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Generalized Kazdan-Warner equations associated with a linear action of a torus on a complex vector space

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Contents

1 Introduction

Kazdan and Warner [KW] studied the following elliptic PDE on a Riemannian manifold (M, g_M) which is now called the *Kazdan-Warner equation* in connection with the prescribed Gaussian curvature problem on a compact real surface:

$$\Delta_{g_M} f + he^f = c, \tag{1.1}$$

where h and c are given real functions over M and $\Delta_{g_M} \coloneqq d^*d$ denotes the geometric Laplacian and f is a solution of (1.1). Although the primary motivation in [KW] to introduce this equation was to solve the prescribed Gaussian curvature problem, the Kazdan-Warner equation itself has been studied in various contexts [Ber, Br1, JT, Mor, MP, Tau].

In this paper, we introduce a generalization of the Kazdan-Warner equation (1.1) which we call the *generalized Kazdan-Warner equation* from the point of view of the moment maps for linear torus actions. The generalized Kazdan-Warner equation is defined as follows: Let $T^d \coloneqq \mathrm{U}(1)^d$ be a compact connected torus of dimension d. We denote by t^d the Lie algebra of T^d. We take a closed connected subtorus K of T^d with Lie algebra k. Let $\iota : k \to t^d$ be the inclusion map and $\iota^* : (t^d)^* \to k^*$ the dual of ι. Let $u_1, \dots, u_d$ be a basis of t^d defined as

$$\begin{aligned}
u_1 \coloneqq &(\sqrt{-1}, 0, \dots, 0),\\
u_2 \coloneqq &(0, \sqrt{-1}, 0, \dots, 0),\\
&\cdots\\
u_d \coloneqq &(0, \dots, 0, \sqrt{-1}).
\end{aligned}$$

We denote by $u^1, \dots, u^d \in (t^d)^*$ the dual basis of $u_1, \dots, u_d$. We fix a metric $(\cdot, \cdot)$ on k^*. Let $a_1, \dots, a_d$ be $\mathbb{R}$-valued functions and w a k^*-valued function over M. Then the generalized Kazdan-Warner equation is the following:

$$\Delta_{g_M} \xi + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} \iota^* u^j = w, \tag{1.2}$$

where ξ is a k^*-valued function on M which is a solution of (1.2). We set $d = 1$ and $K = \mathrm{U}(1)$. Then (1.2) coincides with the Kazdan-Warner equation (1.1). Examples of (1.2) also includes the multi-monopole equation on a Kähler surface [BW] (see also [Doa, HW]) and the Hitchin equation for a diagonal harmonic metric on cyclic Higgs bundles which is called the *two-dimensional periodic Toda lattice with opposite sign* [GL] (see also [AF, Bar1, Bar2, DL, GH, LM1, LM2]). Our equation (1.2) is considered as a generalization of these equations.

We solve the equation (1.2) on any compact Riemannian manifold under the following assumption on $a_1, \ldots, a_d$ and w:

($*$) The given functions $a_1, \ldots, a_d$ and w are all smooth and for each $j = 1, \ldots, d$, a_j is a *non-negative* function and if a_j is not identically 0, then $a_j^{-1}(0)$ is a set of measure 0 and $\log a_j$ is integrable.

Our main theorem is the following:

Theorem 1.1. *Suppose that $a_1, \ldots, a_d$ and w satisfy ($*$) and their domain M is a compact connected manifold. Then the following (1) and (2) are equivalent:*

(1) The generalized Kazdan-Warner equation (1.2) has a C^∞-solution ξ;

(2) The given functions $a_1, \ldots, a_d$ and w satisfy

$$\int_M w \; d\mu_{g_M} \in \sum_{j \in J_a} \mathbb{R}_{>0} \iota^* u^j, \tag{1.3}$$

where J_a denotes $\{j \in \{1, \ldots, d\} \mid a_j \text{ is not identically } 0\}$ and μ_{g_M} denotes the measure induced by g_M.

Moreover if ξ and ξ' are C^∞-solutions of equation (1.2), then $\xi - \xi'$ is a constant which lies in the orthogonal complement of $\sum_{j \in J_a} \mathbb{R} \iota^* u^j$.

Remark 1.2. In the definition of the generalized Kazdan-Warner equation (1.2), we have only used the Lie subalgebra k of the Lie algebra t^d. This means that our equation (1.2) can be defined not only for a compact connected subtorus K of T^d, but also for an arbitrary Lie subalgebra k of t^d. Note that in general, the Lie subgroup which corresponds to a Lie subalgebra k of t^d is not a compact subtorus of T^d. We also note that from a pair $(\mathbb{R}^n, v_1, \ldots, v_d)$ consisting of a real vector space $\mathbb{R}^n$ and its d-generators $v_1, \ldots, v_d \in \mathbb{R}^n$ we have a Lie subalgebra of t^d. Indeed, the dual of the following surjection p defines an embedding of $(\mathbb{R}^n)^*$ to t^d:

$$\begin{aligned} p : (t^d)^* &\longrightarrow \mathbb{R}^n \\ r_1 u^1 + \cdots + r_d u^d &\longmapsto r_1 v_1 + \cdots + r_d v_d. \end{aligned}$$

Conversely, from a Lie subalgebra $k \subseteq t^d$, we have a pair $(k^*, \iota^* u^1, \dots, \iota^* u^d)$ consisting of a real vector space k^* and its d-generators $\iota^* u^1, \dots, \iota^* u^d$. Therefore one can consider that our equation (1.2) is defined for a real vector space $\mathbb{R}^n$ and its generators $v_1, \dots, v_d$.

Remark 1.3. Theorem 1.1 also holds even if the equation (1.2) is defined for a Lie subalgebra $k \subseteq t^d$ whose corresponding Lie subgroup is not a compact subtorus of T^d.

Remark 1.4. The definition of our generalized Kazdan-Warner equation (1.2) comes from the moment map for the diagonal action of a compact torus K on $\mathbb{C}^d$, and the condition (2) of Theorem 1.1 is related to the stability condition of the diagonal action of the complexification $K_{\mathbb{C}}$ of the torus K on $\mathbb{C}^d$. We will explain this in Appendix A.

Remark 1.5. The same condition as (2) is obtained by Baptista [Bap] as the stability condition of the Abelian GLSM.

We note that (1) of Theorem 1.1 immediately implies (2): If (1) holds, by integrating both sides of (1.2), we have

$$\sum_{j=1}^{d} \left(\int_M a_j e^{(\iota^* u^j, \xi)} \, d\mu_{g_M} \right) \iota^* u^j = \int_M w \, d\mu_{g_M}.$$

Hence it suffices to solve equation (1.2) under the assumption of (2) and to prove the uniqueness of the solution up to a constant which lies in the orthogonal complement of a vector subspace $\sum_{j \in J_a} \mathbb{R}\iota^* u^j$. We give a proof of Theorem 1.1 in Section 3 by using a variational method inspired from the theory of moment maps for linear torus actions.

As already mentioned, an important example of our equation (1.2) is the two-dimensional periodic Toda lattice with opposite sign, which is the Hitchin equation for a diagonal harmonic metric on a cyclic Higgs bundle. In Section 2, we give a brief review for Higgs bundles and harmonic bundles and we recall the definition of the cyclic Higgs bundle introduced by Baraglia [Bar1, Bar2]. In Section 4, we give the necessary and sufficient condition for the existence of a diagonal pluriharmonic metric on a G-Higgs bundle on a compact Kähler manifold by applying Theorem 1.1 to the Hermitian-Einstein equation for G-Higgs bundles. In particular, we show the existence of the solution of the two-dimensional periodic Toda lattice with opposite sign on a compact Riemann surface from this point of view. We also show that the pluriharmonic map associated with a diagonal pluriharmonic metric lifts to a pluriharmonic map to G/T, where T is a maximal compact torus of a complex connected semisimple Lie group G.

In Section 5, we extend Theorem 1.1 on compact foliated manifolds. Our motivation for considering the generalized Kazdan-Warner equation on foliated manifolds comes from the recent progress in the study of the gauge theory and the Kobayashi-Hitchin correspondence on foliated manifolds [BH1, BH2, BK1, BK2, KLW, WZ]. In Section 5, on a compact foliated manifold, we show that under the assumption of $(*)$ and (2) of Theorem 1.1, a solution of the generalized Kazdan-Warner equation is a basic function with respect to the foliation if $a_1, \ldots, a_d$ and w are all basic and if the Laplacian Δ_{g_M} preserves the space of basic functions. This implies that the same claim as Theorem 4.1 and Theorem 4.2 in Section 4 also holds for basic Higgs bundles over compact Sasakian manifolds.

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2 Higgs bundles and harmonic bundles

In this section, we give a brief review of Higgs bundles and harmonic bundles on compact Kähler manifolds. Let X be a connected complex manifold. We define a *Higgs bundle* over X as a pair (E, Φ) consisting of a holomorphic vector bundle $E \to X$ and a holomorphic section Φ of $\mathrm{End} E \otimes \bigwedge^{1,0}$ satisfying $\Phi \wedge \Phi = 0$. We call the holomorphic section Φ a *Higgs field*. For a Hermitian metric h on E, we define a connection D as

$$D := \nabla^h + \Phi + \Phi^{*h}, \tag{2.1}$$

where we denote by ∇^h the Chern connection of the Hermitian metric h and by Φ^{*h} the adjoint of the Higgs field Φ with respect to h. We call a Hermitian metric h a *pluriharmonic metric* if the connection D is flat. A pluriharmonic metric h gives a $\pi_1(X)$-equivariant pluriharmonic map $\hat{h} : \tilde{X} \to \mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$, where we denote by $\tilde{X}$ the universal covering space of X. We call a pair (E, Φ, h) consisting of a Higgs bundle (E, Φ) and a pluriharmonic metric h a *harmonic bundle*.

Suppose that X is a compact Kähler manifold with Kähler form ω_X. Let $(E, \Phi) \to X$ be a Higgs bundle over X. Suppose that a Lie group A acts on X preserving the metric ω_X and A acts also on E compatibly with the action on X and preserving a Hermitian metric h_0 on E. Suppose also that there exists a character $\lambda : A \to \mathrm{U}(1)$ such that

$$g \circ \Phi \circ g^{-1} = \lambda(g)\Phi \text{ for all } g \in A.$$

Then we say that a Higgs bundle (E, Φ) is *stable with respect to* A (resp. *semistable with respect to* A) if for any proper non-trivial Φ-invariant saturated subsheaf $\mathcal{F}$ of E which is preserved by A, we have

$$\deg_{\omega_X}(\mathcal{F})/\mathrm{rank}\mathcal{F} < \deg_{\omega_X}(E)/\mathrm{rank}E$$
$$(\text{resp. } \deg_{\omega_X}(\mathcal{F})/\mathrm{rank}\mathcal{F} \leq \deg_{\omega_X}(E)/\mathrm{rank}E).$$

If the action of A is trivial, we simply say that (E, Φ) is stable (resp. semistable) instead of saying that (E, Φ) is stable with respect to A (resp. semistable with respect to A). We also say that (E, Φ) is *polystable* if E is a direct sum of holomorphic subbundles $E_1, \dots, E_m$ such that for each $j = 1, \dots, m$, E_j is preserved by Φ and a Higgs bundle $(E_j, \Phi|_{E_j})$ is stable. The following Theorem 2.1 and Theorem 2.2 is known as the *Kobayashi-Hitchin correspondence* of Higgs bundles and harmonic bundles:

Theorem 2.1. ([Sim1]) *Let* $(E, \Phi) \to (X, \omega_X)$ *be a stable Higgs bundle with respect to a Lie group* A*. Then there exists an* A*-invariant Hermitian metric* h *such that* h *solves the following Hermitian-Einstein equation:*

$$\Lambda_{\omega_X} F_D^{\perp} = 0,$$

where we denote by Λ_{ω_X} *the dual of* $\omega_X \wedge$ *and by* $F_D^{\perp}$ *the trace-free part of the curvature of the connection* D *defined as (2.1).*

Theorem 2.2. ([Sim1]) *A Higgs bundle* (E, Φ) *over a compact Kähler manifold admits a pluriharmonic metric if and only if* (E, Φ) *is polystable and* $c_1(E) = c_2(E) = 0$*.*

We note that Theorem 2.2 is proved in [Hit1] for the case that X is a compact Riemann surface and the rank of E is 2. We also note that the Hermitian-Einstein equation is also called the *Hitchin equation* in the case where the base manifold X is a Riemann surface.

We introduce a notion of G-Higgs bundles and G-harmonic bundles. Let G be a complex connected semisimple Lie group with Lie algebra $\mathfrak{g}$. A G-Higgs bundle over a complex manifold X is a pair (P_G, Φ) consisting of a holomorphic principal G-bundle $P_G \to X$ and a holomorphic section Φ of $\mathrm{ad}(P_G) \otimes \bigwedge^{1,0}$ satisfying $[\Phi \wedge \Phi] = 0$, where we denote by $\mathrm{ad}(P_G)$ the adjoint bundle of P_G, and by $[\cdot, \cdot]$ the Lie bracket of $\mathfrak{g}$. We call Φ a Higgs field. Let $K_0 \subseteq G$ be a maximal compact subgroup of G. We denote by $\mathfrak{k}$ the Lie algebra of K_0. We define a $\mathbb{C}$-antilinear involution $\sigma_{K_0} : \mathfrak{g} \to \mathfrak{g}$ as $\sigma_{K_0}(u + \sqrt{-1}v) \coloneqq u - \sqrt{-1}v$ for $u, v \in \mathfrak{k}$. Let $\sigma : \mathrm{ad}(P_G) \to \mathrm{ad}(P_G)$ be a $\mathbb{C}$-antilinear involution such that for each $x \in X$, $\sigma_x : \mathrm{ad}(P_G)_x \to \mathrm{ad}(P_G)_x$ is conjugate to σ_{K_0}:

$$\sigma_x = \mathrm{Ad}_{g^{-1}} \circ \sigma_{K_0} \circ \mathrm{Ad}_g \text{ for a } g \in G. \tag{2.2}$$

Then there exists a unique K_0-subbundle P_{K_0} of P_G such that $\sigma(u + \sqrt{-1}v) = u - \sqrt{-1}v$ for all $u, v \in \mathrm{ad}(P_{K_0})$. Conversely, from a K_0-subbundle P_{K_0}, we have an involution σ defined as above. Associated with the involution σ, there uniquely exists a connection ∇^σ of P_{K_0} such that $(p_U^* \nabla^\sigma)^{0,1} = \bar{\partial}$ for any holomorphic local trivialization $p_U : U \to P_G|_U$, where we denote by $\overline{\partial}$ the holomorphic structure of P_G. We call the connection ∇^σ the Chern connection of σ. By using the Chern connection and the Higgs field Φ, we define a connection D as

$$D \coloneqq \nabla^\sigma + \Phi - \sigma(\Phi). \tag{2.3}$$

We call a pair (P_G, Φ, σ) consisting of a G-Higgs bundle (P_G, Φ) and an involution σ a G-harmonic bundle if the connection D defined as above is flat.

Example 2.3. Suppose that G is a simply connected complex simple Lie group. We give an important example of a family of G-Higgs bundles introduced by Hitchin [Hit2] which is called the *Hitchin section*. Let X be a compact connected Riemann surface with canonical line bundle $K_X \to X$. Assume that X has genus at least 2. Let $T \subseteq K_0$ be a maximal real torus of K_0 with Lie algebra $\mathfrak{t}$. Let $H \subseteq G$ be the complexification of T with Lie algebra $\mathfrak{h} = \mathfrak{t} \oplus \sqrt{-1}\mathfrak{t}$. We denote by $\Delta \subseteq \mathfrak{h}^*$ the root system and by $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ the root space decomposition of $\mathfrak{g}$. Note that for each $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is a complex 1-dimensional subspace of $\mathfrak{g}$. We fix a base of Δ which is denoted as $\Pi = \{\alpha_1, \ldots, \alpha_l\}$. Let $\epsilon_1, \ldots, \epsilon_l$ be the dual basis of $\alpha_1, \ldots, \alpha_l$. We define a semisimple element x of $\mathfrak{g}$ as follows:

$$x \coloneqq \sum_{i=1}^{l} \epsilon_i. \tag{2.4}$$

Let $B(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the Killing form and $B^*(\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ the dual of $B|_{\mathfrak{h}}$. For each $\alpha \in \Delta$, we define the coroot $h_\alpha \in \mathfrak{h}$ of α as $h_\alpha \coloneqq 2\alpha^*/B^*(\alpha, \alpha)$, where we denote by α^* the dual of α with respect to B^*. Then there exists a basis $(e_\alpha)_{\alpha \in \Delta}$ of $\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ such that

$$\begin{aligned} &e_\alpha \in \mathfrak{g}_\alpha, \\ &[e_\alpha, e_{-\alpha}] = h_\alpha, \\ &\sigma_{K_0}(e_\alpha) = -e_{-\alpha}. \end{aligned}$$

The semisimple element x is denoted as

$$x = \sum_{i=1}^{l} r_i h_{\alpha_i}$$

for some positive $r_1, \ldots, r_l$. We define nilpotent elements e and $\tilde{e}$ of $\mathfrak{g}$ as follows:

$$\begin{aligned} e &\coloneqq \sum_{i=1}^{l} e_{\alpha_i}, \\ \tilde{e} &\coloneqq \sum_{i=1}^{l} r_i e_{-\alpha_i}. \end{aligned}$$

Then we can check that the following holds :

$$[x, e] = e,\ [x, \tilde{e}] = -\tilde{e},\ [e, \tilde{e}] = x.$$

Let $\mathfrak{s}$ be the subalgebra which is spanned by $x, e, \tilde{e}$. Then $\mathfrak{s}$ is one of the principal three dimensional subalgebras introduced by Kostant [Kos]. The adjoint representation of $\mathfrak{s}$ on $\mathfrak{g}$ has l-irreducible subspaces denoted as

$$\mathfrak{g} = \bigoplus_{i=1}^{l} V_i$$

with $\dim V_i = 2m_i + 1$ for integers $m_1, \dots, m_l$ since all eigenvalues of $\mathrm{ad}(x)$ are integer and the kernel of $\mathrm{ad}(x)$ coincides with the Cartan subalgebra $\mathfrak{h}$. We suppose that $m_1 \leq \cdots \leq m_l$. Note that m_l coincides with the height of the highest root δ. We choose a holomorphic line bundle $K_X^{1/2} \to X$ over X such that $K_X^{1/2} \otimes K_X^{1/2} \simeq K_X$. Let $P_{\mathrm{SL}(2,\mathbb{C})}$ be the holomorphic frame bundle of $K_X^{1/2} \oplus K_X^{-1/2}$. The principal three-dimensional subalgebra $\mathfrak{s}$ defines an embedding of $\mathrm{SL}(2, \mathbb{C})$ to G. By using this embedding, we define a holomorphic principal G-bundle P_G as

$$P_G := P_{\mathrm{SL}(2,\mathbb{C})} \times_{\mathrm{SL}(2,\mathbb{C})} G.$$

Then the adjoint bundle $\mathrm{ad}(P_G)$ decomposes as follows:

$$\mathrm{ad}(P_G) = \bigoplus_{m=-m_l}^{m_l} \mathfrak{g}_m \otimes K_X^m,$$

where we denote by $\mathfrak{g} = \bigoplus_{m=-m_l}^{m_l} \mathfrak{g}_m$ the eigenspace decomposition of the adjoint action of x. Let $e_1, \dots, e_l$ be the highest weight vectors of $V_1, \dots, V_l$. We note that for each i, the vector e_i lies in $\mathfrak{g}_{m_i}$. For each $q = (q_1 \dots, q_l) \in \bigoplus_{i=1}^{l} H^0(K_X^{m_i+1})$, we define a Higgs field $\Phi(q) \in H^0(\mathrm{ad}(P_G) \otimes K_X)$ as follows:

$$\Phi(q) := \tilde{e} + q_1 e_1 + \cdots + q_l e_l.$$

Then we have a G-Higgs bundle $(P_G, \Phi(q))$ which is parametrized by a $q = (q_1, \dots, q_l) \in \bigoplus_{i=1}^{l} H^0(K_X^{m_i+1})$. This family of Higgs bundles is called the *Hitchin section*. In [Hit2], Hitchin proved that the Higgs bundle $(P_G, \Phi(q))$ is stable for any $q \in \bigoplus_{i=1}^{l} H^0(K_X^{m_i+1})$. We refer the reader to [Hit2] for more details of the Hitchin section.

Let $(P_G, \Phi(q)) \to X$ be a G-Higgs bundle constructed in Example 2.3. In [Bar1, Bar2] Baraglia introduced the notion of the *cyclic Higgs bundle* defined as below:

Definition 2.4. We say that $(P_G, \Phi(q))$ is a *cyclic Higgs bundle* if the parameter $q = (q_1, \dots, q_l)$ satisfies

$$q_1 = \cdots = q_{l-1} = 0.$$

Note that the cyclic Higgs bundles are special cases of the *cyclotomic Higgs bundles* introduced by Simpson [Sim2]. The harmonic metric of a cyclic Higgs bundle has the following property:

Theorem 2.5. ([Sim2, Bar1]) *Let $(E, \Phi(q)) \to X$ be an $\mathrm{SL}(r, \mathbb{C})$-cyclic Higgs bundle with holomorphic vector bundle*

$$E = K_X^{(r-1)/2} \oplus K_X^{(r-3)/2} \oplus \cdots \oplus K_X^{-(r-3)/2} \oplus K_X^{-(r-1)/2}. \tag{2.5}$$

Then the harmonic metric h associated with $(E, \Phi(q))$ is a diagonal metric with respect to the above decomposition of E.

Remark 2.6. The harmonic metrics of the Hitchin section are *real*, i.e., the following $S : E \to E^*$ is isometric with respect to the metrics:

$$S := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

In particular, the harmonic metric associated with a cyclic Higgs bundle is real, in addition to the property of being diagonal.

Here h is a diagonal metric means that (2.5) is an orthogonal decomposition with respect to the metric h. Theorem 2.5 follows from Theorem 2.1 with an observation that the Higgs field $\Phi(q)$ satisfies $\mathrm{Ad}_g \Phi = \omega \Phi(q)$ for $g = \mathrm{diag}(1, \omega, \dots, \omega^{r-1})$, where we denote by ω the r-th root of unity $e^{2\pi\sqrt{-1}/r}$ and by $\mathrm{diag}(1, \omega, \dots, \omega^{r-1})$ the diagonal matrix whose diagonal entries are $1, \omega, \dots, \omega^{r-1}$. This is observed in [Sim2, Bar1]. In [Bar1], Baraglia proved Theorem 2.5 for general G-cyclic Higgs bundles by using the uniqueness of the solution of the Hitchin equation. The Hitchin equation for a diagonal harmonic metric associated with a cyclic Higgs bundle is called the *two-dimensional periodic Toda lattice with opposite sign* [GL]. We refer the reader to [DL, GL, GH, LM1, LM2] for the recent progress in cyclic Higgs bundles.

3 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. The outline of the proof is as follows. As we have already mentioned, it suffices to solve equation (1.2) under the assumption of (2) and to prove the uniqueness of the solution up to a constant which lies in the orthogonal complement of $\sum_{j\in J_a}\mathbb{R}\iota^* u^j$. We prove this using the variational method. In Definition 3.1, we define a functional E whose critical point is a solution of equation (1.2). Then we show that the functional E is convex. The uniqueness of the solution of the equation follows from the convexity of E. In Lemma 3.5, we show that the functional E is bounded below and moreover the following estimate holds:

$$|\xi|_{L^2} \leq (E(\xi)+C)^2 + C'E(\xi) + C''$$

with some constants C, C' and C''. We use (2) in the proof of Lemma 3.5. Starting from this estimate, we show that the functional E has a critical point by using a method developed in [Br2].

Before starting the proof, we introduce a notation. For a k^*-valued integrable function $\xi : M \to k^*$, we denote by $\bar{\xi}$ the average of ξ:

$$\bar{\xi} := \int_M \xi \ d\mu_{g_M},$$

where we normalized the measure μ_{g_M} so that the total volume is 1:

$$\mathrm{Vol}(M, g_M) := \int_M 1 \ d\mu_{g_M} = 1.$$

We then start the proof of Theorem 1.1.

Definition 3.1. We define a functional $E : L^{2m}_3(M, k^*) \to \mathbb{R}$ by

$$E(\xi) := \int_M \left\{ \frac{1}{2}(d\xi, d\xi) + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} - (w, \xi) \right\} d\mu_{g_M} \text{ for } \xi \in L^{2m}_3(M, k^*).$$

Lemma 3.2. *For each $\xi \in L^{2m}_3(M, k^*)$, the following are equivalent:*

(1) ξ is a critical point of E;

(2) ξ solves equation (1.2).

Moreover if ξ solves equation (1.2), then ξ is a C^∞ function.

Proof. We have the following for each $\eta \in L^{2m}_3(M, k^*)$:

$$\left.\frac{d}{dt}\right|_{t=0} E(\xi + t\eta) = \int_M (\Delta_{g_M}\xi + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} \iota^* u^j - w, \eta)\ d\mu_{g_M}.$$

Therefore (1) and (2) are equivalent. The rest of the proof follows from the elliptic regularity theorem. □

Lemma 3.3. *For each $\xi, \eta \in L^{2m}_3(M, k^*)$ and $t \in \mathbb{R}$, the following holds:*

$$\frac{d^2}{dt^2} E(\xi + t\eta) \geq 0.$$

Moreover the following are equivalent:

(1) There exists a $t_0 \in \mathbb{R}$ such that $\left.\frac{d^2}{dt^2}\right|_{t=t_0} E(\xi + t\eta) = 0$;

(2) $\frac{d^2}{dt^2} E(\xi + t\eta) = 0$ for all $t \in \mathbb{R}$;

(3) η is a constant which is in the orthogonal complement of $\sum_{j \in J_a} \mathbb{R}\iota^* u^j$.

Proof. A direct computation shows that

$$\frac{d^2}{dt^2} E(\xi + t\eta) = \int_M \{(d\eta, d\eta) + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi + t\eta)} (\iota^* u^j, \eta)^2\}\ d\mu_{g_M}.$$

This implies the claim. □

Corollary 3.4. *Let ξ and ξ' be C^∞ solutions of equation (1.2). Then $\xi - \xi'$ is a constant which is in the orthogonal complement of $\sum_{j \in J_a} \mathbb{R}\iota^* u^j$.*

Proof. From Lemma 3.2, we have the following:

$$\left.\frac{d}{dt}\right|_{t=0} E(t\xi + (1-t)\xi') = \left.\frac{d}{dt}\right|_{t=1} E(t\xi + (1-t)\xi') = 0.$$

Then Lemma 3.3 gives the result. □

Let $W \subseteq k^*$ be the vector subspace of k^* which is generated by $(\iota^* u^j)_{j \in J_a}$. Hereafter we assume that $W = k^*$ for simplicity. We can make the assumption since if the vector subspace W is strictly smaller than the vector space k^*, then by restricting the domain of the functional E to the subspace $L^{2m}_3(M, W)$ of $L^{2m}_3(M, k^*)$ we have the same proof as in the case that $W = k^*$.

Lemma 3.5. *The functional E is bounded below. Further there exist non-negative constants C, C' and C'' such that*

$$|\xi|_{L^2} \leq (E(\xi) + C)^2 + C'E(\xi) + C''$$

for all $\xi \in L^{2m}_3(M, k^*)$.

Proof. We have the following estimate of the energy functional E:

$$\begin{aligned}
&E(\xi) \\
&= \int_M \left\{ \frac{1}{2}(d\xi, d\xi) + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} - (w, \xi) \right\} d\mu_{g_M} \\
&= \int_M \left\{ \frac{1}{2}(d\xi, d\xi) - (w, \xi - \bar{\xi}) \right\} d\mu_{g_M} + \sum_{j=1}^{d} \int_M a_j e^{(\iota^* u^j, \xi)} \, d\mu_{g_M} - (\bar{w}, \bar{\xi}) \\
&= \int_M \left\{ \frac{1}{2}(d\xi, d\xi) - (w, \xi - \bar{\xi}) \right\} d\mu_{g_M} + \sum_{j \in J_a} \int_M e^{\log a_j} e^{(\iota^* u^j, \xi)} \, d\mu_{g_M} - (\bar{w}, \bar{\xi}) \\
&\geq \int_M \left\{ \frac{1}{2}(d\xi, d\xi) - (w, \xi - \bar{\xi}) \right\} d\mu_{g_M} + \sum_{j \in J_a} (e^{\int_M \log a_j \ d\mu_{g_M}}) e^{(\iota^* u^j, \bar{\xi})} - (\bar{w}, \bar{\xi})
\end{aligned}$$

for all $\xi \in L^{2m}_3(M, k^*)$, and where the final inequality follows from the Jensen's inequality. Since we have (2), there exist positive numbers $(s_j)_{j \in J_a}$ such that $\bar{w} = \sum_{j \in J_a} s_j \iota^* u^j$. Combining this with the above estimate, we have the following:

$$\begin{aligned}
&E(\xi) \\
&\geq \int_M \left\{ \frac{1}{2}(d\xi, d\xi) - (w, \xi - \bar{\xi}) \right\} d\mu_{g_M} + \sum_{j \in J_a} \left\{ s'_j e^{(\iota^* u^j, \bar{\xi})} - s_j (\iota^* u^j, \bar{\xi}) \right\},
\end{aligned}$$

where we denote by s'_j the coefficient $e^{\int_M \log a_j \ d\mu_{g_M}}$ for each $j \in J_a$. We set E_0 and E_1 as follows:

$$E_0(\xi) \coloneqq \int_M \left\{ \frac{1}{2}(d\xi, d\xi) - (w, \xi - \bar{\xi}) \right\} d\mu_{g_M}$$
$$E_1(\xi) \coloneqq \sum_{j \in J_a} \left\{ s'_j e^{(\iota^* u^j, \bar{\xi})} - s_j(\iota^* u^j, \bar{\xi}) \right\}.$$

We note that E_1 depends only on the average of ξ. Then the Poincaré inequality implies that E_0 is bounded below. We see that E_1 is also bounded below since the following f is bounded below for any $s, s' \in \mathbb{R}_{>0}$:

$$f(y) \coloneqq s'e^y - sy \ \text{ for } y \in \mathbb{R}.$$

Therefore E is bounded below. Further we have the following: for each $\bar{\xi} \neq 0$, we see $\lim_{t\to\infty} E_1(t\bar{\xi}) - |t\bar{\xi}|^{1/2} = \infty$. This implies that $E_1(\xi) - |\bar{\xi}|^{1/2}$ attains a minimum since we have the following:

$$\min_{\xi \in L^{2m}_3(M,k^*)} \{E_1(\xi) - |\bar{\xi}|^{1/2}\} = \min_{|\bar{\xi}|=1} \min_{t \in \mathbb{R}} \{E_1(t\bar{\xi}) - |t\bar{\xi}|^{1/2}\}.$$

In particular, $E_1(\xi) - |\bar{\xi}|^{1/2}$ is bounded below. Therefore there exists a constant C such that

$$|\bar{\xi}| \leq (E(\xi) + C)^2$$

for all $\xi \in L^{2m}_3(M, k^*)$. We also obtain the following estimate for some C' and C'' from the Poincaré inequality:

$$|\xi - \bar{\xi}|_{L^2} \leq C'E(\xi) + C''.$$

Then we have

$$|\xi|_{L^2} \leq |\bar{\xi}| + |\xi - \bar{\xi}|_{L^2} \leq (E(\xi) + C)^2 + C'E(\xi) + C''$$

and this implies the claim. □

Note that the following method was originally developed by Bradlow [Br2]:

Definition 3.6. Let $B > 0$ a positive real number. We define a subset $L_3^{2m}(M, k^*)_B$ of $L_3^{2m}(M, k^*)$ by

$$L_3^{2m}(M, k^*)_B \coloneqq \{\xi \in L_3^{2m}(M, k^*) \mid |\Delta_{g_M}\xi + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} \iota^* u^j - w|^{2m}_{L_1^{2m}} \leq B\}.$$

Then we have the following Lemma 3.7. For the proof of Lemma 3.7, we refer the reader to [Br2, Lemma 3.4.2] and [AG, Proposition 3.1].

Lemma 3.7. *If $E|_{L_3^{2m}(M,k^*)_B}$ attains a minimum at $\xi_0 \in L_3^{2m}(M, k^*)_B$, then ξ_0 is a critical point of E.*

Proof. We define a map $F : L_3^{2m}(M, k^*) \to L_1^{2m}(M, k^*)$ as $F(\xi) = \Delta_{g_M}\xi + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} \iota^* u^j - w$ for $\xi \in L_3^{2m}(M, k^*)$. Then its linearization at ξ is the following:

$$\begin{aligned} (DF)_\xi : L_3^{2m}(M, k^*) &\longrightarrow L_1^{2m}(M, k^*) \\ \eta &\longmapsto \Delta_{g_M}\eta + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} (\iota^* u^j, \eta) \iota^* u^j. \end{aligned}$$

The linearization $(DF)_\xi$ satisfies the following for each $\eta, \eta' \in L_3^{2m}(M, k^*)$:

$$((DF)_\xi(\eta), \eta')_{L^2} = \int_M \left\{ (d\eta, d\eta') + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} (\iota^* u^j, \eta)(\iota^* u^j, \eta') \right\} d\mu_{g_M}, \tag{3.1}$$

where we denote by $(\cdot, \cdot)_{L^2}$ the L^2-inner product. (3.1) says that the linearization $(DF)_\xi$ is a self-adjoint operator with respect to the L^2-inner product. We set $\eta = \eta'$. Then we have

$$((DF)_\xi(\eta), \eta)_{L^2} = \int_M \left\{ (d\eta, d\eta) + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} (\iota^* u^j, \eta)^2 \right\} d\mu_{g_M}. \tag{3.2}$$

From (3.2) we see that the linearization $(DF)_\xi$ is injective since if $\eta \in L_3^{2m}(M, k^*)$ satisfies $(DF)_\xi(\eta) = 0$, then the right hand side of (3.2) must be 0 and thus we have $\eta = 0$. It should be noted that we have used here the assumption that $(\iota^* u^j)_{j \in J_a}$ generates k^*. Furthermore, we see that $(DF)_\xi$

is bijective since it is a formally self-adjoint elliptic operator. Assume that $E|_{L_3^{2m}(M,k^*)_B}$ attains a minimum at $\xi_0 \in L_3^{2m}(M,k^*)_B$. Since $(DF)_{\xi_0}$ is bijective, there uniquely exists an $\eta \in L_3^{2m}(M,k^*)$ such that

$$(DF)_{\xi_0}(\eta) = -F(\xi_0).$$

Assume that ξ_0 is not a critical point of E. Then we have $\eta \neq 0$. Let ξ_t denotes a real line $\xi_0 + t\eta$ parametrized by $t \in \mathbb{R}$. We then have the following:

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} E(\xi_t) &= (F(\xi_0), \eta)_{L^2} \\ &= -((DF)_{\xi_0}(\eta), \eta) \\ &= -\int_M \left\{ (d\eta, d\eta) + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} (\iota^* u^j, \eta)^2 \right\} d\mu_{g_M} < 0.\end{aligned}$$

From this, we see that for a sufficiently small $\epsilon > 0$, $E(\xi_t)$ strictly decreases with increasing $t \in (-\epsilon, \epsilon)$. We also have the following:

$$\left.\frac{d}{dt}\right|_{t=0} F(\xi_t) = (DF)_{\xi_0}(\eta) = -F(\xi_0).$$

This implies that around 0, $|F(\xi_t)|_{L_1^{2m}}^{2m}$ decreases with increasing t :

$$\begin{aligned}&\left.\frac{d}{dt}\right|_{t=0} |F(\xi_t)|_{L_1^{2m}}^{2m} \\ =& \left.\frac{d}{dt}\right|_{t=0} \int_M |dF(\xi_t)|^{2m} \, d\mu_{g_M} + \left.\frac{d}{dt}\right|_{t=0} \int_M |F(\xi_t)|^{2m} \, d\mu_{g_M} \\ =& -2m \int_M |dF(\xi_0)|^{2m} \, d\mu_{g_M} - 2m \int_M |F(\xi_0)|^{2m} \, d\mu_{g_M} < 0.\end{aligned}$$

Further this implies that for a sufficiently small $t > 0$, ξ_t satisfies the following:

$$\begin{aligned}E(\xi_t) &< E(\xi_0), \\ |F(\xi_t)|_{L_1^{2m}}^{2m} &\leq B.\end{aligned}$$

However, this contradicts the assumption that $E|_{L_3^{2m}(M,k^*)_B}$ attains a minimum at $\xi_0 \in L_3^{2m}(M,k^*)_B$. Hence ξ_0 is a critical point of E. $\square$

Therefore the problem reduces to show that $E|_{L_3^{2m}(M,k^*)_B}$ attains a minimum. To see this, we prove the following Lemma 3.8:

Lemma 3.8. *Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of $L_3^{2m}(M,k^*)_B$ such that*

$$\lim_{i\to\infty} E(\xi_i) = \inf_{\eta\in L_3^{2m}(M,k^*)_B} E(\eta).$$

Then we have $\sup_{i\in\mathbb{N}} |\xi_i|_{L_3^{2m}} < \infty$.

Before the proof of Lemma 3.8, we recall the following:

Lemma 3.9. ([LT, pp.72-73]) *Let $f \in C^2(M,\mathbb{R})$ be a non-negative function. If*

$$\Delta_{g_M} f \le C_0 f + C_1$$

holds for some $C_0 \in \mathbb{R}_{\ge 0}$ and $C_1 \in \mathbb{R}$, then there is a positive constant C_2, depending only on g_M and C_0, such that

$$\max_{x\in M} f(x) \le C_2(|f|_{L^1} + C_1).$$

Proof of Lemma 3.8. We first note that the functional space $L_3^{2m}(M,k^*)$ is contained in $C^2(M,k^*)$. We then have the following for each $i \in \mathbb{N}$:

$$\begin{aligned}
\frac{1}{2}\Delta_{g_M}|\xi_i|^2 &= (\Delta_{g_M}\xi_i, \xi_i) - |d\xi_i|^2 \\
&\le (\Delta_{g_M}\xi_i + \sum_{j=1}^{d} a_j \iota^* u^j - w, \xi_i) - (\sum_{j=1}^{d} a_j \iota^* u^j - w, \xi_i) \\
&\le (\Delta_{g_M}\xi_i + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi_i)} \iota^* u^j - w, \xi_i) - (\sum_{j=1}^{d} a_j \iota^* u^j - w, \xi_i),
\end{aligned} \tag{3.3}$$

where we have used the following inequality:

$$y \le ye^y \quad \text{for any } y \in \mathbb{R}.$$

From (3.3), we have

$$\begin{aligned}
&\frac{1}{2}\Delta_{g_M}|\xi_i|^2 \\
&\le |\Delta_{g_M}\xi_i + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi_i)} \iota^* u^j - w||\xi_i| + |\sum_{j=1}^{d} a_j \iota^* u^j - w||\xi_i| \\
&\le C_3|\xi_i|,
\end{aligned} \tag{3.4}$$

for a constant C_3, since we have $L_1^{2m}(M,k^*) \subseteq C^0(M,k^*)$ and the following:

$$|\Delta_{g_M}\xi_i + \sum_{j=1}^d a_j e^{(\iota^* u^j, \xi_i)} \iota^* u^j - w|_{L_1^{2m}}^{2m} \leq B.$$

From (3.4) and an inequality $|\xi_i| \leq \frac{1}{2}(|\xi_i|^2 + 1)$ we have the following:

$$\Delta_{g_M}|\xi_i|^2 \leq C_3|\xi_i|^2 + C_3.$$

Then from Lemma 3.9, there exists a constant C_4 such that

$$\max_{x\in M}\{|\xi_i|^2(x)\} \leq C_4(||\xi_i|^2|_{L^1} + C_3) = C_4(|\xi_i|_{L^2}^2 + C_3).$$

Combining Lemma 3.5, we see that there exists a constant C_5 which does not depend on i such that

$$\max_{x\in M}\{|\xi_i|^2(x)\} \leq C_5. \tag{3.5}$$

From (3.5) and the L^p-estimate we have

$$\begin{aligned}
|\xi_i|_{L_2^{2m}} &\leq C_6|\Delta_{g_M}\xi_i|_{L^{2m}} + C_7|\xi_i|_{L^1} \\
&\leq C_6|\Delta_{g_M}\xi_i + \sum_{j=1}^d a_j e^{(\iota^* u^j, \xi_i)} \iota^* u^j - w|_{L^{2m}} \\
&+ C_6|\sum_{j=1}^d a_j e^{(\iota^* u^j, \xi_i)} \iota^* u^j - w|_{L^{2m}} + C_8 \\
&\leq C_9
\end{aligned} \tag{3.6}$$

for some constants C_6, C_7, C_8 and C_9. Then we have the result since we can repeat the same argument for $|\xi_i|_{L_3^{2m}}$ as in the proof of inequality (3.6). $\square$

Corollary 3.10. *The functional E has a critical point.*

Proof. We take a sequence $(\xi_i)_{i\in\mathbb{N}}$ so that

$$\lim_{i\to\infty} E(\xi_i) = \inf_{\eta\in L_3^{2m}(M,k^*)_B} E(\eta).$$

Then by Lemma 3.8 we have $\sup_{i\in\mathbb{N}} |\xi_i|_{L_3^{2m}} < \infty$, and this implies that there exits a subsequence $(\xi_{i_j})_{j\in\mathbb{N}}$ such that $(\xi_{i_j})_{j\in\mathbb{N}}$ weakly converges a

$\xi_\infty \in L_3^{2m}(M, k^*)_B$. Since the functional E is continuous with respect to the weak topology, we have

$$E(\xi_\infty) = \inf_{\eta \in L_3^{2m}(M,k^*)_B} E(\eta). \tag{3.7}$$

(3.7) says that $E|_{L_3^{2m}(M,k^*)_B}$ attains a minimum at ξ_∞ and thus from Lemma 3.7 we have the result. $\square$

4 Harmonic bundles with diagonal pluriharmonic metrics

Let $(E, \Phi) \to X$ be a Higgs bundle over a complex manifold and suppose that the holomorphic vector bundle E decomposes as $E = L_1 \oplus \cdots \oplus L_r$ with holomorphic line bundles $L_1, \ldots, L_r \to X$. We say that a Hermitian metric h on E is a diagonal pluriharmonic metric with respect to the decomposition if h is a pluriharmonic metric of (E, Φ) and if $E = L_1 \oplus \cdots \oplus L_r$ is an orthogonal decomposition with respect to the metric h. As is described in Theorem 2.5, the harmonic metric associated with a cyclic Higgs bundle is a diagonal metric. In this section, we give the necessary and sufficient condition for the existence of a diagonal pluriharmonic metric of a general G-Higgs bundle over a compact Kähler manifold by applying Theorem 1.1 to the Hermitian-Einstein equation for G-Higgs bundles. We also show that the pluriharmonic map which is associated with a diagonal pluriharmonic metric lifts to a pluriharmonic map to the homogeneous space G/T, where T is a maximal compact torus of G.

Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. We fix a maximal compact subgroup K_0 of G. We denote by $\mathfrak{k}$ the Lie algebra of K_0. We define a $\mathbb{C}$-antilinear involution $\sigma_{K_0} : \mathfrak{g} \to \mathfrak{g}$ as $\sigma_{K_0}(u + \sqrt{-1}v) \coloneqq u - \sqrt{-1}v$ for $u, v \in \mathfrak{k}$. Let T be a maximal real torus of K_0 with Lie algebra $\mathfrak{t}$. Let $H \subseteq G$ be the complexification of T. We denote by $\mathfrak{h} = \mathfrak{t} \oplus \sqrt{-1}\mathfrak{t}$ the Lie algebra of H. Let $\Delta \subseteq \mathfrak{h}^*$ be the root system and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ the root space decomposition. We denote by $B(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the Killing form and by $B^*(\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ the dual of $B|_{\mathfrak{h}}$. For each $\alpha \in \Delta$, we define the coroot h_α of α as $h_\alpha \coloneqq 2\alpha^*/B^*(\alpha, \alpha)$, where we denote by α^* the dual of α with respect to B. Then there exists a basis $(e_\alpha)_{\alpha \in \Delta}$ of $\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ such that

$$\begin{aligned} &e_\alpha \in \mathfrak{g}_\alpha, \\ &[e_\alpha, e_{-\alpha}] = h_\alpha, \\ &\sigma_{K_0}(e_\alpha) = -e_{-\alpha}. \end{aligned}$$

Let $(P_G, \Phi) \to (X, \omega_X)$ be a G-Higgs bundle over a compact connected Kähler manifold (X, ω_X). Suppose that there exists a holomorphic H-subbundle $P_H \subseteq P_G$. Then the adjoint bundle $\mathrm{ad}(P_G)$ decomposes as $\mathrm{ad}(P_G) = (P_H \times_H \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Delta} (P_H \times_H \mathfrak{g}_\alpha)$. We note that $P_H \times_H \mathfrak{h}$ is a trivial bundle whose fiber is $\mathfrak{h}$. We denote by $\Phi = \Phi_0 + \sum_{\alpha \in \Delta} \Phi_\alpha e_\alpha$ the corresponding decomposition of the Higgs field Φ. In the following, we implicitly assume that all of the involutions of $\mathrm{ad}(P_G)$ satisfy (2.2) of Section 2 for each point of X. For each involution σ, we define a connection D as (2.3). We fix an involution σ_0 which preserves $P_H \times_H \mathfrak{h}$. Note that an involution σ_0 preserves $P_H \times_H \mathfrak{h}$ if and only if for each $x \in X$, $(\sigma_0)_x$ is conjugate to σ_{K_0} with respect to an element of H:

$$(\sigma_0)_x = \mathrm{Ad}_{g^{-1}} \circ \sigma_{K_0} \circ \mathrm{Ad}_g \text{ for a } g \in H, \tag{4.1}$$

where we have used a trivialization of P_H. Let $\Lambda_{\omega_X} : \Omega^{p,q}(X) \to \Omega^{p-1,q-1}(X)$ be the adjoint operator of $\omega_X \wedge$. Then the following holds:

Theorem 4.1. *There exists an involution σ preserving $P_H \times_H \mathfrak{h}$ such that σ solves the Hermitian-Einstein equation $\Lambda_{\omega_X} F_D = 0$ if and only if*

(i) Φ satisfies the following:

$$\Lambda_{\omega_X}[\Phi \wedge (-\sigma_0)(\Phi)] \in P_H \times_H \mathfrak{h}; \tag{4.2}$$

(ii) The following holds:

$$-\gamma \in \sum_{\alpha \in \Delta, \Phi_\alpha \neq 0} \mathbb{R}_{>0} h_\alpha, \tag{4.3}$$

where γ is defined to be

$$\gamma = \frac{\sqrt{-1}}{2\pi} \int_X F_{\nabla^{\sigma_0}} \wedge \frac{\omega_X^{n-1}}{(n-1)!}. \tag{4.4}$$

In particular, a G-Higgs bundle satisfying (i) and (ii) is polystable.

Theorem 4.2. *There exists an involution σ preserving $P_H \times_H \mathfrak{h}$ such that (P_G, Φ, σ) is a G-harmonic bundle if and only if (P_G, Φ) satisfies (i) and (ii) of Theorem 4.1 and $c_2(P_G \times_\rho W) = 0$ for a faithful representation $\rho : G \to$* $\mathrm{GL}(W)$.

Theorem 4.3. *Let (P_G, Φ, σ) be a G-harmonic bundle such that σ preserves $P_H \times_H \mathfrak{h}$. Then the pluriharmonic map $\hat{\sigma} : \tilde{X} \to G/K$ associated with σ is a composition of a pluriharmonic map $\hat{\sigma}_T : \tilde{X} \to G/T$ and the projection $\pi : G/T \to G/K$, where we denote by $\tilde{X}$ the universal covering space of X.*

Remark 4.4. Condition (4.2) and the right hand side of (4.4) does not depend on the choice of σ_0 which preserves $P_H \times_H \mathfrak{h}$.

Remark 4.5. The author hopes that we can solve the Hermitian-Einstein equation using the method of the proof of Theorem 1.1 even if we drop the assumption of (4.2) and only assume that the holomorphic vector bundle is a direct sum of holomorphic line bundles. We will show that a stable Higgs bundle whose holomorphic vector bundle is a direct sum of holomorphic line bundles satisfies (4.3) in Appendix B.

Remark 4.6. For the case where $G = \mathrm{SL}(r, \mathbb{C})$, Theorem 4.1 could be derived from [AG]. In this case, condition (4.3) should coincides with the stability of quiver bundles. Also, as is mentioned in Remark 1.4, the same condition (4.3) is obtained in [Bap] as the stability condition of the Abelian GLSM.

Remark 4.7. Theorem 4.3 is proved in [Bar2] for cyclic Higgs bundles on a Riemann surface. See [Bar2, Proposition 2.3.1].

Example 4.8. Consider a cyclic Higgs bundle $(P_G, \Phi(q)) \to X$ over a compact connected Riemann surface X (see Definition 2.4). The Higgs field $\Phi(q)$ is the following:

$$\Phi(q) = \sum_{i=1}^{l} r_i e_{-\alpha_i} + q_l e_l.$$

One can check that $\Phi(q)$ satisfies (4.2). Also, the vector γ is given as

$$\gamma = (g-1)x,$$

where we denote by g the genus of X and x is an element of $\mathfrak{h}$ defined as (2.4). Then $(P_G, \Phi(q))$ satisfies (4.3) if $q_l \neq 0$ since the following holds:

$$\sum_{\alpha \in \{-\alpha_1, \ldots, -\alpha_l, \delta\}} \mathbb{R}_{>0} h_\alpha = \sqrt{-1}\mathfrak{t}. \tag{4.5}$$

The condition (4.3) is satisfied also in the case of $q_l = 0$ since x lies in $\sum_{i=1}^{l} \mathbb{R}_{>0} h_{\alpha_i}$. Note that for a G-Higgs bundle $(P_G, \Phi) \to (X, \omega_X)$ over a compact connected Kähler manifold (X, ω_X), if the Higgs field is of the following form

$$\Phi = \sum_{\alpha \in \{-\alpha_1, \dots, -\alpha_l, \delta\}} \Phi_\alpha e_\alpha$$

and if $\Phi_\alpha \neq 0$ for all $\alpha \in \{-\alpha_1, \dots, -\alpha_l, \delta\}$, then the condition (4.3) is automatically satisfied since we have (4.5).

We then prove Theorem 4.1, Theorem 4.2 and Theorem 4.3. We note that Theorem 4.2 is an immediate consequence of Theorem 4.1 and [Sim1, Proposition 3.4]. So it is enough to prove Theorem 4.1 and Theorem 4.3.

We first prove Theorem 4.1. We can easily check that condition (4.2) is a necessary condition for the existence of an involution σ preserving $P_H \times_H \mathfrak{h}$ which solves the Hermitian-Einstein equation. We suppose (4.2). Let σ be an involution which preserves $P_H \times_H \mathfrak{h}$. Then there exists a smooth function $\Omega : X \to \sqrt{-1}\mathfrak{t}$ such that $\sigma = \mathrm{Ad}_{\mathrm{Exp}(\Omega)} \circ \sigma_0 \circ \mathrm{Ad}_{\mathrm{Exp}(-\Omega)} = \mathrm{Ad}_{\mathrm{Exp}(2\Omega)} \circ \sigma_0$. From the assumption of (4.2), the Hermitian-Einstein equation for σ is the following:

$$\Delta_{\omega_X} \Omega + \sum_{\alpha \in \Delta} |\Phi_\alpha|^2_{\omega_X} e^{2\alpha(\Omega)} h_\alpha = -\sqrt{-1}\Lambda_{\omega_X} F_{\nabla^{\sigma_0}}, \tag{4.6}$$

where we denote by $F_{\nabla^{\sigma_0}}$ the curvature of the Chern connection of σ_0 and by Δ_{ω_X} the geometric Laplacian. We apply Theorem 1.1 to the equation (4.6). In order to apply Theorem 1.1 to the equation (4.6), we must check that for each $\alpha \in \Delta$, if Φ_α is not a zero section of $(P_H \times_H \mathfrak{g}_\alpha) \otimes \bigwedge^{1,0}$, then $\log |\Phi_\alpha|^2_{\omega_X}$ is integrable. This is an immediate consequence of the following lemma:

Lemma 4.9. *Let $U \subseteq \mathbb{C}^n$ be a domain and $f : U \to V$ a holomorphic section of a trivial bundle $V = U \times \mathbb{C}^r \to U$. Then for any smooth Hermitian metric h_V on V, $\log |f|_{h_V} \in L^1_{loc}(U)$.*

Proof. Let $e_1, \dots, e_r : U \to V$ be a holomorphic frame of V. Then f is denoted as $f = f_1 e_1 + \cdots + f_r e_r$. We denote by $\hat{h}_V : U \to \mathrm{Herm}(r)$ the Hermitian matrix valued smooth function whose (i,j) component is $h_V(e_i, e_j)$. Then $\hat{h}_V$ is diagonalized as ${}^t\bar{B}\hat{h}_V B = \mathrm{diag}(\lambda_1, \dots, \lambda_r)$ for a unitary matrix valued function B and positive functions $\lambda_1, \dots, \lambda_r$ over U, where we denote by $\mathrm{diag}(\lambda_1, \dots, \lambda_r)$ the diagonal matrix whose diagonal entries are $\lambda_1, \dots, \lambda_r$. We set $(f_1', \dots, f_r') \coloneqq (f_1, \dots, f_r)B$. Let $F \subseteq U$ be a compact subset of U. We define a positive constant C as $C \coloneqq \min_{1\leq i\leq r}\{\min_{x\in F}\lambda_1(x),\dots,\min_{x\in F}\lambda_r(x)\}$. Then we have

$$\begin{aligned}\log|f|^2_{h_V} &= \log\{\lambda_1|f_1'|^2+\cdots+\lambda_r|f_r'|^2\}\\ &\geq \log\{C|f_1'|^2+\cdots+C|f_r'|^2\}\\ &= \log C+\log\{|f_1'|^2+\cdots+|f_r'|^2\}\\ &= \log C+\log\{|f_1|^2+\cdots+|f_r|^2\}\end{aligned}$$

for each point of F. Since $f_1,\dots,f_r$ are holomorphic functions, $\log\{|f_1|^2+\cdots+|f_r|^2\}$ is a plurisubharmonic function (see [Dem]). In particular, it is in $L^1_{loc}(U)$. This implies the claim. □

Then we prove Theorem 4.1:

Proof of Theorem 4.1. As already remarked, under the assumption of (i), the Hermitian-Einstein equation for an involution σ which preserves $P_H\times_H\mathfrak{h}$ is equation (4.6). Then from Theorem 1.1, equation (4.6) has a smooth solution if and only if (ii) holds. This implies the claim. □

We next prove Theorem 4.3. Note that for a G-harmonic bundle (P_G,Φ,σ), if σ preserves $P_H\times_H\mathfrak{h}$, then the pluriharmonic map $\hat{\sigma}:\tilde{X}\to G/K$ naturally lifts to a map $\hat{\sigma}_T:\tilde{X}\to G/T$ and we prove that this $\hat{\sigma}_T$ is a pluriharmonic map. In the following, we suppose that $G=\mathrm{SL}(r,\mathbb{C})$ for simplicity. Let $T\subseteq\mathrm{SU}(r)$ be the maximal compact torus which consists of all diagonal matrixes of $\mathrm{SU}(r)$. Then the following holds:

Lemma 4.10. *Let $(E,D,h)\to(M,g_M)$ be a harmonic bundle over a Riemannian manifold, where D is a flat connection of a complex vector bundle E and h is a harmonic metric. Suppose that the vector bundle E decomposes as $E=L_1\oplus\cdots\oplus L_r$ with complex line bundles $L_1,\dots,L_r\to M$. Suppose also that h is a diagonal metric with respect to the decomposition and the unitary part ∇^h of the connection D preserves the decomposition of E. Then the natural map $\hat{h}_T:\tilde{M}\to\mathrm{SL}(r,\mathbb{C})/T$ associated with h is a harmonic map, where $\tilde{M}$ is the universal covering space.*

Proof. For each point p of $\tilde{M}$, there exists an open neighborhood V of $\hat{h}_T(p)$ such that there exists an isometric local section $s:V\to\mathrm{SL}(r,\mathbb{C})$ of the projection $\mathrm{SL}(r,\mathbb{C})\to\mathrm{SL}(r,\mathbb{C})/T$. We take an open neighborhood U of p small enough so that $\hat{h}_T(U)\subseteq V$ and we set $\Psi_U\coloneqq(s\circ\hat{h}_T)^{-1}d(s\circ\hat{h}_T)$. We may assume that the diagonal part of Ψ_U vanishes. Then we need to check that the following holds:

$$d^*\Psi_U + g_M([{}^t\overline{\Psi_U} \otimes \Psi_U]) = 0.$$

From the assumption that ∇^h preserves the decomposition of E, we have ${}^t\overline{\Psi_U} = \Psi_U$ and thus $g_M([{}^t\overline{\Psi_U} \otimes \Psi_U]) = 0$. We also see that $d^*\Psi_U = 0$ since h is a harmonic metric. This implies the claim. $\square$

Lemma 4.10 immediately implies Theorem 4.3:

Proof of Theorem 4.3. Let $(E, \Phi, h) \to X$ be a harmonic bundle and suppose that E decomposes as $E = L_1 \oplus \cdots \oplus L_r$ with holomorphic line bundles $L_1, \ldots, L_r \to X$ and suppose also that h is a diagonal pluriharmonic metric with respect to the decomposition. Then the Chern connection ∇^h preserves the decomposition of E and thus for any complex 1-dimensional submanifold $C \subseteq X$, the harmonic bundle over C which is a pull-back of (E, Φ, h) satisfies the assumptions of Lemma 4.10. This implies the claim. $\square$

5 Generalized Kazdan-Warner equations on foliated manifolds

In this section, we extend Theorem 1.1 on foliated manifolds. We refer the reader to [BG, Mol] for basic facts about foliations. Let $(M, \mathcal{F})$ be a connected foliated manifold. We denote by $T\mathcal{F} \subseteq TM$ the integrable distribution associated with the foliation. A differential p-form ϕ is said to be *basic* if ϕ satisfies the following for all $X \in \Gamma(T\mathcal{F})$:

$$i_X \phi = i_X d\phi = 0,$$

where we denote by i_X the interior product. Note that a function f is basic if and only if $Xf = 0$ for all $X \in \Gamma(T\mathcal{F})$. Let $\Omega^p_B(M)$ the space of smooth basic p-forms. The exterior derivative d preserves the space of basic forms:

$$d : \Omega^p_B(M) \to \Omega^{p+1}_B(M).$$

Let g_M be a Riemannian metric on M. We denote by $d^*_B : \Omega^{p+1}_B(M) \to \Omega^p_B(M)$ the L^2-adjoint of $d : \Omega^p_B(M) \to \Omega^{p+1}_B(M)$. We define the *basic Laplacian* $\Delta_B : \Omega^p_B(M) \to \Omega^p_B(M)$ as

$$\Delta_B \coloneqq d^*_B d + d d^*_B.$$

We consider the equation (1.2) on a foliated manifold $(M, \mathcal{F})$. On a foliated manifold $(M, \mathcal{F})$, the following holds:

Theorem 5.1. *Suppose that $a_1, \ldots, a_d$ and w satisfy $(*)$ of Section 1 and their domain M is a compact connected foliated manifold. Suppose also that $a_1, \ldots, a_d$ and w are all basic with respect to the foliation and the Laplacian Δ_{g_M} preserves $\Omega^0_B(M)$. Then the following are equivalent:*

(i) The generalized Kazdan-Warner equation (1.2) has a C^∞-solution ξ.

(ii) The given functions $a_1, \ldots, a_d$ and w satisfy

$$\int_M w \; d\mu_{g_M} \in \sum_{j \in J_a} \mathbb{R}_{>0} \iota^* u^j,$$

where J_a denotes $\{j \in \{1, \ldots, d\} \mid a_j$ is not identically 0$\}$.

(iii) The generalized Kazdan-Warner equation (1.2) has a basic C^∞-solution ξ.

(iv) There exists a basic C^∞-function $\xi : M \to k^*$ which satisfies the following:

$$\Delta_B \xi + \sum_{j=1}^{d} a_j e^{(\iota^* u^j, \xi)} \iota^* u^j = w. \tag{5.1}$$

Moreover if ξ and ξ' are C^∞-solutions of equation (1.2), then $\xi - \xi'$ is a constant which lies in the orthogonal complement of $\sum_{j \in J_a} \mathbb{R}\iota^* u^j$.

Proof. From Theorem 1.1, we see that (i) and (ii) are equivalent and the solution of (1.2) is unique up to a constant which lies in $(\sum_{j\in J_a} \mathbb{R}\iota^* u^j)^\perp$. Clearly, (iii) implies (i). We also see that (iii) and (iv) are equivalent since the Laplacian Δ_{g_M} preserves $\Omega_B^0(M)$ if and only if the following holds for all $f \in \Omega_B^0(M)$:

$$\Delta_{g_M} f = \Delta_B f.$$

Therefore it is enough to show that (ii) implies (iii). We show this by using the variational method. We define a subspace $L^{2m}_{3,b}(M, k^*)$ of $L^{2m}_3(M, k^*)$ as

$$L^{2m}_{3,b}(M, k^*) := \{\xi \in L^{2m}_3(M, k^*) \mid \xi \text{ is a basic function.}\}.$$

Let E be the energy functional defined in Definition 3.1. We define a functional E_b as

$$E_b := E|_{L^{2m}_{3,b}(M,k^*)}.$$

Then we see that E_b has a critical point if and only if there exists a smooth basic solution ξ of (1.2) from the same argument as the proof of Lemma 3.2 and Lemma 3.3. Note that we have used here the assumption that $a_1, \dots, a_d$ and w are all basic and Δ_{g_M} preserves $\Omega_B^0(M)$. We can show that E_b has a critical point under the assumption of (ii) by the same argument as the proof of Theorem 1.1 and thus we see that (ii) implies (iii). $\square$

Remark 5.2. If g_M is a bundle-like metric, then the Laplacian Δ_{g_M} preserves $\Omega_B^0(M)$.

Example 5.3. We show that the transverse Hermitian-Einstein equation for a diagonal harmonic metric on a basic Higgs bundle over a Sasakian manifold is an example of (5.1). Let (M, η, g_η) be a compact connected Sasakian manifold with contact form η and the Webster metric g_η (see [BG, BK1] for Sasakian manifolds). The Sasakian manifold M has a rank 1 foliation defined by the Reeb vector field of η. Let $(E, \Phi) \to M$ be a basic Higgs bundle over M (see [BK1] for the definition of the basic Higgs bundle). We assume that the basic first Chern class $c_{1,B}(E)$ of E vanishes for simplicity. Suppose that E decomposes as $E = L_1 \oplus \cdots \oplus L_r$ with transversely holomorphic line bundles $L_1, \ldots, L_r \to M$. Then the basic Higgs field Φ decomposes as $\Phi = \Phi_0 + \sum_{i,j=1,\ldots,r} \Phi_{i,j}$, where $\Phi_{i,j}$ is a basic holomorphic $(1,0)$-form with values in $L_j^{-1} L_i$ and Φ_0 is the diagonal part. We assume that for each $j = 1, \ldots, r$ there exists a basic Hermitian metric h_j on L_j. We denote by ∇^h the Chern connection of the basic Hermitian metric $h \coloneqq (h_1, \ldots, h_r)$. Let Λ be the adjoint operator of $d\eta\wedge$. We suppose that off-diagonal part of $\Lambda[\Phi \wedge \Phi^{*h}]$ vanishes, where we denote by Φ^{*h} the adjoint of Φ with respect to the metric h. We take $\mathbb{R}$-valued basic functions $f_1, \ldots, f_r$ satisfying $f_1 + \cdots + f_r = 0$ and we set $\xi \coloneqq (f_1, \ldots, f_r)$. Let h' be a basic Hermitian metric on E defined as $h' \coloneqq (e^{f_1} h_1, \ldots, e^{f_r} h_r)$. We also define $\alpha_{i,j} \in (\mathbb{R}^r)^*$ and $h_{\alpha_{i,j}} \in \mathbb{R}^r$ $(i, j = 1, \ldots, r)$ as follows:

$$\alpha_{i,j}(v) \coloneqq v_i - v_j \text{ for } v = (v_1, \ldots, v_r) \in \mathbb{R}^r,$$
$$h_{\alpha_{i,j}} \coloneqq e_i - e_j,$$

where we denote by $e_1, \ldots, e_r$ the canonical basis of $\mathbb{R}^r$. Then the transverse Hermitian-Einstein equation for h' is the following:

$$\Delta_B \xi + \sum_{i,j=1,\ldots,r} |\Phi_{i,j}|^2 e^{2\alpha_{i,j}(\xi)} h_{\alpha_{i,j}} = -\sqrt{-1} \Lambda F_{\nabla^h}, \tag{5.2}$$

where we denote by F_{∇^h} the curvature of the Chern connection ∇^h and by Δ_B the basic Laplacian with respect to the Webster metric g_η. Note that the functions $|\Phi_{i,j}|$ $(i, j = 1, \ldots, r)$ and $-\sqrt{-1}\Lambda F_{\nabla^h}$ are all basic. We also see that if $\Phi_{i,j} \neq 0$, then $\log |\Phi_{i,j}|^2$ is integrable by the same argument as Lemma 4.9. Therefore from Theorem 5.1, we see that equation (5.2) has a solution if and only if the following holds:

$$-\gamma \in \sum_{\substack{i,j=1,\ldots,r, \\ \Phi_{i,j} \neq 0}} \mathbb{R}_{>0} h_{\alpha_{i,j}},$$

where γ is defined to be

$$\gamma \coloneqq \int_M \sqrt{-1}\Lambda F_{\nabla^h} d\mu_{g_M}.$$

Combining the result of [BK1], we see that the same claim as Theorem 4.1 and Theorem 4.2 also holds for basic Higgs bundles over compact Sasakian manifolds. Note that the above construction could be extended for basic G-Higgs bundles over compact Sasakian manifolds.

A Geometric invariant theory and the moment maps for linear torus actions

We give a brief review of the relationship between the geometric invariant theory and the moment maps for linear torus actions. In particular, we clarify the relationship between the condition (2) of Theorem 1.1 and the stability condition of the geometric invariant theory for torus actions. General references for this section are [Dol, Kin, Kir, KN, MFK, Nak, New].

Let K be a closed connected subtorus of a real torus $T^d := \mathrm{U}(1)^d$ with the Lie algebra $k \subseteq t^d$. We denote by $\iota^* : (t^d)^* \to k^*$ the dual map of the inclusion map $\iota : k \to t^d$. Let $u_1, \ldots, u_d$ be a basis of t^d defined by

$$\begin{aligned} u_1 &:= (\sqrt{-1}, 0, \ldots, 0), \\ u_2 &:= (0, \sqrt{-1}, 0, \ldots, 0), \\ &\cdots \\ u_d &:= (0, \ldots, 0, \sqrt{-1}). \end{aligned}$$

We denote by $u^1, \ldots, u^d \in (t^d)^*$ the dual basis of $u_1, \ldots, u_d$. Let $(\cdot, \cdot)$ be the metric on t^d satisfying

$$(u_i, u_j) = \delta_{ij} \quad \text{for all } i, j,$$

where δ_{ij} denotes the Kronecker delta. We also denote by $(\cdot, \cdot)$ the meric on $(t^d)^*$ induced from the metric on t^d. The diagonal action of T^d on $\mathbb{C}^d$ induces an action of K which preserves the Kähler structure of $\mathbb{C}^d$. Let $\mu_K : \mathbb{C}^d \to k^*$ be a moment map for the action of K which is defined by

$$\langle \mu_K(z), v \rangle = \frac{1}{2} g_{\mathbb{R}^{2d}}(\sqrt{-1} v z, z) \quad \text{for } v \in k,$$

where we denote by $g_{\mathbb{R}^{2d}}(\cdot, \cdot)$ the standard metric of $\mathbb{C}^d \simeq \mathbb{R}^{2d}$, and by $\langle \cdot, \cdot \rangle$ the natural coupling. The moment map μ_K is also denoted as

$$\mu_K(z) = -\frac{1}{2} \sum_{j=1}^{d} \iota^* u^j |z_j|^2 \quad \text{for } z = (z_1, \ldots, z_d) \in \mathbb{C}^d.$$

Let $T^d_{\mathbb{C}} := (\mathbb{C}^*)^d$ be the complexification of T^d. We define the exponential map $\mathrm{Exp} : t^d \oplus \sqrt{-1} t^d \to T^d_{\mathbb{C}}$ by

$$\mathrm{Exp}(v + \sqrt{-1} v') = (e^{\sqrt{-1}\langle v + \sqrt{-1} v', u^1 \rangle}, \ldots, e^{\sqrt{-1}\langle v + \sqrt{-1} v', u^d \rangle}).$$

We denote by $K_{\mathbb{C}}$ the complexification of K. Let $k_{\mathbb{Z}} \subseteq k$ be $\ker \operatorname{Exp}|_k$ and $(k_{\mathbb{Z}})^*$ the dual. Note that $(k_{\mathbb{Z}})^*$ is naturally identified with $\sum_{j=1}^d \mathbb{Z}\,(\iota^* u^j/2\pi)$. For each $\alpha \in (k_{\mathbb{Z}})^*$, we define a character $\chi_\alpha : K_{\mathbb{C}} \to \mathbb{C}^*$ by

$$\chi_\alpha(\operatorname{Exp}(v + \sqrt{-1}v')) = e^{2\pi\sqrt{-1}\langle v+\sqrt{-1}v', \alpha\rangle}.$$

Let α be an element of $(k_{\mathbb{Z}})^*$. We define an action of $K_{\mathbb{C}}$ on $\mathbb{C}^d \times \mathbb{C}$ by

$$g \cdot (z, v) := (gz, \chi_\alpha(g)^{-1} v) \text{ for } (z, v) \in \mathbb{C}^d \times \mathbb{C}.$$

Let R_α be the invariant ring for the above action:

$$R_\alpha := \{\hat{f}(x, y) \in \mathbb{C}[x_1, \dots, x_d, y] \mid \hat{f}(g \cdot (x, y)) = \hat{f}(x, y) \text{ for all } g \in K_{\mathbb{C}}\}.$$

By a theorem of Nagata, R_α is finitely generated. For each $n \in \mathbb{Z}_{\geq 0}$, let $R_{\alpha,n}$ be a space of polynomials defined by

$$R_{\alpha,n} := \{f(x) \in \mathbb{C}[x_1, \dots, x_d] \mid f(gx) = \chi_\alpha(g)^n f(x) \text{ for all } g \in K_{\mathbb{C}}\}.$$

Then R_α is naturally identified with $\bigoplus_{n\geq 0} R_{\alpha,n}$.

Definition A.1. Define $\mathbb{C}^d /\!\!/_\alpha K_{\mathbb{C}} := \operatorname{Proj}(\bigoplus_{n\geq 0} R_{\alpha,n})$. This is called the *geometric invariant theory (GIT) quotient.*

Definition A.2. We say $z \in \mathbb{C}^d$ is *α-semistable* if there exists an $f(x) \in R_{\alpha,n}$ with $n \in \mathbb{Z}_{>0}$ such that $f(z) \neq 0$. We denote by $(\mathbb{C}^d)^{\alpha-ss}$ the set of all α-semistable points.

The GIT quotient can be described as follows:

Proposition A.3. *There exists a categorical quotient $\phi : (\mathbb{C}^d)^{\alpha-ss} \to \mathbb{C}^d /\!\!/_\alpha K_{\mathbb{C}}$ which satisfies the following properties: For each $z, z' \in (\mathbb{C}^d)^{\alpha-ss}$, $\phi(z) = \phi(z')$ holds if and only if $\overline{K_{\mathbb{C}} \cdot z} \cap \overline{K_{\mathbb{C}} \cdot z'} \cap (\mathbb{C}^d)^{\alpha-ss} \neq \emptyset$ and further for each $q \in \mathbb{C}^d /\!\!/_\alpha K_{\mathbb{C}}$, $\phi^{-1}(q)$ contains a unique $K_{\mathbb{C}}$-orbit which is closed in $(\mathbb{C}^d)^{\alpha-ss}$.*

For the proof of Proposition A.3, we refer the reader to [Dol, MFK, New]. Note that in Proposition A.3, both of the Euclidean topology and the Zariski topology may be used for the orbit topology. This is guaranteed by the following Proposition:

Proposition A.4. *Let $G \subseteq \mathrm{GL}(n, \mathbb{C})$ be an algebraic subgroup of a general linear group $\mathrm{GL}(n, \mathbb{C})$ over $\mathbb{C}$. Then the following holds for all $p \in \mathbb{C}^n$:*

$$\overline{\overline{G \cdot p}} = \overline{G \cdot p},$$

where we denote by $\overline{\overline{G \cdot p}}$ the Euclidean closure, and by $\overline{G \cdot p}$ the Zariski closure. In particular, $G \cdot p$ is closed with respect to the Euclidean topology if and only if it is closed with respect to the Zariski topology.

We refer the reader to [Mum] for the proof of Proposition A.4. An equivalence relation $\sim$ on $(\mathbb{C}^d)^{\alpha-ss}$ is defined as follows:

$$z \sim z' \Longleftrightarrow \overline{K_{\mathbb{C}} \cdot z} \cap \overline{K_{\mathbb{C}} \cdot z'} \cap (\mathbb{C}^d)^{\alpha-ss} \neq \emptyset \quad \text{for } z, z' \in (\mathbb{C}^d)^{\alpha-ss}.$$

Then by Proposition A.3, $\mathbb{C}^d /\!\!/_{\alpha} K_{\mathbb{C}}$ is identified with $(\mathbb{C}^d)^{\alpha-ss}/\sim$. Moreover for each equivalent class there exists a $z \in (\mathbb{C}^d)^{\alpha-ss}$ such that $K_{\mathbb{C}} \cdot z = (\mathbb{C}^d)^{\alpha-ss} \cap \overline{K_{\mathbb{C}} \cdot z}$ and such a z is unique up to a transformation of $K_{\mathbb{C}}$. We can characterize the α-semistable points as follows:

Proposition A.5. *The following are equivalent for each $z \in \mathbb{C}^d$:*

(i) z is α-semistable;

(ii) α satisfies the following:

$$\alpha \in \sum_{j \in J_z} \mathbb{Q}_{\geq 0}(\iota^* u^j / 2\pi),$$

where J_z denotes $\{j \in \{1, \ldots, d\} \mid z_j \neq 0\}$;

(iii) α is in the cone generated by $(\iota^* u^j / 2\pi)_{j \in J_z}$:

$$\alpha \in \sum_{j \in J_z} \mathbb{R}_{\geq 0}(\iota^* u^j / 2\pi);$$

(iv) For each $v \in \mathbb{C} \backslash \{0\}$, $\overline{K_{\mathbb{C}} \cdot (z, v)}$ does not intersect with $\mathbb{C}^d \times \{0\}$.

Proof. We can show that (i) and (ii) are equivalent by the same argument as in the proof of [Kon, Lemma 3.4]. We also see that (ii) and (iii) are equivalent from the general theory of polyhedral convex cones (see [Ful]). We show that (i) implies (iv). Suppose z is α-semistable. We take an $f \in R_{n,\alpha}$ such that

$n \in \mathbb{Z}_{>0}$ and $f(z) \neq 0$. We define a polynomial $\hat{f}(x,y)$ by $\hat{f}(x,y) \coloneqq y^n f(x)$. Then we have the following:

$$\hat{f}(x,y)|_{\overline{K_{\mathbb{C}} \cdot (z,v)}} \equiv v^n f(z),$$
$$\hat{f}(x,y)|_{\mathbb{C}^d \times \{0\}} \equiv 0,$$

and thus (iv) holds. We then show that (iv) implies (i). Suppose (iv) holds. Then there exists a polynomial $\hat{f}(x,y)$ such that

$$\hat{f}(x,y)|_{\overline{K_{\mathbb{C}} \cdot (z,v)}} \equiv 1,$$
$$\hat{f}(x,y)|_{\mathbb{C}^d \times \{0\}} \equiv 0.$$

The polynomial $\hat{f}(x,y)$ can be written as $\hat{f}(x,y) = yf_1(x) + \cdots + y^m f_m(x)$. Take an $n \in \{1, \ldots, m\}$ such that $f_n(x) \neq 0$. Then we have $f_n \in R_{n,\alpha}$ and $f_n(z) \neq 0$. □

The closed orbits are characterized as follows:

Proposition A.6. *The following are equivalent for each $z \in \mathbb{C}^d$:*

(i) z is α-semistable and the $K_{\mathbb{C}}$-orbit is closed in $(\mathbb{C}^d)^{\alpha-ss}$:

$$K_{\mathbb{C}} \cdot z = \overline{K_{\mathbb{C}} \cdot z} \cap (\mathbb{C}^d)^{\alpha-ss};$$

(ii) α satisfies the following:

$$\alpha \in \sum_{j \in J_z} \mathbb{Q}_{>0}(\iota^* u^j / 2\pi);$$

(iii) α is in the interior of the cone generated by $(\iota^* u^j/2\pi)_{j \in J_z}$:

$$\alpha \in \sum_{j \in J_z} \mathbb{R}_{>0}(\iota^* u^j / 2\pi);$$

(iv) The following holds:

$$\sum_{j \in J_z} \mathbb{R}(\iota^* u^j / 2\pi) = \sum_{j \in J_z} \mathbb{R}_{\geq 0}(\iota^* u^j / 2\pi) + \mathbb{R}_{\geq 0}(-\alpha);$$

(v) For each $v \in \mathbb{C} \backslash \{0\}$, $K_{\mathbb{C}} \cdot (z,v)$ is closed;

(vi) The following holds:

$$\mu_K^{-1}(-\alpha) \cap K_{\mathbb{C}} \cdot z \neq \emptyset.$$

Proof. We see that (ii), (iii) and (iv) are equivalent from the general theory of polyhedral convex cones (see [Ful]). We also see that (iv) and (v) are equivalent from the argument of [Nak, pp.30-31]. We show that (i) implies (v). Suppose (i) holds. By the general theory of algebraic groups, there uniquely exists a closed orbit which is contained in $\overline{K_{\mathbb{C}} \cdot (z, v)}$. Let $K_{\mathbb{C}} \cdot (z', v)$ be such a closed orbit. Then by Proposition A.5, $z' \in (\mathbb{C}^d)^{\alpha-ss}$. We take a sequence $(g_i)_{i \in \mathbb{N}}$ such that

$$(z', v) = \lim_{i \to \infty} g_i \cdot (z, v).$$

Therefore we have $z' = \lim_{i \to \infty} g_i \cdot z$, and thus we see $z' \in \overline{K_{\mathbb{C}} \cdot z} \cap (\mathbb{C}^d)^{\alpha-ss}$. Then (v) holds. We then show that (v) implies (i). Suppose (v) holds. Let $z' \in \overline{K_{\mathbb{C}} \cdot z} \backslash K_{\mathbb{C}} \cdot z$. We take a sequence $(g_i)_{i \in \mathbb{N}}$ so that $z' = \lim_{i \to \infty} g_i \cdot z$. Since $K_{\mathbb{C}} \cdot (z, 1)$ is closed, we have $\lim_{i \to \infty} |\chi_\alpha(g_i)^{-1}| = \infty$. This implies that $\lim_{i \to \infty}(g_i^{-1} z', \chi_\alpha(g_i)) \in \mathbb{C}^d \times \{0\}$ and thus we have $z' \notin (\mathbb{C}^d)^{\alpha-ss}$. We shall prove (iii) and (vi) are equivalent in Proposition A.7. □

The equivalence of (ii) and (iii) of Proposition A.6 holds for any $\lambda \in k^*$:

Proposition A.7. *Let $\lambda \in k^*$ and $z \in \mathbb{C}^d$. We define a functional $l_{\lambda,z} : k \to \mathbb{R}$ by*

$$l_{\lambda,z}(v) := \frac{1}{4} \sum_{j=1}^{d} |z_j|^2 e^{2\langle u^j, v \rangle} - \langle \lambda, v \rangle.$$

Then the following are equivalent:

(i) λ is in the interior of the cone generated by $(\iota^* u^j / 2\pi)_{j \in J_z}$:

$$\lambda \in \sum_{j \in J_z} \mathbb{R}_{>0} \iota^* u^j;$$

(ii) The following holds:

$$\mu_K^{-1}(-\lambda) \cap K_{\mathbb{C}} \cdot z \neq \emptyset;$$

(iii) $l_{\lambda,z}$ *attains a minimum.*

Moreover if v *and* v' *be minimizers of* $l_{\lambda,z}$, *then* $v - v'$ *is in the orthogonal complement of* $\sum_{j\in J_z} \mathbb{R}\iota^* u^j$.

Proof. We assume that $(\iota^* u^j)_{j\in J_z}$ generates k^* for simplicity. Then a direct computation shows that $l_{\lambda,z}$ is strictly convex. We also see that for each $v \in k$, v is a critical point of $l_{\lambda,z}$ if and only if the following holds.

$$\lambda = \frac{1}{2}\sum_{j=1}^{d} e^{2\langle u^j, v\rangle}|z_j|^2 \iota^* u^j.$$

Therefore (ii) and (iii) are equivalent. Clearly, (ii) implies (i). Assume that (i) holds. We show that (iii) holds. From the assumption, there exists a positive numbers $(s_j)_{j\in J_z}$ such that $\lambda = \sum_{j\in J_z} s_j \iota^* u^j$. Then the functional $l_{\lambda,z}$ is denoted as

$$l_{\lambda,z}(v) = \sum_{j\in J_z} (|z_j|^2 e^{2\langle u^j, v\rangle} - s_j\langle u^j, v\rangle).$$

This implies that $\lim_{t\to\infty} l_{\lambda,z}(tv) = \infty$ for each $v \neq 0$ and thus the functional $l_{\lambda,z}$ attains a minimum. □

From Proposition A.3, Proposition A.6 and Proposition A.7, we have the following:

Corollary A.8. *The following map is bijective:*

$$\mu_K^{-1}(-\alpha)/K \longrightarrow (\mathbb{C}^d)^{\alpha-ss}/\sim .$$

B Relationship between condition (4.3) and the stability condition of Higgs bundles

In this section, we explain the relationship between condition (4.3) and the stability condition of Higgs bundles. Let $(E, \Phi) \to (X, \omega_X)$ be a Higgs bundle over a compact connected Kähler manifold (X, ω_X). Suppose that the holomorphic vector bundle E decomposes as $E = L_1 \oplus \cdots \oplus L_r$ with holomorphic line bundles $L_1, \ldots, L_r \to X$. Then the Higgs field Φ decomposes as

$\Phi = \Phi_0 + \sum_{i,j=1,\ldots,r} \Phi_{i,j}$, where $\Phi_{i,j}$ is a holomorphic $(1,0)$-form with values in $L_j^{-1} L_i$ and Φ_0 is the diagonal part. For each $j = 1, \ldots, r$, let m_j be the degree of the holomorphic line bundle L_j with respect to the Kähler form ω_X. Suppose that the Kähler form ω_X represents a rational cohomology of X. Then all of the numbers $m_1, \ldots, m_r$ are rational numbers. We assume that $\deg(E) = m_1 + \cdots + m_r = 0$ for simplicity. We define a vector $\gamma \in \mathbb{Q}^r$ as $\gamma \coloneqq (m_1, \ldots, m_r)$. We also define $\alpha_{i,j} \in (\mathbb{Q}^r)^*$ and $h_{\alpha_{i,j}} \in \mathbb{Q}^r$ $(i, j = 1, \ldots, r)$ as follows:

$$\begin{aligned} &\alpha_{i,j}(v) \coloneqq v_i - v_j \text{ for } v = (v_1, \ldots, v_r) \in \mathbb{Q}^r, \\ &h_{\alpha_{i,j}} \coloneqq e_i - e_j, \end{aligned}$$

where we denote by $e_1, \ldots, e_r$ the canonical basis of $\mathbb{R}^r$. Then the following holds:

Proposition B.1. *The following (i) and (ii) holds:*

(i) Suppose that (E, Φ) is semistable. Then the vector γ satisfies the following:

$$-\gamma \in \sum_{\substack{i,j=1,\ldots,r, \\ \Phi_{i,j} \neq 0}} \mathbb{Q}_{\geq 0} h_{\alpha_{i,j}}. \tag{B.1}$$

(ii) Suppose that (E, Φ) is stable. Then the vector γ satisfies the following:

$$-\gamma \in \sum_{\substack{i,j=1,\ldots,r, \\ \Phi_{i,j} \neq 0}} \mathbb{Q}_{> 0} h_{\alpha_{i,j}}. \tag{B.2}$$

Remark B.2. There are many ways to decompose the vector bundle E into a direct sum of holomorphic line bundles. For example, suppose that the rank of E is 2 and E decomposes as $E = L_1 \oplus L_2$ and there exists a non-trivial holomorphic bundle map $f : L_1 \to L_2$. Then $E = L_1' \oplus L_2$ is a new decomposition, where we defined a holomorphic line bundle L_1' as $L_1' \coloneqq \{(v_1, v_2) \in L_1 \oplus L_2 \mid v_2 = f(v_1)\}$. Theorem B.1 holds for any decomposition of E.

Before starting the proof of Proposition B.1, we make some preparations. Let T be the maximal torus of $\mathrm{SU}(r)$ which consists of all diagonal matrix of $\mathrm{SU}(r)$. We denote by $\mathfrak{t}$ the Lie algebra of T. Let $H \subseteq \mathrm{SL}(r, \mathbb{C})$ be the complexification of T with Lie algebra $\mathfrak{h} = \mathfrak{t} \oplus \sqrt{-1}\mathfrak{t}$. We define a lattice $\mathfrak{h}_{\mathbb{Z}}$ of $\sqrt{-1}\mathfrak{t}$ as $\mathfrak{h}_{\mathbb{Z}} := \{\mathrm{diag}(n_1, \dots, n_r) \mid n_1, \dots, n_r \in \mathbb{Z}, n_1 + \cdots + n_r = 0\}$. Note that $\mathfrak{h}_{\mathbb{Z}}$ coincides with the kernel of the exponential map $\mathrm{Exp} : \mathfrak{h} \to H$. We regard the vector γ and the vectors $h_{\alpha_{i,j}}$ $(i, j = 1, \dots, r)$ as elements of $\mathfrak{h}_{\mathbb{Q}} := \mathfrak{h}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ by identifying the canonical basis $e_1, \dots, e_r$ with $\mathrm{diag}(1, \dots, 0), \dots, \mathrm{diag}(0, \dots, 1)$. Let $B(\cdot, \cdot)$ be the Killing form defined as $B(u, v) := 2r\mathrm{Tr}(uv)$ for $u, v \in \mathfrak{h}$. We denote by $\gamma^* \in \mathfrak{h}_{\mathbb{Q}}^*$ be the dual of the vector γ with respect to the Killing form B. We take a positive integer n so that $n\gamma^* \in \mathfrak{h}_{\mathbb{Z}}^*$. Then we define an element $\gamma^\vee \in \mathfrak{h}_{\mathbb{Z}}^*$ as $\gamma^\vee := -n\gamma^*$. Also, we define a character $\chi_{\gamma^\vee} : H \to \mathbb{C}^*$ as $\chi_{\gamma^\vee}(\mathrm{Exp}(v)) := e^{\langle v, \gamma^\vee \rangle}$ for $v \in \mathfrak{h}$, where we denote by $\langle \cdot, \cdot \rangle$ the coupling between $\mathfrak{h}$ and $\mathfrak{h}^*$. We define an action of the algebraic torus H on $H^0(\mathrm{End}E \otimes \bigwedge^{1,0})$ as follows:

$$g \cdot (\theta, z) := (\theta_0 + \sum_{i,j=1,\dots,r} g_j^{-1} g_i \theta_{i,j}, \chi_{\gamma^\vee}^{-1}(g) z) \text{ for } (\theta, z) \in H^0(\mathrm{End}E \otimes \bigwedge^{1,0}) \times \mathbb{C},$$

where $g = \mathrm{diag}(g_1, \dots, g_r)$ is an element of H and we denote by θ_0 and $\theta_{i,j}$ $(i, j = 1, \dots, r)$ the diagonal component and the (i, j)-component of θ, respectively. Then we start the proof of Proposition B.1.

Proof of Proposition B.1. We first show that (i) holds. From Proposition A.5, (B.1) holds if and only if for each $z \neq 0$, the closure of $H \cdot (\Phi, z)$ does not intersects $\mathbb{C} \times \{0\}$. Furthermore, this is equivalent to that $\gamma^\vee(s) \geq 0$ holds for any $s \in \sqrt{-1}\mathfrak{t}$ such that

$$\Phi \in H^0(\bigoplus_{(i,j) \in A_s} L_j^{-1} L_i \otimes \bigwedge^{1,0}), \tag{B.3}$$

where we defined A_s as $A_s := \{(i, j) \mid i, j \in \{1, \dots, r\}, \alpha_{i,j}(s) \geq 0\}$. Suppose that (E, Φ) is semistable. We show that $\gamma^\vee(s) \geq 0$ holds for any $s \in \sqrt{-1}\mathfrak{t}$ satisfying (B.3). Let $s = \mathrm{diag}(s_1, \dots, s_r) \in \sqrt{-1}\mathfrak{t}$ such that (B.3) holds. We may assume that $s_1 \geq \cdots \geq s_r$. From (B.3), the Higgs field Φ preserves the following sequence of subbundles:

$$0 \subsetneq E_{j_1} \subsetneq E_{j_2} \subseteq \cdots \subsetneq E_{j_k} \subsetneq E,$$

where we defined $E_j \subseteq E$ as $E_j := L_1 \oplus \cdots \oplus L_j$ and $1 \leq j_1 < \cdots < j_k < r$ are all the elements of $J_s := \{j \in \{1, \dots, r-1\} \mid s_j > s_{j+1}\}$. Then we see that $\gamma^\vee(s) \geq 0$ since we have

$$\begin{aligned}\gamma^\vee(s) &= -2rn\mathrm{Tr}(\gamma s) \\ &= -2rn(m_1 s_1 + \cdots + m_r s_r) \\ &= -2rn(m_1(s_1 - s_2) + (m_1 + m_2)(s_2 - s_3) + \cdots \\ &\quad + (m_1 + \cdots + m_{r-1})(s_{r-1} - s_r)) \\ &= -2rn\sum_{j \in J_s}(s_j - s_{j+1})\deg(E_j) \end{aligned} \tag{B.4}$$

and (E, Φ) is semistable. Therefore (i) holds. We next show that (ii) holds. The proof of (ii) is similar. From Proposition A.6, we see that (B.2) holds if and only if $H \cdot (\Phi, z)$ is closed for each $z \neq 0$. Furthermore, this is equivalent to that the following holds for all $s \in \sqrt{-1}\mathfrak{t}$ such that (B.3) holds:

- The element s of $\sqrt{-1}\mathfrak{t}$ satisfies $\gamma^\vee(s) \geq 0$ and if $\gamma^\vee(s) = 0$, then Φ lies in

  $$H^0(\bigoplus_{(i,j) \in A_s^0} L_j^{-1} L_i \otimes \bigwedge^{1,0}),$$

  where A_s^0 is defined as $A_s^0 := \{(i,j) \mid i, j \in \{1, \ldots, r\}, \alpha_{i,j}(s) = 0\}$.

Suppose that (E, Φ) is stable. Then from (B.4), we see that $\gamma^\vee(s) > 0$ any $s \in \sqrt{-1}\mathfrak{t}$ such that (B.3) holds, and thus we have (B.2). This gives the result. $\square$

Remark B.3. It is known that the stability condition of G-Higgs bundles is formulated by using the parabolic subgroups of the complex reductive Lie group G and their characters (see [GGM]). This definition is based on the work of Ramanathan [Ram] which generalized the theorem of Narasimhan and Seshadri to the case of general principal G-bundles. In the above proof of Proposition B.1, the definition of the stability condition of Higgs bundles formulated in the language of parabolic subgroups and parabolic subalgebras is used.

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