partial_t$, $\partial_+ - \partial_- = 2\partial_r$.

Next we review several estimates relevant to the free wave equation

$$\begin{cases} \Box\phi = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \phi(0) = \phi_0, & x \in \mathbb{R}^2. \\ \partial_t\phi(0) = \phi_1, & \end{cases} \tag{2.2.4}$$

Lemma 2.2.1. *For $\phi_0, \phi_1 \in C_0^\infty(\mathbb{R}^2)$ and $\alpha \in \mathbb{Z}_+^3$, there is a positive constant $C = C_\alpha(\phi_0, \phi_1)$ such that the smooth solution ϕ to* (2.2.4) *satisfies*

$$|\partial^\alpha \phi(t,x)| \le C\langle t+|x|\rangle^{-1}\langle t-|x|\rangle^{-|\alpha|-1/2} \tag{2.2.5}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^2$.

Lemma 2.2.2. *For $\phi_0, \phi_1 \in C_0^\infty(\mathbb{R}^2)$, there is a positive constant $C = C(\phi_0, \phi_1)$ such that the smooth solution ϕ to* (2.2.4) *satisfies*

$$\left| |x|^{1/2}\partial\phi(t,x) - \hat{\omega}(x)(\partial_\sigma \mathcal{F}_0[\phi_0,\phi_1])(|x|-t,\omega) \right| \le C\langle t+|x|\rangle^{-1}\langle t-|x|\rangle^{-1/2} \tag{2.2.6}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^2 \setminus \{0\}$, where $\hat{\omega}(x) = (-1, x_1/|x|, x_2/|x|)$.

Lemma 2.2.3. *For $\phi_0, \phi_1 \in C_0^\infty(\mathbb{R}^2)$, there is a positive constant $C = C(\phi_0, \phi_1)$ such that*

$$|\partial_\sigma \mathcal{F}_0[\phi_0,\phi_1](\sigma,\omega)| \le C\langle\sigma\rangle^{-3/2} \tag{2.2.7}$$

for $(\sigma,\omega) \in \mathbb{R} \times \mathbb{S}^1$.

For the proof of Lemmas 2.2.1, 2.2.2 and 2.2.3, see Section 3 in [34].

We close this section with the basic estimates for the global small amplitude solution u to (2.1.1)–(2.1.2) under (**Ag**). According to Section 3 in [39], we already know the following estimates.

Lemma 2.2.4. *Let $k \ge 4$, $0 < \mu < 1/10$ and $0 < (8k+7)\nu < \mu$. If $\varepsilon > 0$ is suitably small, then the solution u to* (2.1.1)–(2.1.2) *satisfies*

$$|u(t,x)|_{k+1} \le C\varepsilon\langle t+|x|\rangle^{-1/2+\mu}, \tag{2.2.8}$$

$$|\partial u(t,x)| \le C\varepsilon\langle t+|x|\rangle^{-1/2}\langle t-|x|\rangle^{\mu-1}, \tag{2.2.9}$$

$$|\partial u(t,x)|_k \le C\varepsilon\langle t+|x|\rangle^{-1/2+\nu}\langle t-|x|\rangle^{\mu-1}, \tag{2.2.10}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^2$ and

$$\|\partial u(t)\|_k \le C\varepsilon(1+t)^{\mu-\nu} \tag{2.2.11}$$

for $t \ge 0$, where C is a positive constant independent of ε.

2.3 The John–Hörmander reduction

In this section, we are going to make reductions of the problem to along the approach exploited in [40], [38], [39], [35]. The essential idea is based on John[30] and Hörmander [22] concerning detailed lifespan estimates for quadratic quasilinear wave equations in three space dimensions.

Let $u = (u_j)_{1\le j\le N}$ be a smooth solution to (2.1.1)–(2.1.2) on $[0,\infty)\times\mathbb{R}^2$. Since f and g are compactly-supported, we can take $R > 0$ such that $\operatorname{supp} f \cup \operatorname{supp} g \subset \{x \in \mathbb{R}^2; |x| \le R\}$. Then, by the finite propagation property, we have

$$\operatorname{supp} u(t,\cdot) \subset \{x \in \mathbb{R}^2; |x| \le t + R\} \tag{2.3.1}$$

for $t \ge 0$. We define $U = (U_j(t,x))_{1\le j\le N}$ by $U_j(t,x) = \mathcal{D}(|x|^{1/2}u_j(t,x))$, $1 \le j \le N$, $\mathcal{D} = -2^{-1}\partial_-$. We also introduce $H = (H_j(t,x))_{1\le j\le N}$ by

$$H_j = \frac{1}{2}\left(\frac{1}{t}F^{\mathrm{c,red}}(\omega,U) - r^{1/2}F(\partial u)\right) - \frac{1}{8r^{3/2}}(4\Omega^2+1)u_j.$$

By (2.2.1), we have

$$\partial_+ U_j(t,x) = \frac{-1}{2t}F_j^{\mathrm{c,red}}(\omega, U(t,x)) + H_j(t,x). \tag{2.3.2}$$

We introduce the following lemmas associated with U and H:

Lemma 2.3.1. *There exists a positive constant C such that*

$$\left||x|^{1/2}\partial u(t,x) - \hat{\omega}(x)U(t,x)\right| \le C\langle t+|x|\rangle^{-1/2}|u(t,x)|_1$$

for $(t,x) \in \Lambda_\infty := \{(t,x) \in [0,\infty)\times\mathbb{R}^2; |x| \ge t/2 \ge 1\}$.

It follows from (2.2.2) and (2.2.3). See Corollary 3.3 in [40] for more detail of the proof.

Lemma 2.3.2. *Under the quadratic null condition and (2.3.1), there exists a positive constant C which may depend on R such that*

$$|H(t,x)| \le Ct^{-1/2}\big(|\partial u| + \langle t+|x|\rangle^{-1}|u|_1\big)^2|u|_1 + Ct^{-3/2}|u|_2 \tag{2.3.3}$$

for $(t,x) \in \Lambda_{\infty,R} := \{(t,x) \in \Lambda_\infty\,;\, |x| \le t+R\}$.

For the proof, see Lemma 2.8 in [39].

These lemmas tell us that $\hat{\omega}U$ can be regarded as a good approximation of $r^{1/2}\partial u$ and H can be regarded as a remainder if u decays fast near the light cone. From (2.2.8), (2.2.9), (2.3.3) and Lemma 2.3.1, we obtain

$$\begin{aligned}|U(t,x)| &\le \left||x|^{1/2}\partial u(t,x)\right| + \left||x|^{1/2}\partial u(t,x) - \hat{\omega}U(t,x)\right| \\ &\le C\varepsilon\langle t-|x|\rangle^{\mu-1}\end{aligned} \tag{2.3.4}$$

and

$$\begin{aligned}|H(t,x)| &\le C\varepsilon^2 t^{-1/2}\langle t+|x|\rangle^{\mu-1}\langle t-|x|\rangle^{\mu-1} + C\varepsilon t^{-3/2}\langle t+|x|\rangle^{\mu-1/2} \\ &\le C\varepsilon t^{2\mu-3/2}\langle t-|x|\rangle^{-\mu-1/2}\end{aligned} \tag{2.3.5}$$

for $(t,x) \in \Lambda_{\infty,R}$. Note that the weights $|x|^{-1}$, t^{-1}, $(1+t)^{-1}$, $\langle t+|x|\rangle^{-1}$ are equivalent to each other on $\Lambda_{\infty,R}$. Indeed we have

$$\langle t+|x|\rangle^{-1} \le |x|^{-1} \le 2t^{-1} \le 3(1+t)^{-1} \le 3(R+2)\langle t+|x|\rangle^{-1}.$$

Now we make the final reduction. We set

$$\Sigma = \{(t,x) \in [0,\infty) \times \mathbb{R}^2; |x| \ge t/2 = 1 \text{ or } |x| = t/2 \ge 1\}$$

and $t_{0,\sigma} = \max\{2, -2\sigma\}$. Then, since the half line $\{(t,(t+\sigma)\omega)\ ;\ t \ge 0\}$ meets Σ at the point $(t_{0,\sigma}, (t_{0,\sigma}+\sigma)\omega)$ for each $(\sigma,\omega) \in \mathbb{R} \times \mathbb{S}^1$, we can see that

$$\Lambda_{\infty,R} = \bigcup_{(\sigma,\omega)\in(-\infty,R]\times\mathbb{S}^1} \{\ (t,(t+\sigma)\omega)\ ;\ t \ge t_{0,\sigma}\}.$$

We also note that there exists a positive constant c_0 depending only on R such that

$$c_0^{-1}\langle\sigma\rangle \le t_{0,\sigma} \le c_0\langle\sigma\rangle \tag{2.3.6}$$

for $\sigma \in (-\infty,R]$. We set $V_j(t;\sigma,\omega) = U_j(t,(t+\sigma)\omega)$ and $K_j(t;\sigma,\omega) = H_j(t,(t+\sigma)\omega)$ for $(t;\sigma,\omega) \in [t_{0,\sigma},\infty) \times \mathbb{R} \times \mathbb{S}^1$, $1 \le j \le N$. Then we can rewrite (2.3.2) as

$$\partial_t V_j(t) = \frac{-1}{2t} F_j^{\mathrm{c,red}}(\omega, V(t)) + K_j(t), \tag{2.3.7}$$

which we call the *profile equation.* It follows from (2.3.4) and (2.3.5) that

$$|V(t;\sigma,\omega)| \le C\varepsilon\langle\sigma\rangle^{\mu-1} \tag{2.3.8}$$

and

$$|K(t;\sigma,\omega)| \le C\varepsilon\langle\sigma\rangle^{-\mu-1/2} t^{2\mu-3/2} \tag{2.3.9}$$

for $(t,\sigma,\omega) \in [t_{0,\sigma},\infty) \times (-\infty,R] \times \mathbb{S}^1$.

2.4 Proof of Theorem 2.1.1

In this section, we are going to prove Theorem 2.1.1. We always assume that the quadratic null condition and (**Ag**) are satisfied but the cubic null condition is violated in this section.

2.4.1 A detailed pointwise estimate under (Ag) for single case

This subsection is devoted to a detailed pointwise estimate for the solution to (2.1.4) under the quadratic null condition and (**Ag**). The goal of this subsection is the following lemma.

Lemma 2.4.1. *Let $0 < \mu < 1/10$. Assume that the quadratic null condition and* (**Ag**) *are satisfied. If ε is suitably small, there exists a positive constant C, not depending on ε, such that the solution u to* (2.1.4) *satisfies*

$$|\partial u(t, r\omega)| \le \frac{C\varepsilon}{\sqrt{t}} \min\left\{ \frac{1}{\sqrt{P(\omega)\varepsilon^2 \log t}}, \frac{1}{\langle t-r \rangle^{1-\mu}} \right\} \tag{2.4.1}$$

for $(t, r, \omega) \in [2, \infty) \times [0, \infty) \times \mathbb{S}^1$.

Proof. By the definition of $P(\omega)$, we have $F^{\mathrm{c,red}}(\omega, Y) = P(\omega)Y^3$. In virtue of (2.3.7), we see that (2.1.4) is reduced to

$$\partial_t V(t) = -\frac{P(\omega)}{2t} V(t)^3 + K(t). \tag{2.4.2}$$

To investigate the asymptotics for $V(t)$, let us also recall the following useful lemma due to Matsumura.

Lemma 2.4.2. *Let $C_0 > 0$, $C_1 \ge 0$, $p > 1$, $q > 1$ and $t_0 \ge 2$. Suppose that a function $\Phi(t)$ satisfies*

$$\frac{d\Phi}{dt}(t) \le -\frac{C_0}{t} |\Phi(t)|^p + \frac{C_1}{t^q}$$

for $t \ge t_0$. Then we have

$$\Phi(t) \le \frac{C_2}{(\log t)^{p^*-1}}$$

for $t \ge t_0$, where p^* is the Hölder conjugate of p (i.e., $1/p + 1/p^* = 1$), and

$$C_2 = \frac{1}{\log 2} \left((\log t_0)^{p^*} \Phi(t_0) + C_1 \int_2^\infty \frac{(\log \tau)^{p^*}}{\tau^q} d\tau \right) + \left(\frac{p^*}{C_0 p} \right)^{p^*-1}.$$

For the proof, see Lemma 4.1 of [39].

Let $(\sigma, \omega) \in (-\infty, R] \times \mathbb{S}^1$ be fixed, and we set $\Phi(t) = \Phi(t; \sigma, \omega) = P(\omega)V(t; \sigma, \omega)^2$ for $t \ge t_{0,\sigma}$. It follows from (2.3.8), (2.3.9) and (2.4.2) that

$$\begin{aligned}
\partial_t \Phi(t) &= 2P(\omega)V(t)\partial_t V(t) \\
&= -\frac{P(\omega)^2}{t} V(t)^4 + 2P(\omega)V(t)K(t) \\
&\le -\frac{1}{t}\Phi(t)^2 + \frac{C_* \varepsilon^2}{t^{3/2-2\mu} \langle \sigma \rangle^{3/2}}
\end{aligned}$$

with some $C_* > 0$ not depending on σ, ω and ε. Therefore we can apply Lemma 2.4.2 with $p = 2$, $q = 3/2 - 2\mu$ and $t_0 = t_{0,\sigma}$ to obtain

$$0 \le \Phi(t;\sigma,\omega) \le \frac{M(\sigma,\omega)}{\log t},$$

where

$$M(\sigma,\omega) = \frac{1}{\log 2}\left((\log t_{0,\sigma})^2 P(\omega) V(t_{0,\sigma};\sigma,\omega)^2 + \frac{C_*\varepsilon^2}{\langle\sigma\rangle^{3/2}}\int_2^\infty \frac{(\log\tau)^2}{\tau^{3/2-2\mu}}d\tau\right)+1.$$

By virtue of (2.3.6) and (2.3.8), we see that $M(\sigma,\omega)$ can be dominated by a positive constant not depending on σ, ω and ε. Therefore we deduce that

$$|V(t;\sigma,\omega)| \le \sqrt{\frac{\Phi(t;\sigma,\omega)}{P(\omega)}} \le \frac{C}{\sqrt{P(\omega)\log t}}$$

for $(t,\sigma,\omega) \in [t_{0,\sigma},\infty)\times(-\infty,R]\times\mathbb{S}^1$. By Lemma 2.3.1 and (2.2.8), we have

$$\begin{aligned} r^{1/2}|\partial u(t,r\omega)| &\le \sqrt{2}|V(t;r-t,\omega)| + \left|r^{1/2}\partial u(t,r\omega) - \hat{\omega}U(t,r\omega)\right| \\ &\le \frac{C}{\sqrt{P(\omega)\log t}} + \frac{C\varepsilon}{\langle t+r\rangle^{1-\mu}} \end{aligned}$$

for $(t,r\omega)\in\Lambda_{\infty,R}$, whence

$$|\partial u(t,r\omega)| \le \frac{C}{\sqrt{rP(\omega)\log t}}\left(1 + \frac{\varepsilon\sqrt{P(\omega)\log t}}{t^{1-\mu}}\right) \le \frac{C\varepsilon}{\sqrt{t}}\cdot\frac{1}{\sqrt{P(\omega)\varepsilon^2\log t}}$$

for $(t,r\omega)\in\Lambda_{\infty,R}$. Piecing together this with (2.2.9), we arrive at the desired estimate (2.4.1) in the case of $(t,r\omega)\in\Lambda_{\infty,R}$. It is much easier to derive the bound for $|\partial u(t,r\omega)|$ in the case of $(t,r\omega)\notin\Lambda_{\infty,R}$ (indeed it follows from (2.2.9) only), so we skip it here. □

2.4.2 Key lemmas

This subsection is devoted to two important lemmas which play key roles in our analysis. Throughout this subsection, we suppose that $\Psi(\theta)$ is a real-valued function on $[0,2\pi]$ which can be written as a (finite) linear combination of the terms $\cos^{p_1}\theta\sin^{p_2}\theta$ with p_1, $p_2\in\mathbb{Z}_{\ge 0}$.

Lemma 2.4.3. *If $\Psi(\theta)\ge 0$ for all $\theta\in[0,2\pi]$, then we have either of the following three assertions:*

(**a**) $\Psi(\theta) = 0$ *for all* $\theta\in[0,2\pi]$.

(**b**) $\Psi(\theta) > 0$ *for all* $\theta\in[0,2\pi]$.

(**c**) *There exist positive integers m, $\nu_1, \dots, \nu_m$, points $\theta_1, \dots, \theta_m \in [0, 2\pi]$, and positive constants $c_1, \dots, c_m$ such that*

- $\Psi(\theta) > 0$ *for* $\theta \in [0, 2\pi] \backslash \{\theta_1, \dots, \theta_m\}$,
- $\Psi(\theta) = (\theta - \theta_j)^{2\nu_j}\big(c_j + o(1)\big)$ *as* $\theta \to \theta_j$ *for each* $j = 1, \dots, m$.

Proof. Let $\mathcal{N}$ be the set of zeros of Ψ on $[0, 2\pi]$. It is easy to see that the case $\mathcal{N} = \emptyset$ corresponds to the case (**b**) in the statement. Next we consider the case of $\sharp\mathcal{N} = \infty$. It follows from the Bolzano-Weierstrass theorem that $\mathcal{N}$ has an accumulation point. This is impossible unless Ψ vanishes identically on $[0, 2\pi]$ since Ψ is a trigonometric polynomial. (Indeed, through the standard identification of $\mathbb{S}^1$ with $\mathcal{C} := \{z \in \mathbb{C}\,;\, |z| = 1\}$, we may regard $\Psi(\theta)$ as a function $\psi(z)$ in the form $\sum_{k=-L}^{L} a_k z^k$ on $\mathcal{C}$, which is analytic in a neighborhood of $\mathcal{C}$ in $\mathbb{C}$. Then the identity theorem implies $\psi(z)$ vanishes identically on $\mathcal{C}$, so does $\Psi(\theta)$ on $[0, 2\pi]$.) Therefore we have (**a**). What remains is the case where $0 < \sharp\mathcal{N} < \infty$. In this case we can write $\mathcal{N}$ as $\{\theta_1, \dots, \theta_m\}$ with $m = \sharp\mathcal{N}$. Note that $\Psi(\theta) > 0$ for $\theta \in [0, 2\pi]\backslash\mathcal{N}$. Now let us focus on local behavior of $\Psi(\theta)$ near the point θ_j. We observe that we can take $\kappa_j \in \mathbb{Z}_{>0}$ such that $\Psi^{(l)}(\theta_j) = 0$ for $l \leq \kappa_j - 1$ and $\Psi^{(\kappa_j)}(\theta_j) \neq 0$. By the Taylor expansion, we have

$$\begin{aligned}\Psi(\theta) &= \sum_{l \leq \kappa_j} \frac{\Psi^{(l)}(\theta_j)}{l!}(\theta - \theta_j)^l + O((\theta - \theta_j)^{\kappa_j + 1}) \\ &= (\theta - \theta_j)^{\kappa_j}\left(\frac{\Psi^{(\kappa_j)}(\theta_j)}{\kappa_j!} + o(1)\right)\end{aligned}$$

as $\theta \to \theta_j$. By the assumption that Ψ is non-negative, we see that κ_j must be an even integer and $\Psi^{(\kappa_j)}(\theta_j)$ must be strictly positive. Therefore we arrive at the case (**c**) by setting $c_j = \Psi^{(\kappa_j)}(\theta_j)/(\kappa_j!)$ and $\nu_j = \kappa_j/2$. □

Lemma 2.4.4. *Assume that Ψ satisfies* (**c**). *We set $\nu = \max\{\nu_1, \dots, \nu_m\}$. Then, for $0 < \gamma < 1/(2\nu)$, we have*

$$\int_0^{2\pi} \frac{d\theta}{\Psi(\theta)^\gamma} < \infty.$$

Proof. We consider only the case where $\theta_j \neq 0,\ 2\pi$ for $j = 1, \dots, m$. The other case can be also shown by minor modifications. We take positive constants δ_j $(j = 1, \dots, m)$ so small that the intervals $J_j = (\theta_j - \delta_j, \theta_j + \delta_j)$ satisfy

$$J_j \cap J_k = \emptyset \quad \text{for} \quad 1 \leq j < k \leq m$$

and

$$\Psi(\theta) \geq \frac{c_j}{2}(\theta - \theta_j)^{2\nu_j} \quad \text{for} \quad \theta \in J_j.$$

We also set

$$K = [0, 2\pi] \setminus \bigcup_{j=1}^{m} J_j.$$

Since K is compact, we can take $M > 0$ such that $\Psi(\theta) \geq M$ for $\theta \in K$. So it follows that

$$\int_K \frac{d\theta}{\Psi(\theta)^\gamma} \leq \frac{2\pi}{M^\gamma} < \infty.$$

On the other hand, since $2\gamma\nu_j \leq 2\gamma\nu < 1$, we have

$$\int_{J_j} \frac{d\theta}{\Psi(\theta)^\gamma} \leq \left(\frac{2}{c_j}\right)^\gamma \int_{-\delta_j}^{\delta_j} \frac{d\theta}{|\theta|^{2\gamma\nu_j}} < \infty$$

for $j = 1, \ldots, m$. Summing up, we arrive at

$$\int_0^{2\pi} \frac{d\theta}{\Psi(\theta)^\gamma} = \int_K \frac{d\theta}{\Psi(\theta)^\gamma} + \sum_{j=1}^{m} \int_{J_j} \frac{d\theta}{\Psi(\theta)^\gamma} < \infty,$$

as desired. □

2.4.3 Proof of Theorem 2.1.1

Now we are ready to prove Theorem 2.1.1. As mentioned in Section 2.1, we already know that the conclusion is true under ($\mathbf{A}_+$). Since we assume that the quadratic null condition and (**Ag**) are satisfied but the cubic null condition is violated, we see that the case (**a**) in Lemma 2.4.3 is excluded by $\Psi(\theta) = P(\cos\theta, \sin\theta)$, whence it satisfies (**b**) or (**c**). Therefore, by Lemma 2.4.4, there exists $0 < \lambda < 1/4$ such that

$$\int_0^{2\pi} \frac{d\theta}{P(\cos\theta, \sin\theta)^{2\lambda}} < \infty.$$

With this λ, we choose μ such that

$$0 < \mu < \min\left\{\frac{1}{10}, \frac{1-4\lambda}{2-4\lambda}\right\}.$$

Let $t \geq 2$ from now on. By Lemma 2.4.1, we have

$$\begin{aligned} |\partial u(t, r\omega)| &\leq \frac{C\varepsilon}{\sqrt{t}} \left(\frac{1}{\sqrt{P(\omega)\varepsilon^2 \log t}}\right)^{2\lambda} \left(\frac{1}{\langle t-r\rangle^{1-\mu}}\right)^{1-2\lambda} \\ &= \frac{C\varepsilon}{(\varepsilon^2 \log t)^\lambda} \cdot \frac{1}{P(\omega)^\lambda} \cdot \frac{1}{\sqrt{t}\langle t-r\rangle^{(1-\mu)(1-2\lambda)}} \end{aligned} \tag{2.4.3}$$

for $(t, r, \omega) \in [2, \infty) \times (0, \infty) \times \mathbb{S}^1$. Next we set $\rho(t) = (\varepsilon^2 \log t)^{\frac{2\lambda}{1-2\mu}}$. For small $\varepsilon > 0$, we have $0 < \rho(t) < t$, and thus $0 < t + R - \rho(t) \le t + R$. Then we can split

$$2\|u(t)\|_E^2 = \int_{|x| \le t+R-\rho(t)} |\partial u(t,x)|^2 dx + \int_{t+R-\rho(t) \le |x| \le t+R} |\partial u(t,x)|^2 dx$$
$$=: I_1(t) + I_2(t).$$

We also note that

$$r/t \le (t+R)/t \le 1 + R/2 \quad \text{for} \quad 0 \le r \le t + R,$$

and

$$0 < \rho(t) \le R + t - r \le (1+R)\langle t - r\rangle \quad \text{for} \quad 0 \le r \le t + R - \rho(t).$$

By using the polar coordinates, we deduce from (2.4.1) and (2.4.3) that

$$\begin{aligned} I_1(t) &\le \int_0^{2\pi} \int_0^{t+R-\rho(t)} \left(\frac{C\varepsilon}{\sqrt{t}\langle t - r\rangle^{1-\mu}} \right)^2 r dr d\theta \\ &\le C\varepsilon^2 \int_0^{t+R-\rho(t)} \frac{r dr}{t\langle t - r\rangle^{2-2\mu}} \\ &\le C\varepsilon^2 \int_0^{t+R-\rho(t)} \frac{dr}{(R + t - r)^{2-2\mu}} \\ &\le \frac{C\varepsilon^2}{\rho(t)^{1-2\mu}} \end{aligned}$$

and

$$\begin{aligned} I_2(t) &\le \frac{C\varepsilon^2}{(\varepsilon^2 \log t)^{2\lambda}} \left(\int_0^{2\pi} \frac{d\theta}{P(\cos\theta, \sin\theta)^{2\lambda}} \right) \left(\int_{t+R-\rho(t)}^{t+R} \frac{r dr}{t\langle t - r\rangle^{2(1-\mu)(1-2\lambda)}} \right) \\ &\le \frac{C\varepsilon^2}{(\varepsilon^2 \log t)^{2\lambda}} \int_{\mathbb{R}} \frac{d\sigma}{\langle \sigma \rangle^{2(1-\mu)(1-2\lambda)}}, \end{aligned}$$

respectively. Since $2(1-\mu)(1-2\lambda) > 1$, we see that the integral in the last line converges. Eventually we obtain

$$\|u(t)\|_E^2 \le \frac{C\varepsilon^2}{\rho(t)^{1-2\mu}} + \frac{C\varepsilon^2}{(\varepsilon^2 \log t)^{2\lambda}} \le \frac{C\varepsilon^2}{(\varepsilon^2 \log(t+2))^{2\lambda}}.$$

We also have

$$\|u(t)\|_E^2 \le C\varepsilon^2 \int_0^{t+R} \frac{r dr}{t\langle t - r\rangle^{2-2\mu}} \le C\varepsilon^2 \int_{\mathbb{R}} \frac{d\sigma}{\langle \sigma \rangle^{2-2\mu}} \le C\varepsilon^2$$

by (2.4.1). Summing up, we arrive at the desired estimate. □

2.4.4 Remarks on the decay rates

It is worthwhile to mention the exponent λ appearing in Theorem 2.1.1. In view of the argument in Subsection 2.4.3, we can see that λ is determined by ν coming from Lemma 2.4.4. To be more precise, we can take $\lambda = 1/(4\nu) - \delta$ with arbitrarily small $\delta > 0$, and 2ν is the maximum of the vanishing order of zeros of $\Psi(\theta) = P(\cos\theta, \sin\theta)$.

Now, let us compute ν for the examples of $F^{\mathrm{c}}(\partial u)$ raised in Remark 2.1.1.

(1) We first focus on $F^{\mathrm{c}}(\partial u) = -(\partial_1 u)^2 \partial_t u$. Since $\Psi(\theta) = \cos^2\theta$, we can check that

$$\Psi(\theta) = (\theta - \pi/2)^2(1 + o(1)) \quad (\theta \to \pi/2),$$
$$\Psi(\theta) = (\theta - 3\pi/2)^2(1 + o(1)) \quad (\theta \to 3\pi/2),$$

and $\Psi(\theta) > 0$ when $\theta \neq \pi/2, 3\pi/2$. These tell us that $\nu = 1$, and thus we have $\|u(t)\|_E = O((\log t)^{-1/4+\delta})$ as $t \to \infty$, where $\delta > 0$ can be arbitrarily small.

(2) In the case of $F^{\mathrm{c}}(\partial u) = -(\partial_1 u)^2(\partial_t u + \partial_2 u)$, we see that $\Psi(\theta) = \cos^2\theta(1 - \sin\theta)$, and its zeros are $\theta = \pi/2$ and $3\pi/2$. Near these points, we have

$$\Psi(\theta) = (\theta - \pi/2)^4(1/2 + o(1)) \quad (\theta \to \pi/2)$$

and

$$\Psi(\theta) = (\theta - 3\pi/2)^2(2 + o(1)) \quad (\theta \to 3\pi/2).$$

Hence $\nu = \max\{2, 1\} = 2$, from which it follows that $\|u(t)\|_E$ decays like $O((\log t)^{-1/8+\delta})$ as $t \to \infty$ with arbitrarily small $\delta > 0$.

(3) For $F^{\mathrm{c}}(\partial u) = -(\partial_t u + \partial_2 u)^3$, we have $\Psi(\theta) = (1 - \sin\theta)^3$. This vanishes only when $\theta = \pi/2$, and it holds that

$$\Psi(\theta) = (\theta - \pi/2)^6(1/8 + o(1)) \quad (\theta \to \pi/2).$$

Therefore $\nu = 3$, and this implies that $\|u(t)\|_E = O((\log t)^{-1/12+\delta})$ as $t \to \infty$ with $0 < \delta \ll 1/12$.

Remark 2.4.1. It is not certain whether these decay rates are the best or not. It may be an interesting problem to specify the optimal rates for the energy decay.

2.5 Proof of Theorems 2.1.2 and 2.1.3

In this section, we prove Theorems 2.1.2 and 2.1.3. The key of our proof is to specify the asymptotic behavior of solutions to the profile equation (2.3.7) in the case of (2.1.5)

2.5.1 Asymptotics of solutions to the profile equation for a system case

We focus on large-time behavior of solutions to the profile equation associated with (2.1.5). In virtue of (2.3.7), we can rewrite (2.1.5) as

$$\begin{cases} \partial_t V_1(t) = \dfrac{-1}{2t} V_1(t) V_2(t)^2 + K_1(t), \\ \partial_t V_2(t) = \dfrac{-1}{2t} V_1(t)^2 V_2(t) + K_2(t). \end{cases} \tag{2.5.1}$$

The goal of this subsection is to show the following.

Proposition 2.5.1. *Let $V = (V_j(t;\sigma,\omega))_{j=1,2}$ be the solutions to (2.5.1). There exists $V^+ = (V_j^+(\sigma,\omega))_{j=1,2} \in L^2(\mathbb{R}\times\mathbb{S}^1)$ such that*

$$\lim_{t\to\infty} \int_{\mathbb{R}} \int_{\mathbb{S}^1} \left| \chi_t(\sigma) V(t;\sigma,\omega) - V^+(\sigma,\omega) \right|^2 \, dS_\omega d\sigma = 0, \tag{2.5.2}$$

where $\chi_t : \mathbb{R} \to \mathbb{R}$ is a bump function satisfying $\chi_t(\sigma) = 1$ for $\sigma > -t$ and $\chi_t(\sigma) = 0$ for $\sigma \le -t$.

Before we start the proof of Proposition 2.5.1, we introduce a following lemma on ODE which is used in our proof.

Lemma 2.5.1. *Let $t_0 \ge 0$ be given. For λ, $Q \in C \cap L^1([t_0,\infty))$, assume that $y(t)$ satisfies*

$$\frac{dy}{dt}(t) = \lambda(t) y(t) + Q(t)$$

for $t \ge t_0$. Then we have

$$|y(t) - y^+| \le C_3 \int_t^\infty \left(|y^+||\lambda(\tau)| + |Q(\tau)| \right) d\tau$$

for $t \ge t_0$, where

$$C_3 = \exp\left(\int_{t_0}^\infty |\lambda(\tau)| \, d\tau \right)$$

and

$$y^+ = y(t_0) e^{\int_{t_0}^\infty \lambda(\tau)\, d\tau} + \int_{t_0}^\infty Q(s) e^{\int_s^\infty \lambda(\tau)\, d\tau} \, ds.$$

Proof. Put

$$\Phi(t;s) = \exp\left(\int_s^t \lambda(\tau) \, d\tau \right)$$

for s, $t \in [t_0,\infty]$. Then we see that

$$y(t) = \Phi(t;t_0) y(t_0) + \int_{t_0}^t \Phi(t;s) Q(s) \, ds = \Phi(t;\infty) y^+ - \int_t^\infty \Phi(t;s) Q(s) \, ds.$$

We also note that $|\Phi(s;t)| \le C_3$ and that

$$|\Phi(t;\infty) - 1| \le C_3 \int_t^\infty |\lambda(\tau)|\, d\tau.$$

Therefore we obtain

$$\begin{aligned} |y(t) - y^+| &\le |\Phi(t;\infty) - 1||y^+| + \int_t^\infty |\Phi(t;s)||Q(s)|\, ds \\ &\le C_3|y^+| \int_t^\infty |\lambda(\tau)|\, d\tau + C_3 \int_t^\infty |Q(\tau)|\, d\tau, \end{aligned}$$

as desired. □

Proof of Proposition 2.5.1. We first show the pointwise convergence of $V(t;\sigma,\omega)$ as $t \to +\infty$. We note that (2.3.1) implies $V(t;\sigma,\omega) = 0$ if $\sigma \ge R$. In what follows, we fix $(\sigma,\omega) \in (-\infty, R] \times \mathbb{S}^1$ and introduce

$$\rho(t) = \rho(t;\sigma,\omega) := V_1(t;\sigma,\omega)K_1(t;\sigma,\omega) - V_2(t;\sigma,\omega)K_2(t;\sigma,\omega)$$

so that

$$\frac{1}{2}\partial_t\Big((V_1(t))^2 - (V_2(t))^2\Big) = V_1(t)\partial_t V_1(t) - V_2(t)\partial_t V_2(t) = \rho(t).$$

It follows from (2.3.8) and (2.3.9) that

$$\begin{aligned} |\rho(t)| &\le \sum_{j=1}^{2} |V_j(\tau;\sigma,\omega)K_j(\tau;\sigma,\omega)| \\ &\le C\varepsilon^2 \langle\sigma\rangle^{-3/2} t^{2\mu-3/2}. \end{aligned} \tag{2.5.3}$$

Thus, by (2.3.6) and (2.5.3), we have

$$\begin{aligned} \int_{t_{0,\sigma}}^\infty |\rho(\tau;\sigma,\omega)|\, d\tau &\le \int_{t_{0,\sigma}}^\infty C\varepsilon^2\langle\sigma\rangle^{-3/2}\tau^{2\mu-3/2}\, d\tau \\ &\le C\varepsilon^2\langle\sigma\rangle^{-3/2}(t_{0,\sigma})^{2\mu-1/2} \\ &\le C\varepsilon^2\langle\sigma\rangle^{2\mu-2}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &(V_1(t;\sigma,\omega))^2 - (V_2(t;\sigma,\omega))^2 \\ &= (V_1(t_{0,\sigma};\sigma,\omega))^2 - (V_2(t_{0,\sigma};\sigma,\omega))^2 + 2\int_{t_{0,\sigma}}^t \rho(\tau;\sigma,\omega)\, d\tau \\ &= m(\sigma,\omega) - r(t;\sigma,\omega) \end{aligned} \tag{2.5.4}$$

for $t \geq t_{0,\sigma}$, where

$$m(\sigma, \omega) := (V_1(t_{0,\sigma}; \sigma, \omega))^2 - (V_2(t_{0,\sigma}; \sigma, \omega))^2 + 2\int_{t_{0,\sigma}}^{\infty} \rho(\tau; \sigma, \omega)\, d\tau \quad (2.5.5)$$

and

$$r(t) = r(t; \sigma, \omega) := 2\int_{t}^{\infty} \rho(\tau; \sigma, \omega)\, d\tau.$$

Note that

$$|m| \leq |V(t_{0,\sigma})|^2 + C\int_{t_{0,\sigma}}^{\infty} |\rho(\tau)|\, d\tau \leq C\varepsilon^2 \langle\sigma\rangle^{2\mu-2}$$

and

$$|r(t)| \leq C\int_{t}^{\infty} |\rho(\tau)|\, d\tau \leq C\varepsilon^2 \langle\sigma\rangle^{-3/2} t^{2\mu-1/2}. \quad (2.5.6)$$

Now we divide the argument into three cases according to the sign of $m(\sigma, \omega)$ as follows.

Case 1: $m(\sigma, \omega) > 0$**.** First we focus on the asymptotics for $V_2(t)$. By (2.3.8), (2.3.9), (2.5.1), (2.5.4) and (2.5.6), we have

$$\begin{aligned} \partial_t V_2(t) &= \frac{-1}{2t} V_2(t)^3 - \frac{m}{2t} V_2(t) + \frac{r(t)}{2t} V_2(t) + K_2(t) \\ &\leq \frac{-1}{2t} V_2(t)^3 - \frac{m}{2t} V_2(t) + C\varepsilon \langle\sigma\rangle^{-\mu-1/2} t^{2\mu-3/2}, \end{aligned}$$

whence

$$\begin{aligned} \frac{1}{2}\partial_t \left(t^m V_2(t)^2\right) &= t^m V_2(t) \left(\partial_t V_2(t) + \frac{m}{2t} V_2(t)\right) \\ &\leq t^m \left(\frac{-1}{2t} V_2(t)^4 + C\varepsilon^2 \langle\sigma\rangle^{-3/2} t^{2\mu-3/2}\right) \\ &\leq C\varepsilon^2 \langle\sigma\rangle^{-3/2} t^{2\mu+m-3/2}. \end{aligned}$$

Integration in t leads to

$$\begin{aligned} t^m V_2(t)^2 - (t_{0,\sigma})^m V_2(t_{0,\sigma})^2 &\leq C\varepsilon^2 \langle\sigma\rangle^{-3/2} \int_{t_{0,\sigma}}^{t} \tau^{2\mu+m-3/2} d\tau \\ &\leq C\varepsilon^2 \langle\sigma\rangle^{-3/2} (t_{0,\sigma})^{2\mu+m-1/2} \\ &\leq C\varepsilon^2 \langle\sigma\rangle^{2\mu+m-2} \end{aligned}$$

for $t \geq t_{0,\sigma}$. Therefore we deduce that

$$|V_2(t)| \leq C\varepsilon \langle\sigma\rangle^{\mu+m/2-1} t^{-m/2}. \quad (2.5.7)$$

In particular, $V_2(t) \to 0$ as $t \to +\infty$. Next we turn our attentions to the asymptotics for $V_1(t)$. Since $V_1(t)$ solves $\partial_t V_1(t) = \lambda(t)V_1(t) + Q(t)$ with $\lambda(t) = -V_2(t)^2/t$ and $Q(t) = K_1(t;\sigma,\omega)$, we can apply Lemma 2.5.1 to $V_1(t)$. Then we have

$$|V_1(t) - W_1^+| \le C\int_t^\infty \left(\frac{|W_1^+||V_2(\tau)|^2}{\tau} + |K_1(\tau)|\right) d\tau,$$

where

$$W_1^+ = W_1^+(\sigma,\omega) := V_1(t_{0,\sigma};\sigma,\omega)e^{-\int_{t_{0,\sigma}}^\infty V_2(\tau;\sigma,\omega)^2\frac{d\tau}{\tau}} + \int_{t_{0,\sigma}}^\infty K_1(s;\sigma,\omega)e^{-\int_s^\infty V_2(\tau;\sigma,\omega)^2\frac{d\tau}{\tau}}\, ds.$$

By (2.3.8), (2.3.9) and (2.5.7), we have

$$|W_1^+| \le |V_1(t_{0,\sigma})| + \int_{t_{0,\sigma}}^\infty |K_1(s)|ds \le C\varepsilon\langle\sigma\rangle^{\mu-1} \tag{2.5.8}$$

and

$$\begin{aligned}\int_t^\infty \left(\frac{|W_1^+||V_2(\tau)|^2}{\tau} + |K_1(\tau)|\right) d\tau &\le C\int_t^\infty \left(\frac{\varepsilon^3\langle\sigma\rangle^{3\mu+m-3}}{\tau^{1+m}} + \frac{\varepsilon\langle\sigma\rangle^{-\mu-1/2}}{\tau^{3/2-2\mu}}\right) d\tau \\ &\le \frac{C\varepsilon^3\langle\sigma\rangle^{3\mu+m-3}}{mt^m} + \frac{C\varepsilon\langle\sigma\rangle^{-\mu-1/2}}{t^{1/2-2\mu}}.\end{aligned}$$

Therefore we conclude that $V_1(t) \to W_1^+$ as $t \to +\infty$.

Case 2: $m(\sigma,\omega) < 0$**.** Similarly to the previous case, we have

$$\lim_{t\to\infty} |V_1(t;\sigma,\omega)| = 0, \qquad \lim_{t\to\infty} |V_2(t;\sigma,\omega) - W_2^+(\sigma,\omega)| = 0,$$

where

$$W_2^+(\sigma,\omega) := V_2(t_{0,\sigma};\sigma,\omega)e^{-\int_{t_{0,\sigma}}^\infty V_1(\tau;\sigma,\omega)^2\frac{d\tau}{\tau}} + \int_{t_{0,\sigma}}^\infty K_2(s;\sigma,\omega)e^{-\int_s^\infty V_1(\tau;\sigma,\omega)^2\frac{d\tau}{\tau}}\, ds.$$

Remark that $|W_2^+| \le C\varepsilon\langle\sigma\rangle^{\mu-1}$.

Case 3: $m(\sigma,\omega) = 0$**.** By (2.3.7), (2.3.8), (2.3.9), (2.5.4) and (2.5.6), we have

$$\begin{aligned}\partial_t\left(V_1(t)^2\right) &= \frac{-1}{t}V_1(t)^4 - \frac{r(t)}{t}V_1(t)^2 + 2V_1(t)K_1(t) \\ &\le \frac{-1}{t}(V_1(t))^4 + C\varepsilon^2\langle\sigma\rangle^{-3/2}t^{2\mu-3/2}\end{aligned}$$

for $t \geq t_{0,\sigma}$. Thus we can apply Lemma 2.4.2 with $\Phi(t) = V_1(t)^2$ to obtain

$$|V_1(t)| \leq \frac{C}{\sqrt{\log t}} \to 0 \qquad (t \to +\infty).$$

Also (2.5.4) gives us $|V_2(t)| = \sqrt{V_1(t)^2 + r(t)} \to 0$ as $t \to \infty$.

Summing up the three cases above, we deduce that $V(t;\sigma,\omega)$ converges as $t \to +\infty$ for each fixed $(\sigma,\omega) \in \mathbb{R} \times \mathbb{S}^1$. In order to show (2.5.2), we set

$$V_1^+(\sigma,\omega) := \begin{cases} W_1^+(\sigma,\omega) & (m(\sigma,\omega) > 0)\,, \\ 0 & (m(\sigma,\omega) \leq 0)\,, \end{cases}$$

$$V_2^+(\sigma,\omega) := \begin{cases} 0 & (m(\sigma,\omega) \geq 0)\,, \\ W_2^+(\sigma,\omega) & (m(\sigma,\omega) < 0)\,, \end{cases}$$

and $V^+(\sigma,\omega) = (V_j^+(\sigma,\omega))_{j=1,2}$ for $(\sigma,\omega) \in \mathbb{R} \times \mathbb{S}^1$. Then, by virtue of (2.5.8), we have $V^+ \in L^2(\mathbb{R} \times \mathbb{S}^1)$ and

$$\left|\chi_t(\sigma)V(t;\sigma,\omega) - V^+(\sigma,\omega)\right|^2 \leq C\varepsilon^2 \langle\sigma\rangle^{2\mu-2} \in L^1(\mathbb{R} \times \mathbb{S}^1)$$

for all $t \geq t_{0,\sigma}$. Moreover, it holds that

$$\lim_{t\to\infty} \left|\chi_t(\sigma)V(t;\sigma,\omega) - V^+(\sigma,\omega)\right|^2 = 0$$

for each fixed $(\sigma,\omega) \in \mathbb{R} \times \mathbb{S}^1$. Consequently, Lebesgue's dominated convergence theorem yields (2.5.2). □

2.5.2 Proof of Theorem 2.1.2

We are going to prove Theorem 2.1.2. First we recall the following useful lemma.

Lemma 2.5.2 ([32] Theorem 2.1)**.** *For* $\phi \in C\left([0,\infty);\dot{H}^1\right) \cap C^1\left([0,\infty);L^2\right)$, *the following two assertions* (i) *and* (ii) *are equivalent:*

(i) *There exists* $(\phi_0^+,\phi_1^+) \in \dot{H}^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ *such that*

$$\lim_{t\to\infty} \|\phi(t) - \phi^+(t)\|_E = 0,$$

where $\phi^+ \in C\left([0,\infty);\dot{H}^1(\mathbb{R}^2)\right) \cap C^1\left([0,\infty);L^2(\mathbb{R}^2)\right)$ *is a unique solution to* $\Box\phi^+ = 0$, $\phi^+(0) = \phi_0^+$, $\partial\phi^+(0) = \phi_1^+$.

(ii) *There exists* $\Phi = \Phi(\sigma,\omega) \in L^2(\mathbb{R} \times \mathbb{S}^1)$ *such that*

$$\lim_{t\to\infty} \|\partial\phi(t,\cdot) - \hat{\omega}(\cdot)\Phi^\sharp(t,\cdot)\|_{L^2(\mathbb{R}^2)} = 0,$$

where $\Phi^\sharp(t,x) = |x|^{-1/2}\Phi(|x| - t, x/|x|)$.

By virtue of this lemma, to prove that u_1 is asymptotically free, it is sufficient to show

$$\lim_{t\to\infty} \|\partial u_1(t,\cdot) - \hat{\omega}(\cdot)V_1^{+,\sharp}(t,\cdot)\|_{L^2(\mathbb{R}^2)} = 0 \tag{2.5.9}$$

for $V_1^+(\sigma,\omega)$ obtained in Proposition 2.5.1. To prove (2.5.9), we split

$$\begin{aligned}
&\|\partial u_1(t,\cdot) - \hat{\omega}(\cdot)V_1^{+,\sharp}(t,\cdot)\|_{L^2(\mathbb{R}^2)}^2 \\
=&\int_{\mathbb{R}^2} |\partial u_1(t,x) - \hat{\omega}(x)|x|^{-1/2}V_1^+(|x|-t, x/|x|)|^2 dx \\
\le& 2\int_{\mathbb{R}^2\setminus\Lambda_\infty} |\partial u_1(t,x) - \hat{\omega}(x)|x|^{-1/2}V_1(t;|x|-t, x/|x|)|^2\, dx \\
&+ 2\int_{\Lambda_\infty} |\partial u_1(t,x) - \hat{\omega}(x)|x|^{-1/2}V_1(t;|x|-t, x/|x|)|^2\, dx \\
&+ 2\int_0^\infty \int_{\mathbb{S}^1} |\hat{\omega}(r\omega)V_1(t;r-t,\omega) - \hat{\omega}(r\omega)V_1^+(r-t,\omega)|^2\, dS_\omega dr \\
=&: J_1(t) + J_2(t) + J_3(t).
\end{aligned}$$

To show the decay for $J_1(t)$, we note that $\langle t+|x|\rangle \le C\langle t-|x|\rangle$ on $\mathbb{R}^2\setminus\Lambda_\infty$. Then (2.2.9) and (2.3.8) imply

$$\begin{aligned}
J_1(t) &\le C\varepsilon^2 \int_{\mathbb{R}^2\setminus\Lambda_\infty} \left(\langle t-|x|\rangle^{-1}\langle t+|x|\rangle^{2\mu-2} + |x|^{-1}\langle t-|x|\rangle^{2\mu-2}\right) dx \\
&\le C\varepsilon^2 \int_{\mathbb{R}^2\setminus\Lambda_\infty} |x|^{-1}\langle t+|x|\rangle^{2\mu-2}\, dx \\
&\le C\varepsilon^2 \int_0^\infty \int_{\mathbb{S}^1} (1+t+r)^{2\mu-2} dS_\omega\, dr \\
&\le C\varepsilon^2 (1+t)^{2\mu-1}.
\end{aligned}$$

As for $J_2(t)$, we see from Lemma 2.3.1 and (2.2.8) that

$$\begin{aligned}
J_2(t) &= 2\int_{\Lambda_\infty} |x|^{-1} \left||x|^{1/2}\partial u_1(t,x) - \hat{\omega}(x)\mathcal{D}\left(|x|^{1/2}u_1(t,x)\right)\right|^2 dx \\
&\le C\int_{\Lambda_\infty} |x|^{-1}\langle t+|x|\rangle^{-1}|u(t,x)|_1^2\, dx \\
&\le C\varepsilon^2 \int_{\mathbb{R}^2} |x|^{-1}\langle t+|x|\rangle^{2\mu-2}\, dx \\
&\le C\varepsilon^2 (1+t)^{2\mu-1}.
\end{aligned}$$

Finally, we note that

$$\begin{aligned}J_3(t) &\le C\int_0^\infty \int_{\mathbb{S}^1} \left|V_1(t; r-t,\omega) - V_1^+(r-t,\omega)\right|^2 \, dS_\omega dr \\ &\le C\int_{-t}^\infty \int_{\mathbb{S}^1} \left|V_1(t;\sigma,\omega) - V_1^+(\sigma,\omega)\right|^2 \, dS_\omega d\sigma \\ &\le C\int_{\mathbb{R}} \int_{\mathbb{S}^1} \left|\chi_t(\sigma)V_1(t;\sigma,\omega) - V_1^+(\sigma,\omega)\right|^2 \, dS_\omega d\sigma.\end{aligned}$$

Then, by using Lebesgue's dominated convergence theorem, it follows from (2.5.2) that we obtain $\lim_{t\to\infty} J_3(t) = 0$.

Piecing them together, we arrive at (2.5.9). Similarly we have

$$\lim_{t\to\infty} \|\partial u_2(t,\cdot) - \hat{\omega}(\cdot)V_2^{+,\sharp}(t,\cdot)\|_{L^2(\mathbb{R}^2)} = 0,$$

where V_2^+ is from Proposition 2.5.1. With the aid of Lemma 2.5.2, we conclude that u_2 is also asymptotically free. □

2.5.3 Leading term of $m(\sigma,\omega)$

According to Subsection 2.5.1, the function $m(\sigma,\omega)$ is closely related to the vanishing of the scattering state. We summarize the result in the proof of Proposition 2.5.1 as following lemma.

Lemma 2.5.3. *Let $V^+ = (V_1^+, V_2^+)$ be in Proposition 2.5.1 and $m = m(\sigma,\omega)$ be the function defined by (2.5.5). Then the following holds for each $(\sigma,\omega) \in \mathbb{R}\times\mathbb{S}^1$:*

- *$m(\sigma,\omega) > 0$ implies $V_1^+(\sigma,\omega) \neq 0$ and $V_2^+(\sigma,\omega) = 0$;*
- *$m(\sigma,\omega) < 0$ implies $V_1^+(\sigma,\omega) = 0$ and $V_2^+(\sigma,\omega) \neq 0$;*
- *$m(\sigma,\omega) = 0$ implies $V_1^+(\sigma,\omega) = V_2^+(\sigma,\omega) = 0$.*

From this lemma, it is natural to expect that the better understanding of $m(\sigma,\omega)$ bring us more precise information on the energy decay. Therefore, we focus on $m(\sigma,\omega)$ to prove Theorem 2.1.3. In the rest of this subsection, we are going to specify the leading term of $m(\sigma,\omega)$.

Lemma 2.5.4. *Let $0 < \mu < 1/10$. Then we have*

$$m(\sigma,\omega) = \varepsilon^2\left((\partial_\sigma \mathcal{F}_1(\sigma,\omega))^2 - (\partial_\sigma \mathcal{F}_2(\sigma,\omega))^2\right) + O(\varepsilon^{5/2-2\mu})$$

uniformly in $(\sigma,\omega) \in \mathbb{R}\times\mathbb{S}^1$ as $\varepsilon \to +0$.

Proof. First we note that $t_{0,\sigma}$ in (2.5.5) can be replaced by $t_{1,\sigma} := \max\{\varepsilon^{-1}, -2\sigma\}$ since we have

$$\begin{aligned}
&\big(V_1(t_{1,\sigma})^2 - V_2(t_{1,\sigma})^2\big) - \big(V_1(t_{0,\sigma})^2 - V_2(t_{0,\sigma})^2\big) \\
&\qquad = 2\int_{t_{0,\sigma}}^{t_{1,\sigma}} \left(V_1(\tau)\partial_t V_1(\tau) - V_2(\tau)\partial_t V_2(\tau)\right)\, d\tau \\
&\qquad = 2\int_{t_{0,\sigma}}^{t_{1,\sigma}} \rho(\tau;\sigma,\omega)\, d\tau. \qquad (2.5.10)
\end{aligned}$$

It follows from (2.5.3), we also obtain

$$\begin{aligned}
\left|\int_{t_{1,\sigma}}^{\infty} \rho(\tau;\sigma,\omega)\, d\tau\right| &\le C\varepsilon^2 \langle\sigma\rangle^{-3/2} \int_{\varepsilon^{-1}}^{\infty} \tau^{2\mu-3/2}\, d\tau \\
&\le C\varepsilon^{5/2-2\mu}. \qquad (2.5.11)
\end{aligned}$$

From (2.5.5), (2.5.10) and (2.5.11), we get

$$\left|m(\sigma,\omega) - \big((V_1(t_{1,\sigma};\sigma,\omega))^2 - (V_2(t_{1,\sigma};\sigma,\omega))^2\big)\right| \le C\varepsilon^{5/2-2\mu}$$

for $(\sigma,\omega) \in \mathbb{R}\times\mathbb{S}^1$. Thus, to prove Lemma 2.5.4, it suffices to show

$$V_j(t_{1,\sigma};\sigma,\omega) = \varepsilon\partial_\sigma \mathcal{F}_j(\sigma,\omega) + O(\varepsilon^{2-\mu}) \qquad (2.5.12)$$

as $\varepsilon \to +0$ uniformly in $(\sigma,\omega) \in \mathbb{R}\times\mathbb{S}^1$ for $j = 1, 2$. The rest part of this subsection is devoted to the proof of (2.5.12). We divide the argument into the following two cases.

Case 1: $\sigma \le -1/(2\varepsilon)$**.** If we assume $|x| \le t/2$ and $t \ge \varepsilon^{-1}$, we have $\varepsilon^{-1} \le t \le \langle t+|x|\rangle \le C\langle t-|x|\rangle$. It follows from (2.2.8) and (2.2.9) that

$$\begin{aligned}
|U(t,x)| &\le C|x|^{-1/2}\,|u(t,x)| + C|x|^{1/2}\,|\partial u(t,x)| \\
&\le C\varepsilon|x|^{-1/2}\langle t+|x|\rangle^{-1/2+\mu} + C\varepsilon|x|^{1/2}\langle t+|x|\rangle^{-1/2}\langle t-|x|\rangle^{\mu-1} \\
&\le C\varepsilon^{2-\mu}
\end{aligned}$$

for $|x| \le t/2$ and $t \ge \varepsilon^{-1}$. Then we obtain

$$|V(t;\sigma,\omega)| = |U(t,(t+\sigma)\omega)| \le C\varepsilon^{2-\mu} \qquad (2.5.13)$$

for $t+\sigma \le t/2$ and $t \ge \varepsilon^{-1}$. In the case $\varepsilon^{-1} \le -2\sigma$, we have $t_{1,\sigma}+\sigma = t_{1,\sigma}/2$ and $t_{1,\sigma} \ge \varepsilon^{-1}$. Therefore, from (2.2.7), (2.5.13) and $|\sigma| \ge 1/(2\varepsilon)$, we get

$$\begin{aligned}
|V_j(t_{1,\sigma};\sigma,\omega) - \varepsilon\partial_\sigma\mathcal{F}_j(\sigma,\omega)| &\le |V_j(t_{1,\sigma};\sigma,\omega)| + \varepsilon|\partial_\sigma\mathcal{F}_j(\sigma,\omega)| \\
&\le C\varepsilon^{2-\mu} + C\varepsilon\langle\sigma\rangle^{-3/2} \\
&\le C\varepsilon^{2-\mu}.
\end{aligned}$$

Case 2: $\sigma > -1/(2\varepsilon)$**.** For $j = 1, 2$, let $u_j^0 = u_j^0(t,x)$ be the solution to the free wave equation $\Box u_j^0 = 0$ with the initial data $u_j^0(0) = f_j$, $\partial_t u_j^0(0) = g_j$ and we put $u_j^1(t,x) := u_j(t,x) - \varepsilon u_j^0(t,x)$, so that u_j^1 solves

$$\begin{aligned}&\Box u_j^1(t,x) = F_j(\partial u), && (t,x) \in (0,\infty) \times \mathbb{R}^2,\\ &u_j^1(0,x) = \partial u_j^1(0,x) = 0, && x \in \mathbb{R}^2.\end{aligned}$$

We also define $U^l(t,x) = \mathcal{D}(|x|^{1/2} u^l(t,x))$ and $V^l(t;\sigma,\omega) := U^l(t,(t+\sigma)\omega)$, for $l = 0, 1$, respectively. It follows from (2.2.5) and (2.2.6) that

$$\begin{aligned}&\left|U_j^0(t,x) - \partial_\sigma \mathcal{F}_j(|x| - t, x/|x|)\right|\\ \le&\frac{1}{2}\sum_{a=0}^{2} \left||x|^{1/2}\partial_a u_j^0(t,x) - \omega_a \partial_\sigma \mathcal{F}_j(|x|-t,\omega)\right| + \frac{1}{4|x|^{1/2}}|u_j^0(t,x)|\\ \le&C\langle t+|x|\rangle^{-1}\langle t-|x|\rangle^{-1/2} + C|x|^{-1/2}\langle t+|x|\rangle^{-1/2}\langle t-|x|\rangle^{-1/2}\\ \le&C\varepsilon\end{aligned}$$

for $|x| \ge 1/(2\varepsilon)$. Hence we get

$$\left|V_j^0(t;\sigma,\omega) - \partial_\sigma \mathcal{F}_j(\sigma,\omega)\right| \le C\varepsilon \tag{2.5.14}$$

for $t+\sigma \ge 1/(2\varepsilon)$. We next consider the estimate for V^1. Note that we have $(\Gamma^\alpha \phi, \partial_t \Gamma^\alpha \phi)|_{t=0} \in \left(C_0^\infty(\mathbb{R}^2)\right)^2$ and $\|\Gamma^\alpha \phi(0)\|_{L^\infty(\mathbb{R}^2)}$, $\|\partial_t \Gamma^\alpha \phi(0)\|_{L^\infty(\mathbb{R}^2)} = O(\varepsilon^3)$ for $\alpha \in \mathbb{Z}_+^7$ if $\phi(t,x)$ satisfies $\Box\phi = N(\partial\phi)$ with a cubic nonlinear term $N(\partial\phi)$ and $(\phi, \partial_t\phi)|_{t=0} \in \left(C_0^\infty(\mathbb{R}^2)\right)^2$. By using (2.2.10), (2.2.11) and the standard energy method for $\Gamma^\alpha u^1$ with $|\alpha| \le 2$, we obtain

$$\begin{aligned}\|\partial u^1(t)\|_2 &\le C\varepsilon^3 + C\int_0^t \||\partial u(\tau,\cdot)|_1\|_{L^\infty}^2 \,\|\partial u(\tau)\|_2\, d\tau\\ &\le C\varepsilon^3 + C\varepsilon^3 \int_0^{\varepsilon^{-1}} (1+\tau)^{-1+\mu+\nu}\, d\tau\\ &\le C\varepsilon^3 + C\varepsilon^3 (1+\varepsilon^{-1})^{\mu+\nu}\\ &\le C\varepsilon^{3-\mu-\nu}\end{aligned}$$

for $0 \le t \le \varepsilon^{-1}$. Then, by the Klainerman-Sobolev inequality, we get

$$\langle t+|x|\rangle^{1/2}|\partial u^1(t,x)| \le C\varepsilon^{3-\mu-\nu} \tag{2.5.15}$$

for $0 \le t \le \varepsilon^{-1}$, $x \in \mathbb{R}^2$. It follows from (2.2.5) and (2.2.8) that

$$\begin{aligned}|x|^{-1/2}\left|u^1(t,x)\right| &\le |x|^{-1/2}\left(|u(t,x)| + \varepsilon|u^0(t,x)|\right)\\ &\le |x|^{-1/2}\Big(C\varepsilon\langle t+|x|\rangle^{\mu-1/2} + C\varepsilon\langle t+|x|\rangle^{-1/2}\langle t-|x|\rangle^{-1/2}\Big)\\ &\le C\varepsilon^{2-\mu}\end{aligned} \tag{2.5.16}$$

for $|x| \geq 1/(2\varepsilon)$. From (2.5.15) and (2.5.16), we get

$$\begin{aligned}\left|U^1(t,x)\right| &\leq C|x|^{1/2}|\partial u^1(t,x)| + C|x|^{-1/2}|u^1(t,x)| \\ &\leq C\varepsilon^{3-\mu-\nu} + C\varepsilon^{2-\mu} \\ &\leq C\varepsilon^{2-\mu}\end{aligned}$$

for $|x| \geq 1/(2\varepsilon)$, $0 \leq t \leq \varepsilon^{-1}$. Therefore, we obtain

$$\left|V^1(t;\sigma,\omega)\right| \leq C\varepsilon^{2-\mu} \tag{2.5.17}$$

for $t+\sigma \geq 1/(2\varepsilon)$, $0 \leq t \leq \varepsilon^{-1}$. When $\varepsilon^{-1} > -2\sigma$, we have $t_{1,\sigma} = \varepsilon^{-1}$ and $t_{1,\sigma} + \sigma > t_{1,\sigma}/2 = 1/(2\varepsilon)$. Thus, by (2.5.14) and (2.5.17), we get

$$\begin{aligned}&|V_j(t_{1,\sigma};\sigma,\omega) - \varepsilon\partial_\sigma\mathcal{F}_j(\sigma,\omega)| \\ \leq&|V_j^1(t_{1,\sigma};\sigma,\omega)| + \varepsilon|V_j^0(t_{1,\sigma};\sigma,\omega) - \partial_\sigma\mathcal{F}_j(\sigma,\omega)| \\ \leq&C\varepsilon^{2-\mu}.\end{aligned}$$

Combining the two cases above, we arrive at the desired expression (2.5.12). This completes the proof of Lemma 2.5.4. □

2.5.4 Proof of Theorem 2.1.3

Now we are ready to prove Theorem 2.1.3. We put

$$E = \{(\sigma,\omega) \in \mathbb{R}\times\mathbb{S}^1; |\partial_\sigma\mathcal{F}_1(\sigma,\omega)| > |\partial_\sigma\mathcal{F}_2(\sigma,\omega)|\}.$$

By (2.1.8), E is a non-empty open set. Hence we can take a bounded open set $\mathcal{M}$ in $\mathbb{R}$ and an open set $\mathcal{N}$ in $\mathbb{S}^1$ such that $\sigma^* \in \mathcal{M}$, $\omega^* \in \mathcal{N}$ and $\overline{\mathcal{M}\times\mathcal{N}} \subset E$, where $\overline{\mathcal{M}\times\mathcal{N}}$ denotes the closure of $\mathcal{M}\times\mathcal{N}$ in $\mathbb{R}\times\mathbb{S}^1$. Now we put $F = \overline{\mathcal{M}\times\mathcal{N}}$ and

$$C_1 = \min_{(\sigma,\omega)\in F}\left((\partial_\sigma\mathcal{F}_1(\sigma,\omega))^2 - (\partial_\sigma\mathcal{F}_2(\sigma,\omega))^2\right).$$

Then we see that F is compact, and thus $C_1 > 0$. By Lemma 2.5.4, we have

$$m(\sigma,\omega) \geq C_1\varepsilon^2 - C\varepsilon^{5/2-2\mu} > 0$$

for $(\sigma,\omega) \in F$, if $\varepsilon > 0$ is small enough. Thus Lemma 2.5.3 implies $V_1^+(\sigma,\omega) \neq 0$ for $(\sigma,\omega) \in F$, whence $\|V_1^+\|_{L^2(F)} > 0$. By virtue of (2.5.9), we can take $T_1 > 0$ such that

$$\|\partial u_1(t,\cdot) - \hat{\omega}(\cdot)V_1^{+,\#}(t,\cdot)\|_{L^2(\mathbb{R}^2)} < \frac{1}{\sqrt{2}}\|V_1^+\|_{L^2(F)}$$

for $t > T_1$. Therefore we have

$$
\begin{aligned}
\|u_1(t)\|_E &\geq \left(\frac{1}{2}\int_{\mathbb{R}^2} |\hat{\omega}(x)V_1^{+,\#}(t,x)|^2\,dx\right)^{1/2} \\
&\quad - \left(\frac{1}{2}\int_{\mathbb{R}^2} |\partial u_1(t,x) - \hat{\omega}(x)V_1^{+,\#}(t,x)|^2\,dx\right)^{1/2} \\
&= \|V_1^{+,\#}(t,\cdot)\|_{L^2(\mathbb{R}^2)} - \frac{1}{\sqrt{2}}\|\partial u_1(t,\cdot) - \hat{\omega}(\cdot)V_1^{+,\#}(t,\cdot)\|_{L^2(\mathbb{R}^2)} \\
&\geq \|V_1^+\|_{L^2(F)} - \frac{1}{2}\|V_1^+\|_{L^2(F)} \\
&= \frac{1}{2}\|V_1^+\|_{L^2(F)}
\end{aligned}
$$

for $t > T_1$. Consequently, we arrive at the desired estimate

$$\lim_{t\to+\infty} \|u_1(t)\|_E \geq \frac{1}{2}\|V_1^+\|_{L^2(F)} > 0.$$

Interchanging the roles of u_1 and u_2, we also have $\lim_{t\to+\infty} \|u_2(t)\|_E > 0$. $\square$

Chapter 3

Asymptotic behavior of solutions to nonlinear Schrödinger equations with weakly dissipative structure

3.1 Introduction and results

This chapter is based on the joint works [53], [54] and [55] with Chunhua Li, Yuji Sagawa and Hideaki Sunagawa. In this chapter, we deal with cubic nonlinear Schrödinger equations. We first study the initial value problem

$$\begin{cases} \mathcal{L}u = N(u, \partial_x u), & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{3.1.1}$$

where $\mathcal{L} := i\partial_t + \frac{1}{2}\partial_x^2$, $u = (u(t, x))$ is a $\mathbb{C}$-valued unknown function on $[0, \infty) \times \mathbb{R}$. φ is a prescribed $\mathbb{C}$-valued function on $\mathbb{R}$ which belongs to suitable weighted Sobolev space and is suitably small in its norm. We assume that the nonlinear term $N(u, \partial_x u)$ is a cubic homogeneous polynomial in $(u, \overline{u}, \partial_x u, \overline{\partial_x u})$ with complex coefficients.

First of all, let us summarize the backgrounds briefly. As is well-known, cubic nonlinearity gives a critical situation when we consider large time behavior of solutions to the nonlinear Schrödinger equation in $\mathbb{R}$. In general, cubic nonlinearity should be regarded as a long-range perturbation. For example, according to Hayashi–Naumkin [11], the small data solution $u(t, x)$ to

$$\mathcal{L}u = \lambda |u|^2 u \tag{3.1.2}$$

with $\lambda \in \mathbb{R}\backslash\{0\}$ behaves like

$$u(t, x) = \frac{1}{\sqrt{it}} \alpha^{\pm}(x/t) e^{i\{\frac{x^2}{2t} - \lambda |\alpha^{\pm}(x/t)|^2 \log t\}} + o(t^{-1/2}) \quad \text{in } L^{\infty}(\mathbb{R}_x)$$

as $t \to \pm\infty$, where $\alpha^{\pm}$ is a suitable $\mathbb{C}$-valued function on $\mathbb{R}$. An important consequence of this asymptotic expression is that the solution to (3.1.2) decays like $O(|t|^{-1/2})$ uniformly in $x \in \mathbb{R}$, while it does not behave like the free solution (unless $\lambda = 0$). In other words, the additional logarithmic correction in the phase reflects a typical long-range character of the cubic nonlinear Schrödinger equations in one space dimension. If $\lambda \in \mathbb{C}$ in (3.1.2), another kind of long-range effect can be observed. For instance, according to [68] (see also [47], [28], [9], etc.), the small data solution $u(t,x)$ to (3.1.2) decays like $O(t^{-1/2}(\log t)^{-1/2})$ in $L^{\infty}(\mathbb{R}_x)$ as $t \to +\infty$ if $\operatorname{Im}\lambda < 0$. This gain of additional logarithmic time decay should be interpreted as another kind of long-range effect. There are various extensions of these results. In the previous works [51] and [52], several structural conditions on the nonlinearity have been introduced under which the small data global existence holds for a class of cubic nonlinear Schrödinger systems in $\mathbb{R}$, and large time asymptotic behavior of the global solutions have also been investigated (see also [42], [67], [41] and the references cited therein for related works).

What we can expect for general cubic nonlinear Schrödinger equations in $\mathbb{R}$ is the lower estimate for the lifespan T_ε in the form $T_\varepsilon \geq \exp(c/\varepsilon^2)$ with some $c > 0$, and this is best possible in general (see [44] for an example of small data blow-up). More precise information on the lower bound is available under the restriction

$$N(e^{i\theta}, 0) = e^{i\theta} N(1,0), \qquad \theta \in \mathbb{R}. \tag{3.1.3}$$

According to [66] (see also [70]), if we assume (3.1.3) and the initial condition in (3.1.1) is replaced by $u(0,x) = \varepsilon\psi(x)$ with $\psi \in H^3 \cap H^{2,1}$, then it holds that

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq \frac{1}{2 \sup_{\xi \in \mathbb{R}} (|\mathcal{F}\psi(\xi)|^2 \operatorname{Im}\nu(\xi))} \tag{3.1.4}$$

with the convention $1/0 = +\infty$, where the function $\nu : \mathbb{R} \to \mathbb{C}$ is defined by

$$\nu(\xi) = \frac{1}{2\pi i} \oint_{|z|=1} N(z, i\xi z) \frac{dz}{z^2}. \tag{3.1.5}$$

Note that (3.1.3) excludes just the worst terms u^3, $|u|^2\overline{u}$, $\overline{u}^3$. As pointed out in [13], [14], [16], [17], [18], [60], [61], etc., these three terms make the situation much more complicated. We do not intend to pursue this case here. We always assume (3.1.3) in what follows.

In view of the right-hand side in (3.1.4), it may be natural to expect that the sign of $\operatorname{Im}\nu(\xi)$ has something to do with global behavior of small data solutions to (3.1.1). In fact, it has been pointed out in [66] that typical results on small data global existence and large-time asymptotic behavior for (3.1.1) under (3.1.3) can be summarized in terms of $\operatorname{Im}\nu(\xi)$ as follows.

(i) The small data global existence holds in $H^3 \cap H^{2,1}$ under the condition

$$\operatorname{Im} \nu(\xi) \le 0, \quad \xi \in \mathbb{R}. \tag{$\mathbf{A}$}$$

(ii) If the inequality in ($\mathbf{A}$) is replaced by the equality, i.e.,

$$\operatorname{Im} \nu(\xi) = 0, \quad \xi \in \mathbb{R}, \tag{$\mathbf{A}_0$}$$

then the solution has a logarithmic phase correction in the asymptotic profile, i.e., it holds that

$$u(t,x) = \frac{1}{\sqrt{t}} \alpha^+(x/t) \exp\left(\frac{ix^2}{2t} - i|\alpha^+(x/t)|^2 \operatorname{Re} \nu(x/t) \log t \right) + o(t^{-1/2})$$

as $t \to +\infty$ uniformly in $x \in \mathbb{R}$, where $\alpha^+(\xi)$ is a suitable $\mathbb{C}$-valued function of $\xi \in \mathbb{R}$.

(iii) If the inequality in ($\mathbf{A}$) is strict, i.e.,

$$\sup_{\xi \in \mathbb{R}} \operatorname{Im} \nu(\xi) < 0, \tag{$\mathbf{A}_+$}$$

then the solution gains an additional logarithmic time decay $\|u(t)\|_{L^\infty} = O((t \log t)^{-1/2})$.

For more details on each case, see the references cited in Section 1 of [66]. As for the large time behavior in the sense of L^2_x under ($\mathbf{A}$), it is not difficult to see that ($\mathbf{A}_+$) implies $\lim_{t\to+\infty} \|u(t)\|_{L^2} = 0$, whereas ($\mathbf{A}_0$) implies $\lim_{t\to+\infty} \|u(t)\|_{L^2} \neq 0$ for generic initial data of small amplitude. However, it is not clear whether L^2-decay occurs or not in the other cases (even for a simple example such as $N(u, \partial_x u) = -i|\partial_x u|^2(u + \partial_x u) + \partial_x(u^3)$, for which we have

$$\nu(\xi) = \frac{1}{2\pi i} \oint_{|z|=1} (-i\xi^2(1+i\xi)|z|^2 z + 3i\xi z^3) \frac{dz}{z^2} = -i\xi^2 + \xi^3$$

and $\operatorname{Im} \nu(\xi) = -\xi^2$). Despite the recent progress of studies on dissipative nonlinear Schrödinger equations ([9], [19], [27], [28], [37], [42], [45], [46], [47], [51], [52], [68], etc.), questions on decay/non-decay in L^2_x without ($\mathbf{A}_+$) have not been addressed in the previous works. The first aim of this chapter is to fill in the missing piece between ($\mathbf{A}_+$) and ($\mathbf{A}_0$), that is, to investigate L^2-decay property of global solutions to (3.1.1) under (3.1.3) and ($\mathbf{A}$) without ($\mathbf{A}_+$) and ($\mathbf{A}_0$). The first result in this chapter is as follows.

Theorem 3.1.1. *Suppose that $\varepsilon = \|\varphi\|_{H^3 \cap H^{2,1}}$ is sufficiently small. Assume that* (3.1.3) *and* ($\mathbf{A}$) *are satisfied but* ($\mathbf{A}_0$) *is violated. Then, for any $\delta > 0$,* *there exists a positive constant* C *such that the global solution* u *to* (3.1.1) *satisfies*

$$\|u(t)\|_{L^2} \le \frac{C\varepsilon}{(1+\varepsilon^2 \log(t+2))^{1/4-\delta}}$$

for $t \ge 0$.

Remark 3.1.1. Under (3.1.3) and ($\mathbf{A}_+$), we can show the global solution to (3.1.1) has the stronger L^2-decay of order $O((\log t)^{-3/8+\delta})$ with arbitrarily small $\delta > 0$ by the same method. However, the decay order $O((\log t)^{-3/8+\delta})$ is certainly not optimal. For the detail, see Remark 3.2.1 below.

Our second target in this chapter is a system of cubic nonlinear Schrödinger equations which can be comparable to (2.1.5). Let us focus on

$$\begin{cases} \mathcal{L}u_1 = -i|u_2|^2 u_1, \\ \mathcal{L}u_2 = -i|u_1|^2 u_2, \end{cases} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \tag{3.1.6}$$

with the initial condition

$$u_j(0,x) = \varphi_j^0(x), \qquad x \in \mathbb{R},\ j = 1,2, \tag{3.1.7}$$

where $\varphi^0 = (\varphi_1^0(x), \varphi_2^0(x))$ is a given $\mathbb{C}^2$-valued function of $x \in \mathbb{R}$ which belongs to an appropriate weighted Sobolev space and satisfies a suitable smallness condition.

There are various extensions of the conditions ($\mathbf{A}$), ($\mathbf{A}_+$) and ($\mathbf{A}_0$) for the system case (see e.g. [51], [52], [42], [67], [41]). In the previous works [51] and [52], several structural conditions on the nonlinearity have been introduced under which the small data global existence holds for a class of cubic nonlinear Schrödinger systems in $\mathbb{R}$, and large time asymptotic behavior of the global solutions have also been investigated (see also [42], [67], [41] and the references cited therein for related works). We do not state these conditions here, but we only point out that the small data global existence for (3.1.6) follows from the results of [51] and [42] but the large time asymptotic behavior of solutions is not covered by these results.

We note that the system (3.1.6) possesses two conservation laws

$$\frac{d}{dt}\Big(\|u_1(t)\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2\Big) = -4\int_{\mathbb{R}} |u_1(t,x)|^2 |u_2(t,x)|^2\,dx$$

and

$$\frac{d}{dt}\Big(\|u_1(t)\|_{L^2}^2 - \|u_2(t)\|_{L^2}^2\Big) = 0.$$

However, these are not enough to say something about the large time asymptotics for $u(t)$, and this is not trivial at all. To the author's knowledge, there are no previous results which cover the asymptotic behavior of solutions to (3.1.6)–(3.1.7). From the viewpoint of conservation laws, there are a lot of similarities between (3.1.6) and (2.1.5). It has been shown in the previous chapter that global solutions to (2.1.5) with small data behaves like solutions to the free wave equations, but there is a strong restriction in the profiles. Although the approach of the previous chapter does not use the conservation laws directly, it may be natural to expect that an analogous phenomenon can be observed for solutions to (3.1.6). We are going to reveal it.

Before stating the results, let us introduce a notation. We set $\mathcal{U}(t) = \exp(i\frac{t}{2}\partial_x^2)$, so that $\mathcal{U}(t)\phi =: w(t)$ solves the free Schrödinger equation $\mathcal{L}w = 0$ with $w(0) = \phi$.

Theorem 3.1.2. *Suppose that* $\varphi^0 = (\varphi_1^0, \varphi_2^0) \in H^2 \cap H^{1,1}$ *and* $\varepsilon = \|\varphi^0\|_{H^2\cap H^{1,1}}$ *is suitably small. Let* $u = (u_1, u_2) \in C([0,\infty); H^2 \cap H^{1,1})$ *be the solution to* (3.1.6)–(3.1.7)*. Then there exists* $\varphi^+ = (\varphi_1^+, \varphi_2^+) \in L^2$ *with* $\hat{\varphi}^+ = (\hat{\varphi}_1^+, \hat{\varphi}_2^+) \in L^\infty$ *such that*

$$\lim_{t\to+\infty} \|u_j(t) - \mathcal{U}(t)\varphi_j^+\|_{L^2} = 0, \quad j = 1, 2.$$

Moreover we have

$$\hat{\varphi}_1^+(\xi) \cdot \hat{\varphi}_2^+(\xi) = 0, \quad \xi \in \mathbb{R}. \tag{3.1.8}$$

Remark 3.1.2. We emphasize that (3.1.8) should be regarded as a consequence of non-trivial long-range nonlinear interactions because such a phenomenon does not occur in the usual short-range situation. To complement this point, we will give auxiliary results on the final state problem for (3.1.6) in Section 3.6.

To investigate more precise information on φ^+, we put a small parameter ε in front of the initial data to distinguish information on the amplitude from the others, that is, we replace the initial condition (3.1.7) by

$$u_j(0, x) = \varepsilon\psi_j(x), \quad j = 1, 2, \tag{3.1.9}$$

where $\psi_j \in H^2 \cap H^{1,1}$ is independent of ε. We have following criteria for (non-)triviality of the scattering state $\varphi^+ = (\varphi_1^+, \varphi_2^+)$ for (3.1.6)–(3.1.9).

Theorem 3.1.3. *We put* $\varphi_j^+ = \lim_{t\to+\infty} \mathcal{U}(-t)u_j(t)$ *in* L^2*,* $j = 1, 2$*, for the global solution* $u = (u_1, u_2)$ *to* (3.1.6)–(3.1.9)*, whose existence is guaranteed by Theorem 3.1.2. Assume that there exist points* $\xi^* \in \mathbb{R}$ *and* $\xi_* \in \mathbb{R}$ *such that*

$$|\hat{\psi}_1(\xi^*)| > |\hat{\psi}_2(\xi^*)| \tag{3.1.10}$$

and

$$|\hat{\psi}_1(\xi_*)| < |\hat{\psi}_2(\xi_*)|, \tag{3.1.11}$$

respectively. Then, we have $\|\varphi_1^+\|_{L^2} > 0$ *and* $\|\varphi_2^+\|_{L^2} > 0$ *for sufficiently small* ε*.*

Theorem 3.1.4. *Assume that*

$$|\hat{\psi}_1(\xi)| > |\hat{\psi}_2(\xi)| \tag{3.1.12}$$

for all $\xi \in \mathbb{R}$*. Then, for sufficiently small* ε*,* φ_2^+ *vanishes almost everywhere on* $\mathbb{R}$*, while* $\|\varphi_1^+\|_{L^2} > 0$*.*

It follows from Theorems 3.1.2 and 3.1.3 that both $u_1(t)$ and $u_2(t)$ behave like non-trivial free solutions as $t \to +\infty$. In particular, we see that L^2 decay does not occur for $u_1(t)$ and $u_2(t)$ under (3.1.10) and (3.1.11). To the contrary, Theorem 3.1.4 tells us that only the second component $u_2(t)$ is dissipated as $t \to \infty$ in the sense of L^2 under (3.1.12). We emphasize again that such phenomena do not occur in the usual short-range settings. In this sense, the dynamics for the system (3.1.6) is much more delicate than that for the single Schrödinger equation with dissipative cubic nonlinear terms.

Remark 3.1.3. It is worthwhile to note that the presence of $-i$ in the right-hand sides of (3.1.6) is essential for our result. Indeed, if we drop $-i$ from the right-hand sides of (3.1.6) (that is, $\mathcal{L}u_1 = |u_2|^2u_1$ and $\mathcal{L}u_2 = |u_1|^2u_2$), we can show that the solutions have logarithmic phase corrections as in the single case (3.1.2) with $\lambda \in \mathbb{R}\backslash\{0\}$ (see e.g. [71] for detail).

Remark 3.1.4. Theorems 3.1.2, 3.1.3 and 3.1.4 concern only the forward Cauchy problem (i.e., for $t > 0$). In the backward case, the small data global existence may fail in general. See [65] and the references cited therein for more information and the related works on this issue.

3.2 Proof of Theorem 3.1.1

This section is devoted to the proof of Theorem 3.1.1. The argument will be divided into four steps.

Step 1: We begin with the following elementary lemma, whose proof is skipped.

Lemma 3.2.1. *Let* $p(\xi)$ *be a real polynomial with* $\deg p \le 3$*. If* $p(\xi) \ge 0$ *for all* $\xi \in \mathbb{R}$*, then we have either of the following three assertions.*

- **(a)** $p(\xi)$ *vanishes identically on* $\mathbb{R}$*.*
- **(b)** $\inf_{\xi \in \mathbb{R}} p(\xi) > 0$*.*
- **(c)** *There exist* $c_0 > 0$ *and* $\xi_0 \in \mathbb{R}$ *such that* $p(\xi) = c_0(\xi - \xi_0)^2$*.*

For $\nu(\xi)$ given by (3.1.5), we put $p(\xi) = -\operatorname{Im}\nu(\xi)$. Since we assume that ($\mathbf{A}$) is satisfied but ($\mathbf{A}_0$) is violated, we see that the case (**a**) in Lemma 3.2.1 is excluded. Note also that (**b**) is equivalent to ($\mathbf{A}_+$). Now, let us turn our attentions to the admissible range of the parameter θ for convergence of the integral

$$I_\theta = \int_{\mathbb{R}} \frac{d\xi}{p(\xi)^\theta \langle\xi\rangle^{4-4\theta}} \tag{3.2.1}$$

under (**c**) or (**b**). In the case (**c**), we have

$$I_\theta = c_0^{-\theta} \int_{\mathbb{R}} \frac{d\xi}{|\xi-\xi_0|^{2\theta}\langle\xi\rangle^{4-4\theta}} < \infty$$

for $\theta < 1/2$. In the case (**b**), we have

$$I_\theta \le \bigl(\inf_{\xi\in\mathbb{R}} p(\xi)\bigr)^{-\theta} \int_{\mathbb{R}} \frac{d\xi}{\langle\xi\rangle^{4-4\theta}} < \infty$$

for $\theta < 3/4$.

Step 2: Next we summarize the basic estimates for the global solution u to (3.1.1). First we write $\mathcal{J} = x + it\partial_x$. We note the important commutation relations $[\partial_x, \mathcal{J}] = 1$, $[\mathcal{L}, \mathcal{J}] = 0$. We also have

$$\mathcal{J} = \mathcal{U}(t)x\mathcal{U}(-t). \tag{3.2.2}$$

We set $\alpha(t,\xi) = \mathcal{F}\bigl[\mathcal{U}(-t)u(t,\cdot)\bigr](\xi)$ for the solution u to (3.1.1). According to the previous works ([12], [19], [66], etc.), we already know the following estimates.

Lemma 3.2.2. *Let $\varepsilon = \|\varphi\|_{H^3\cap H^{2,1}}$ be suitably small. Assume that* (3.1.3) *and* ($\mathbf{A}$) *are fulfilled. Then, the solution u to* (3.1.1) *satisfies*

$$|\alpha(t,\xi)| \le \frac{C\varepsilon}{\langle\xi\rangle^2} \tag{3.2.3}$$

for $t \ge 0$, $\xi \in \mathbb{R}$, and

$$\|u(t)\|_{H^3} + \|\mathcal{J}u(t)\|_{H^2} \le C\varepsilon(1+t)^\gamma \tag{3.2.4}$$

for $t \ge 0$, where $0 < \gamma < 1/12$.

The following lemma has been obtained in [66] (see also [12]). We write $\alpha_{(\zeta)}(t,\xi) = \alpha(t,\xi/\zeta)$ for $\zeta \in \mathbb{R}\backslash\{0\}$.

Lemma 3.2.3. *Under the assumption* (3.1.3), *we have*

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)(1-\partial_x^2)N(u,u_x) =&(1+\xi^2)\frac{\nu(\xi)}{t}|\alpha|^2\alpha + \frac{\xi e^{i\frac{t}{3}\xi^2}}{t}\mu_1(\xi)\alpha_{(3)}^3\\ &+\frac{\xi e^{i\frac{2t}{3}\xi^2}}{t}\mu_2(\xi)\big(\overline{\alpha_{(-3)}}\big)^3\\ &+\frac{\xi e^{it\xi^2}}{t}\mu_3(\xi)|\alpha_{(-1)}|^2\overline{\alpha_{(-1)}}+R, \qquad (3.2.5)\end{aligned}$$

where $\nu(\xi)$ *is given by* (3.1.5), $\mu_1(\xi)$, $\mu_2(\xi)$, $\mu_3(\xi)$ *are polynomials in* ξ *of degree at most* 4, *and* $R(t,\xi)$ *satisfies*

$$|R(t,\xi)| \le \frac{C}{t^{5/4}}\big(\|u(t)\|_{H^3} + \|\mathcal{J}u(t)\|_{H^2}\big)^3 \qquad (3.2.6)$$

for $t \ge 1$, $\xi \in \mathbb{R}$.

For the proof, see Lemma 4.3 in [66]. By (3.2.4) and (3.2.6), we have

$$|R(t,\xi)| \le \frac{C\varepsilon^3}{t^{1+\kappa}} \qquad (3.2.7)$$

for $t \ge 1$, $\xi \in \mathbb{R}$, where $\kappa = 1/4 - 3\gamma > 0$. This indicates that R can be regarded as a remainder in (3.2.5). We also observe that one ξ pops up in front of the oscillating factors in (3.2.5). This is the point where (3.1.3) plays a crucial role. As for the role of $\nu(\xi)$, the first term of the right-hand side in (3.2.5) tells us that $\nu(\xi)$ is responsible for the contribution from the gauge-invariant part in N.

Step 3: We are going to make some reductions. The goal in this step is to derive the ordinary differential equation (3.2.10) (with $\xi \in \mathbb{R}$ regarded as a parameter).

Let $t \ge 2$ from now on. By the relation $\mathcal{L} = \mathcal{U}(t)i\partial_t\mathcal{U}(-t)$ and Lemma 3.2.3, we have

$$\begin{aligned}i\partial_t\alpha(t,\xi) &= \mathcal{F}\mathcal{U}(-t)\mathcal{L}u\\ &= \langle\xi\rangle^{-2}\mathcal{F}\mathcal{U}(-t)(1-\partial_x^2)N(u,u_x)\\ &= \frac{\nu(\xi)}{t}|\alpha(t,\xi)|^2\alpha(t,\xi) + \eta(t,\xi) + \langle\xi\rangle^{-2}R(t,\xi), \qquad (3.2.8)\end{aligned}$$

where

$$\eta(t,\xi) = \frac{\xi e^{it\xi^2/3}}{t}\frac{\mu_1(\xi)}{\langle\xi\rangle^2}\alpha_{(3)}^3 + \frac{\xi e^{i2t\xi^2/3}}{t}\frac{\mu_2(\xi)}{\langle\xi\rangle^2}\overline{\alpha_{(-3)}}^3 + \frac{\xi e^{it\xi^2}}{t}\frac{\mu_3(\xi)}{\langle\xi\rangle^2}|\alpha_{(-1)}|^2\overline{\alpha_{(-1)}}.$$

It follows from (3.2.3), (3.2.7) and (3.2.8) that

$$|\partial_t\alpha(t,\xi)| \le \frac{C\langle\xi\rangle^3}{t}\left(\frac{C\varepsilon}{\langle\xi\rangle^2}\right)^3 + \frac{C\varepsilon^3}{t^{1+\kappa}\langle\xi\rangle^2} \le \frac{C\varepsilon^3}{t\langle\xi\rangle^2}.$$

Also, by using the identity

$$\begin{aligned}\frac{\xi e^{i\omega t\xi^2}}{t}f(t,\xi) &= \frac{\xi\partial_t(te^{i\omega t\xi^2})}{t(1+i\omega t\xi^2)}f(t,\xi)\\ &= i\partial_t\left(\frac{-i\xi e^{i\omega t\xi^2}}{1+i\omega t\xi^2}f(t,\xi)\right) - te^{i\omega t\xi^2}\partial_t\left(\frac{\xi f(t,\xi)}{t(1+i\omega t\xi^2)}\right)\end{aligned}$$

and the inequality

$$\sup_{\xi\in\mathbb{R}}\frac{|\xi|^a}{|1+i\omega t\xi^2|} \le \frac{C}{(|\omega|t)^{a/2}}$$

for $0 \le a \le 2$, we see that $\eta(t,\xi)$ can be split into

$$\eta = i\partial_t\sigma_1 + \sigma_2; \quad |\sigma_1(t,\xi)| \le \frac{C\varepsilon^3}{t^{1/2}\langle\xi\rangle^4}, \quad |\sigma_2(t,\xi)| \le \frac{C\varepsilon^3}{t^{3/2}\langle\xi\rangle^4}. \qquad (3.2.9)$$

With this σ_1, we set $\beta(t,\xi) = \alpha(t,\xi) - \sigma_1(t,\xi)$. Then it follows from (3.2.8) that

$$i\partial_t\beta(t,\xi) = \frac{\nu(\xi)}{t}|\beta(t,\xi)|^2\beta(t,\xi) + \rho(t,\xi), \qquad (3.2.10)$$

where

$$\begin{aligned}\rho(t,\xi) =& \frac{\nu(\xi)}{t}\left(|\alpha|^2\alpha - |\beta|^2\beta\right) + \sigma_2 + \langle\xi\rangle^{-2}R\\ =& \frac{\nu(\xi)}{t}\left(2|\alpha|^2\sigma_1 + \alpha^2\overline{\sigma_1} - 2\alpha|\sigma_1|^2 - \overline{\alpha}\sigma_1^2 + |\sigma_1|^2\sigma_1\right) + \sigma_2 + \langle\xi\rangle^{-2}R.\end{aligned}$$

By (3.2.3), (3.2.7) and (3.2.9), we have

$$\begin{aligned}|\rho(t,\xi)| &\le \frac{C\langle\xi\rangle^3}{t}\left(\frac{C\varepsilon}{\langle\xi\rangle^2} + \frac{C\varepsilon^3}{t^{1/2}\langle\xi\rangle^4}\right)^2\frac{C\varepsilon^3}{t^{1/2}\langle\xi\rangle^4} + \frac{C\varepsilon^3}{t^{3/2}\langle\xi\rangle^4} + \frac{C\varepsilon^3}{t^{1+\kappa}\langle\xi\rangle^2}\\ &\le \frac{C\varepsilon^3}{t^{1+\kappa}\langle\xi\rangle^2}.\end{aligned}$$

Remember that $0 < \kappa < 1/4$.

Roughly speaking, what we have seen so far is that the solution u to (3.1.1) under (3.1.3) can be expressed as

$$u = \mathcal{U}(t)\mathcal{F}^{-1}\beta + \cdots$$

with

$$i\partial_t\beta = \frac{\nu(\xi)}{t}|\beta|^2\beta + \cdots,$$

where the terms "$+\cdots$" are expected to be harmless. By this reason it would be fair to call (3.2.10) the *profile equation* associated with (3.1.1) under (3.1.3). The original idea of this reduction is due to Hayashi–Naumkin [11].

Final step: We set $\Phi(t,\xi) = p(\xi)|\beta(t,\xi)|^2$ with $p(\xi) = -\operatorname{Im}\nu(\xi)$. Note that $\Phi(t,\xi) \geq 0$ by (**A**). It follows from (3.2.10) that

$$\begin{aligned}
\partial_t \Phi(t,\xi) &= 2p(\xi)\operatorname{Im}\left(\overline{\beta(t,\xi)} i\partial_t \beta(t,\xi)\right) \\
&= 2p(\xi)\left(\frac{\operatorname{Im}\nu(\xi)}{t}|\beta(t,\xi)|^4 + \operatorname{Im}\left(\overline{\beta(t,\xi)}\rho(t,\xi)\right)\right) \\
&\leq -\frac{2p(\xi)^2}{t}|\beta(t,\xi)|^4 + C\langle\xi\rangle^3 \frac{C\varepsilon}{\langle\xi\rangle^2}\frac{C\varepsilon^3}{t^{1+\kappa}\langle\xi\rangle^2} \\
&\leq -\frac{2}{t}\Phi(t,\xi)^2 + \frac{C\varepsilon^4}{t^{1+\kappa}\langle\xi\rangle},
\end{aligned}$$

where $\kappa \in (0, 1/4)$. We also note that (3.2.3) yields

$$\Phi(2,\xi) \leq C\langle\xi\rangle^3 \left(\frac{C\varepsilon}{\langle\xi\rangle^2}\right)^2 \leq \frac{C\varepsilon^2}{\langle\xi\rangle}.$$

Therefore we can apply Lemma 2.4.2 with $q = 2$ and $s = 1 + \kappa$ to obtain

$$0 \leq \Phi(t,\xi) \leq \frac{C}{\log t},$$

whence

$$\begin{aligned}
|\alpha(t,\xi)| &\leq \sqrt{\frac{\Phi(t,\xi)}{p(\xi)}} + |\sigma_1(t,\xi)| \\
&\leq \frac{C}{\sqrt{p(\xi)\log t}}\left(1 + \varepsilon^3 \frac{\sqrt{p(\xi)}}{\langle\xi\rangle^4}\sqrt{\frac{\log t}{t}}\right) \\
&\leq \frac{C\varepsilon}{\sqrt{p(\xi)\varepsilon^2 \log t}}.
\end{aligned}$$

Interpolating this with (3.2.3), we deduce that

$$|\alpha(t,\xi)| \leq \frac{C\varepsilon}{(\varepsilon^2 \log t)^{\theta/2}} \frac{1}{p(\xi)^{\theta/2}\langle\xi\rangle^{2-2\theta}}$$

for $\theta \in [0,1]$. By the L^2-unitarity of $\mathcal{U}(t)$ and $\mathcal{F}$, we have

$$\|u(t)\|_{L^2}^2 = \|\alpha(t)\|_{L^2}^2 \leq \frac{C\varepsilon^2}{(\varepsilon^2 \log t)^\theta} I_\theta \tag{3.2.11}$$

for $0 \leq \theta < \frac{1}{2}$, where I_θ is given by (3.2.1). Therefore we can take $\theta = 1/2 - 2\delta$ with $\delta > 0$ to see that

$$\|u(t)\|_{L^2} \leq \frac{C\varepsilon}{(\varepsilon^2 \log t)^{1/4-\delta}}.$$

Also we obtain $\|u(t)\|_{L^2} \le C\varepsilon$ by taking $\theta = 0$ in (3.2.11). Piecing them together, we arrive at the desired estimate. $\square$

Remark 3.2.1. If we assume (3.1.3) and the stronger condition ($\mathbf{A}_+$), we can choose $\theta = 3/4 - 2\delta$ in (3.2.11) because ($\mathbf{A}_+$) implies (**b**) in Lemma 3.2.1 and thus the admissible range for θ in (3.2.11) becomes $0 \le \theta < \frac{3}{4}$. That is the reason why $\|u(t)\|_{L^2}$ decays like $O((\log t)^{-3/8+\delta})$ under ($\mathbf{A}_+$). It is not certain whether this rate is the best or not. Indeed, it is possible to improve the exponent from $-3/8 + \delta$ to $-1/2$ if there exists a positive constant C_* such that

$$\operatorname{Im} \nu(\xi) \le -C_* \langle \xi \rangle^2, \quad \xi \in \mathbb{R} \tag{$\mathbf{A}_{++}$}$$

(cf. Theorem 2.3 in [51]). A typical example of N satisfying ($\mathbf{A}_{++}$) is $-i|u + \partial_x u|^2 u$.

It may be an interesting problem to specify the optimal L^2-decay rates for the solutions to (3.1.1) under (3.1.3) and ($\mathbf{A}$) (with or without ($\mathbf{A}_+$)).

3.3 Preliminaries for Theorems 3.1.2, 3.1.3 and 3.1.4

In this section, we collect several inequalities and basic estimates which are useful in the proof of Theorems 3.1.2, 3.1.3 and 3.1.4.

3.3.1 Basic estimates

Let $u = (u_1, u_2)$ be a smooth solution to (3.1.6)–(3.1.7) on $[0, \infty) \times \mathbb{R}$. We define $\alpha = (\alpha_1, \alpha_2)$ by

$$\alpha_j(t, \xi) = \mathcal{F}\Big[\mathcal{U}(-t) u_j(t, \cdot)\Big](\xi) \tag{3.3.1}$$

for $j = 1, 2$. Then it follows from (3.1.6) that

$$\partial_t \alpha_1 = -i\mathcal{F}\mathcal{U}(-t)\mathcal{L}u_1 = -\mathcal{F}\mathcal{U}(-t)(|u_2|^2 u_1) = -\frac{1}{t}|\alpha_2|^2 \alpha_1 + R_1, \tag{3.3.2}$$

where

$$R_1 = \frac{1}{t}|\alpha_2|^2 \alpha_1 - \mathcal{F}\mathcal{U}(-t)\big[|u_2|^2 u_1\big].$$

Similarly we have

$$\partial_t \alpha_2 = -\frac{1}{t}|\alpha_1|^2 \alpha_2 + R_2, \tag{3.3.3}$$

where

$$R_2 = \frac{1}{t}|\alpha_1|^2\alpha_2 - \mathcal{F}\mathcal{U}(-t)\big[|u_1|^2u_2\big].$$

Concerning estimates for $R = (R_1, R_2)$, we have the following estimate.

Lemma 3.3.1. *Let R be as above. For $t \geq 1$, we have*

$$|R(t,\xi)| \leq \frac{C}{t^{5/4}\langle\xi\rangle}\big(\|u(t)\|_{H^1} + \|\mathcal{J}u(t)\|_{H^1}\big)^3.$$

This estimate is not a new one (see e.g. Lemma 5.2 in [51]). For the convenience, we will prove it in Subsection 3.3.2, below.

Next we review the basic estimates for global solutions u to (3.1.6)–(3.1.7). From the argument of [51], we already know the following result.

Lemma 3.3.2. *Let $0 < \gamma < 1/12$. Suppose that $\varepsilon = \|\varphi^0\|_{H^2\cap H^{1,1}}$ is suitably small. Then the solution u to* (3.1.6)–(3.1.7) *satisfies*

$$\|u(t)\|_{H^2} + \|\mathcal{J}u(t)\|_{H^1} \leq C\varepsilon\langle t\rangle^{\gamma}, \tag{3.3.4}$$

$$\|u(t)\|_{L^\infty} \leq C\varepsilon\langle t\rangle^{-1/2} \tag{3.3.5}$$

for $t \geq 0$ and

$$|\alpha(t,\xi)| \leq C\varepsilon\langle\xi\rangle^{-1} \tag{3.3.6}$$

for $t \geq 0$, $\xi \in \mathbb{R}$, where α is given by (3.3.1).

It follows from Lemmas 3.3.1 and 3.3.2 that we obtain

$$|R(t,\xi)| \leq \frac{C\varepsilon^3}{t^{5/4-3\gamma}\langle\xi\rangle} \tag{3.3.7}$$

for $t \geq 1$. Roughly speaking, this means that the evolution of $\alpha = (\alpha_1, \alpha_2)$ could be characterized by

$$\partial_t\alpha_1 = -\frac{1}{t}|\alpha_2|^2\alpha_1, \quad \partial_t\alpha_2 = -\frac{1}{t}|\alpha_1|^2\alpha_2$$

up to the harmless remainders. We also note that $u(t) = \mathcal{U}(t)\mathcal{F}^{-1}\alpha(t)$. This point of view, whose original idea goes back to Hayashi–Naumkin [11], is the key of our approach.

3.3.2 Proof of Lemma 3.3.1

We give a proof of Lemma 3.3.1. For this purpose, we introduce some notations. We define the operators $\mathcal{M}(t)$, $\mathcal{D}(t)$ and $\mathcal{W}(t)$ by

$$\big(\mathcal{M}(t)\phi\big)(x) = e^{i\frac{x^2}{2t}}\phi(x), \big(\mathcal{D}(t)\phi\big)(x) = (it)^{-1/2}\phi\left(\frac{x}{t}\right), \mathcal{W}(t)\phi = \mathcal{F}\mathcal{M}(t)\mathcal{F}^{-1}\phi,$$

so that $\mathcal{U}(t) = \exp(i\frac{t}{2}\partial_x^2)$ is decomposed into

$$\mathcal{U}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}\mathcal{M}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{W}(t)\mathcal{F}. \tag{3.3.8}$$

An important estimate is

$$\|(\mathcal{W}(t)-1)\phi\|_{L^\infty} + \|(\mathcal{W}(t)^{-1}-1)\phi\|_{L^\infty} \le Ct^{-1/4}\|\phi\|_{H^1}, \tag{3.3.9}$$

which comes from the Gagliardo-Nirenberg inequality

$$\|\phi\|_{L^\infty} \le C\|\phi\|_{L^2}^{1/2}\|\partial_x\phi\|_{L^2}^{1/2}$$

and the inequality

$$|e^{i\theta}-1| \le C|\theta|^\sigma \qquad (\theta\in\mathbb{R},\ 0\le\sigma\le 1) \tag{3.3.10}$$

with $\theta = x^2/(2t), \sigma = 1/2$. Note also that

$$\|\mathcal{W}(t)\mathcal{F}\mathcal{U}(-t)\phi\|_{H^1} + \|\mathcal{W}(t)^{-1}\mathcal{F}\mathcal{U}(-t)\phi\|_{H^1} \le C(\|\phi\|_{L^2} + \|\mathcal{J}\phi\|_{L^2}) \tag{3.3.11}$$

and

$$\|\mathcal{F}\mathcal{U}(-t)[\phi_1\phi_2\phi_3]\|_{L^\infty} \le C\|\phi_1\|_{L^2}\|\phi_2\|_{L^2}\|\phi_3\|_{L^\infty}, \tag{3.3.12}$$

where the constant C is independent of t (see e.g., [51] for the proof). In what follows, we will occasionally omit "(t)" from $\mathcal{M}(t)$, $\mathcal{D}(t)$, $\mathcal{W}(t)$ if it causes no confusion.

Let α be given by (3.3.1). By (3.3.8), we have

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)\big[|u_2|^2u_1\big] &= \mathcal{W}^{-1}\mathcal{D}^{-1}\mathcal{M}^{-1}\big[|\mathcal{M}\mathcal{D}\mathcal{W}\alpha_2|^2\mathcal{M}\mathcal{D}\mathcal{W}\alpha_1\big]\\ &= \frac{1}{t}\mathcal{W}^{-1}\big[|\mathcal{W}\alpha_2|^2\mathcal{W}\alpha_1\big],\end{aligned}$$

whence

$$\begin{aligned}R_1 &= \frac{1}{t}\Big(|\alpha_2|^2\alpha_1 - \mathcal{W}^{-1}\big[|\mathcal{W}\alpha_2|^2\mathcal{W}\alpha_1\big]\Big)\\ &= \frac{1}{t}(1-\mathcal{W}^{-1})\big[|\mathcal{W}\alpha_2|^2\mathcal{W}\alpha_1\big] + \frac{1}{t}|\mathcal{W}\alpha_2|^2(1-\mathcal{W})\alpha_1\\ &\quad + \frac{1}{t}(\mathcal{W}\alpha_2)(\overline{(1-\mathcal{W})\alpha_2})\alpha_1 + \frac{1}{t}((1-\mathcal{W})\alpha_2)\overline{\alpha_2}\alpha_1.\end{aligned}$$

Therefore (3.3.9), (3.3.11), (3.3.12) and the Sobolev imbedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ lead to

$$|R_1(t,\xi)| \le Ct^{-5/4}(\|u\|_{L^2} + \|\mathcal{J}u\|_{L^2})^3. \tag{3.3.13}$$

Next we observe that

$$\begin{aligned}
i\xi R_1 &= \frac{i\xi}{t}|\alpha_2|^2\alpha_1 - \mathcal{F}\mathcal{U}(-t)\Big[\partial_x\big(|u_2|^2u_1\big)\Big] \\
&= \frac{1}{t}\Big(\alpha_2^{(1)}\overline{\alpha_2}\alpha_1 - \mathcal{W}^{-1}\big[(\mathcal{W}\alpha_2^{(1)})(\overline{\mathcal{W}\alpha_2})\mathcal{W}\alpha_1\big]\Big) \\
&\quad + \frac{1}{t}\Big(\alpha_2\overline{\alpha_2^{(1)}}\alpha_1 - \mathcal{W}^{-1}\big[(\mathcal{W}\alpha_2)(\overline{\mathcal{W}\alpha_2^{(1)}})\mathcal{W}\alpha_1\big]\Big) \\
&\quad + \frac{1}{t}\Big(\alpha_2\overline{\alpha_2}\alpha_1^{(1)} - \mathcal{W}^{-1}\big[(\mathcal{W}\alpha_2)(\overline{\mathcal{W}\alpha_2})\mathcal{W}\alpha_1^{(1)}\big]\Big),
\end{aligned}$$

where $\alpha_j^{(1)} = i\xi\alpha_j$. Then we see as before that

$$|\xi R_1(t,\xi)| \le Ct^{-5/4}(\|u\|_{H^1} + \|\mathcal{J}u\|_{H^1})^3. \tag{3.3.14}$$

The desired estimate for R_1 follows immediately from (3.3.13) and (3.3.14). The estimate for R_2 can be shown in the same way. □

3.4 Proof of Theorem 3.1.2

In this section, we will prove Theorem 3.1.2. The main step of the proof is to show the following.

Proposition 3.4.1. *Let* $\alpha = (\alpha_1(t,\xi), \alpha_2(t,\xi))$ *be given by* (3.3.1) *for the solution* $u = (u_1, u_2)$ *to* (3.1.6) *satisfying the assumptions of Theorem 3.1.2. There exists* $\alpha^+ = (\alpha_1^+(\xi), \alpha_2^+(\xi)) \in L^2 \cap L^\infty$ *such that*

$$\lim_{t\to+\infty} \|\alpha_j(t) - \alpha_j^+\|_{L^2} = 0 \tag{3.4.1}$$

for $j = 1, 2$. *Moreover we have* $\alpha_1^+(\xi)\cdot\alpha_2^+(\xi) = 0$ *for* $\xi \in \mathbb{R}$.

Once this proposition is obtained, we can derive Theorem 3.1.2 immediately by setting $\varphi_j^+ = \mathcal{F}^{-1}\alpha_j^+$. Indeed we have

$$\|u_j(t) - \mathcal{U}(t)\varphi_j^+\|_{L^2} = \|\mathcal{F}(\mathcal{U}(-t)u_j(t) - \varphi_j^+)\|_{L^2} = \|\alpha_j(t) - \alpha_j^+\|_{L^2} \to 0$$

as $t \to +\infty$.

In the rest of this section, we will prove Proposition 3.4.1. Note that many parts of the arguments below are similar to those in Subsection 2.5.1, though we need several modifications to fit for the present situation.

Proof of Proposition 3.4.1. We first show the pointwise convergence of $\alpha(t,\xi)$ as $t \to +\infty$. We fix $\xi \in \mathbb{R}$ and introduce

$$\rho(t,\xi) = 2\,\mathrm{Re}\Big[\overline{\alpha_1(t,\xi)}R_1(t,\xi) - \overline{\alpha_2(t,\xi)}R_2(t,\xi)\Big].$$

Then it follows from (3.3.2) and (3.3.3) that

$$\partial_t\Big(|\alpha_1(t,\xi)|^2 - |\alpha_2(t,\xi)|^2\Big) = 2\,\mathrm{Re}\Big[\overline{\alpha_1}\partial_t\alpha_1 - \overline{\alpha_2}\partial_t\alpha_2\Big] = \rho(t,\xi).$$

Also (3.3.6) and (3.3.7) lead to

$$\begin{aligned}\int_2^\infty |\rho(\tau,\xi)|\, d\tau &\le C\int_2^\infty |\alpha(t,\xi)||R(t,\xi)|\, d\tau \\ &\le C\int_2^\infty \varepsilon^4\langle\xi\rangle^{-2}\tau^{3\gamma-5/4}\, d\tau \\ &\le C\varepsilon^4\langle\xi\rangle^{-2}\end{aligned}$$

for $0<\gamma<1/12$. Therefore we obtain

$$\begin{aligned}|\alpha_1(t,\xi)|^2-|\alpha_2(t,\xi)|^2 &= |\alpha_1(2,\xi)|^2-|\alpha_2(2,\xi)|^2+\int_2^t \rho(\tau,\xi)\, d\tau \\ &= m(\xi)-r(t,\xi), \end{aligned} \tag{3.4.2}$$

where

$$m(\xi)=|\alpha_1(2,\xi)|^2-|\alpha_2(2,\xi)|^2+\int_2^\infty \rho(\tau,\xi)\, d\tau \tag{3.4.3}$$

and

$$r(t,\xi)=\int_t^\infty \rho(\tau,\xi)\, d\tau$$

for $t\ge 2$. Note that

$$|m(\xi)|\le|\alpha(2,\xi)|^2+\int_2^\infty|\rho(\tau,\xi)|\, d\tau\le C\varepsilon^2\langle\xi\rangle^{-2}$$

and

$$|r(t,\xi)|\le\int_t^\infty|\rho(\tau,\xi)|\, d\tau\le C\varepsilon^4\langle\xi\rangle^{-2}t^{3\gamma-1/4} \tag{3.4.4}$$

for $0<\gamma<1/12$. Now we divide the argument into three cases according to the sign of $m(\xi)$ as follows.

Case 1: $m(\xi)>0$**.**

First we focus on the asymptotics for α_2. By (3.4.2), we can rewrite (3.3.3) as

$$\partial_t\alpha_2(t,\xi)=-\frac{1}{t}|\alpha_2(t,\xi)|^2\alpha_2(t,\xi)-\frac{m(\xi)}{t}\alpha_2(t,\xi)+\frac{r(t,\xi)}{t}\alpha_2(t,\xi)+R_2(t,\xi)$$

for $t\ge 2$. So we have

$$\partial_t\big(|\alpha_2(t,\xi)|^2\big)=2\,\mathrm{Re}\Big(\overline{\alpha_2}\partial_t\alpha_2\Big)\le-\frac{2m(\xi)}{t}|\alpha_2(t,\xi)|^2+C\varepsilon^4\langle\xi\rangle^{-2}t^{3\gamma-5/4}$$

for $t\ge 2$, whence

$$\partial_t\left(t^{2m(\xi)}|\alpha_2(t,\xi)|^2\right)\le C\varepsilon^4\langle\xi\rangle^{-2}t^{3\gamma+2m(\xi)-5/4}.$$

Now we choose $\varepsilon > 0$ so small that

$$m(\xi) \le C\varepsilon^2 \le \frac{1}{16}(1-12\gamma)$$

is satisfied. Then we have

$$3\gamma + 2m(\xi) - \frac{5}{4} \le -1 - \frac{3}{2}\left(\frac{1}{12} - \gamma\right) < -1.$$

Hence integration in t leads to

$$\begin{aligned} t^{2m(\xi)}|\alpha_2(t,\xi)|^2 - 2^{2m(\xi)}|\alpha_2(2,\xi)|^2 &\le C\varepsilon^4\langle\xi\rangle^{-2}\int_2^t \tau^{3\gamma+2m(\xi)-5/4}d\tau \\ &\le C\varepsilon^4\langle\xi\rangle^{-2} \end{aligned}$$

for $t \ge 2$. Therefore we see that

$$|\alpha_2(t,\xi)| \le C\varepsilon\langle\xi\rangle^{-1}t^{-m(\xi)}. \tag{3.4.5}$$

In particular, $\alpha_2(t,\xi) \to 0$ as $t \to +\infty$. Next we turn our attentions to the asymptotics for α_1. Since (3.3.2) can be viewed as

$$\partial_t\alpha_1(t) = \lambda(t)\alpha_1(t) + Q(t)$$

with $\lambda(t) = -|\alpha_2(t,\xi)|^2/t$ and $Q(t) = R_1(t,\xi)$, we can apply Lemma 2.5.1 to obtain

$$|\alpha_1(t,\xi) - \beta_1^+(\xi)| \le C\int_t^\infty \left(\frac{|\beta_1^+(\xi)||\alpha_2(\tau,\xi)|^2}{\tau} + |R_1(\tau,\xi)|\right)d\tau$$

for $t \ge 2$, where

$$\beta_1^+(\xi) = \alpha_1(2,\xi)e^{-\int_2^\infty |\alpha_2(\tau,\xi)|^2\frac{d\tau}{\tau}} + \int_2^\infty R_1(s,\xi)e^{-\int_s^\infty |\alpha_2(\tau,\xi)|^2\frac{d\tau}{\tau}}ds.$$

By (3.3.6), (3.3.7) and (3.4.5), we have

$$|\beta_1^+(\xi)| \le |\alpha_1(2,\xi)| + \int_2^\infty |R_1(s,\xi)|ds \le C\varepsilon\langle\xi\rangle^{-1} \tag{3.4.6}$$

and

$$\begin{aligned} \int_t^\infty \left(\frac{|\beta_1^+(\xi)||\alpha_2(\tau,\xi)|^2}{\tau} + |R_1(\tau,\xi)|\right)d\tau &\le C\int_t^\infty \left(\frac{\varepsilon^3\langle\xi\rangle^{-3}}{\tau^{1+2m(\xi)}} + \frac{\varepsilon^3\langle\xi\rangle^{-1}}{\tau^{5/4-3\gamma}}\right)d\tau \\ &\le \frac{C\varepsilon^3\langle\xi\rangle^{-3}}{2m(\xi)t^{2m(\xi)}} + \frac{C\varepsilon^3\langle\xi\rangle^{-1}}{t^{1/4-3\gamma}} \end{aligned}$$

for $t \ge 2$. Therefore we conclude that $\alpha_1(t,\xi) \to \beta_1^+(\xi)$ as $t \to +\infty$.

Case 2: $m(\xi) < 0$.

Similarly to the previous case, we have

$$\lim_{t\to+\infty} |\alpha_1(t,\xi)| = 0, \qquad \lim_{t\to+\infty} |\alpha_2(t,\xi) - \beta_2^+(\xi)| = 0$$

for each fixed $\xi \in \mathbb{R}$, where

$$\beta_2^+(\xi) = \alpha_2(2,\xi)e^{-\int_2^\infty |\alpha_1(\tau,\xi)|^2 \frac{d\tau}{\tau}} + \int_2^\infty R_2(s,\xi)e^{-\int_s^\infty |\alpha_1(\tau,\xi)|^2 \frac{d\tau}{\tau}}\, ds.$$

Remark that $|\beta_2^+(\xi)| \le C\varepsilon\langle\xi\rangle^{-1}$.

Case 3: $m(\xi) = 0$**.**

By (3.3.2), (3.3.6), (3.3.7), (3.4.2) and (3.4.4), we have

$$\begin{aligned}\partial_t\left(|\alpha_1(t,\xi)|^2\right) &\le -\frac{2}{t}|\alpha_1(t,\xi)|^4 - \frac{2r(t,\xi)}{t}|\alpha_1(t,\xi)|^2 + 2|\alpha_1(t,\xi)||R_1(t,\xi)| \\ &\le -\frac{2}{t}|\alpha_1(t,\xi)|^4 + C\varepsilon^4\langle\xi\rangle^{-2}t^{3\gamma-5/4}\end{aligned}$$

for $t \ge 2$, and $0 < \gamma < 1/12$. Thus we can apply Lemma 2.4.2 with $\Phi(t) = |\alpha_1(t,\xi)|^2$ to obtain

$$|\alpha_1(t,\xi)| \le \frac{C}{(\log t)^{1/2}} \to 0 \qquad (t \to +\infty).$$

Also (3.4.2) gives us $|\alpha_2(t,\xi)| = \sqrt{|\alpha_1(t,\xi)|^2 + r(t,\xi)} \to 0$ as $t \to +\infty$.

Summing up the three cases above, we deduce that $\alpha(t,\xi)$ converges as $t \to +\infty$ for each fixed $\xi \in \mathbb{R}$. To obtain (3.4.1), we set

$$\alpha_1^+(\xi) := \begin{cases} \beta_1^+(\xi) & (m(\xi) > 0), \\ 0 & (m(\xi) \le 0), \end{cases} \qquad \alpha_2^+(\xi) := \begin{cases} 0 & (m(\xi) \ge 0), \\ \beta_2^+(\xi) & (m(\xi) < 0), \end{cases}$$

and $\alpha^+(\xi) = (\alpha_1^+(\xi), \alpha_2^+(\xi))$ for $\xi \in \mathbb{R}$, where $\beta_1^+(\xi)$ and $\beta_2^+(\xi)$ are shown in Cases 1 and 2, respectively. Then it is obvious that $\alpha_1^+(\xi)\cdot\alpha_2^+(\xi) = 0$ for $\xi \in \mathbb{R}$. Also, by virtue of (3.4.6), we have $\alpha^+ \in L^2 \cap L^\infty(\mathbb{R})$ and

$$\left|\alpha(t,\xi) - \alpha^+(\xi)\right|^2 \le C\varepsilon^2\langle\xi\rangle^{-2} \in L^1(\mathbb{R})$$

for $t \ge 2$. Moreover, it holds that

$$\lim_{t\to+\infty} \left|\alpha(t,\xi) - \alpha^+(\xi)\right|^2 = 0$$

for each fixed $\xi \in \mathbb{R}$. Therefore Lebesgue's dominated convergence theorem yields (3.4.1). □

3.5 Proof of Theorems 3.1.3 and 3.1.4

In this section, we are going to prove Theorems 3.1.3 and 3.1.4. From the proof of Proposition 3.4.1, vanishing of the scattering state $\varphi^+ = (\varphi_1^+, \varphi_2^+)$ can be characterized by the sign of the function $m(\xi)$. Let us summarize it as follows.

Proposition 3.5.1. *We put* $\varphi_j^+ = \lim_{t\to+\infty} \mathcal{U}(-t)u_j(t)$ *in* L^2, $j = 1, 2$, *for the global solution* $u = (u_1, u_2)$ *to* (3.1.6)–(3.1.7), *whose existence is guaranteed by Theorem 3.1.2. Let* m *be the function defined by* (3.4.3). *Then the followings hold for each* $\xi \in \mathbb{R}$*:*

- $m(\xi) > 0$ *implies* $\hat{\varphi}_1^+(\xi) \neq 0$ *and* $\hat{\varphi}_2^+(\xi) = 0$*;*
- $m(\xi) < 0$ *implies* $\hat{\varphi}_1^+(\xi) = 0$ *and* $\hat{\varphi}_2^+(\xi) \neq 0$*;*
- $m(\xi) = 0$ *implies* $\hat{\varphi}_1^+(\xi) = \hat{\varphi}_2^+(\xi) = 0$.

Proposition 3.5.1 gives us more precise information than (3.1.8) and the function $m(\xi)$ plays an important role in it. This indicates that better understanding of $m(\xi)$ will bring us more precise information on the scattering state.

3.5.1 Leading term of $m(\xi)$

The key of our proof of Theorems 3.1.3 and 3.1.4 is the following lemma, which specifies the leading term of $m(\xi)$ for sufficiently small initial data.

Lemma 3.5.1. *Let* m *be the function given by* (3.4.3) *with the initial condition* (3.1.7) *replaced by* (3.1.9). *We have*

$$m(\xi) = \varepsilon^2 \left(|\hat{\psi}_1(\xi)|^2 - |\hat{\psi}_2(\xi)|^2\right) + O(\varepsilon^4)$$

as $\varepsilon \to +0$ *uniformly in* $\xi \in \mathbb{R}$.

Proof. By (3.4.3) and (3.4.4), we have

$$\sup_{\xi\in\mathbb{R}} \left| m(\xi) - \left(|\alpha_1(2,\xi)|^2 - |\alpha_2(2,\xi)|^2\right) \right| \leq C\varepsilon^4.$$

Therefore, it suffices to show

$$\alpha_j(2,\xi) = \varepsilon\hat{\psi}_j(\xi) + O(\varepsilon^3) \tag{3.5.1}$$

as $\varepsilon \to +0$, uniformly in $\xi \in \mathbb{R}$ for $j = 1, 2$.

We put $N_1(u) = |u_2|^2 u_1$ and $N_2(u) = |u_1|^2 u_2$. Then it follows from (3.2.2), (3.3.4), (3.3.5), the relation $\mathcal{L} = \mathcal{U}(t) i\partial_t \mathcal{U}(-t)$ and the Sobolev embedding that

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}} \big|\alpha_j(2,\xi) - \varepsilon\hat{\psi}_j(\xi)\big| &\le C\big\|\mathcal{U}(-2)u_j(2,\cdot) - u_j(0,\cdot)\big\|_{H^{0,1}} \\
&\le C\int_0^2 \|\mathcal{U}(-\tau)N_j(u(\tau))\|_{H^{0,1}}\, d\tau \\
&\le C\int_0^2 \|u(\tau)\|_{L^\infty}^2 \big(\|u(\tau)\|_{L^2} + \|\mathcal{J}u(\tau)\|_{L^2}\big)\, d\tau \\
&\le C\varepsilon^3
\end{aligned}$$

as desired. □

3.5.2 Proof of Theorem 3.1.3

We put $V = \{\xi \in \mathbb{R} \,|\, |\hat{\psi}_1(\xi)| > |\hat{\psi}_2(\xi)|\}$. By (3.1.10), we see that V is a non-empty open set. Now we take $r > 0$ so small that the closed interval $I = [\xi^* - r, \xi^* + r]$ is included in V, and we put

$$C_1 = \min_{\xi\in I} \big(|\hat{\psi}_1(\xi)|^2 - |\hat{\psi}_2(\xi)|^2\big).$$

Then we have $C_1 > 0$, and Lemma 3.5.1 gives us

$$m(\xi) \ge C_1\varepsilon^2 - C\varepsilon^4 > 0$$

for $\xi \in I$, if $\varepsilon > 0$ is small enough. By Proposition 3.5.1, we have $\hat{\varphi}_1^+(\xi) \ne 0$ for $\xi \in I$. Therefore we obtain

$$\|\varphi_1^+\|_{L^2} \ge \|\hat{\varphi}_1^+\|_{L^2(I)} > 0.$$

Similarly, (3.1.11) yields $\|\varphi_2^+\|_{L^2} > 0$. □

3.5.3 Proof of Theorem 3.1.4

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a cut-off function satisfying $\chi(\xi) = 1$ ($|\xi| \le 1$) and $\chi(\xi) = 0$ ($|\xi| \ge 2$). For given $\delta > 0$, we can choose $q \ge 1$ so large that $\|(1-\chi_q)\hat{\varphi}_2^+\|_{L^2} < \delta$, where $\chi_q(\xi) = \chi(\xi/q)$. With this q, we put

$$C_2 = \min_{|\xi|\le 2q} \big(|\hat{\psi}_1(\xi)|^2 - |\hat{\psi}_2(\xi)|^2\big).$$

Then we have $C_2 > 0$, because of (3.1.12). So it follows from Lemma 3.5.1 that

$$m(\xi) \ge C_2\varepsilon^2 - C\varepsilon^4 > 0$$

for $|\xi| \le 2q$, if $\varepsilon > 0$ is small enough. By Proposition 3.5.1, we deduce that $\chi_q(\xi)\hat{\varphi}_2^+(\xi) = 0$ for all $\xi \in \mathbb{R}$. Therefore

$$\|\varphi_2^+\|_{L^2} = \|(1-\chi_q)\hat{\varphi}_2^+\|_{L^2} < \delta.$$

Since δ can be taken arbitrarily small, this means that φ_2^+ vanishes almost everywhere on $\mathbb{R}$. □

3.6 Final state problem for (3.1.6)

To complement Remark 3.1.2, we give two auxiliary results on the final state problem for (3.1.6), that is, finding a solution $u = (u_1, u_2)$ to (3.1.6) which satisfies

$$\lim_{t\to+\infty} \|u_j(t) - \mathcal{U}(t)\psi_j^+\|_{L^2} = 0, \quad j = 1, 2, \tag{3.6.1}$$

for a prescribed final state $\psi^+ = (\psi_1^+, \psi_2^+)$. Roughly speaking, the propositions below imply that (3.6.1) holds if and only if

$$\hat{\psi}_1^+(\xi) \cdot \hat{\psi}_2^+(\xi) = 0, \quad \xi \in \mathbb{R}. \tag{3.6.2}$$

This indicates that our problem must be distinguished from the usual short-range situation, because (3.6.1) should hold in the short-range case regardless of whether (3.6.2) is true or not (see e.g. [10]).

The precise statements are as follows.

Proposition 3.6.1. *Let $T_0 \ge 1$ be given, and let u be a solution to* (3.1.6) *for $t \ge T_0$ satisfying*

$$\sup_{t\ge T_0}\Big(t^{-\gamma}\|\mathcal{U}(-t)u(t)\|_{H^{1,1}} + \|\mathcal{F}\mathcal{U}(-t)u(t)\|_{L^\infty}\Big) < \infty \tag{3.6.3}$$

with some $\gamma \in (0, 1/12)$. If there exists $\psi^+ \in L^2$ with $\hat{\psi}^+ \in L^\infty$ such that (3.6.1) *holds, then we must have* (3.6.2).

Proposition 3.6.2. *Suppose that ψ^+ satisfies $\hat{\psi}^+ \in H^{0,s} \cap L^\infty$ with some $s > 1$, and that $\delta = \|\hat{\psi}^+\|_{L^\infty}$ is suitably small. If* (3.6.2) *holds, then there exist $T \ge 1$ and a unique solution u to* (3.1.6) *for $t \ge T$ satisfying $\mathcal{U}(-t)u \in C([T,\infty); H^{0,1})$ and* (3.6.1).

We are going to give a proof of them. Note that the arguments below are essentially the same as those given in Section 5 of [50]. We also remark that the regularity assumptions in these propositions are certainly not optimal. It may be possible to relax them (see e.g. [15]), but that is out of the present purpose.

Proof of Proposition 3.6.1. In what follows, we write $N_1(v) = |v_2|^2 v_1$, $N_2(v) = |v_1|^2 v_2$ and $N(v) = (N_1(v), N_2(v))$ for $v = (v_1, v_2)$. Let α be given by (3.3.1) for the solution u to (3.1.6). Then, similarly to (3.3.2), we have

$$\partial_t \alpha_j(t,\xi) = -\frac{1}{t} N_j(\hat{\psi}^+(\xi)) + S_j(t,\xi) + R_j(t,\xi), \quad j = 1, 2,$$

where

$$S_j(t,\xi) = \frac{1}{t}\Big(N_j(\hat{\psi}^+(\xi)) - N_j(\alpha(t,\xi))\Big)$$

and

$$R_j(t,\xi) = \frac{1}{t} N_j(\alpha(t,\xi)) - \mathcal{F}\Big[\mathcal{U}(-t) N_j(u(t,\cdot))\Big](\xi).$$

Now we shall argue by contradiction. If (3.6.2) is not true, then we can take $\eta > 0$ such that $\|N_j(\hat{\psi}^+)\|_{L^2} \geq \eta$ for $j = 1, 2$. By (3.6.3) and Lemma 3.3.1, we have $\|R_j(t)\|_{L^2} \leq C t^{-5/4+3\gamma}$ for $t \geq T_0$. We also note that

$$\begin{aligned}\|S_j(t)\|_{L^2} &= \frac{1}{t}\|N_j(\hat{\psi}^+) - N_j(\mathcal{F}\mathcal{U}(-t)u(t))\|_{L^2} \\ &\leq \frac{C}{t}\big(\|\hat{\psi}\|_{L^\infty}^2 + \|\mathcal{F}\mathcal{U}(-t)u(t)\|_{L^\infty}^2\big)\|\mathcal{F}(\psi^+ - \mathcal{U}(-t)u(t))\|_{L^2} \\ &\leq \frac{C}{t}\|\mathcal{U}(t)\psi^+ - u(t)\|_{L^2},\end{aligned}$$

whence, by (3.6.1), we can take $T^* \geq T_0$ such that $\|S_j(t)\|_{L^2} \leq \eta/(2t)$ for $t \geq T^*$. Summing up, we obtain

$$\begin{aligned}&\|\mathcal{U}(-2t)u_j(2t) - \mathcal{U}(-t)u_j(t)\|_{L^2} \\ =&\|\alpha_j(2t) - \alpha_j(t)\|_{L^2} \\ \geq&\eta \int_t^{2t} \frac{d\tau}{\tau} - \int_t^{2t} \|S_j(\tau)\|_{L^2}\, d\tau - \int_t^{2t} \|R_j(\tau)\|_{L^2}\, d\tau \\ \geq&\frac{\eta}{2}\log 2 - C t^{-1/4+3\gamma}\end{aligned}$$

for $t \geq T^*$. Letting $t \to \infty$, we have

$$0 = \|\psi_j^+ - \psi_j^+\|_{L^2} \geq \frac{\eta}{2}\log 2 > 0,$$

which is the desired contradiction. □

Proof of Proposition 3.6.2. With $T \geq 1$ to be fixed, let us introduce the function space

$$\mathfrak{X}_T = \Big\{\phi = (\phi_1(t,x), \phi_2(t,x)) \;\Big|\; \mathcal{U}(-t)\phi(t,\cdot) \in C([T,\infty); H^{0,1})\Big\}$$

and the norm

$$\|\phi\|_{\mathfrak{X}_T} = \sup_{t\in[T,\infty)} \big(t^{\mu+1/2}\|\phi(t)\|_{L^2} + t^{\mu}\|\mathcal{J}\phi(t)\|_{L^2}\big),$$

where $0 < \mu < (s_0 - 1)/2$ and $s_0 = \min\{2, s\}$. For $v = (v_1, v_2) \in \mathfrak{X}_T$, we set

$$\Phi_j[v](t) = \mathcal{U}(t)\psi_j^+ - \int_t^\infty \mathcal{U}(t-\tau)N_j(v(\tau))\,d\tau, \quad j = 1, 2, \tag{3.6.4}$$

and $\Phi[v] = (\Phi_1[v], \Phi_2[v])$. We also put $w^\sharp(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}\psi^+$, $w^\flat(t) = \mathcal{U}(t)\psi^+ - w^\sharp(t)$, $\kappa = \|\psi^+\|_{H^{0,s_0}}$ and

$$\mathfrak{Y}_T = \left\{\phi \in \mathfrak{X}_T \;\middle|\; \|\phi - w^\sharp\|_{\mathfrak{X}_T} \le \kappa\right\}.$$

Since (3.6.2) yields $N(\hat{\psi}^+) = 0$, it follows from (3.3.8) that

$$\mathcal{U}(-\tau)N(w^\sharp(\tau)) = \frac{1}{\tau}\mathcal{M}(\tau)^{-1}\mathcal{F}^{-1}N(\hat{\psi}^+) = 0. \tag{3.6.5}$$

We observe the basic estimates for $w^\sharp(t)$ and $w^\flat(t)$:

$\|w^\sharp(t)\|_{L^\infty} = t^{-1/2}\|\hat{\psi}^+\|_{L^\infty} = \delta t^{-1/2}$,
$\|w^\sharp(t)\|_{L^2} = \|\psi^+\|_{L^2} \le \kappa$,
$\|w^\flat(t)\|_{L^2} \le \|(\mathcal{M}(t) - 1)\psi^+\|_{L^2} \le Ct^{-s_0/2}\|\psi^+\|_{H^{0,s_0}} \le C\kappa t^{-s_0/2}$,
$\|\mathcal{J}w^\sharp(t)\|_{L^2} = \|x\psi^+\|_{L^2} \le \kappa$,
$\|\mathcal{J}w^\flat(t)\|_{L^2} \le \|x(\mathcal{M}(t) - 1)\psi^+\|_{L^2} \le Ct^{-(s_0-1)/2}\|\psi^+\|_{H^{0,s_0}} \le C\kappa t^{-(s_0-1)/2}$,

where we have used (3.2.2), (3.3.8) and (3.3.10) with $\sigma = s_0/2$ or $(s_0 - 1)/2$.

Now we are going to show that Φ is a contraction mapping on $\mathfrak{Y}_T$ by choosing δ and T appropriately. Let $v \in \mathfrak{Y}_T$. By using (3.6.5), we rewrite (3.6.4) as

$$\Phi[v](t) - w^\sharp(t) = -\int_t^\infty \mathcal{U}(t-\tau)\Big(N(v(\tau)) - N(w^\sharp(\tau))\Big)\,d\tau + w^\flat(t).$$

It follows from the inequality $\|\phi\|_{L^\infty} \le Ct^{-1/2}\|\phi\|_{L^2}^{1/2}\|\mathcal{J}\phi\|_{L^2}^{1/2}$ that

$$\|v(t) - w^\sharp(t)\|_{L^\infty} \le Ct^{-1/2}\|v(t) - w^\sharp(t)\|_{L^2}^{1/2}\|\mathcal{J}(v(t) - w^\sharp(t))\|_{L^2}^{1/2} \le C\kappa t^{-3/4-\mu}.$$

So we have

$$\|v(t)\|_{L^\infty} \le \|w^\sharp(t)\|_{L^\infty} + \|v(t) - w^\sharp(t)\|_{L^\infty} \le C(\delta + \kappa T^{-1/4-\mu})t^{-1/2}.$$

Therefore

$$\begin{aligned}
&\|\Phi[v](t) - w^\sharp(t)\|_{L^2} \\
\le& C\int_t^\infty (\|v(\tau)\|_{L^\infty}^2 + \|w^\sharp(\tau)\|_{L^\infty}^2)\|v(\tau) - w^\sharp(\tau)\|_{L^2}\,d\tau + C\kappa t^{-s_0/2} \\
\le& C(\delta + \kappa T^{-1/4-\mu})^2\kappa\int_t^\infty \frac{d\tau}{\tau^{3/2+\mu}} + C\kappa T^{-(s_0-1)/2+\mu}t^{-1/2-\mu} \\
\le& C(\delta^2 + \kappa^2 T^{-1/2-2\mu} + T^{-(s_0-1)/2+\mu})\kappa t^{-1/2-\mu}.
\end{aligned} \tag{3.6.6}$$

Also, because of the estimate

$$
\begin{aligned}
&\|\mathcal{J}(N(v(t)) - N(w^\sharp(t)))\|_{L^2} \\
\le& C(\|v\|_{L^\infty}^2 + \|w^\sharp\|_{L^\infty}^2)\|\mathcal{J}(v - w^\sharp)\|_{L^2} \\
&+ C(\|v\|_{L^\infty} + \|w^\sharp\|_{L^\infty})(\|\mathcal{J}v\|_{L^2} + \|\mathcal{J}w^\sharp\|_{L^2})\|v - w^\sharp\|_{L^\infty} \\
\le& C(\delta^2 + \kappa^2 T^{-1/2-2\mu})\kappa t^{-1-\mu} \\
&+ C(\delta + \kappa T^{-1/4-\mu})t^{-1/2} \cdot C(\kappa t^{-\mu} + \kappa) \cdot C\kappa t^{-3/4-\mu} \\
\le& C(\delta^2 + \kappa^2 T^{-1/2-\mu} + \delta\kappa T^{-1/4})\kappa t^{-1-\mu},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\|\mathcal{J}(\Phi[v](t) - w^\sharp(t))\|_{L^2} \\
\le& \int_t^\infty \|\mathcal{J}(N(v(\tau)) - N(w^\sharp(\tau)))\|_{L^2}\, d\tau + C\kappa t^{-(s_0-1)/2} \\
\le& C(\delta^2 + \kappa^2 T^{-1/2-\mu} + \delta\kappa T^{-1/4})\kappa \int_t^\infty \frac{d\tau}{\tau^{1+\mu}} + C\kappa T^{-(s_0-1)/2+\mu} t^{-\mu} \\
=& C(\delta^2 + \kappa^2 T^{-1/2-\mu} + \delta\kappa T^{-1/4} + T^{-(s_0-1)/2+\mu})\kappa t^{-\mu}. \qquad (3.6.7)
\end{aligned}
$$

Combining (3.6.6) and (3.6.7), we arrive at

$$
\|\Phi[v] - w^\sharp\|_{\mathfrak{X}_T} \le \underbrace{C(\delta^2 + \kappa^2 T^{-1/2-\mu} + \delta\kappa T^{-1/4} + T^{-(s_0-1)/2+\mu})}_{(*)}\kappa.
$$

Hence we have $\Phi[v] \in \mathfrak{Y}_T$ if we choose δ so small and T so large that the term $(*)$ does not exceed 1. Next we take v, $\tilde{v} \in \mathfrak{Y}_T$. Then we have

$$
\Phi[v](t) - \Phi[\tilde{v}](t) = -\int_t^\infty \mathcal{U}(t-\tau)\Big(N(v(\tau)) - N(\tilde{v}(\tau))\Big)\, d\tau
$$

and we can see as before that

$$
\|\Phi[v] - \Phi[\tilde{v}]\|_{\mathfrak{X}_T} \le \frac{1}{2}\|v - \tilde{v}\|_{\mathfrak{X}_T}
$$

by choosing δ and T suitably. Therefore $\Phi : \mathfrak{Y}_T \to \mathfrak{Y}_T$ is a contraction mapping, and thus, admits a unique fixed point. In other words, there exists $u \in \mathfrak{Y}_T$ such that

$$
u(t) = \mathcal{U}(t)\psi^+ - \int_t^\infty \mathcal{U}(t-\tau)N(u(\tau))\, d\tau,
$$

which gives the desired solution to (3.1.6) for $t \ge T$. Moreover we have

$$
\begin{aligned}
\|u(t) - \mathcal{U}(t)\psi^+\|_{L^2} &\le \|u(t) - w^\sharp(t)\|_{L^2} + \|w^\flat(t)\|_{L^2} \\
&\le \kappa t^{-1/2-\mu} + C\kappa t^{-s_0/2} \\
&\to 0
\end{aligned}
$$

as $t \to +\infty$. This completes the proof of Proposition 3.6.2. □

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